Transient-state Feasibility Set Approximation of Power Networks Against Disturbances

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Power Network: Efficiency & Robustness

Efficiency

1. Economic Dispatch
2. Optimal Power Flow

...

Robustness

1. Voltage Collapse
2. Cascading Failure

...

Efficiency

1. Economic Dispatch
2. Optimal Power Flow

...

Robustness

1. Voltage Collapse
2. Cascading Failure

...

How to identify the **disturbances** under which
(a) **frequencies** of buses stay within safe bounds, and
(b) **power flows** of transmission lines stay within safe bounds?

Outline

1. Problem Statement
   - Linearized Power Network Dynamics
   - Disturbance Modeling
2. Equivalent Transformation
   - Time Domain Solution
   - Set Decomposition
3. Approximation of the Feasibility Set
   - Outer Approximations
   - Inner Approximations
4. Simulations

Linearized Power Network Dynamics

$$\begin{bmatrix} \dot{\Lambda}(t) \\ \dot{\Omega}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m\times m} & D \\ -M^{-1}D^T Y_b & -M^{-1}E \end{bmatrix} \begin{bmatrix} \Lambda(t) \\ \Omega(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_m \\ M^{-1}P(t) \end{bmatrix}$$

$\Lambda = [\lambda_1, \lambda_2, \dots \lambda_m]^T \in \mathbb{R}^m$— **angle difference vector**
$\Omega = [\omega_1, \omega_2, \dots \omega_n]^T \in \mathbb{R}^n$— **frequency vector**
$M \in \mathbb{R}^{n\times n}$— inertia matrix
$E \in \mathbb{R}^{n\times n}$— damping/droop parameter matrix
$Y_b \in \mathbb{R}^{m\times m}$— susceptance matrix
$P = [p_1, p_2, \dots p_n]^T \in \mathbb{R}^n$— power injection vector
($Y_b\Lambda = [f_1, f_2, \dots f_m]^T \in \mathbb{R}^m$— **power flow vector**)

Disturbance Modeling

Power network dynamics

$$\begin{bmatrix} \dot{\Lambda}(t) \\ \dot{\Omega}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m\times m} & D \\ -M^{-1}D^T Y_b & -M^{-1}E \end{bmatrix} \begin{bmatrix} \Lambda(t) \\ \Omega(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_m \\ M^{-1}P(t) \end{bmatrix}$$

Disturbance Modeling

Power network dynamics

$$\begin{bmatrix} \dot{\Lambda}(t, K) \\ \dot{\Omega}(t, K) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m\times m} & D \\ -M^{-1}D^T Y_b & -M^{-1}E \end{bmatrix} \begin{bmatrix} \Lambda(t, K) \\ \Omega(t, K) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_m \\ M^{-1}P(t, K) \end{bmatrix}$$

$P(t, K) = P_0(t) + \bar{P}(t, K)$
$P_0(t) \in \mathbb{R}^n$: scheduled power injection
$\bar{P}(t, K) \in \mathbb{R}^n$: power disturbance

Disturbance Modeling

Power network dynamics

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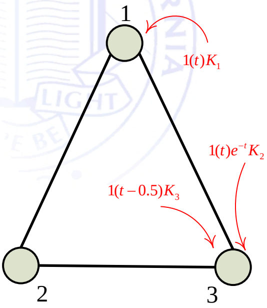
$P(t,K) = P_0(t) + \bar{P}(t,K)$
$P_0(t) \in \mathbb{R}^n$: scheduled power injection
$\bar{P}(t,K) \in \mathbb{R}^n$: power disturbance

