Perspectives on Coupled Oscillators: Geometry, Analysis and Computation

Francesco Bullo

Department of Mechanical Engineering
Center for Control, Dynamical Systems & Computation
University of California at Santa Barbara
http://motion.me.ucsb.edu

Synchronization in Natural and Engineering Systems: Open Problems in Modeling, Analysis, and Control

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Acknowledgments

Saber Jafarpour
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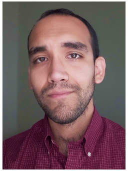

Pedro Cisneros-Velarde
UIUC

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Outline

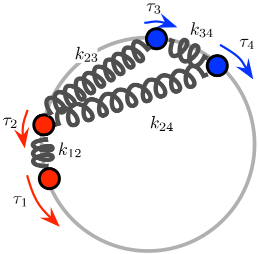
1. Recent progress
   - Elastic and flow networks on the torus
   - Cutset spaces
   - Geometric graph theory on the n-torus
   - Convexity, monotonicity, and contraction theory
   - Multistability in phase-coupled oscillators
   - Sync threshold: Approximate inverse via series methods
   - Sync threshold: gap between necessary and sufficient conditions
   - State-space oscillators
2. Open Problems

#1: Elastic and flow networks on the torus

$$\omega_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

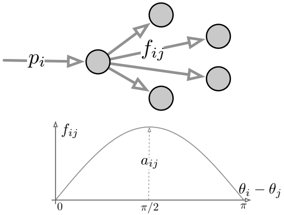
Spring network

- $\omega_i = \tau_i$: torque at i
- $a_{ij} = k_{ij}$: spring stiffness i, j
- $\sin(\theta_i - \theta_j)$: modulation
- elastic energy $\mathcal{E} = \sum_{ij}(1 - \cos(\theta_i - \theta_j))$

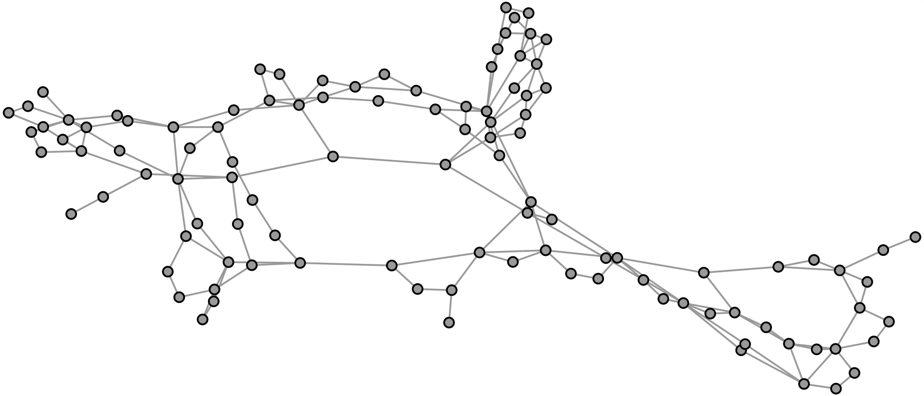


Power network

- $\omega_i = p_i$: injected power
- a_{ij} : max power flow i, j
- $\sin(\theta_i - \theta_j)$: modulation
- KCL flow conservation and Ohm's flow law



