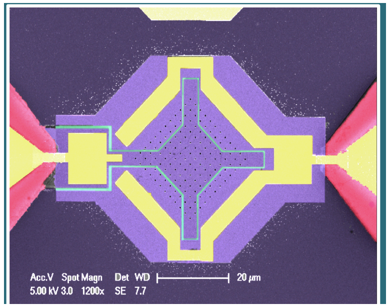
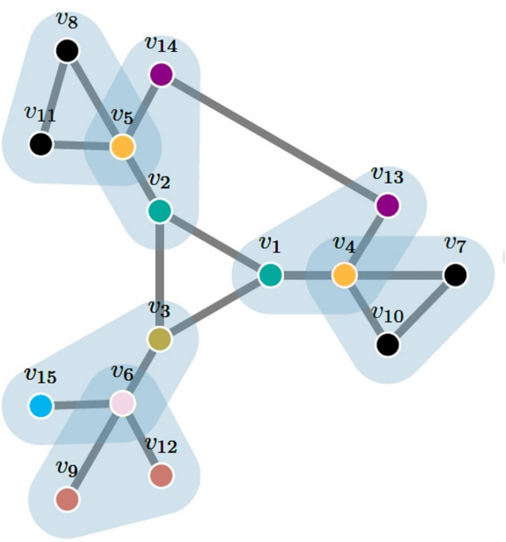
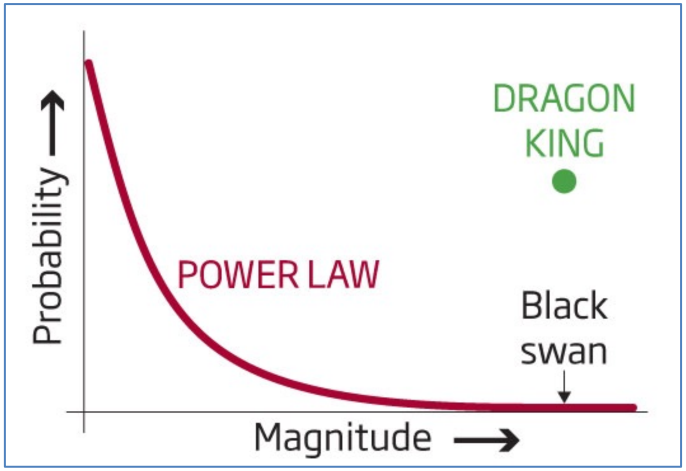
Decoupled states, hypergraphs, and Dragon Kings





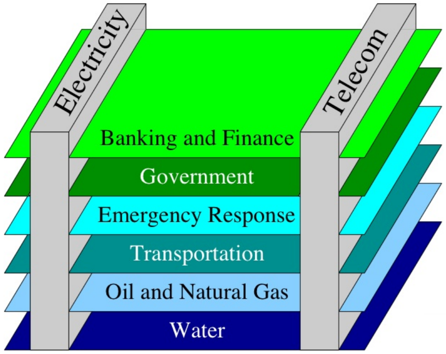


Raissa M. D'Souza
University of California, Davis

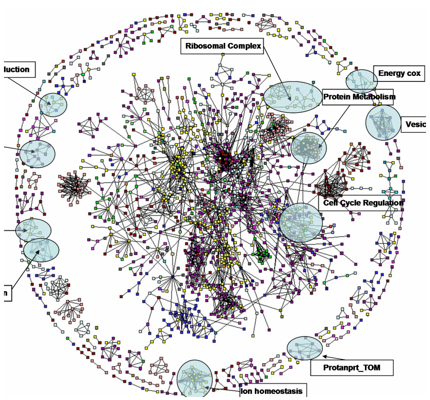
COMPLEXITY SCIENCE HUB VIENNA

SANTA FE INSTITUTE

Structure and function of interdependent networks

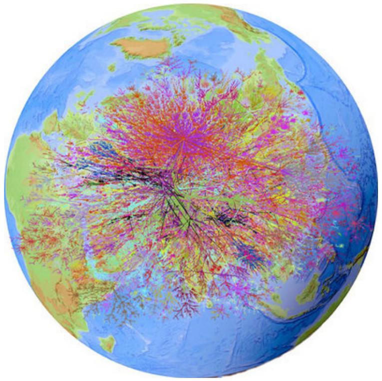


Critical Infrastructure



Biological & Ecological networks

Social networks: Economics & Epidemics

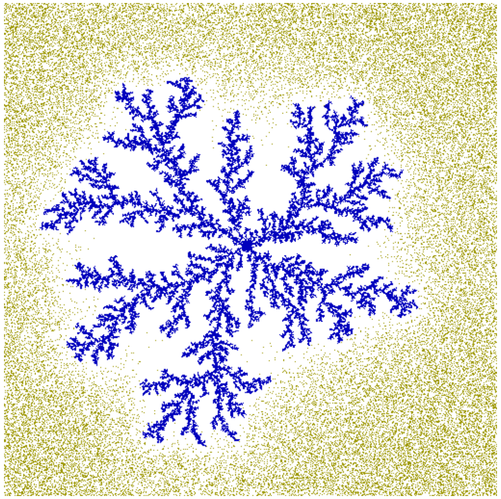


Information and Communication technology

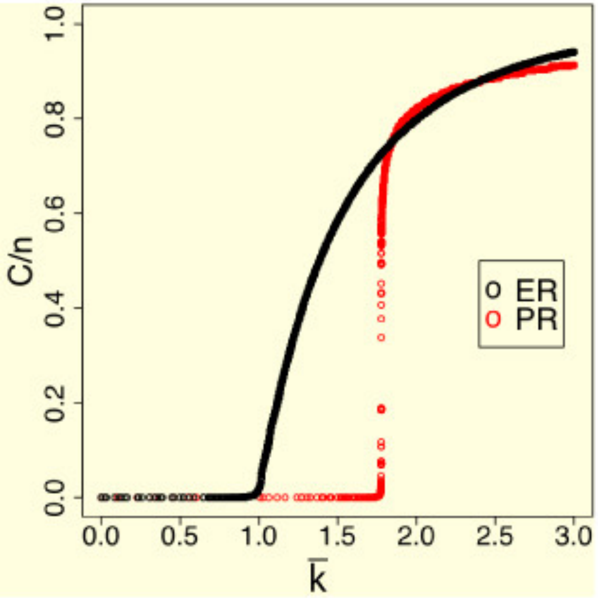
Each network is a complex system with emergent behaviors

What are emergent collective behaviors?

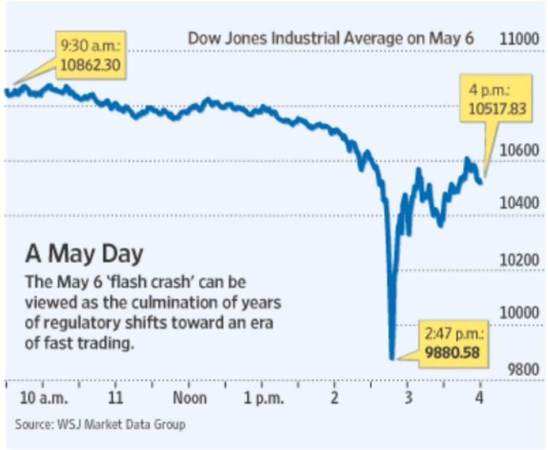
Behaviors not predicted a priori from the constituent equations of motion.



Synchronization and pattern formation.



Phase transitions
"**Tipping points**"



Cascading failures.

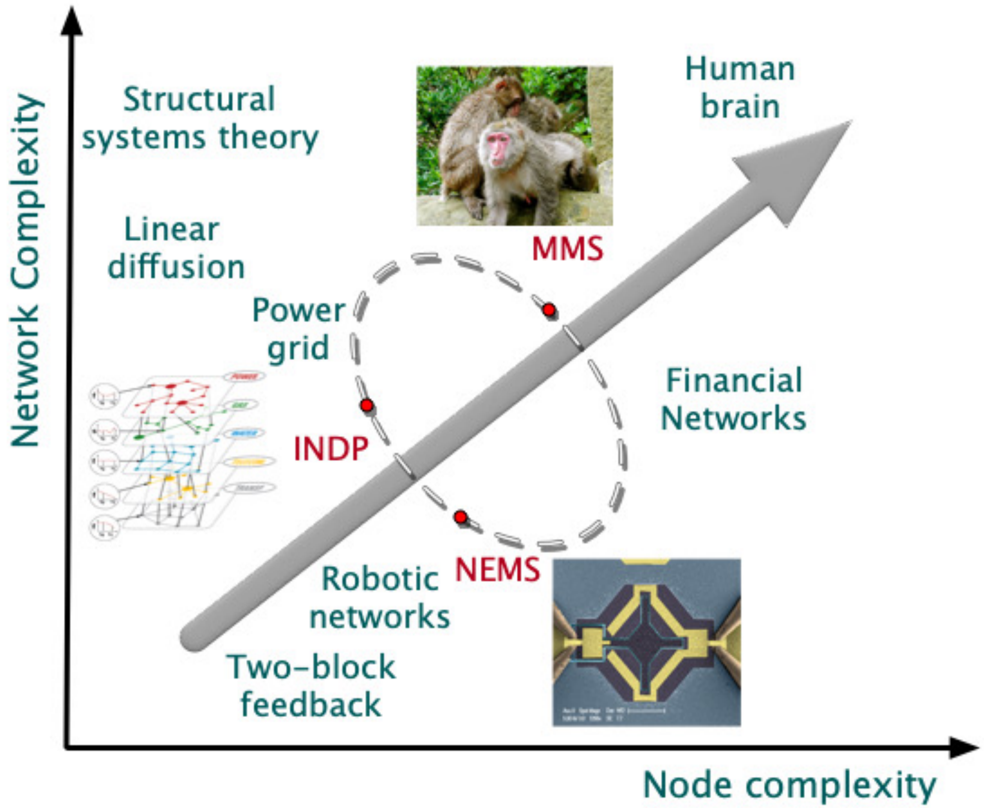
Where is the complexity?

Statistical Physics

In the network structure?

In the nodal dynamics?

In both!



Control theory and non-linear dynamics

Today’s topics:

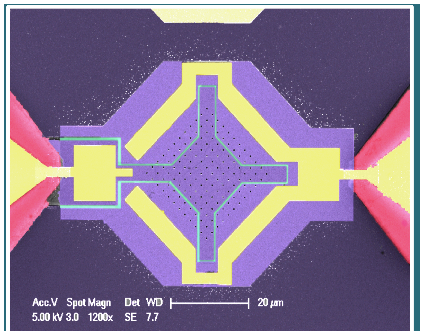
- **Decoupled synchronized states**
  - Interplay of nodal dynamics and network structure

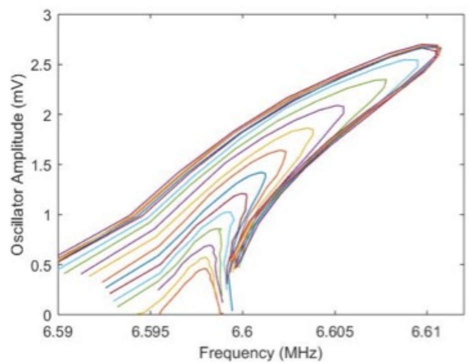
- **Cluster synchronization on hypergraphs**
  - Systems with higher-order coupling, beyond dyadic
  - The projection onto dyadic matrices is sufficient for analyzing full synchronization, but not cluster synchronization.
  - Formulation in terms of node- and edge-clusters.

