The Effect of Delayed Side Information on Fundamental Limitations of Disturbance Attenuation

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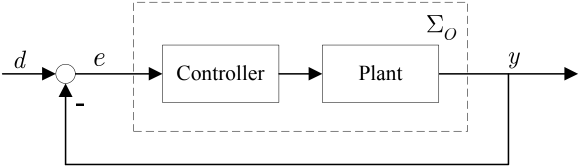
Disturbance attenuation in discrete-time feedback systems



Measure of disturbance attenuation performance at frequency ω:

$$S_{d,e}(\omega) = \sqrt{\Phi_e(\omega)/\Phi_d(\omega)}$$

- $\Phi_x(\omega)$ denotes the power spectral density of a wide sense stationary stochastic process x

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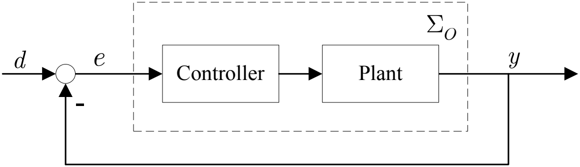

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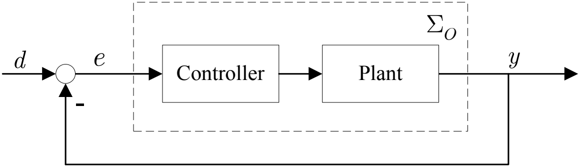

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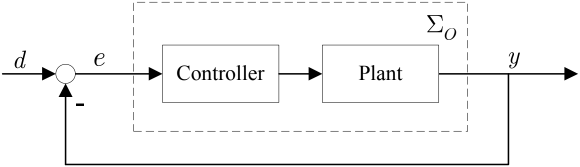

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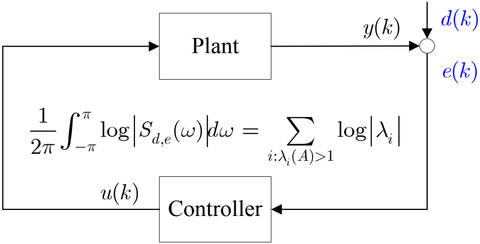
- If the controller is linear time-invariant, $S_{d,e}$ is the transfer function between d and e
- Small $S_{d,e}(\omega)$ implies good disturbance attenuation performance
- However, it is in general not possible to make $S_{d,e}(\omega)$ small at all frequencies

Classical Bode integral formula (DT, SISO, LTI)



$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left|S_{d,e}(\omega)\right| d\omega = \sum_{i:\lambda_i(A)>1} \log\left|\lambda_i\right|$$

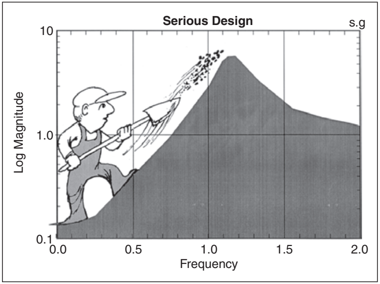


Figure: 1989 Bode lecture: respect the unstable, Gunter Stein

- Open-loop dynamics $\rightarrow$ achievable closed-loop performance.
- Controller can only *shape* the sensitivity integral.
- Important for controller design reference.

(Limited) literature review on Bode integral formula

- Bode (1945): **Continuous, SISO, LTI, stable plant**
- Freudenberg and Looze (1985): **Unstable plant**
- Freudenberg and Looze (1988), Chen and Nett (1995), Chen (2000), Ishii, Okano, and Hara (2011): **MIMO system**
- Iglesias (2001,2002), Sandberg and Bernhardsson (2005): **Time-varying system**
- Zhang and Iglesias (2003), Martins and Dahleh (2008), Yu and Mehta (2010): **Nonlinear control**
- Martins, Dahleh, and Doyle (2007): **Bode integral formula with disturbance preview**
- Zhao and Gupta (2014): **DT linear periodic systems**

Preview side information improves disturbance rejection

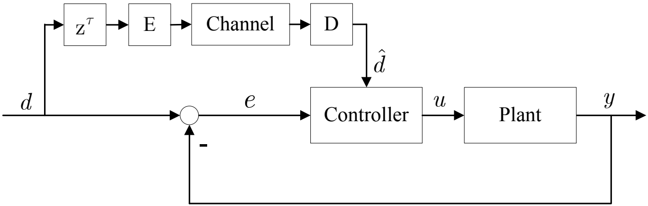


Figure: Preview side information at the controller improves closed-loop disturbance rejection (**Martins, Dahleh, and Doyle (2007)**).

