Event-triggered stabilization of linear systems under bounded *bit* rates

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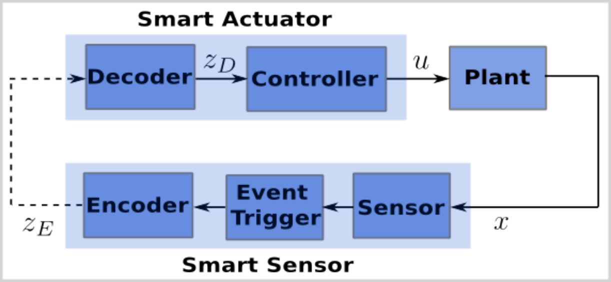
Networked control systems



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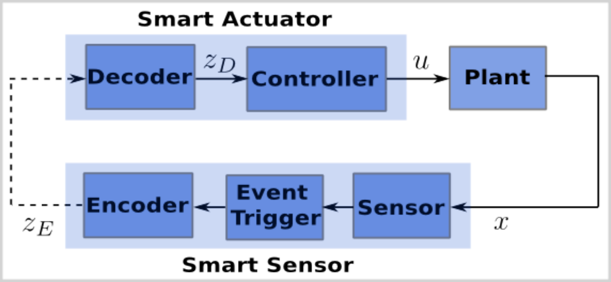
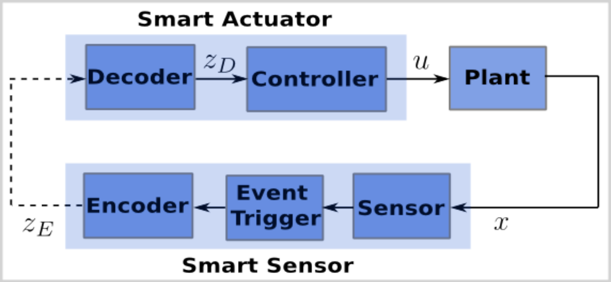
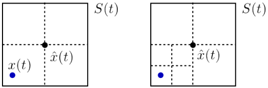


- **When to transmit:**
  Event-triggered strategies
  - A trigger function encodes the control goal
  - Transmissions occur only when necessary
  - Better use of resources than time-triggered

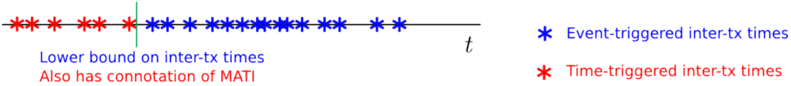
Networked control systems



- **What to transmit:**
  Information-theory based data rate theorems
  - Quite successful in the discrete-time setting
  - Tight necessary and sufficient data rates are available



Unanswered questions



Event-triggered control:

- What is the average inter-tx time?

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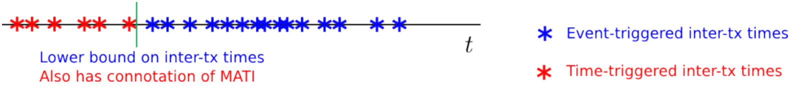

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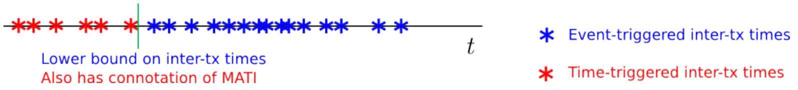

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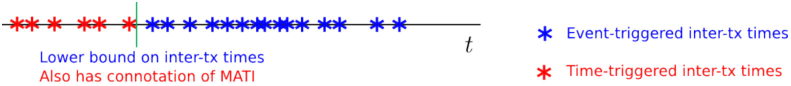

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- There is still a lot of scope for work in the continuous-time setting
- How to design controllers with specified performance (e.g. convergence rate)?