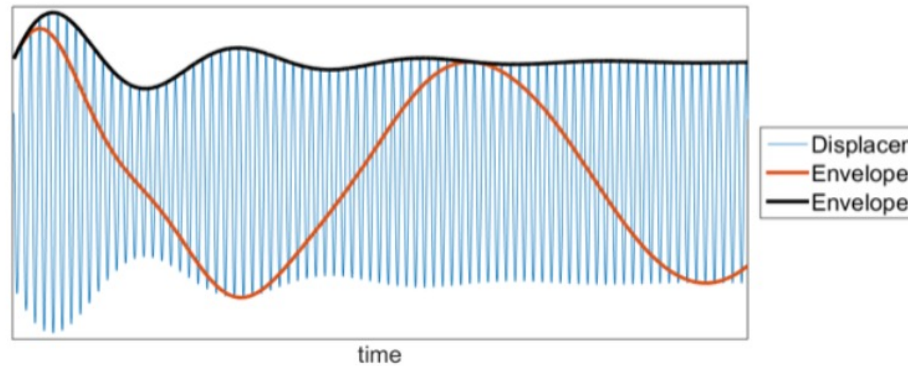
- **Cascading dynamics on oscillator networks**
  - The BTW sandpile model meets Kuramoto

Decoupled states: Phase-amplitude oscillators

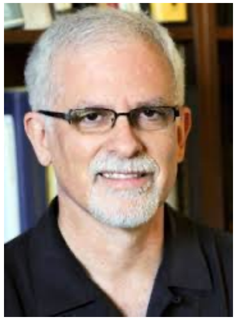
Nanoelectromechanical membrane, with a "Duffing"-like non-linearity







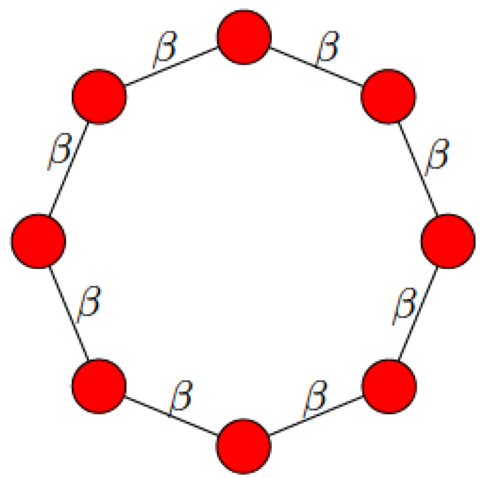
Described by slow-time envelope dynamics $A_i(t)$



Experimental collaboration with Micheal Roukes and Matt Matheny at Caltech

ARO MURI No. W911NF-13-1-0340

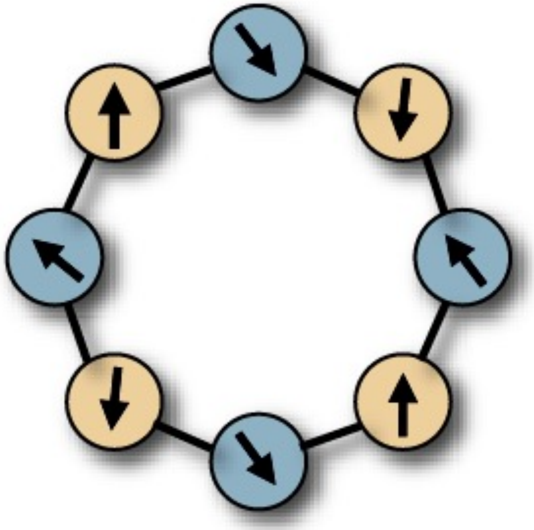
8-node of ring NEMs oscillators

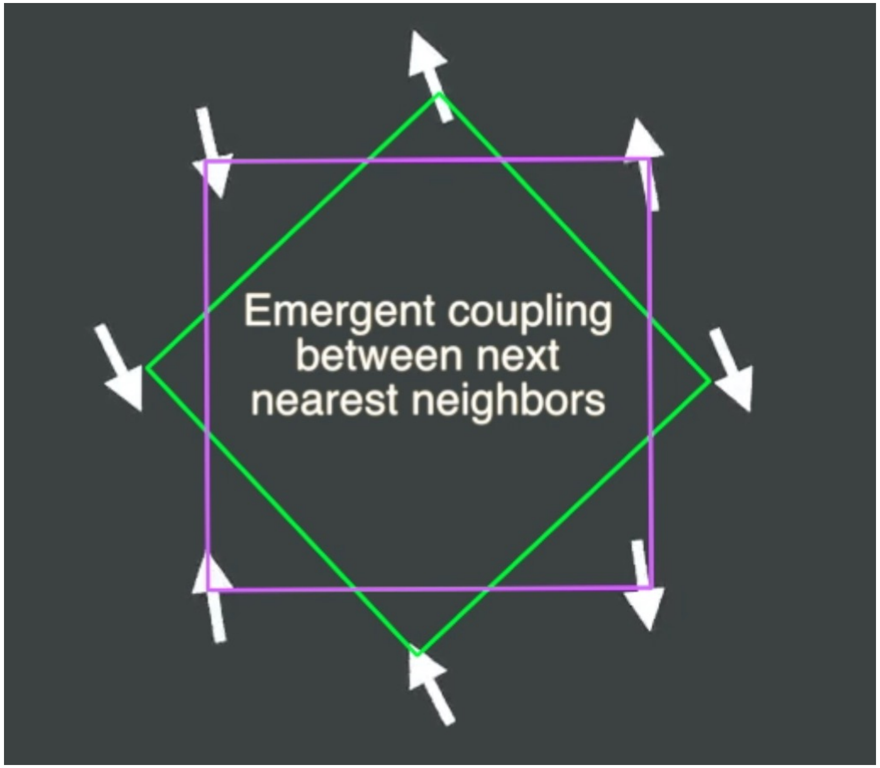


$$\frac{dA_i}{dT} = \underbrace{-\frac{1}{2}A_i}_{\text{Dissipation}} + \overbrace{i\alpha|A_i|^2 A_i}^{\text{Duffing Nonlinearity}} + \underbrace{\frac{A_i}{2|A_i|}}_{\text{Self Feedback}} + \overbrace{\frac{i\beta}{2}\left(A_{i+1} - 2A_i + A_{i-1}\right)}^{\text{Coupling Feedback}}$$

Decoupled NN with emergent NNN order

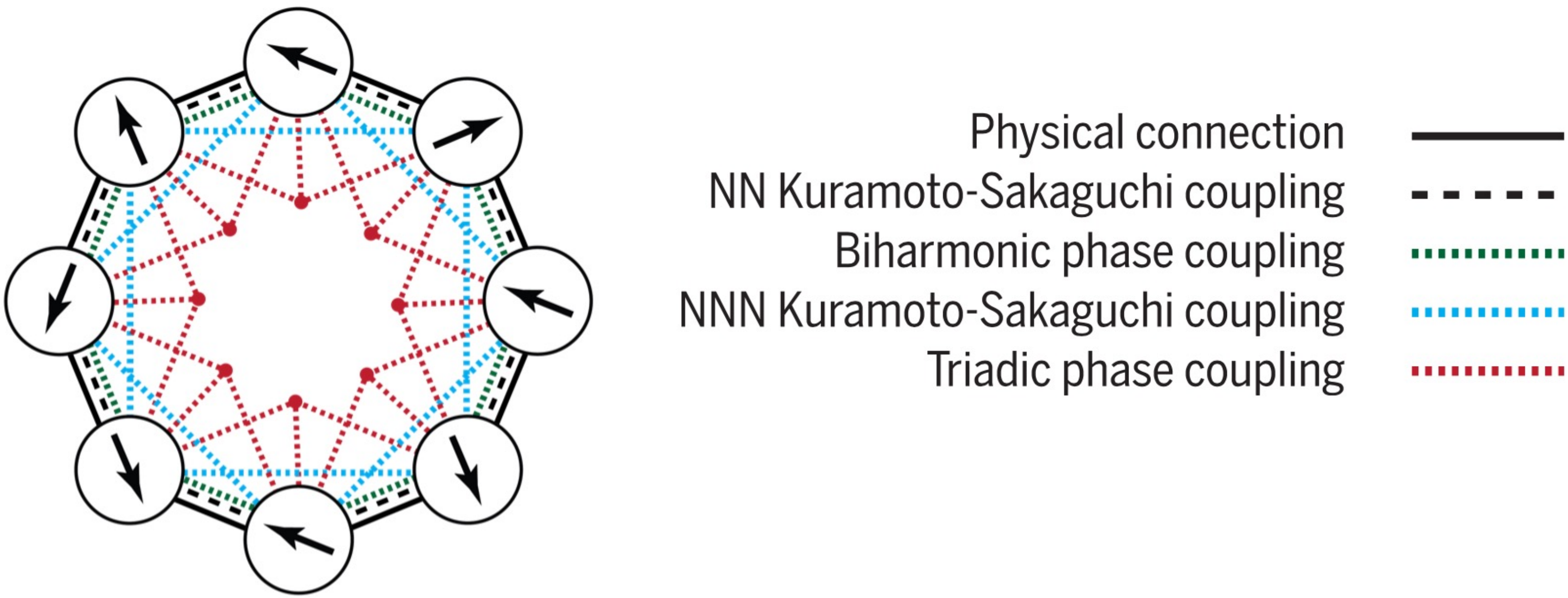
Interplay of nodal dynamics and coupling structure lead to decoupled states on ring of N=4m, m ∈ $\mathbb{Z}$.





Average $|A_i|=1$

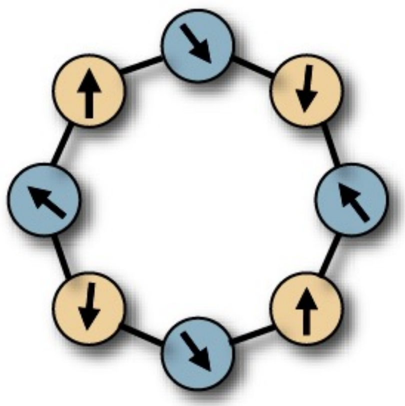
Emergent couplings of higher order



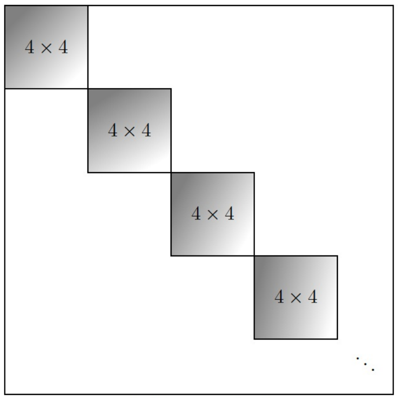
Matheny et al., “Exotic states in a simple network of nanoelectromechanical oscillators”, ***Science***, 363**,** March 8, 2019.

Linear stability calculations

Symmetry subgroups of nodal dynamics and coupling structure constrain the Jacobian:



Ring of 4m, $m \in \mathbb{Z}$



(a) Our method

$$D_k = \frac{1}{2}\begin{bmatrix} -1 & -\beta(1-\zeta^{-k})\sin\psi & 0 & \beta(1-\zeta^{-k})\cos\psi \\ \beta(1-\zeta^{k})\sin\psi & -1 & \beta(1-\zeta^{k})\cos\psi & 0 \\ 4\alpha & -\beta(1-\zeta^{-k})\cos\psi & 0 & -\beta(1-\zeta^{-k})\sin\psi \\ -\beta(1-\zeta^{k})\cos\psi & 4\alpha & \beta(1-\zeta^{k})\sin\psi & 0 \end{bmatrix}$$

- Unstable for phase-only oscillators.
- Stable for phase-amplitude oscillators.
- Although average $|A_i| = 1$, fluctuations are necessary to stabilize the system!