#2: Cutset spaces



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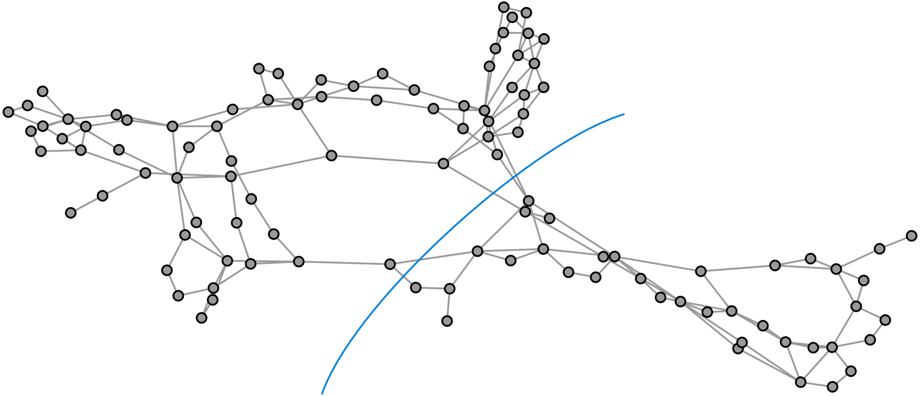
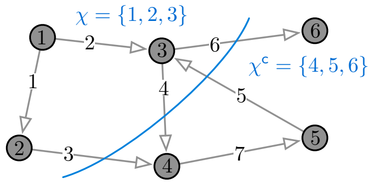
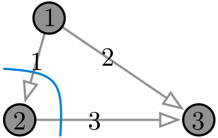
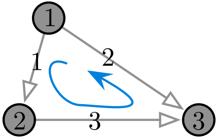




$$v_\chi = \begin{bmatrix} 0 & 0 & +1 & +1 & -1 & +1 & 0 \end{bmatrix}^\top$$





$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\mathrm{Im}(B^\top)}_{\substack{\text{cutset space} \\ \text{flow vectors}}} \oplus \underbrace{\mathrm{Ker}(B\mathcal{A})}_{\substack{\text{weighted cycle space} \\ \text{cycle vectors}}}$$

$\mathcal{P} = B^\top L^\dagger B \mathcal{A}$ = cutset projection operator — onto $\mathrm{Im}(B^\top)$ parallel to $\mathrm{Ker}(B\mathcal{A})$

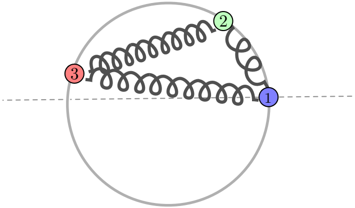
1. if G unweighted, then $\mathcal{P}$ is orthogonal and $\|\mathcal{P}\|_2 = 1$
2. if G acyclic, then $\mathcal{P} = I_m$ and $\|\mathcal{P}\|_p = 1$
3. if G uniform complete or ring, then $\|\mathcal{P}\|_\infty = 2(n-1)/n \leq 2$
4. if θ is the minimal angle between the cutset space and the cycle space of G, then $\sin(\theta) = \|\mathcal{P}\|_2^{-1}$
5. if $R_{\text{eff}} \in \mathbb{R}^{n \times n}$ are effective resistances, then $\mathcal{P} = -\frac{1}{2} B^\top R_{\text{eff}} B \mathcal{A}$
6. ...

#3: Winding numbers and partitions

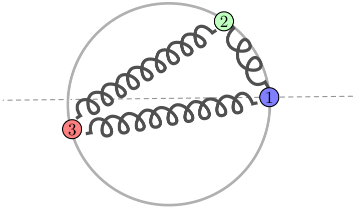
Given a cycle $\sigma = (1, \dots, n_\sigma)$ and orientation

1. **winding number of $\theta \in \mathbb{T}^n$ along σ**

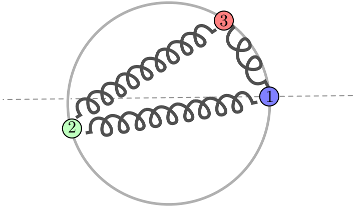
   $=$ number of times the **shortest-arc path wraps around torus**



$w = 0$



$w = +1$



$w = -1$

2. given basis $\sigma_1, \dots, \sigma_r$ for cycles, **winding vector of θ** is

$$w(\theta) = (w_{\sigma_1}(\theta), \dots, w_{\sigma_r}(\theta))$$

Theorem: Kirchhoff angle law on $\mathbb{T}^n$

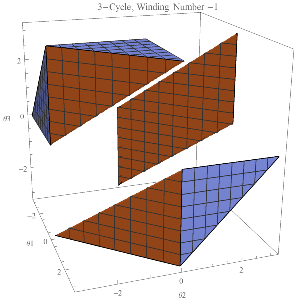
winding number is at most $\pm \lfloor n_\sigma / 2 \rfloor - 1$