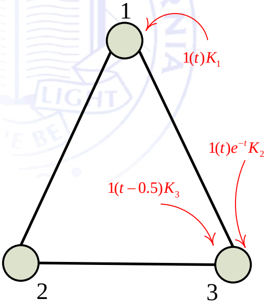
Disturbance model

$$\bar{P}(t,K) = B_K D_{\zeta(t)} K$$

Example

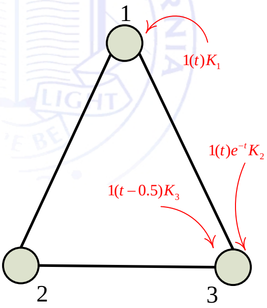


Example



$$\bar{P}(t,K) = \begin{bmatrix} 1(t)K_1 \\ 0 \\ 1(t)e^{-t}K_2 + 1(t-0.5)K_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1(t) & 0 & 0 \\ 0 & 1(t)e^{-t} & 0 \\ 0 & 0 & 1(t-0.5) \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}$$
$$= B_K \operatorname{diag}\left\{1(t)\ 1(t)e^{-t}\ 1(t-0.5)\right\} K$$
$$= B_K D_{\zeta(t)} K$$

Example



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$$= B_K \operatorname{diag}\left\{1(t) \ 1(t)e^{-t} \ 1(t-0.5)\right\} K$$

$$= B_K D_{\zeta(t)} K$$

$\bar{P}(t,K) =$ "location $\times$ trajectory form $\times$ amplitude"

Problem Statement

Power network dynamics

$$\begin{bmatrix} \dot{\Lambda}(t,K) \\ \dot{\Omega}(t,K) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m\times m} & D \\ -M^{-1}D^TY_b & -M^{-1}E \end{bmatrix} \begin{bmatrix} \Lambda(t,K) \\ \Omega(t,K) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_m \\ M^{-1}P(t,K) \end{bmatrix}$$

Problem Statement

Power network dynamics

$$\begin{bmatrix}\dot{\Lambda}(t,K)\\ \dot{\Omega}(t,K)\end{bmatrix}=\begin{bmatrix}\mathbf{0}_{m\times m} & D\\ -M^{-1}D^TY_b & -M^{-1}E\end{bmatrix}\begin{bmatrix}\Lambda(t,K)\\ \Omega(t,K)\end{bmatrix}+\begin{bmatrix}\mathbf{0}_m\\ M^{-1}\big(P_0(t)+B_KD_{\zeta(t)}K\big)\end{bmatrix}$$

For a given $0 \leqslant t_1 < t_2$, find all K's that guarantee:

1. *Transient-state frequency bound:* $\Omega^{\min} \leqslant \Omega(t,K) \leqslant \Omega^{\max}$, $\forall t \in [t_1, t_2]$
2. *Transient-state power flow bound:* $F^{\min} \leqslant Y_b\Lambda(t,K) \leqslant F^{\max}$, $\forall t \in [t_1, t_2]$

Problem Statement

Power network dynamics

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$$\Psi \triangleq \Big\{K \in \mathbb{R}^s \mid \Omega^{\min} \leqslant \Omega(t,K) \leqslant \Omega^{\max},\ F^{\min} \leqslant Y_b\Lambda(t,K) \leqslant F^{\max},\ \forall t \in [t_1,t_2]\Big\}$$

Ψ:(**transient-state**) **feasibility set**

Goal: Characterize Ψ!

Outline

Time Domain Solution

$$\begin{bmatrix} \dot{\Lambda}(t,K) \\ \dot{\Omega}(t,K) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m\times m} & D \\ -M^{-1}D^T Y_b & -M^{-1}E \end{bmatrix} \begin{bmatrix} \Lambda(t,K) \\ \Omega(t,K) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_m \\ M^{-1}\left(P_0(t) + B_K D_{\zeta(t)} K\right) \end{bmatrix}$$

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$$\Updownarrow$$

$$\dot{x}(t,K) = Ax(t,K) + \begin{bmatrix} \mathbf{0}_m \\ M^{-1}\left(P_0(t) + B_K D_{\zeta(t)} K\right) \end{bmatrix}$$

Time Domain Solution

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$$\dot{x}(t,K) = Ax(t,K) + \begin{bmatrix} \mathbf{0}_m \\ M^{-1}\left(P_0(t) + B_K D_{\zeta(t)} K\right) \end{bmatrix}$$