Emenheiser, et al., arXiv:2010.09131

Admissibility and stability of decoupled states in general

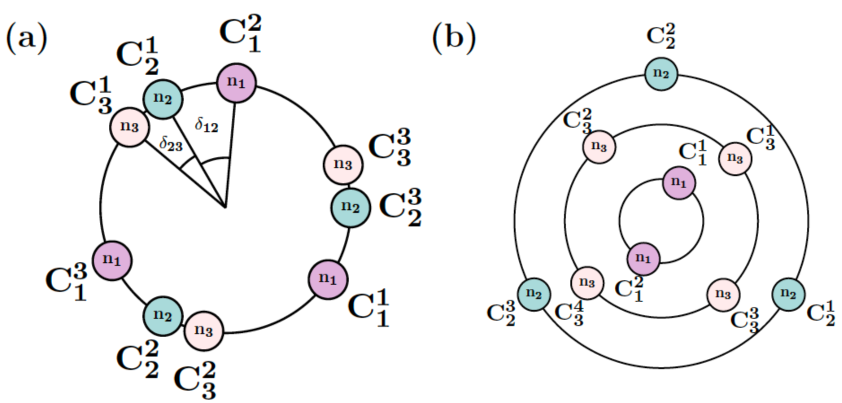
PHYSICAL REVIEW RESEARCH **2**, 043261 (2020)

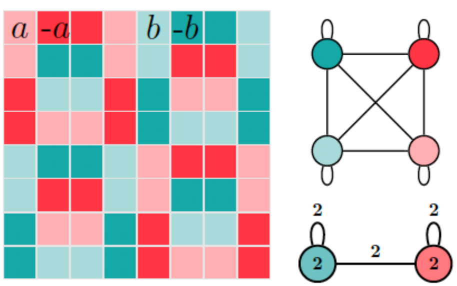
Decoupled synchronized states in networks of linearly coupled limit cycle oscillators

Anastasiya Salova[1,2,*] and Raissa M. D'Souza[2,3,4]

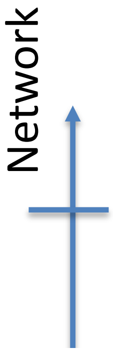
[1]*Department of Physics and Astronomy, University of California, Davis, California 95616, USA*
[2]*Complexity Sciences Center, University of California, Davis, California 95616, USA*
[3]*Department of Computer Science and Department of Mechanical and Aerospace Engineering, University of California, Davis, California 95616, USA*
[4]*Santa Fe Institute, Santa Fe, New Mexico 87501, USA*

(Received 26 June 2020; accepted 27 October 2020; published 19 November 2020)

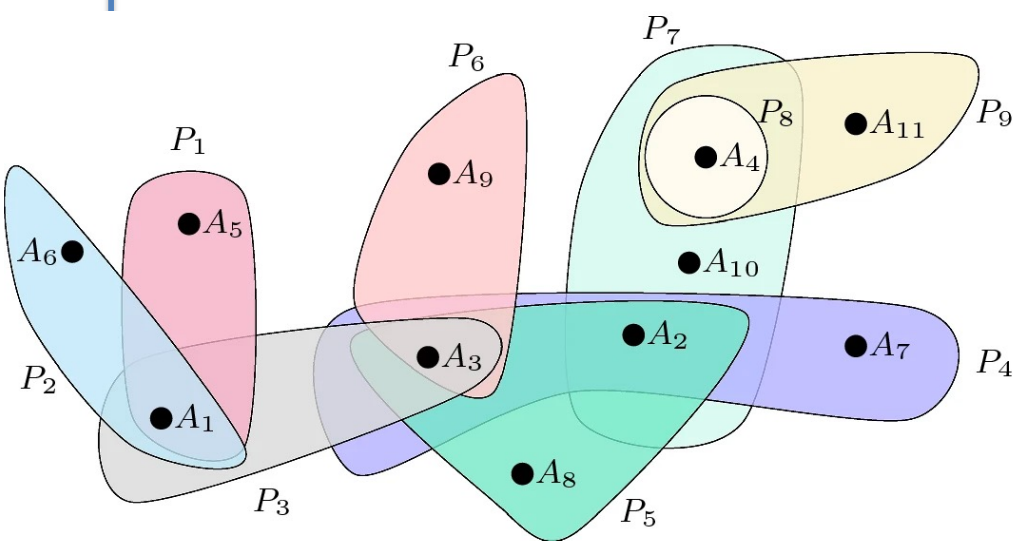




Hypergraphs – beyond dyadic coupling



Cluster synchronization on hypergraphs



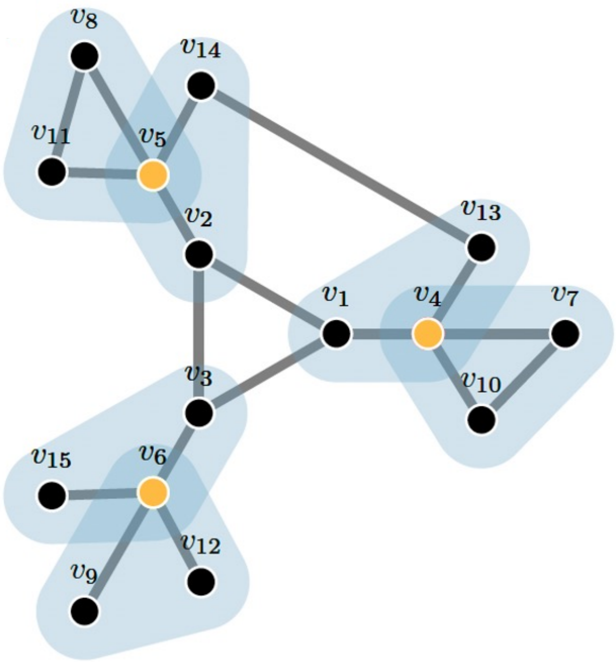
Example higher-order interactions:
- Chemical reactions
- Co-authorship networks

- **Hypergraph coupling**: Hyperedges representing higher-order interactions (dyadic, triadic, etc.)
- **Challenge**: Hyperedges of all order contribute to the dynamics and the stability calculations.

Cluster synchronization on hypergraphs

Anastasiya Salova, R.D., arXiv:2101.05464

Anastasiya Salova, R.D., arXiv:2107.13712

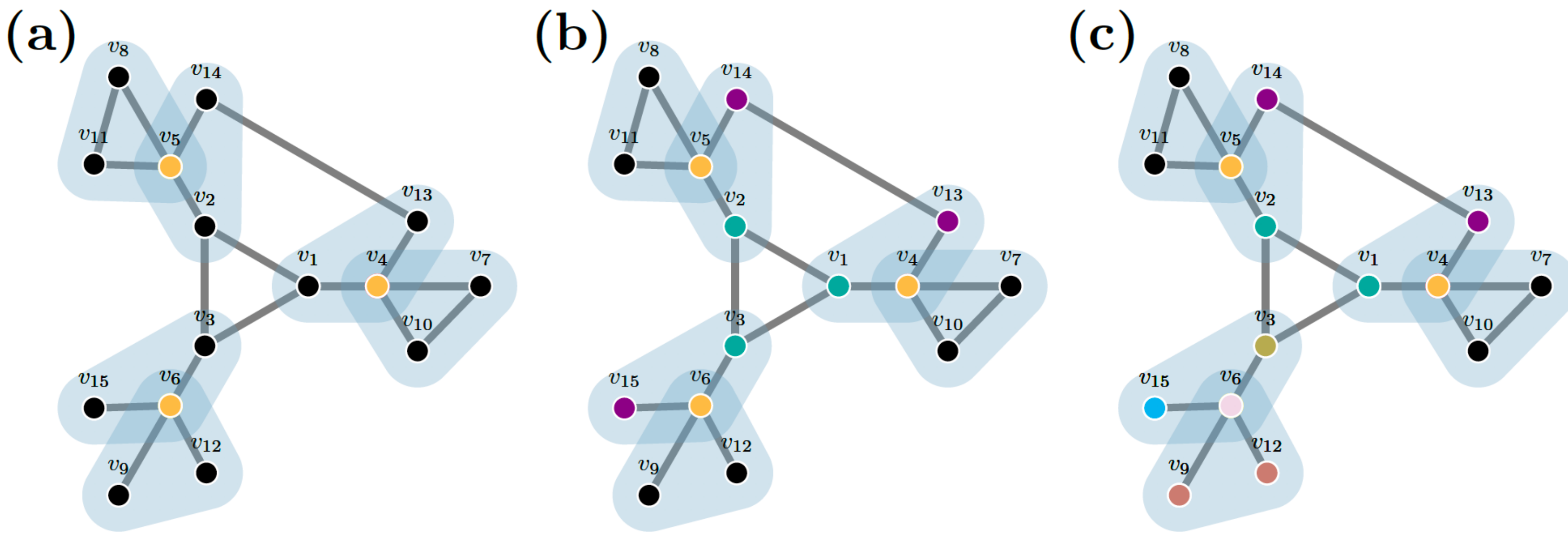


2 cluster state

Cluster synchronization:

Nodes can be divided up into distinct groups where the dynamical trajectories within a group are identical, but distinct from all the other groups.

The hypergraph may support multiple states

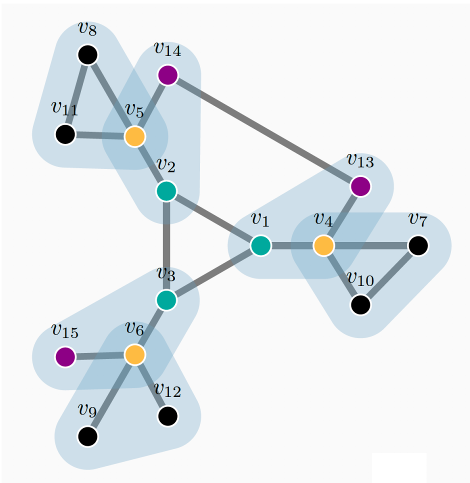


2 cluster state | 4 cluster state | 8 cluster state

- How do we find the admissible clusters?
- How do we calculate their stability properties?

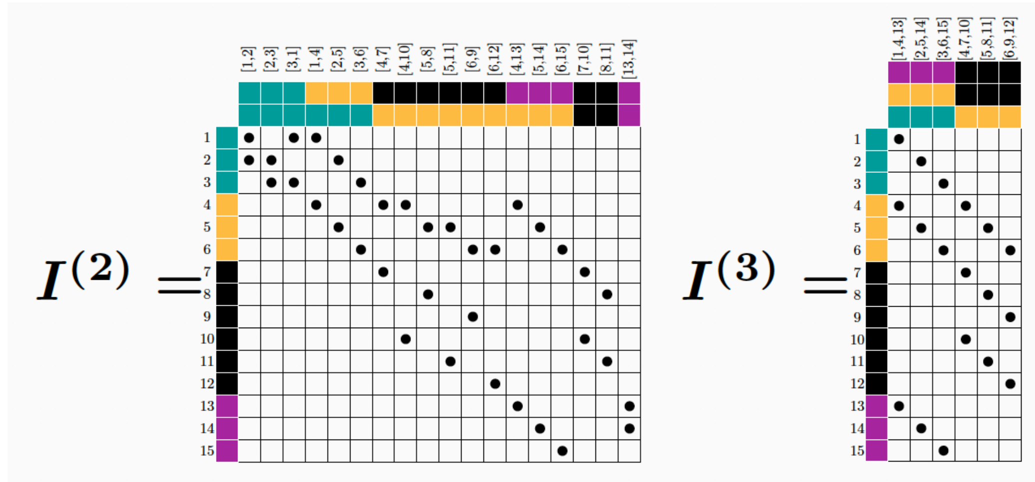
Hypergraph: Incidence matrices of all orders

4 cluster state



Dyadic

Triadic



Incidence matrices

$$[I^{(m)}]_{i,e} = 1 \text{ if } e \in \mathcal{E}_i^{(m)}$$

(Can equivalently formulate in terms of adjacency tensors $A^{(m)}$)

e.g., Battiston et al., *Phys. Reports* 2020.

Node evolution:

$$\dot{x}_i = F(x_i) + \sum_{m=2}^{d} \sigma^{(m)} \sum_{e \in \mathcal{E}^{(m)}} [I^{(m)}]_{i,e} G^{(m)}(x_i, x_{e \setminus i})$$

Dyadic projection of the hypergraph

Projecting the interactions for each order *m* onto a dyadic adjacency matrix

$$\mathcal{A}^{(m)} = I^{(m)}[I^{(m)}]^T - \mathcal{D}^{(m)}$$

$$\text{where } [\mathcal{D}^{(m)}]_{ii} = \sum_j I_{ij}^{(m)}$$

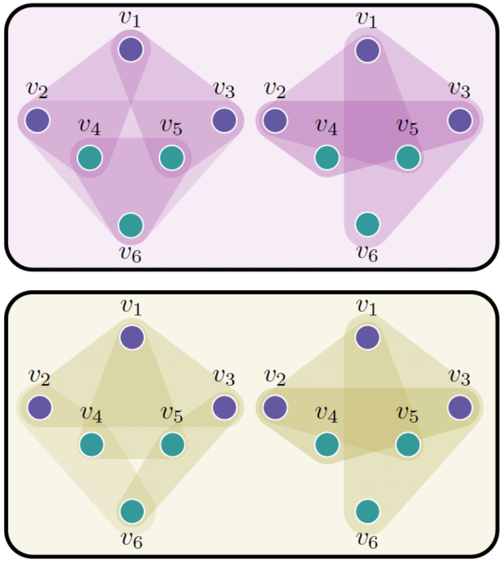
Useful in analyzing the admissibility and stability of full synchronization:

- Gambuzza, et al., *Nature Communications* 12(1), 2021.
- Lucas, et al., *Phys. Rev. Research* 2:033410, 2020.
- Carletti, et al., *J of Phys: Complexity* 1(3):035006, 2020.
- Ferraz de Arruda et al., *Communications Physics*, 4(1), 2021.