Theorem: Winding partition For each possible winding vector u, define

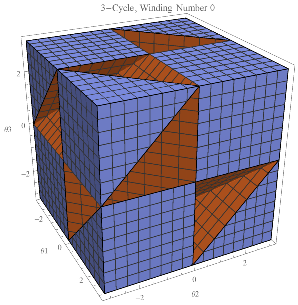
$$\mathsf{WindingCell}(u) := \{\theta \in \mathbb{T}^n \mid w(\theta) = u\}$$

Then

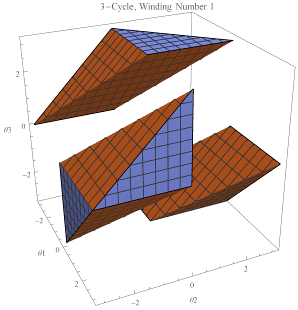
$$\mathbb{T}^n = \cup_u \mathsf{WindingCell}(u)$$



$w = -1$



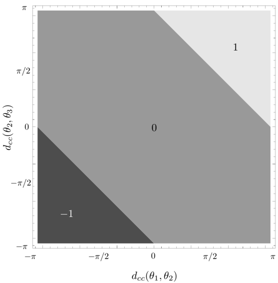
$w = 0$

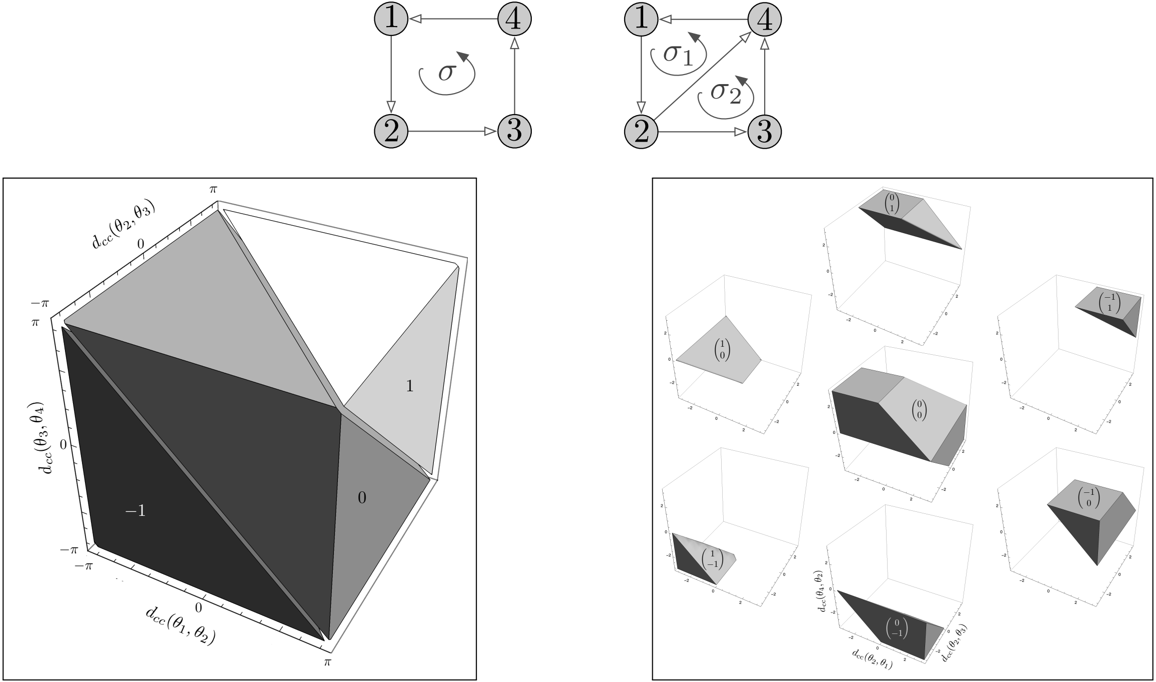


$w = +1$

Theorem: Reduced cell is convex polytope

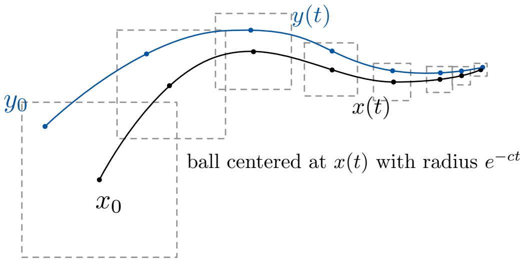
- each winding cell is connected and invariant under rotation
- **bijection:**
  reduced winding cell $\longleftrightarrow$ open convex polytope





1
2
3
4
σ
σ1
σ2
π
dcc(θ2, θ3)
0
−π
π
dcc(θ3, θ4)
0
−π
−π
dcc(θ1, θ2)
0
π
1
0
−1
(0 1)
(1 0)
(−1 1)
(0 0)
(1 −1)
(−1 0)
(0 −1)
dcc(θ4, θ2)
dcc(θ2, θ1)
dcc(θ2, θ3)

#4: Analysis: Convexity, monotonicity, and contraction theory¡

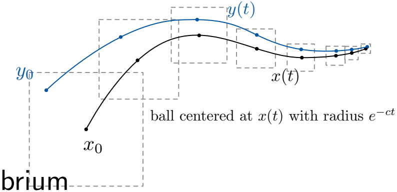


1. V is **strongly convex** with parameter m
2. $-\operatorname{grad} V$ is m-**strongly contracting**, that is