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{d,e}(\omega) d\omega \geq \sum_{i:|\lambda_i(A)|>1} \log|\lambda_i(A)| - C$$

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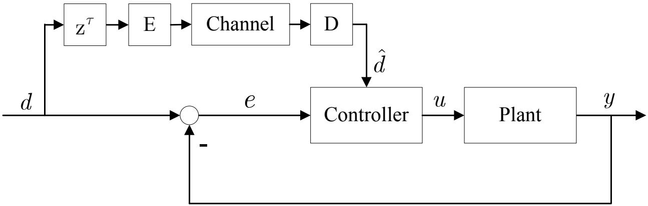

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What about delayed side information?

Can DSI improve disturbance rejection?

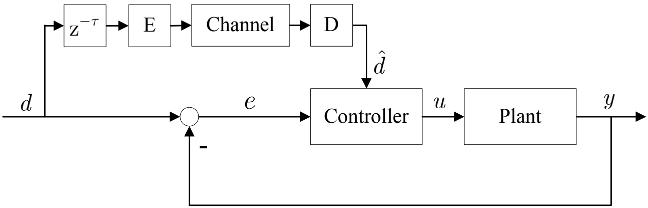


Figure: Feedback system configuration when the controller has delayed side information.

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{d,e}(\omega) d\omega \geq \quad ?$$

Intuitively, delayed side information about an i.i.d. disturbance process is not useful since it contains no information about the current or future disturbance.

Can DSI improve disturbance rejection?

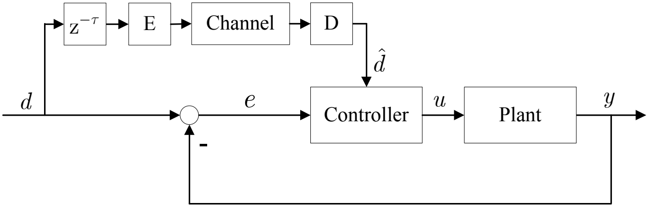


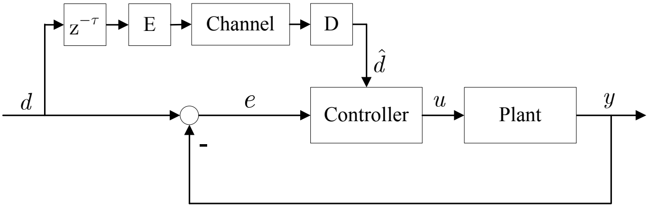
Figure: Feedback system configuration when the controller has DSI.

However, we will show that DSI improves disturbance rejection if the plant is unstable

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{d,e}(\omega) d\omega \geq \Big(\sum_{i:|\lambda_i(A)|>1} \log |\lambda_i(A)| - C \Big)^+$$

where $(x)^+ \triangleq \max(x, 0)$ and C represents the Shannon capacity of the side channel.

Problem setup



Plant:

$$\begin{bmatrix} x(k+1) \\ y(k) \end{bmatrix} = \begin{bmatrix} A & B \\ H & 0 \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}$$

where $x(k) \in \mathbb{R}^n$, $u(k), y(k), e(k) \in \mathbb{R}, \quad \forall k \in \mathbb{Z}^+$.

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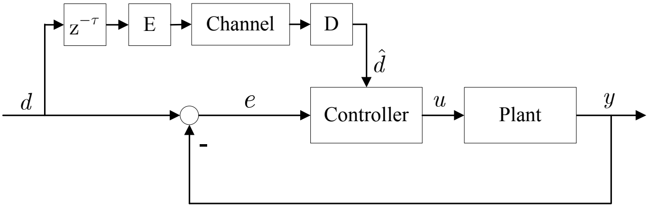

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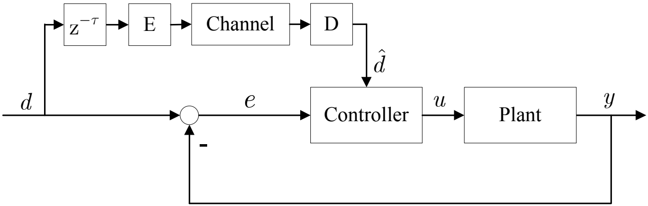
Controller:

$$u(k) = f_k(k, \hat{\mathbf{d}}^k, e^k)$$

where f_k is a time-varying, possibly nonlinear, function.