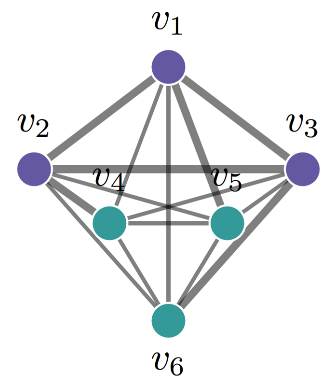
But not sufficient for more complex dynamics like cluster synchronization on general hypergraphs:

The projection is not always unique

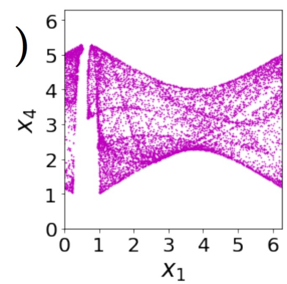
Two non-isomorphic hypergraphs
(with only triadic interactions)

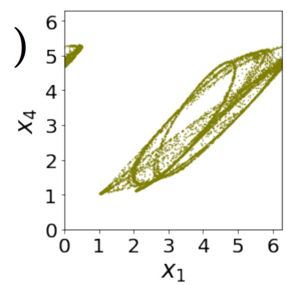


Same projection, $\mathcal{A}^{(3)}$



Distinct dynamical processes:





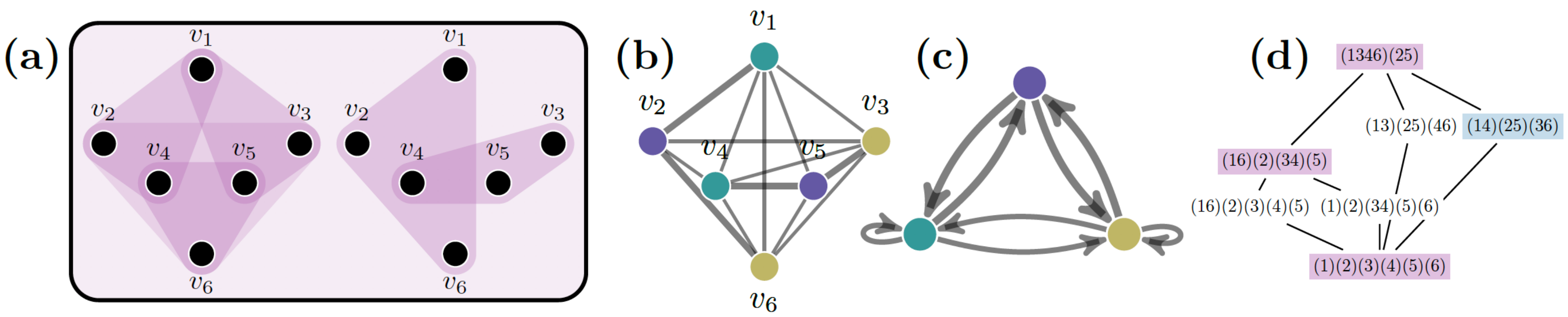
$$F(x_i^t) = \alpha \frac{1-\cos(x_i^t)}{2} + \frac{\pi}{6},$$

$$G^{(3)}(x_j^t, x_k^t) = \sigma^{(3)} \frac{1-\cos(x_j^t + x_k^t)}{2}$$

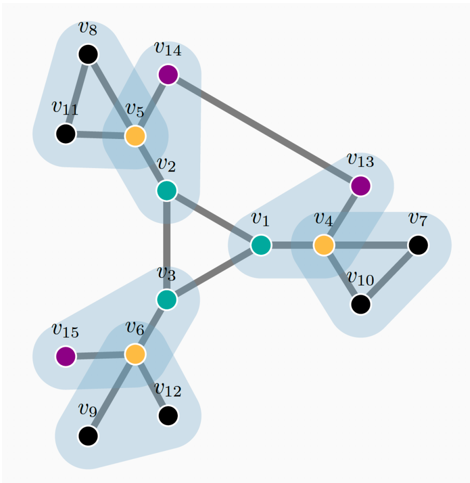
Symmetries of the projection versus the hypergraph

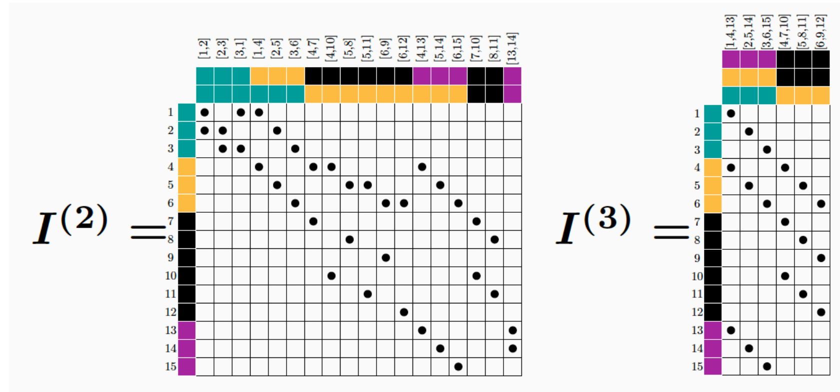
Structural symmetries of the system (orbital partitions) let us determine many of the admissible states.

Not all the symmetries of the projection (b) are symmetries of the original hypergraph (a)



The solution – organize by edge clusters





6 types of dyadic edge-clusters

2 types of triadic edge-clusters

For *K*-cluster state

Node clusters: C_1 to C_K

Node cluster trajectories: s_1 to s_K

For each order *m* there are K_m different edge clusters

Edge clusters: $C_1^{(m)}$ to $C_{Km}^{(m)}$ for each order m

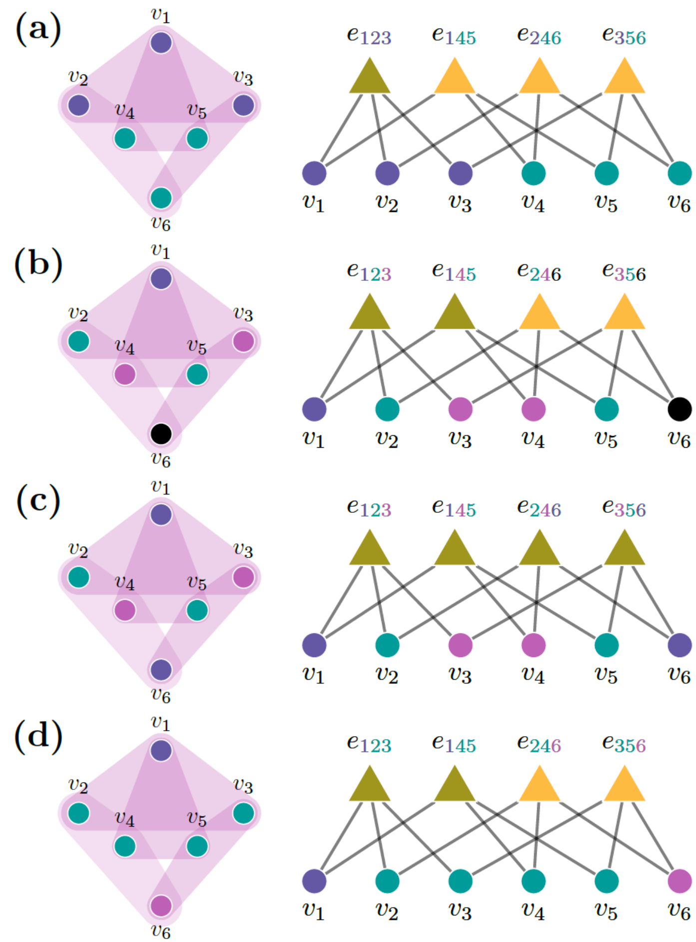
Edge cluster trajectories:

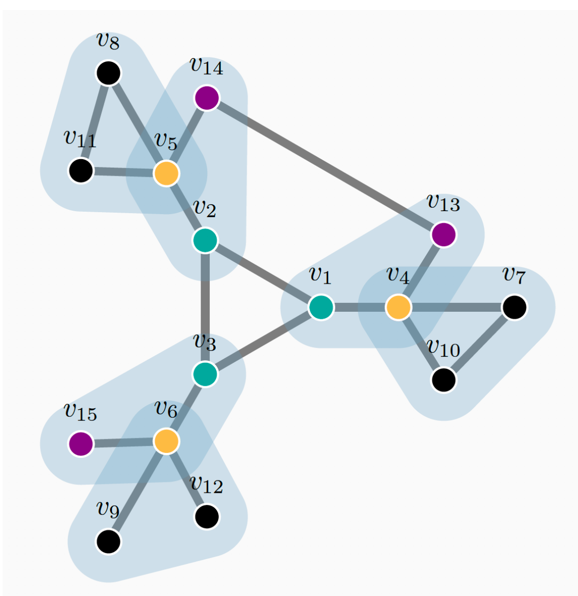
$$s_{C_1^{(m)}}, s_{C_2^{(m)}}, \ldots, s_{C_{K_m}^{(m)}}$$

Admissible patterns of cluster synchronization

Equitable partitions of node and edge clusters:

The partition is equitable if each node in a given node cluster gets the same input from each edge cluster over all edge orders





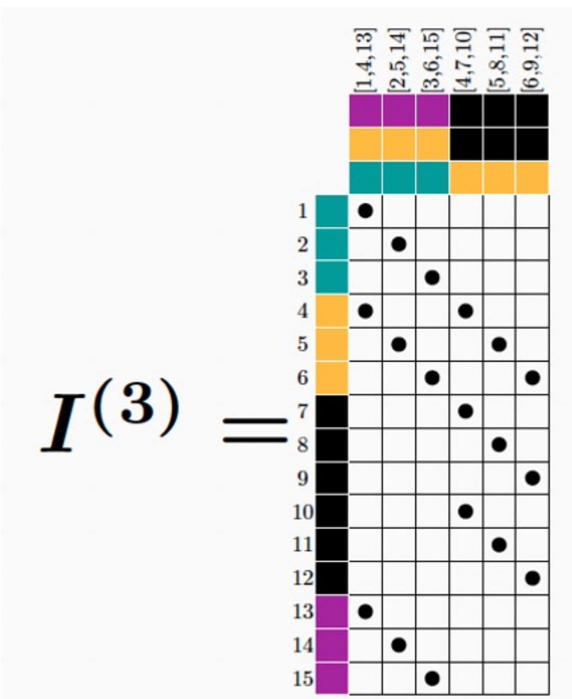
External equitable partitions if Laplacian-like coupling

Stability calculations

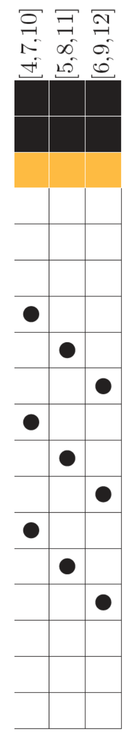
Instead of defining one single projection $A^{(m)}$ for each order m, define a projected adjacency matrix **for each *edge cluster* of order m>2**

$$\mathcal{A}_k^{(m)} = I_k^{(m)}[I_k^{(m)}]^T - \mathcal{D}_k^{(m)}$$

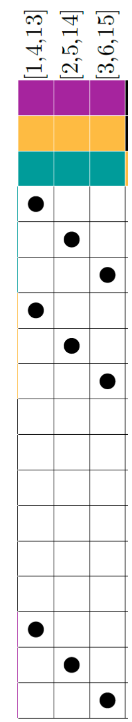
k denotes the k-th, m-th order edge cluster



$I^{(3)}{}_1 =$



$I^{(3)}{}_2 =$



Stability calculations

Jacobian organized by edge-clusters and their trajectories

$$\delta\dot{x} = \Bigg(\sum_{k=1}^{K} E_k \otimes JF(s_k) - \sum_{m=2}^{d} \sigma^{(m)}.$$

$$\Bigg[\sum_{k=1}^{K_m} \sum_{l \in \{C_k^{(m)}\}} \sum_{p \in \{C_k^{(m)} \setminus l\}} E_l \mathcal{A}_k^{(m)} E_p \otimes JG^{(m)}(s_l, s_p, s_{C_k^{(m)} \setminus l,p})$$

$$+ \sum_{k=1}^{K_m} \sum_{l \in \{C_k^{(m)}\}} E_l \mathcal{D}_k^{(m)} \otimes JG^{(m)}(s_l, s_{C_k^{(m)} \setminus l}) \Bigg] \Bigg) \delta x.$$