$$\big(-\operatorname{grad} V(x) + \operatorname{grad} V(y) \big)^\top (x-y) \leq -m\|x-y\|_2^2$$

1. F is a **monotone operator** (or a **coercive operator**) with parameter m,
2. $-$F is m-**strongly contracting**

search for contraction properties
design engineering systems to be contracting



Highly ordered **transient** and **asymptotic** behavior:

1. time-invariant F: unique globally exponential stable equilibrium
   two natural Lyapunov functions
2. periodic F: contracting system entrain to periodic inputs
3. accurate numerical integration and equilibrium computation
4. contractivity rate is natural measure/indicator of robust stability
   - input-to-state stability
   - finite input-state gain
   - contraction margin wrt unmodeled dynamics
   - input-to-state stability under delayed dynamics

#5: Multistable Sync = global partition + local contraction

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

in each winding cell

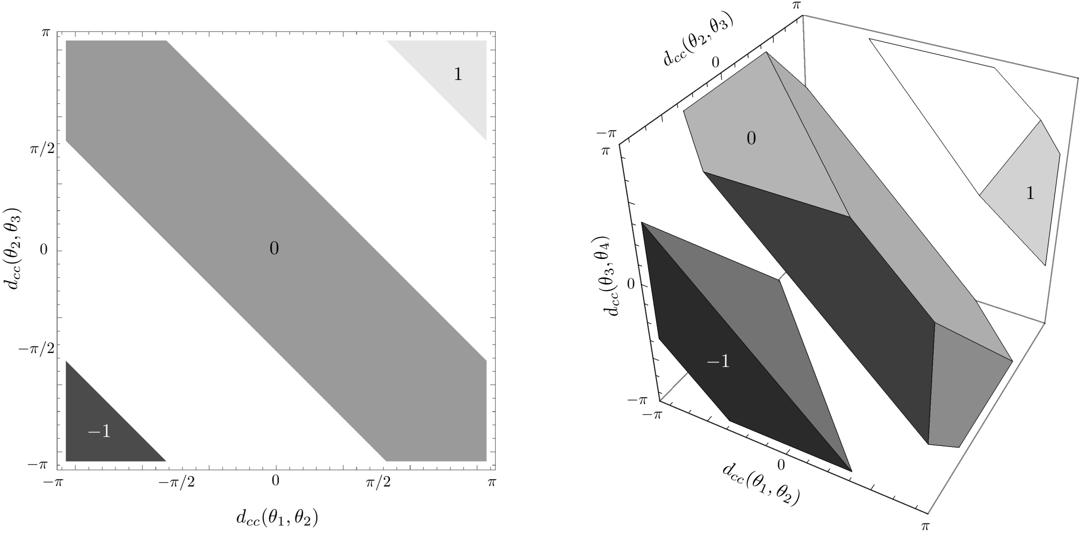
1. $\dot{\theta} = -\operatorname{grad} \mathcal{E}(\theta)$, where