Assumptions



- The closed-loop system is mean-square stable.
- The disturbance process d is a zero-mean Gaussian process with i.i.d. r.v. $d(k)$. The plant's initial condition $\mathbf{x}(0)$ is a zero-mean r.v. with finite differential entropy, and independent of d.

DSI is useful for unstable plants

Theorem (DSI can reduce the log integral of sensitivity)

Denote the transfer function from the disturbance d to the error e by $S_{d,e}$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{d,e}(\omega) d\omega \geq \Big(\sum_{i:|\lambda_i(A)|>1} \log|\lambda_i(A)| - C\Big)^{+}.$$

- Unlike PSI, the contribution of DSI to the disturbance attenuation performance is upper bounded by $\sum_{i:|\lambda_i(A)|>1} \log|\lambda_i(A)|$.

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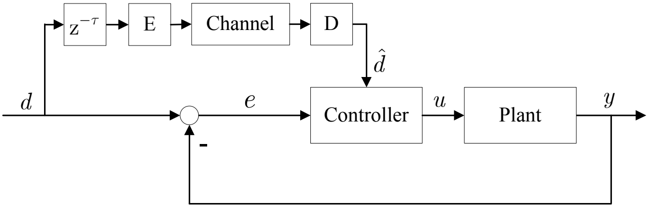
- Unlike PSI, the contribution of DSI to the disturbance attenuation performance is upper bounded by $\sum_{i:|\lambda_i(A)|>1} \log|\lambda_i(A)|$.
- DSI can only help to stabilize the open-loop system but cannot reduce the controller's uncertainty about the disturbance.

DSI can help to stabilize an unstable plant



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- The power in e comes from 2 sources: disturbance d and stabilizing information about $x(0)$.

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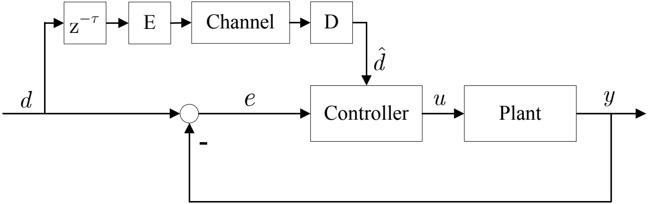

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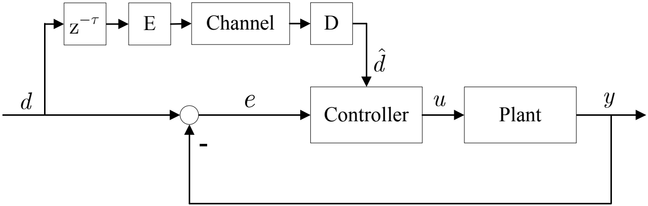
- The power in e comes from 2 sources: disturbance d and stabilizing information about $x(0)$.
- Even if $\hat{d}$ is independent of $x(0)$, it can still help to stabilize the system by providing *conditional information* about the initial condition given e $(I(\hat{\mathbf{d}}^k;\mathbf{x}(0)|\mathbf{e}^k) > 0)$.

Lower bound on log integral of sensitivity is tight



Consider the scalar plant

$$\mathbf{x}(k+1) = a\mathbf{x}(k) + \mathbf{u}(k), \quad \mathbf{y}(k) = \mathbf{x}(k),$$

for some $|a| > 1$ and channel capacity $C > \log |a|$ bits/sec.

Let the side channel transmit $\mathbf{d}(0)$ at every time step k, so that the controller has an increasingly better estimate $\hat{\mathbf{d}}_0(k)$ of $\mathbf{d}(0)$.

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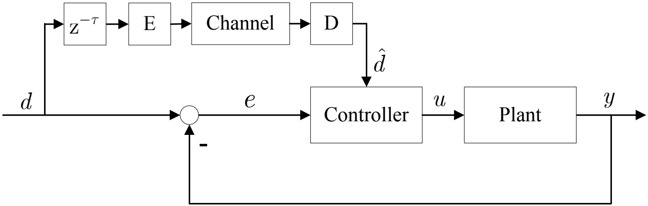
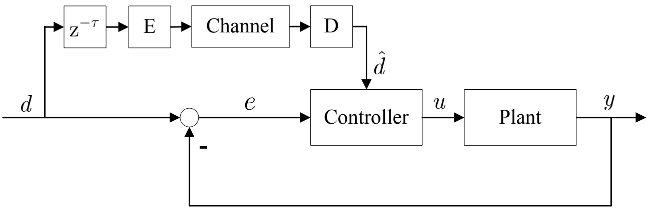