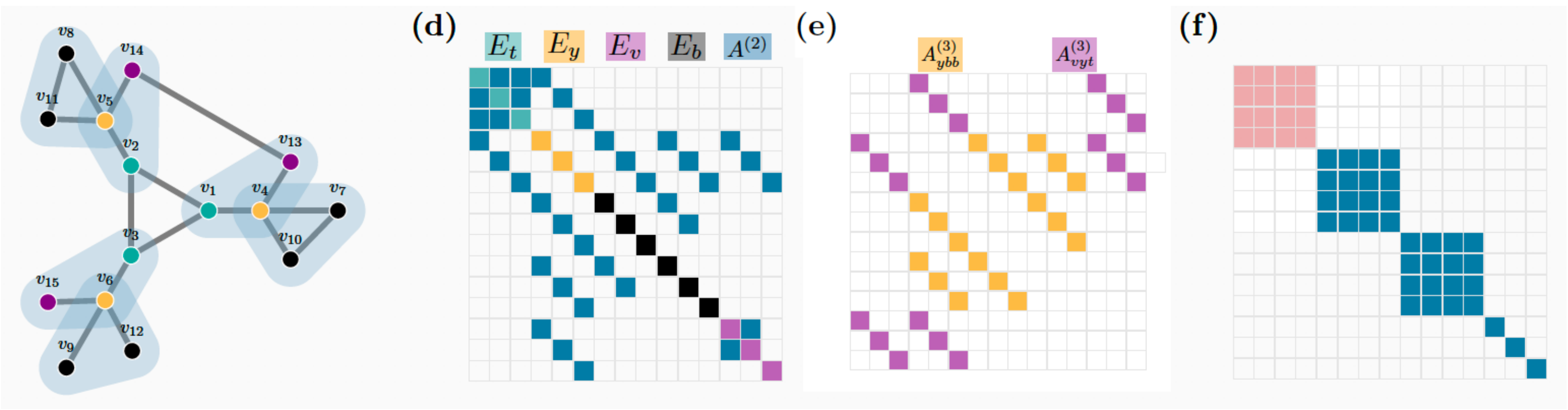
E_j – cluster indicator matrix for which nodes are in node cluster j.

All node clusters (first term) and **all edge clusters of all orders** (second term) contribute to linear stability calculation.

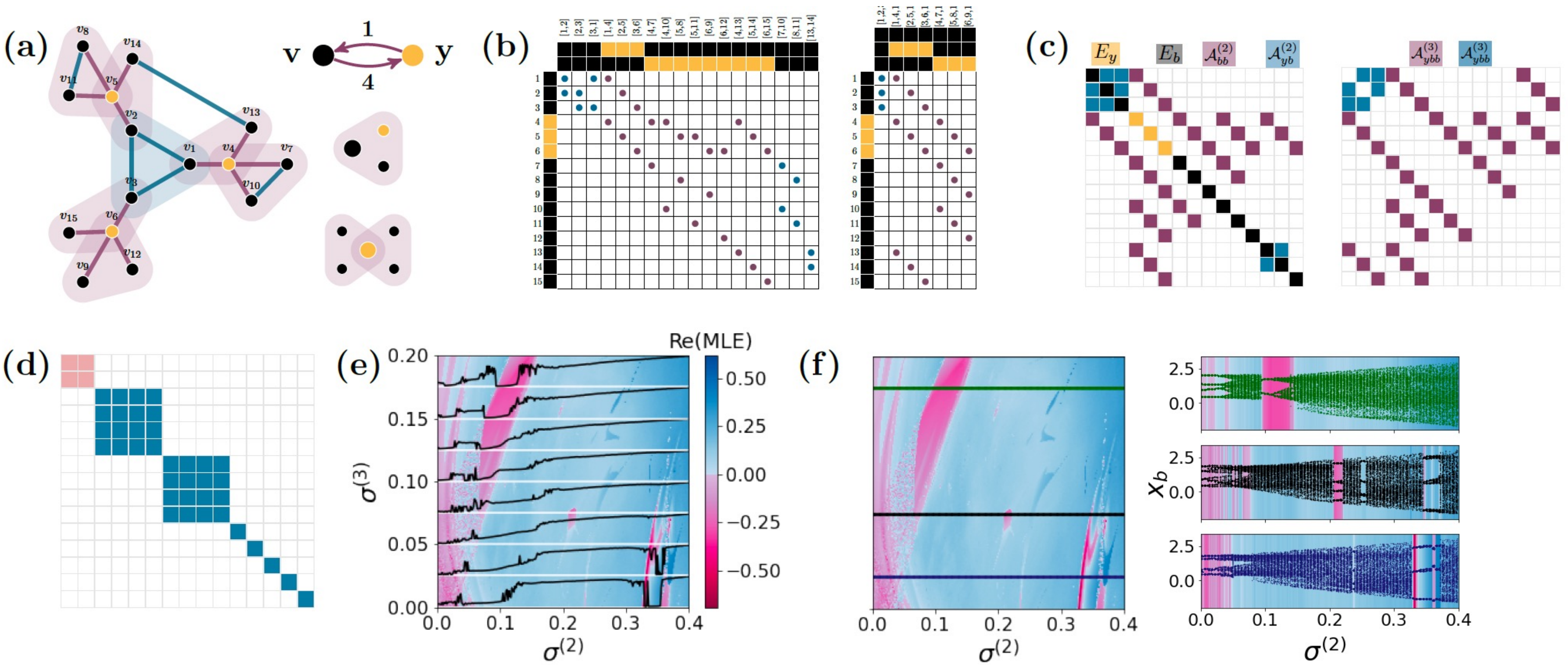
Stability calculations, cont

Simultaneously block-diagonalizing this set of matrices block-diagonalizes the Jacobian.

$$\{E_1, \ldots, E_K, \mathcal{A}^{(2)}, \mathcal{A}_1^{(3)}, \ldots, \mathcal{A}_{K_3}^{(3)}, \ldots, \mathcal{A}_1^{(d)}, \ldots, \mathcal{A}_{K_d}^{(d)}\}$$



Example: Layered hypergraph (multiple edge types)



Anastasiya Salova, R.D., arXiv:2101.05464

Anastasiya Salova, R.D., arXiv:2107.13712

https://github.com/asalova/hypergraph-cluster-sync

Oscillator networks exhibit cascading failures

Image © extremetech

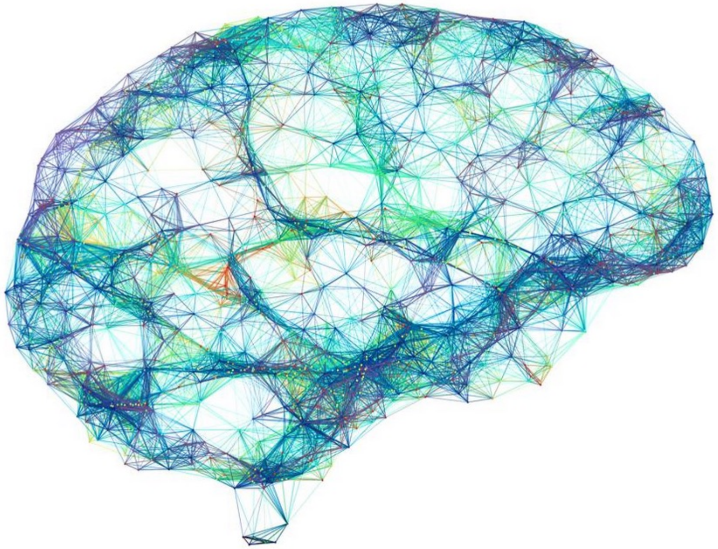


Image © Forbes

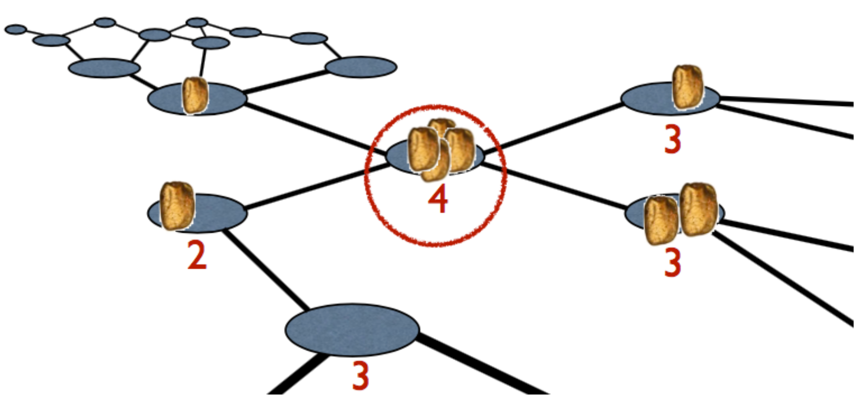
BTW sandpile model used to model power grid and brain networks

Self-organized criticality

Bak-Tang-Wiesenfeld *PRL* 1987: self-organized criticality

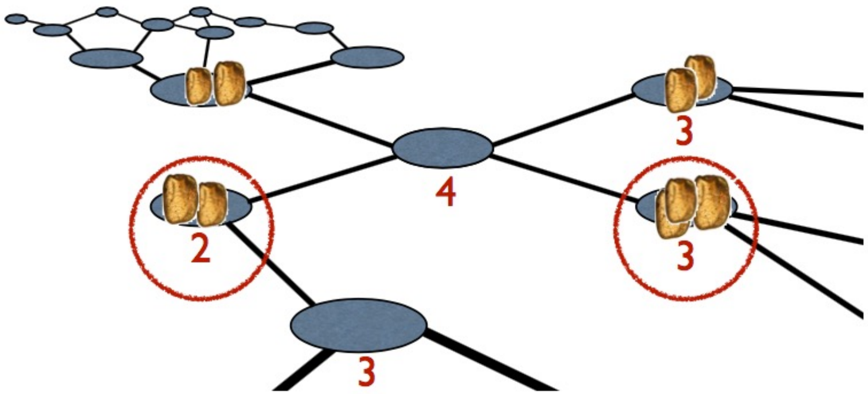
Sandpile model on networks

- Start with a network
- Drop units of load randomly on nodes
- Each node has a threshold.
  Here = degree.
- Load on a node ≥ threshold
  ⇒ node topples, moves load to neighbors



Sandpile models on networks

- Start with a network
- Drop <u>units of load</u> randomly on nodes
- Each node has a threshold.
  Here = degree.
- Load on a node ≥ threshold
  ⇒ node topples, moves load to neighbors
- Neighbors may topple. Etc.
  Cascade (or avalanche) of topplings.



Power-law distribution of avalanche sizes, $P(s) \sim s^{-3/2}$

Self-organized criticality

Power law tails (Universal behavior)

Extreme events often referred to as "Black Swans"

This scaling behavior is robust on networks. (Goh et al. PRL 03, Phys. A 2004/2005, PRE 2005. PLRGs with $2 < \gamma < 3$ not mean-field.)

Power law tails seem to characterize the sizes of electrical blackouts, financial fluctuations, neuronal avalanches, earthquakes, landslides, overspill in water reservoirs, forest fires and solar flares.

[1] I. Dobson, B. A. Carreras, V. E. Lynch, and D. E. Newman, Chaos **17**, 026103 (2007).
[2] X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley, Nature **423**, 267 (2003).
[3] J. M. Beggs and D. Plenz, J, Neurosci. **23**, 11167 (2003).
[4] D. E. Juanico and C. Monterola, J. Phys. A **40**, 9297 (2007).
[5] T. Ribeiro, M. Copelli, F. Caixeta, and H. Belchior, PLoS ONE **5**, e14129 (2010).
[6] A. Saichev and D. Sornette, Phys. Rev. E **70**, 046123 (2004).
[7] S. Hergarten, Natural Hazards and Earth System Sciences **3**, 505 (2003).
[8] G. L. Mamede, N. A. M. Araujo, C. M. Schneider, J. C. de Araújo, and H. J. Herrmann, Proc. Natl. Acad. Sci. U.S.A. **109**, 7191 (2012).
[9] P. Sinha-Ray and H. J. Jensen, Phys. Rev. E **62**, 3216 (2000).
[10] B. D. Malamud, G. Morein, and D. L. Turcotte, Science **281**, 1840 (1998).
[11] E. T. Lu and R. J. Hamilton, Astrophys. J. **380**, L89 (1991).
[12] M. Paczuski, S. Boettcher, and M. Baiesi, Phys. Rev. Lett. **95**, 181102 (2005).
[13] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).

SOC in power grids and the brain?