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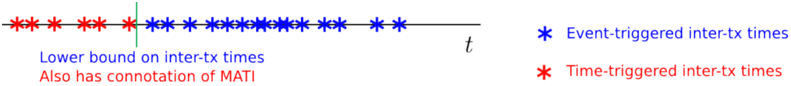

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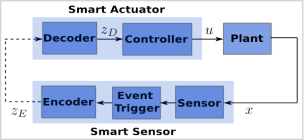
The two themes have complementary strengths

System description



Plant dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t), \quad u(t) = K\hat{x}(t)$$

$$\|v(t)\|_2 \leq \nu, \ \forall t \in [0, \infty)$$

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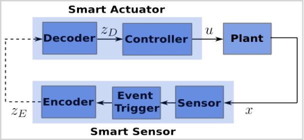

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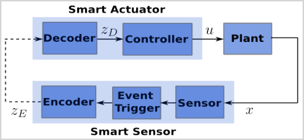

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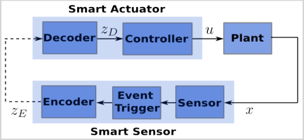

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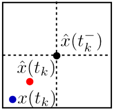

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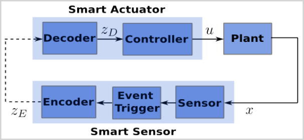

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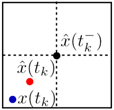

Closed loop flow, for $t \in [r_k, r_{k+1})$

$$\dot{x}(t) = \bar{A}x(t) - BKx_e(t) + v(t), \qquad \bar{A} \triangleq A + BK$$

$$\dot{x}_e(t) = Ax_e(t) + v(t), \qquad x_e \triangleq x - \hat{x} \text{ (encoding error)}$$

Quantization and coding (instant communication)

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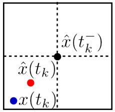
Then, $\|x_e(t)\|_\infty \leq d_e(t)$, for all $t \geq t_0$

$d_e(t_k^-)$ defines the quantization domain at time t_k

bits used to quantize at time t_k is np_k



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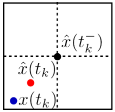

Non-instant communication: more involved

Control objective

Suppose $\bar{A} = A + BK$ is Hurwitz $\iff$ $P\bar{A} + \bar{A}^T P = -Q$

Lyapunov function: $x \mapsto V(x) = x^T P x$

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Desired performance function: $V_d(t) = (V_d(t_0) - V_0)e^{-\beta(t-t_0)} + V_0$

Performance objective: ensure $b(t) \triangleq \frac{V(x(t))}{V_d(t)} \leq 1$, for all $t \geq t_0$

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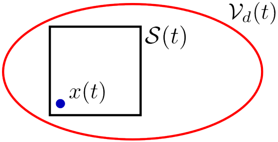
Design objective:

- Design event-triggered communication policy that recursively determines $\{t_k\}$ and np_k
- Ensure a uniform positive lower bound for $\{t_k - t_{k-1}\}_{k \in \mathbb{N}}$
- Ensure np_k is upper bounded by the given "channel capacity"
- Quantify the average data rate

Necessary data rate (non-state-triggered transmissions)



Set $\mathcal{S}(t)$ must lie within the set
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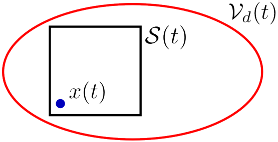

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Number of bits necessary to be transmitted between t_0 and t to meet the control goal:

$$\mathcal{B}(t, t_0) \geq \left(\operatorname{tr}(A) + \frac{n\beta}{2}\right) \log_2(e)(t - t_0) + \log_2\left(\frac{\operatorname{vol}(\mathcal{S}(t_0))}{c_P(V_d(t_0))^{\frac{n}{2}}}\right)$$

$$R_{\mathrm{as}} \triangleq \lim_{t\to\infty} \frac{\mathcal{B}(t, t_0)}{t - t_0} \geq \left(\operatorname{tr}(A) + \frac{n\beta}{2}\right) \log_2(e)$$

Assuming all eigenvalues of A have real parts greater than $-\beta$.