- The encoder/decoder pair is such that

$$E(\|\mathbf{d}(0) - \hat{\mathbf{d}}_0(k)\|^2) \leq 2^{-2Ck} E(\|\mathbf{d}(0)\|^2).$$

Lower bound on log integral of sensitivity is tight



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- Use the control law

$$\mathbf{u}(k) = \begin{cases} a(\hat{\mathbf{d}}_0(0) - \mathbf{e}(0)), & k = 0, \\ a^{k+1}(\hat{\mathbf{d}}_0(k) - \hat{\mathbf{d}}_0(k-1)), & k \geq 1. \end{cases}$$

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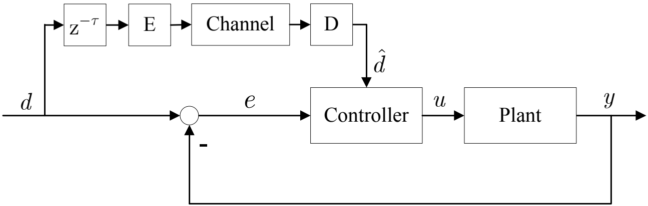

- The corresponding closed-loop dynamics is given by

$$\mathbf{x}(k) = a^k(\hat{\mathbf{d}}_0(k-1) - \mathbf{d}(0)).$$

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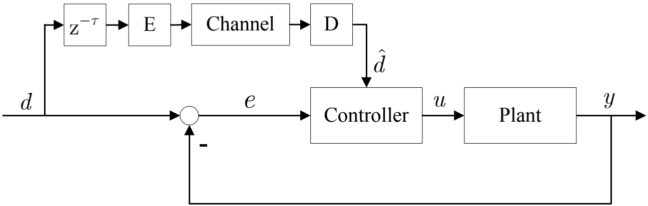

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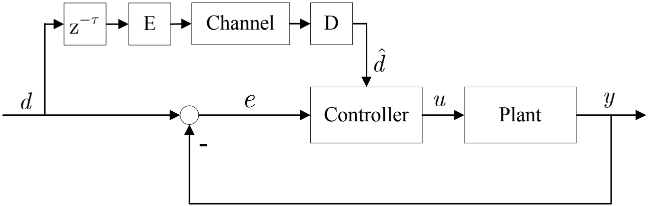
- Based on the above computation, it follows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|S_{\mathbf{d},\mathbf{e}}(\omega)|d\omega = 0 = \left(\log|a| - C, 0\right)^+,$$

and the lower bound is achieved for any $C > \log|a|$.

Conclusion and future direction



Even delayed side information can help to attenuate the disturbance if the plant is unstable, i.e., the log integral of sensitivity can be reduced at most $\sum_{i:|\lambda_i(A)|>1} \log|\lambda_i(A)|$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log S_{d,e}(\omega)d\omega \geq \Big(\sum_{i:|\lambda_i(A)|>1} \log|\lambda_i(A)| - C\Big)^{+}$$

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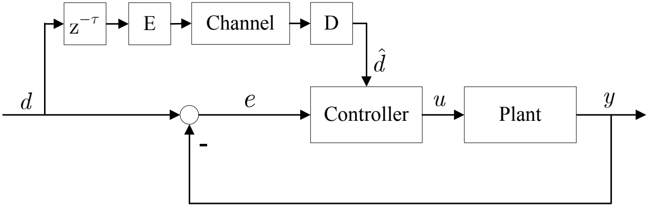
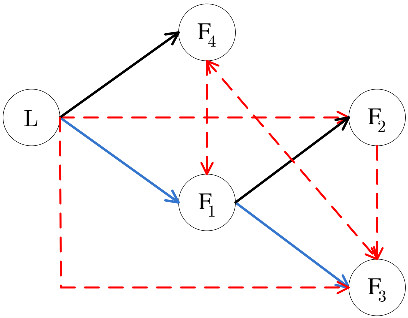


Future work 1: study the effect of DSI on the log integral of **complementary sensitivity function**

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log S_{d,y}(\omega)d\omega \geq \quad ?$$

Conclusion and future direction



Future work 2: study network control system where the side information is a mix of preview and delayed side information.