$$\mathcal{E}(\theta) = \sum\nolimits_{ij} (1 - \cos(\theta_i - \theta_j)) + \omega^\top \theta$$

2. Hessian $\mathcal{E}(\theta) = -$Cosine-Laplacian$(\theta) \preceq 0$
3. Hessian $\mathcal{E}(\theta) \preceq 0$ on the **cohesive subset** $|\theta_i - \theta_j| \leq \pi/2$
4. modulo the symmetry, the dynamics is strongly contracting

Theorem:

1. each winding cell has at most one cohesive equilibrium
2. contraction algorithm to decide/compute in each winding cell



π
π/2
0
−π/2
−π
−π
−π/2
0
π/2
π
$d_{cc}(\theta_1, \theta_2)$
$d_{cc}(\theta_2, \theta_3)$
1
0
−1
$d_{cc}(\theta_2, \theta_3)$
$d_{cc}(\theta_3, \theta_4)$
$d_{cc}(\theta_1, \theta_2)$

(a) (b)

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j + \phi_{ij})$$

same properties, by robustness of contracting dynamics

#6: Sync threshold: Approximate inverse via series methods

Projection onto to cutset space: $z = B^\top L^\dagger \omega$ and $x = B^\top \theta$
synchrony equilibrium equation is

$$z = \mathcal{P} \sin(x)$$

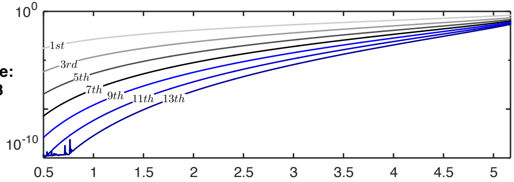
Given input z, unique solution is

$$x = \sum_{i=0}^{\infty} A_{2i+1}(z), \qquad A_1(z) = z \qquad = B^\top L^\dagger \omega$$

$$A_3(z) = \mathcal{P}\left(\frac{1}{3!} z^{\circ 3}\right)$$

$$A_5(z) = \mathcal{P}\left(\frac{3}{3!} A_3(z) \circ z^{\circ 2} - \frac{1}{5!} z^{\circ 5}\right) \qquad \dots$$

Test case: IEEE 118



#7: Sync threshold: gap between necessary and sufficient conditions

$$z = \mathcal{P}\sin(x)$$

given a norm, define

$$\alpha(\mathcal{P}) := \text{min amplification factor of } \big(\mathcal{P}\,\mathrm{diag}[\mathrm{sinc}(x)]\big) \quad < \quad \|\mathcal{P}\|$$

Theorem: Sufficient Cohesive equilibrium angles exist if, in some norm,

$$\|B^\top L^\dagger \omega\| \leq \alpha(\mathcal{P})$$

Necessary Equilibrium angles do not exist if, in some norm

$$\|\mathcal{P}\| \leq \|B^\top L^\dagger \omega\|$$

Considering only first order term in expansion $\iff$ $\alpha_\infty(\mathcal{P}) \approx 1$ (PNAS '13)