Image © extremetech

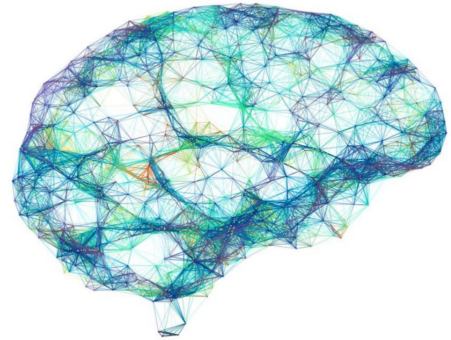


Image © Forbes

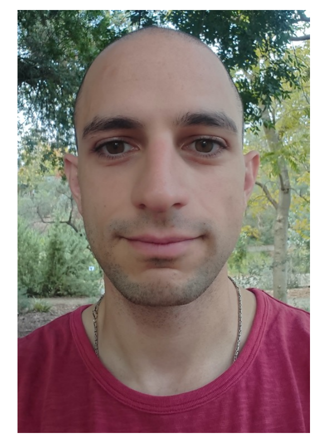
But this neglects the oscillatory nature of the nodes!

Sandpile cascades on oscillator networks: the BTW model meets Kuramoto

Guram Mikaberidze[1, a)] and Raissa M. D'Souza[2, 3]
[1)]*Department of Mathematics, University of California, Davis, CA, 95616, USA*
[2)]*University of California, Davis, CA, 95616, USA*
[3)]*Santa Fe Institute, Santa Fe, NM, 87501, USA*

arXiv:2112.00104

Initial goal: Leverage interaction to maximize synchronization and minimize large cascades.



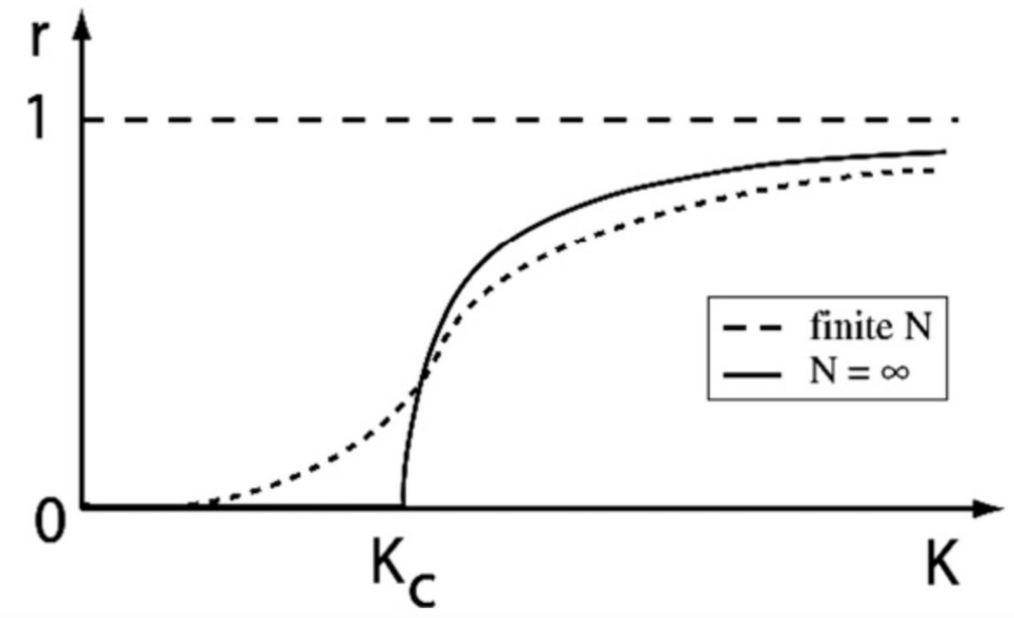
Guram Mikaberidze

Oscillator dynamics: The Kuramoto model

$$\dot{\phi}_i(t) = \omega_i + k \sum_{j \in \mathcal{N}_i} \sin(\phi_j(t) - \phi_i(t))$$

Time evolution of the phase of oscillator *i*

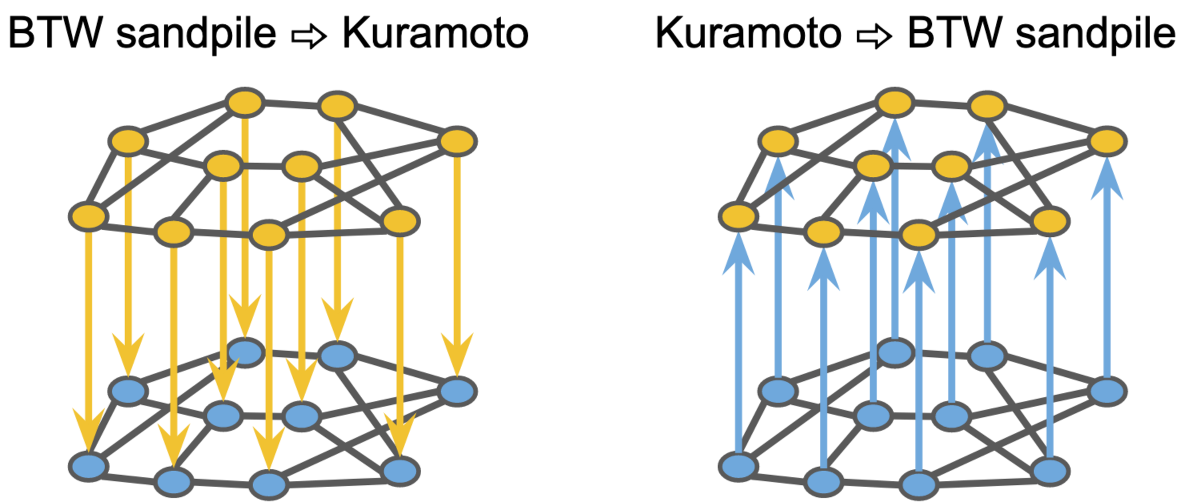
Synchronization phase transition at critical coupling



Coupled BTW-KM dynamics. Each node has:

- Phase ϕ_i (KM)
- Capacity c_i (BTW)
- Load s_i (BTW)

Coupled BTW-KM



BTW -> KM

- If a node topples during a cascade its phase is reset at random at the end of the cascade.

KM -> BTW

- Assume a node out-of-sync with its neighbors is more vulnerable so lower its capacity to hold load.
  - This creates ***endogenous*** cascade seeds.

Explicit dynamical rules

Rules of the model

1. **KM:** KM runs for time interval *ΔT*:

$$\dot{\phi}_i(t) = \omega + k \sum_{j \in \mathcal{N}_i} \sin(\phi_j(t) - \phi_i(t))$$

2. **KM→BTW:** Update SP capacities:

$$c_i = c^0 - \alpha(l - r_i)$$

3. **BTW:** Add unit load to SP and cascade

4. **BTW→KM:** Randomize phases of toppled nodes

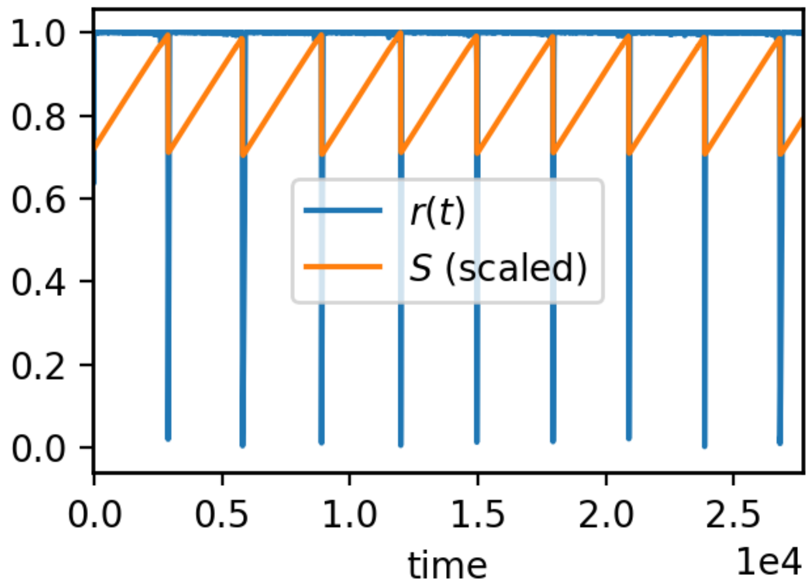
- Infinite separation of time scales:
  cascades are infinitely faster

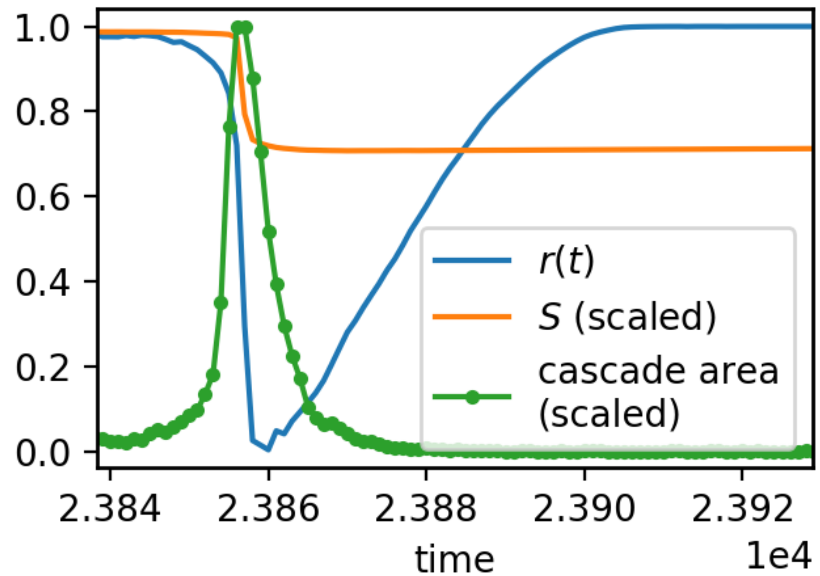
- Uniform oscillator frequencies (for now)

Emergent periodic oscillations

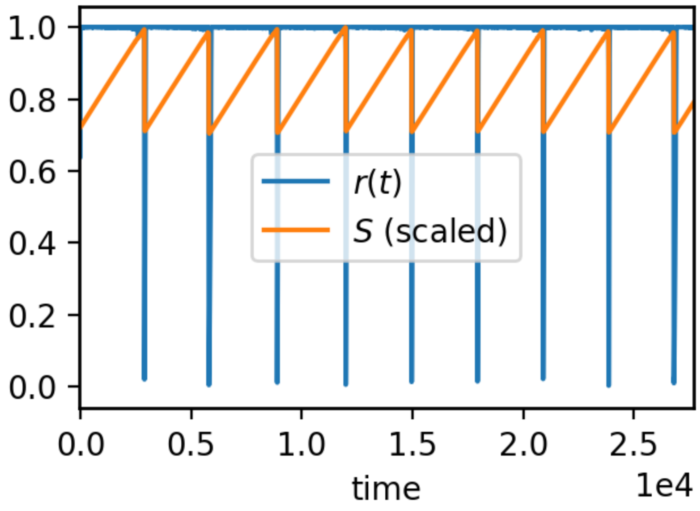
3-regular random graphs

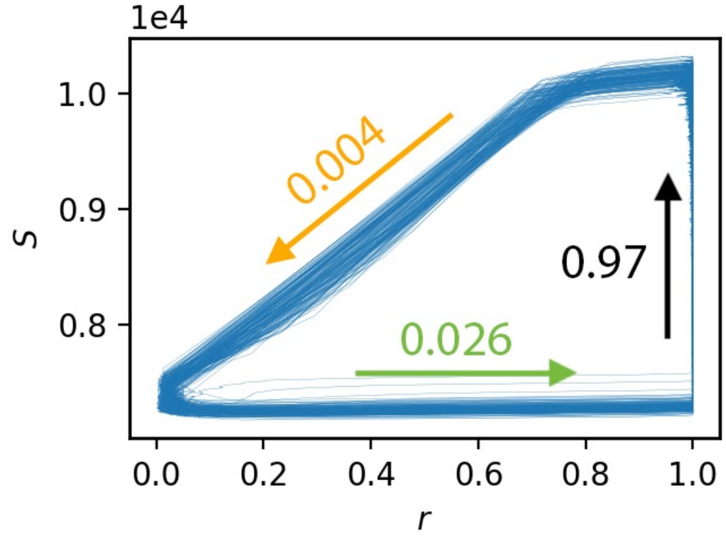
- *r(t)* is the Kuramoto order parameter
- *S* is the total load on the system