Control with arbitrary finite communication rate

Theorem

Assuming control goal is met with continuous and unquantized feedback, let

$$t_{k+1} = \min\left\{t \geq t_k \,:\, b(t) \geq 1,\ \dot{b}(t) \geq 0\right\}, \quad b(t) = \frac{V(x(t))}{V_d(t)}$$

$$np_k \geq n\underline{p_k} \triangleq n\left\lceil \log_2\left(\frac{d_e(t_k^-)}{c\sqrt{V_d(t_k)}}\right)\right\rceil, \quad np_k : \#\ \textit{bits sent at}\ t_k$$

Then

- *Inter-transmission times have a uniform positive lower bound,*
- $V(x(t)) \leq V_d(t)$ *for all* $t \geq t_0$

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No uniform bound on $\underline{p_k}$: for special initial conditions $\underline{p_k}$ can be arbitrarily large

Upper bound on the sufficient data rate

Corollary

If no disturbances, then for any $k \in \mathbb{N}$,

$$n(\underline{p_k} + \sum_{i=1}^{k-1} p_i) \leq n\left(\|A\|_\infty + \frac{\beta}{2}\right) \log_2(e)(t_k - t_0) + n \log_2\left(\frac{d_e(t_0)}{c\sqrt{V_d(t_0)}}\right) + n.$$

- Linear dependence on $t_k - t_0$
- Similar to the necessary data rate (e.g. $\mathrm{tr}(A) \to n\|A\|_\infty$)
- If more bits than sufficient are transmitted in the past, ($p_i > \underline{p_i}$ for some $i < k$), then fewer bits are sufficient at t_k
- For any $k \in \mathbb{N}$, if $t_k - t_{k-1}$ is bounded, then so is $\underline{p_k}$
- Data rate is bounded even though "communication rate" ($\underline{p_k}$) is not uniformly bounded

Control under bounded channel capacity

Channel-trigger function:

$$h_{\mathrm{ch}}(t) \triangleq \frac{d_e(t)}{c\sqrt{V_d(t)}\rho_T(b(t))}, \qquad \rho_T(b) \triangleq \frac{(w+\theta)(1-b)}{W(e^{(w+\theta)T}-1)} + 1,$$

$T > 0$ is a fixed design parameter.

Interpretation: $n\log_2(h_{\mathrm{ch}}(t))$ is a sufficient number of bits that, if transmitted at time t, ensures $b = \frac{V(x(t))}{V_d(t)} \leq 1$ for the next $TT = \min\{\Gamma_1(1,1), T\}$ units of time.

Control under bounded channel capacity

Theorem

Suppose all previous assumptions hold and that $h_{\text{ch}}(t_0) \le 2^{\bar{p}}$, *where* $n\bar{p}$ *is the upper bound on the number of bits that can be sent per transmission. Let*

$$t_{k+1} = \min\{t \ge t_k : b(t) \ge 1,\ \dot{b}(t) \ge 0 \ \text{OR}\ \frac{h_{\text{ch}}(t)}{2^{\bar{p}}} \ge 1\}$$

$$np_k \ge n\underline{p_k} \triangleq n\left\lceil \log_2\left(h_{\text{ch}}(t_k^-)\right)\right\rceil, \quad np_k : \#\ \textit{bits sent at}\ t_k$$

Then

- $\underline{p_1} \le \bar{p}$. *Further for each* $k \in \mathbb{N}$, *if* $p_k \in \mathbb{N} \cap [\underline{p_k}, \bar{p}]$, *then* $\underline{p_{k+1}} \le \bar{p}$.
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Non-instant communication: given an upper bound on the maximum communication time, T_M, main idea is to anticipate the threshold crossing of $b(t)$ and $\frac{h_{\text{ch}}(t)}{2^{\bar{p}}}$ well ahead.