Test Case	State of the Art Empirical Results on IEEE Test Cases ratio of test prediction to numerical computation			
	$\|\cdot\|_2$	$\|\cdot\|_\infty$	$\alpha_\infty(\mathcal{P}) \approx 1$ approximate	numerical α_∞ (fmincon)
IEEE 9	16.5 %	73.7 %	92.1 %	85.1 %†
IEEE 14	8.3 %	59.4 %	83.1 %	81.3 %†
IEEE RTS 24	3.9 %	53.4 %	89.5 %	89.5 %†
IEEE 30	2.7 %	55.7 %	85.5 %	85.5 %†
IEEE 118	0.3 %	43.7 %	85.9 %	—*
IEEE 300	0.2 %	40.3 %	99.8 %	—*
Polish 2383	0.1 %	29.1 %	82.8 %	—*

† *fmincon* with 100 randomized initial conditions

* *fmincon* does not converge

#8: State-space oscillators

Coupled networks of:

1. Stuart-Landau oscillator
2. FitzHugh–Nagumo neurons
3. Rössler chaotic oscillators
4. Lienard oscillators (Van Der Pol)
5. Biological Goodwin models
6. . . .

semi-contraction theory

$$\dot{x}_i = f(t, x_i) - \sum_{j=1}^{n} a_{ij}(x_i - x_j), \quad i \in \{1, \dots, n\}$$

synchronization as function of

1. growth rate of the internal dynamics
2. strength of the diffusive coupling
3. heterogeneity of oscillators

Theorem: semi-contraction sufficient condition

If in some norm

$$\text{osLip}(f) < \lambda_2(L)$$

then

1. semi-contraction rate $\lambda_2(L) - \text{osLip}(f)$,
2. synchronization $\lim_{t\to\infty} \|x_i - x_j\| = 0$ for every i, j

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1. Recent progress
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2. Open Problems

Our recent work

1. S. Jafarpour, E. Y. Huang, and F. Bullo. Synchronization of Kuramoto oscillators: Inverse Taylor expansions.
   SIAM Journal on Control and Optimization, 57(5):3388–3412, 2019.
   doi:10.1137/18M1216262

2. S. Jafarpour and F. Bullo. Synchronization of Kuramoto oscillators via cutset projections.
   IEEE Transactions on Automatic Control, 64(7):2830–2844, 2019.
   doi:10.1109/TAC.2018.2876786

3. S. Jafarpour, E. Y. Huang, K. D. Smith, and F. Bullo. Flow and elastic networks on the n-torus: Geometry, analysis and computation.
   SIAM Review, 64(1):59–104, 2022.
   doi:10.1137/18M1242056

4. S. Jafarpour, P. Cisneros-Velarde, and F. Bullo. Weak and semi-contraction for network systems and diffusively-coupled oscillators.
   IEEE Transactions on Automatic Control, 67(3):1285–1300, 2022.
   doi:10.1109/TAC.2021.3073096

5. R. Delabays, S. Jafarpour, and F. Bullo. Multistability and paradoxes in lossy oscillator networks.
   Submitted, February 2022.
   URL: **https://arxiv.org/pdf/2202.02439.pdf**

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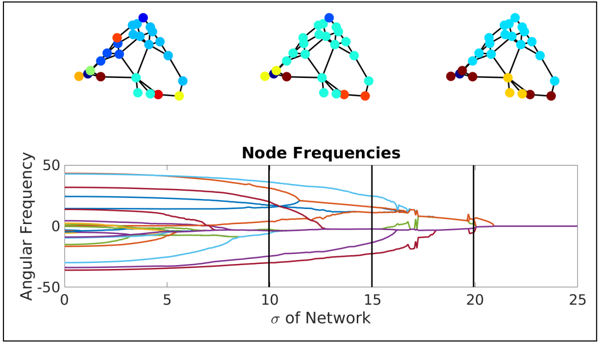
Future research