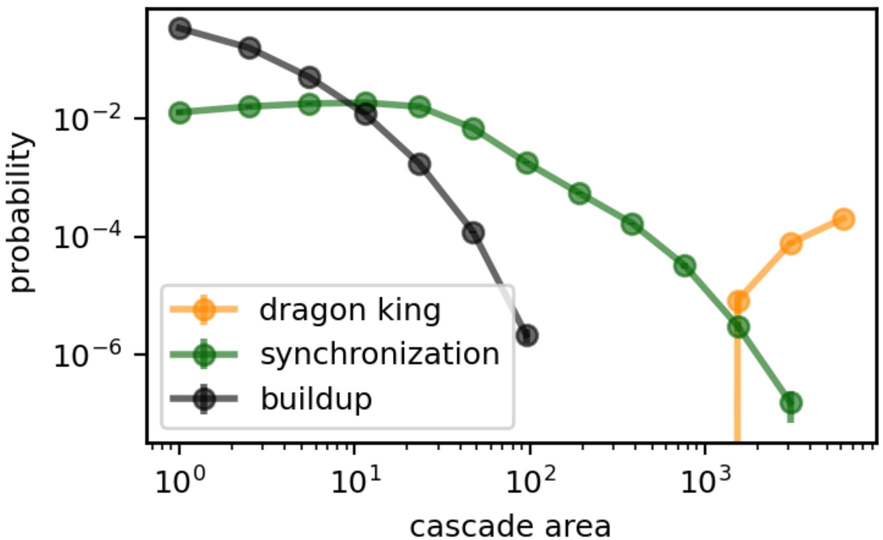


Emergent 3-phase oscillations



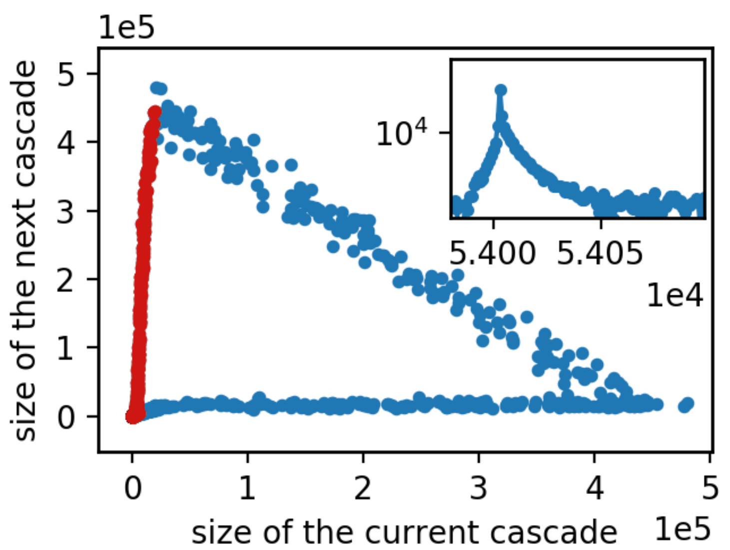


Cascade size distribution in each of the three phases



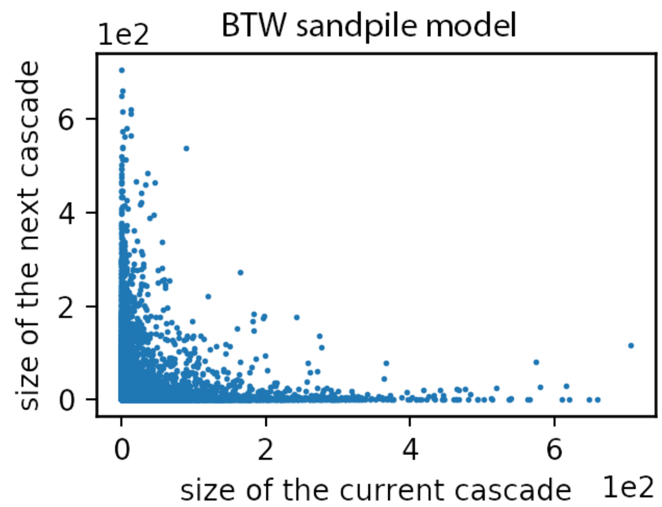
Self-amplifying cascades kick off a DK

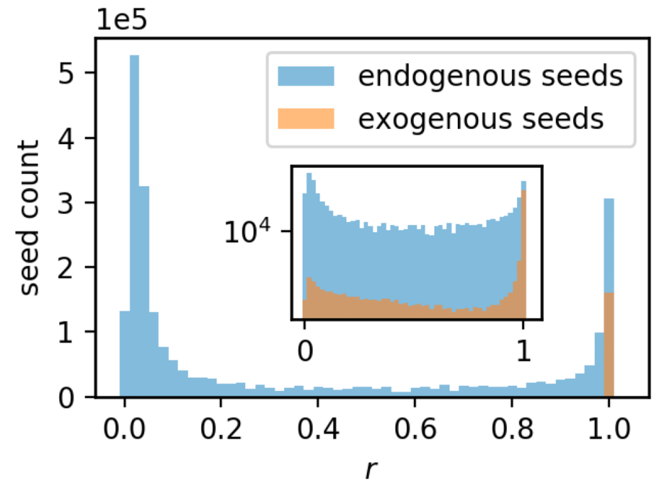
At the tipping point, a large cascade desynchronizes many nodes, causing an even larger cascade at the next step.



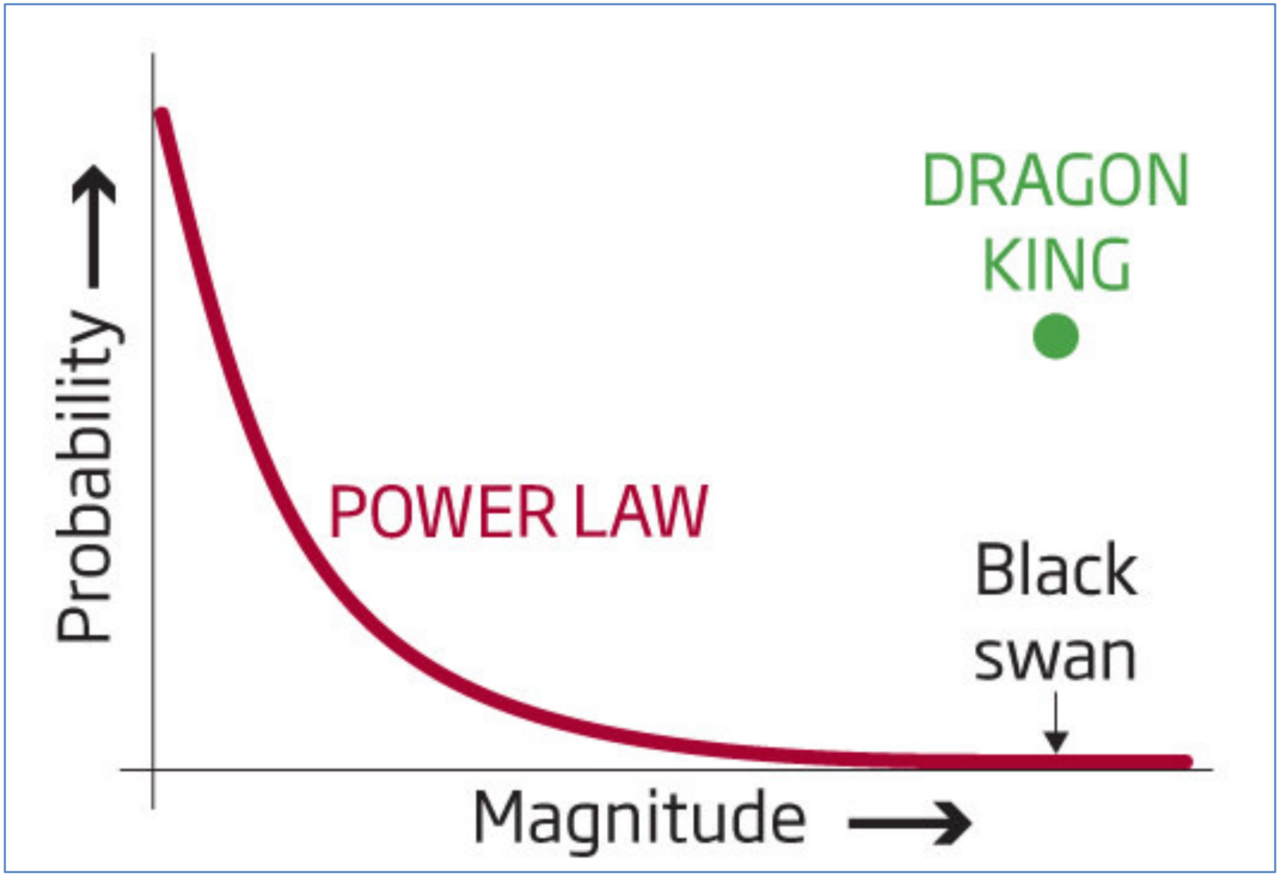
$$s(t) \propto \exp(\frac{a-1}{\Delta T}t)$$

Exponential growth in subsequent size:
"cascade of cascades"





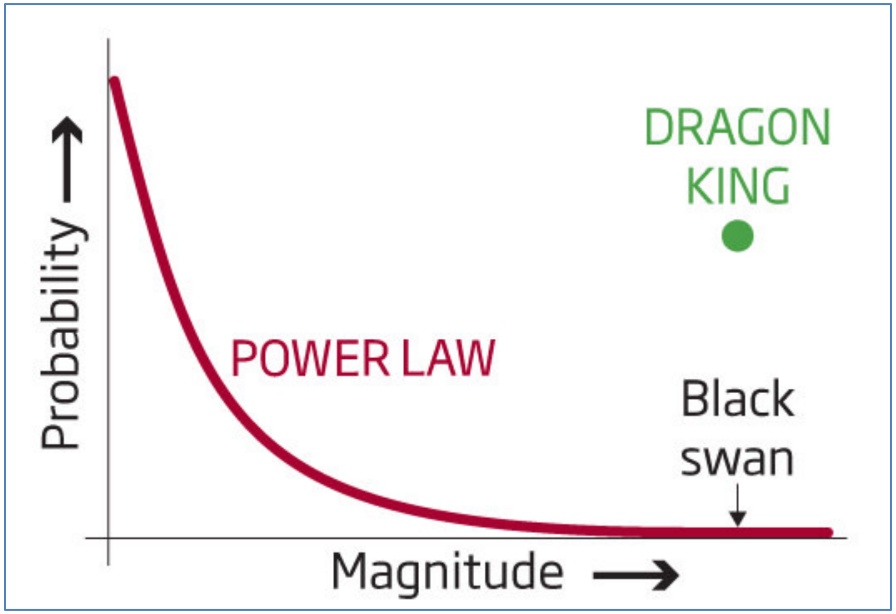
Beyond “Black Swans” -> Dragon Kings



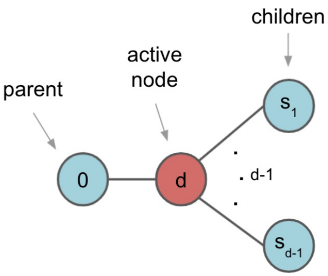
Poorly understood, massive events caused by nonlinear amplifying mechanisms. (Introduced by D. Sornette, 2009.)