Upper bound on the sufficient data rate

Corollary (Non-instant communication, disturbance)

Let $\bar{\theta} = \|A\|_\infty + \frac{\beta}{2}$. *For any* $k \in \mathbb{N}$,

$$\underline{p_k} \leq \log_2\left(\frac{e^{\bar{\theta}T_M}}{\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-)) - \alpha(T_M)}\right) + 1 + \log_2\left(\frac{e^{\bar{\theta}(t_k - t_0)}}{\prod_{j=1}^{k-1} 2^{p_j}}\epsilon(t_0) + \sum_{i=0}^{k-1}\prod_{j=i+1}^{k-1}\frac{e^{\bar{\theta}T_j}}{2^{p_j}}\alpha(T_i)\right).$$

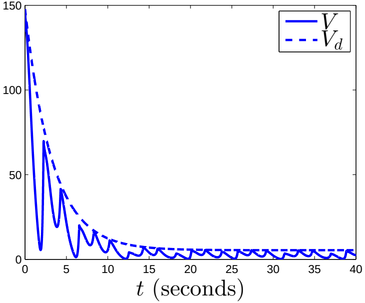
Corollary (Non-instant communication, no disturbance)

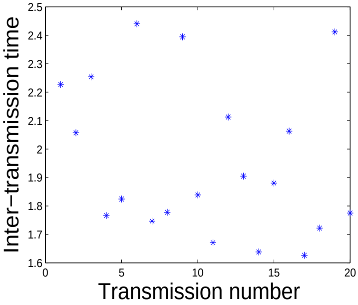
Let $\bar{\theta} = \|A\|_\infty + \frac{\beta}{2}$. *For any* $k \in \mathbb{N}$,

$$n\left(\underline{p_k} + \sum_{i=1}^{k-1} p_i\right) \leq n\left[\log_2\left(\frac{e^{\bar{\theta}T_M}}{\rho_T(\tilde{b}(T_M, b(t_k^-), \epsilon(t_k^-))}\right) + 1 + \bar{\theta}\log_2(e)(t_k - t_0) + \log_2(\epsilon(t_0))\right].$$

- In the general case, only an implicit characterization
- Effect of non-instant communication (through T_M) has only a "transient" effect on sufficient data rate
- If no disturbance and instant communication ($T_M = 0$), then we recover the data rate of the basic implementation

Simulation results: 2D linear system





Non-instantaneous communication, with disturbance, $\bar{p} = 20$.

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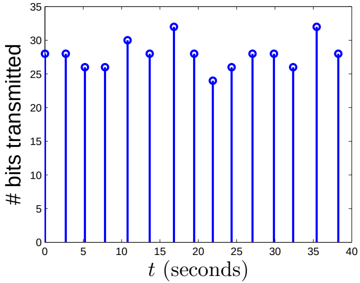

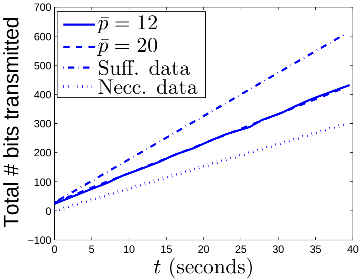

Instant communication and no disturbance

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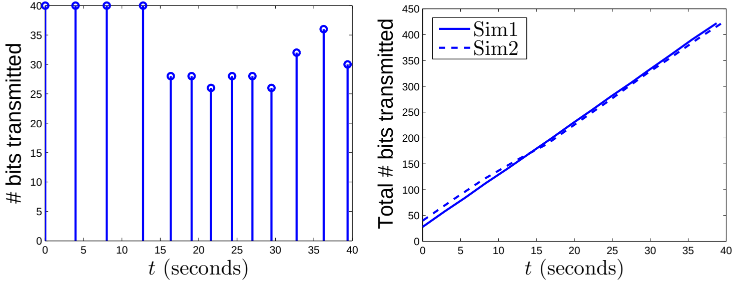

(a) (b)

Non-instantaneous communication without disturbance and $\bar{p} = 20$, (a) shows the number of bits on each transmission for "Sim2" (b) shows a comparison of the interpolated total number of bits transmitted in "Sim1,2".

Conclusions

Contribution:

- Fusion of complementary strengths of event-triggered control and information-theoretic control
- Stabilization with prescribed convergence rate
- Control under bounded and specified channel capacity
- Instantaneous and non-instantaneous transmissions
- Analysis of average data rate

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Future work:

- Overcoming the assumption on synchronized encoder and decoder in non-instant communication
- Efficient quantization and coding schemes
- Stochastic time varying channels

Thank You

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