1. Fundamental theory of phased-coupled oscillators
2. Fundamental theory of state-space-coupled oscillators
3. Applications in energy systems
4. Applications in machine learning and scientific computing

Fundamental theory of phased-coupled oscillators

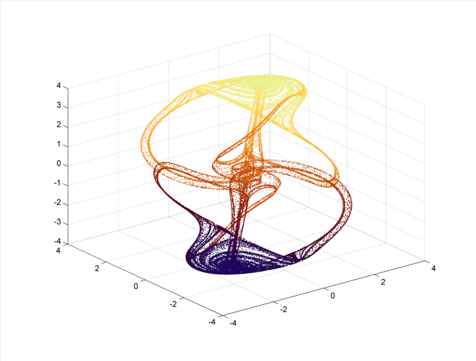
1. outside cohesive set: signed graphs, symbolic dynamics, ...
2. non-monotone phase couplings and and higher-order dynamics
3. analysis and computation of cluster sync and bifurcation diagram



B. Gilg. *Critical Coupling and Synchronized Clusters in Arbitrary Networks of Kuramoto Oscillators.*
PhD thesis, Arizona State University, 2018

Fundamental theory of state-space-coupled oscillators

1. sharpest sync conditions for benchmarks
2. transverse contraction
3. fractal attractors via α-contraction theory



C. Wu, R. Pines, M. Margaliot, and J.-J. E. Slotine.
Generalization of the multiplicative and additive compounds of square matrices and contraction in the Hausdorff dimension.
IEEE Transactions on Automatic Control, 2022.
doi:10.1109/TAC.2022.3162547

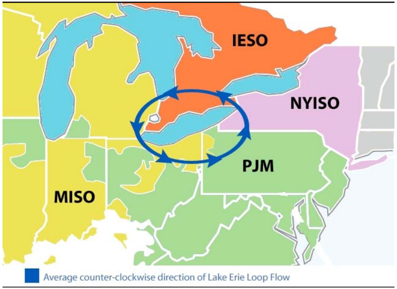
Applications in energy systems

1. understanding multi-stability in power flows
2. thick torus conjecture for active/reactive power flow and for OPF
3. paradoxes in lossy networks

Practical observations:

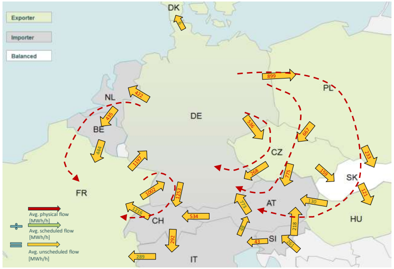
sometimes undesirable power flows around loops

sometimes sizable difference between predicted and actual power flows



New York Independent System Operator, **Lake Erie Loop Flow Mitigation**, Technical Report, 2008

Figure 8: Average unscheduled flows for the years 2011 and 2012, MWh/h[8]



Source: THEMA Consulting Group, based on data from 16 TSOs

THEMA Consulting Group, **Loop-flows - Final advice**, Technical Report prepared for the European Commission, 2013

Applications in machine learning and scientific computing

1. oscillator-based computing
2. nanotech allows contruction of massively-parallel analog fast low-power devices
   CMOS, spin torque nano-oscillators (spintronics), MEMS resonators, optomechanical crystal cavities, ...
3. Example applications:
   1. NP-complete computing
   2. associative memory
   3. reservoir computing

- J. Von Neumann. Non-linear capacitance or inductance switching, amplifying, and memory organs, December 1957. US Patent 2,815,488
- M. H. Matheny et al. Exotic states in a simple network of nanoelectromechanical oscillators. *Science*, 363(6431), 2019. doi:10.1126/science.aav7932
- G. Csaba and W. Porod. Coupled oscillators for computing: A review and perspective. *Applied Physics Reviews*, 7(1):011302, 2020. doi:10.1063/1.5120412

Conclusions

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