Dragon Kings

- Bubbles in financial markets; sizes of cities; failures in engineered systems & nuclear accidents, etc.
- Self-amplifying mechanism, endogenous nature
- Far more likely than Black Swans and equally massive
- Theory in its infancy:
  - Conjecture: needs homogeneous elements with large coupling
- Dragon kings have predictability

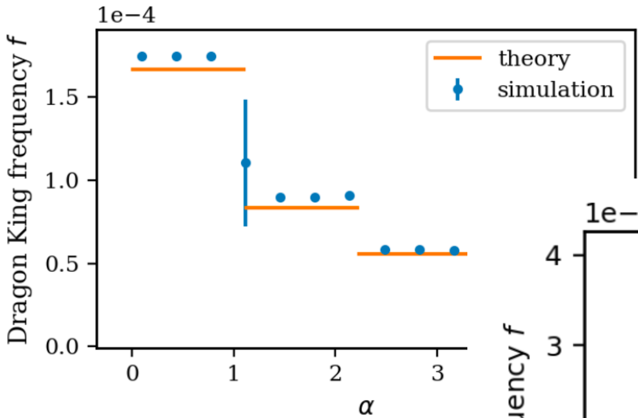


Analytic calculations

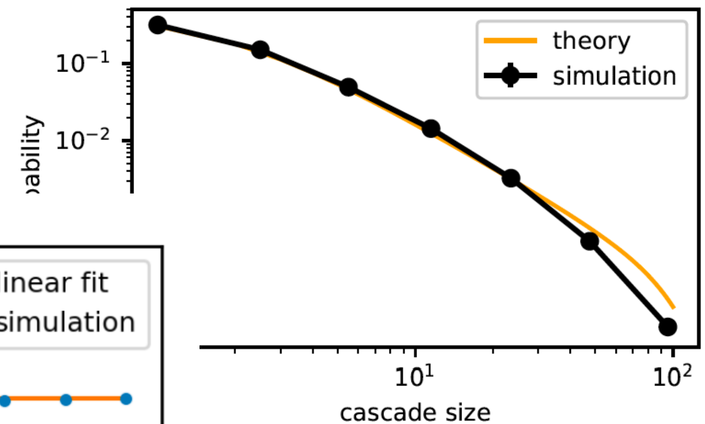


$$(S_{\mathrm{TP}}, r_{\mathrm{TP}}) = \left(\frac{\lfloor c^0 - \alpha(l-1) \rfloor N d}{2(d-1)}, 1 \right)$$

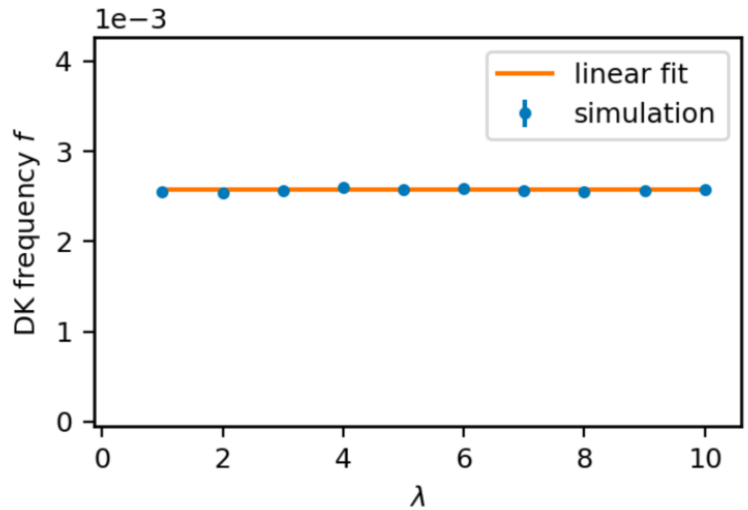
(1) The location of the tipping point



(2) The frequency of the



(3) The cascade distribution in the buildup phase



(4) The thermodynamic limit

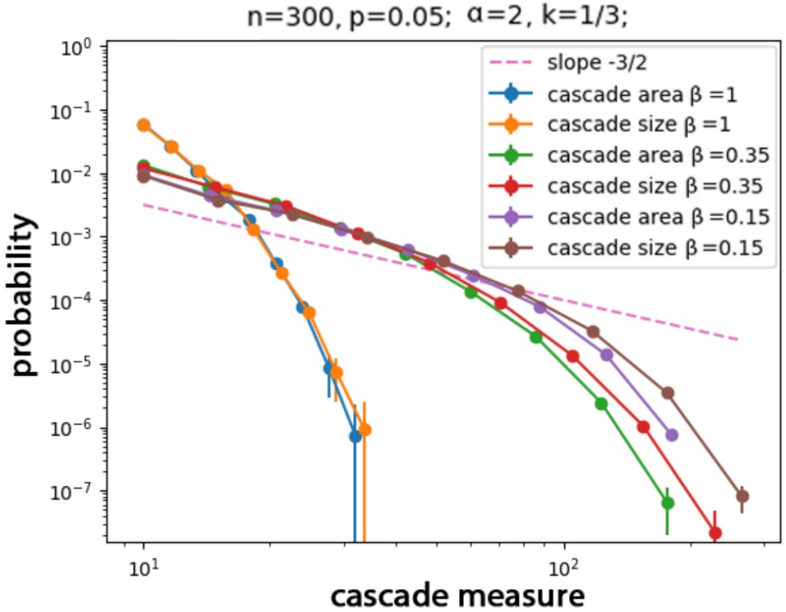
Guram Mikabiredze and R.D., arXiv:2112.00104

Next steps 1: Different coupling from BTW to Kuramoto

Let oscillator frequency depend inversely on load – **more load spins slower**

Natural frequency varies with load
(motivation: power grid with inertia)

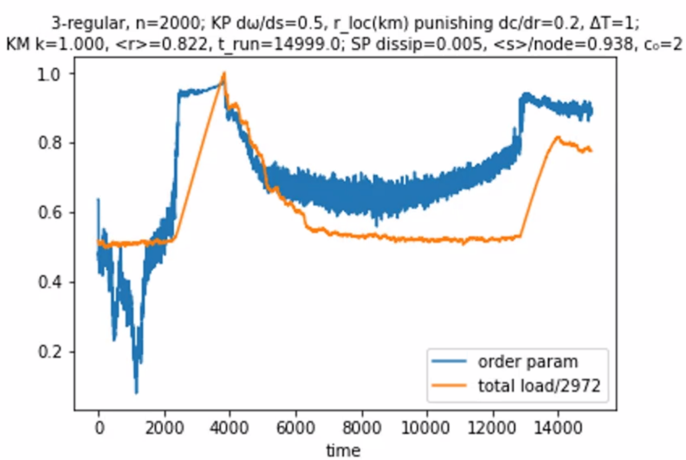
1. **KM:** KM runs for time period *ΔT*:
$$\dot{\phi}_i(t) = \omega_i + k \sum_{j \in \mathcal{N}_i} \sin(\phi_j(t) - \phi_i(t))$$
2. **KM→BTW:** Update SP capacities:
$$c_i = c_i^0 - \alpha(l - r_{(i)})$$
3. **BTW:** Add unit load to SP and cascade.
4. **BTW→KM:** Update KM intrinsic frequencies:
$$\omega_i = \omega_i^0 + \beta s_i$$

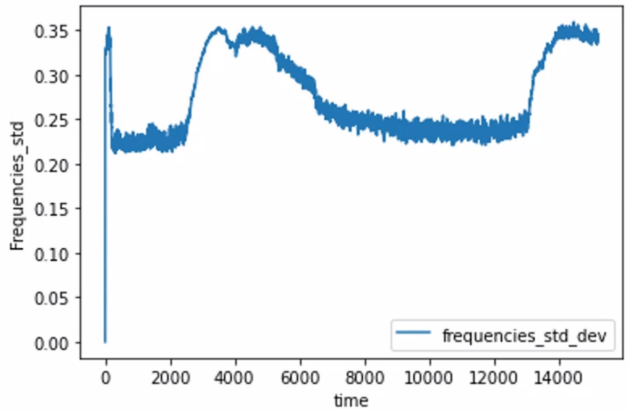


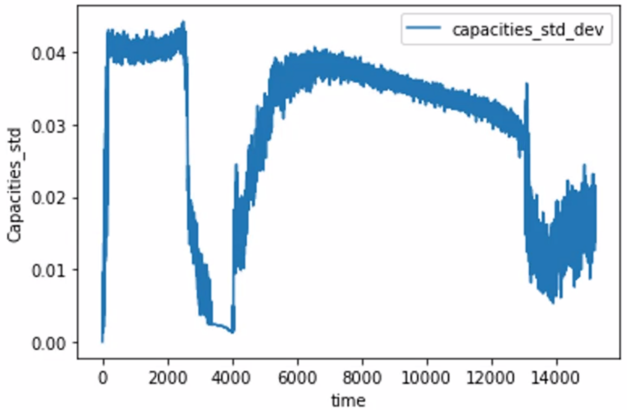
Next steps 1: Different coupling from BTW to Kuramoto

- Local sync drives capacities higher
- Higher capacity which causes higher dispersion in frequencies
- Frequency dispersion destabilizes the synchrony.

Some temporal plots : w=0, dw_ds=0.5, dc_dr=0.2

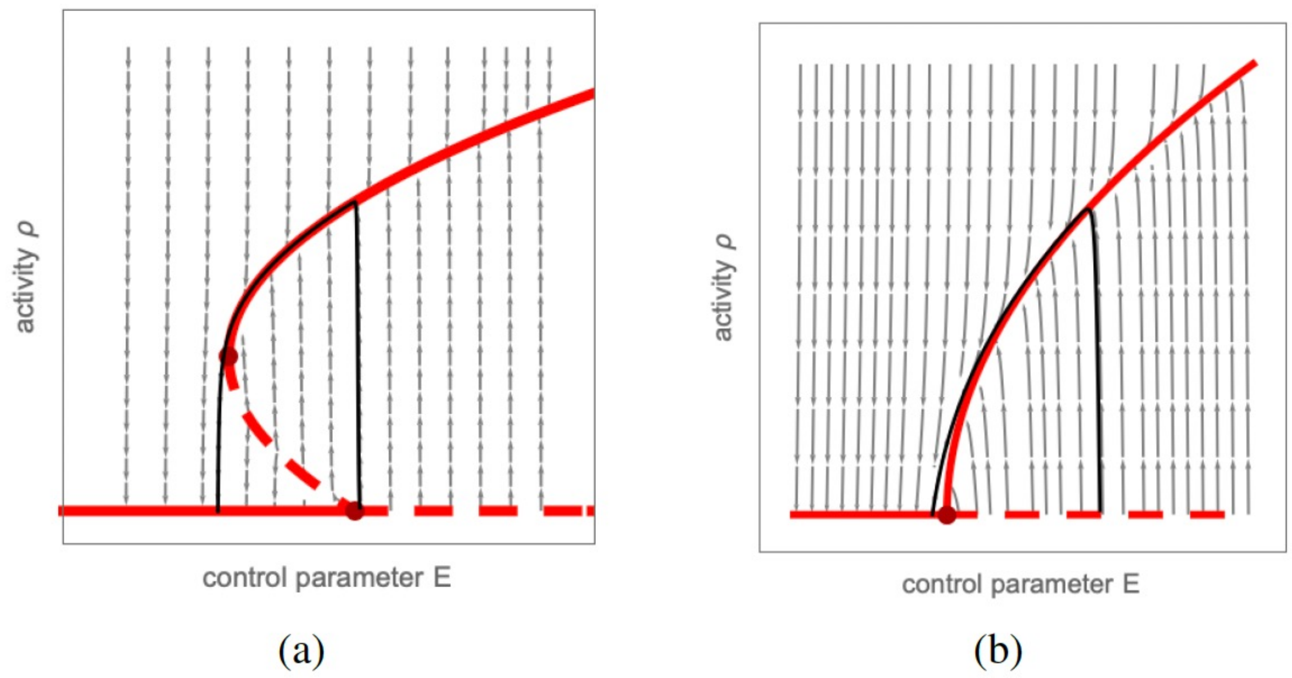




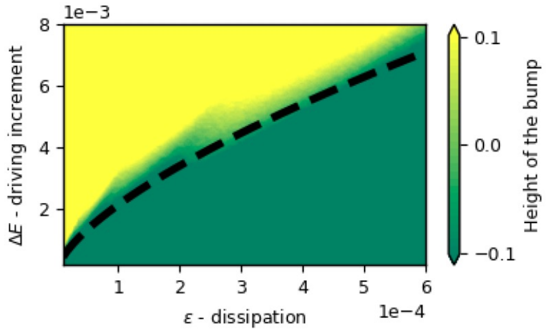


Next steps 2: Theory of Dragon Kings

Dragon Kings as an absorbing-state phase transition.
Slow driving and small dissipation cause self-organized DK cycles.



Self-organization around a (a) 1st order and a (b) second order phase transition.



More opportunities?

- **Complex co-evolving networks**:
  - New phenomena & emergent timescales
  - Long-time oscillations – opportunity to self-organize (e.g., beta-waves in neuronal networks, correlations among generators in the power grid).
  - Modeling layered networks with inhibitory and excitatory layers (e.g., the visual cortex)

- **Distributed control**:
  - How are existing sandpile control schemes disrupted by the oscillatory degree of freedom?
  - Interventions to nodal frequencies for sandpile control (e.g., explosive sync reveals interplay of nodal frequency distribution and network topology).

Conclusions/Opportunities

Interplay of nodal dynamics and network structure

- Decoupled synchronized states

Hypergraph dynamics:

- Higher order interactions
- Organizing by edge clusters organizes calculations and simplifies stability calculations.

Cascades on oscillator networks

- Coupling the oscillatory and cascading dynamics leads to new emergent behaviors and emergent timescales.