$\Updownarrow$ Solve first-order ODE

$$x(t,K) = S(t) + V(t)K$$

where

$$S(t) \triangleq e^{At}x_0 + \int_0^t e^{A(t-\tau)} \begin{bmatrix} \mathbf{0}_m \\ M^{-1}P_0(\tau) \end{bmatrix} \mathrm{d}\tau, \quad V(t) \triangleq \int_0^t e^{A(t-\tau)} \begin{bmatrix} \mathbf{0}_m \\ M^{-1}B_K D_{\zeta(\tau)} \end{bmatrix} \mathrm{d}\tau$$

Equivalent Transformation

$$\Psi \triangleq \left\{ K \in \mathbb{R}^s \mid \Omega^{\min} \leqslant \Omega(t, K) \leqslant \Omega^{\max},\ F^{\min} \leqslant Y_b \Lambda(t, K) \leqslant F^{\max},\ \forall t \in [t_1, t_2] \right\}$$

Equivalent Transformation

$$\Psi \triangleq \Big\{K \in \mathbb{R}^s \mid \Omega^{\min} \leqslant \Omega(t, K) \leqslant \Omega^{\max},\ F^{\min} \leqslant Y_b \Lambda(t, K) \leqslant F^{\max},\ \forall t \in [t_1, t_2]\Big\}$$

$$\Updownarrow$$

$$\Psi = \big\{K \in \mathbb{R}^s \mid x^{\min} \leqslant S(t) + V(t)K \leqslant x^{\max},\ \forall t \in [t_1, t_2]\big\}$$

where

$$x^{\max} \triangleq \begin{bmatrix} \Omega^{\max} \\ Y_b^{-1} F^{\max} \end{bmatrix},\ x^{\min} \triangleq \begin{bmatrix} \Omega^{\min} \\ Y_b^{-1} F^{\min} \end{bmatrix}$$

Set Decomposition

$$\Psi = \left\{ K \in \mathbb{R}^s \mid x^{\min} \leqslant S(t) + V(t)K \leqslant x^{\max},\ \forall t \in [t_1, t_2] \right\}$$

Set Decomposition

$$\begin{aligned}\Psi &= \left\{K \in \mathbb{R}^s \mid x^{\min} \leqslant S(t) + V(t)K \leqslant x^{\max},\ \forall t \in [t_1, t_2]\right\} \\ &= \bigcap_{t_1 \leqslant t \leqslant t_2} \left\{K \in \mathbb{R}^s \mid x^{\min} \leqslant S(t) + V(t)K \leqslant x^{\max}\right\}\end{aligned}$$

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$\Rightarrow$ Ψ contains **infinitely many** constraints $\Rightarrow$ **Approximation**

Set Decomposition

$$\Psi = \left\{ K \in \mathbb{R}^s \mid x^{\min} \leqslant S(t) + V(t)K \leqslant x^{\max},\ \forall t \in [t_1, t_2] \right\}$$

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$$= \bigcap_{i=1,2,\ldots n+m} \left\{K \in \mathbb{R}^s \mid x_i^{\min} \leqslant [S(t)]_i + [V(t)]_i K \leqslant x_i^{\max},\ \forall t \in [t_1, t_2]\right\}$$
$$\triangleq \bigcap_{i=1,2,\ldots n+m} \Psi_i$$

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$$\triangleq \bigcap_{i=1,2,\ldots n+m} \Psi_i$$

Approximation of $\Psi_i \Rightarrow$ Approximation of Ψ

Outline

From Vector to Scalar

$$\Psi_i \triangleq \left\{K \in \mathbb{R}^s \mid x_i^{\min} \leqslant [S(t)]_i + [V(t)]_i K \leqslant x_i^{\max},\ \forall t \in [t_1, t_2]\right\}$$

From Vector to Scalar

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$$\Uparrow$$

$$\Sigma \triangleq \left\{K \in \mathbb{R}^s \mid y^{\min} \leqslant y(t, K) \leqslant y^{\max},\ \forall t \in [t_1, t_2]\right\}$$

where $y(t, K)$ is some scalar signal

From Vector to Scalar

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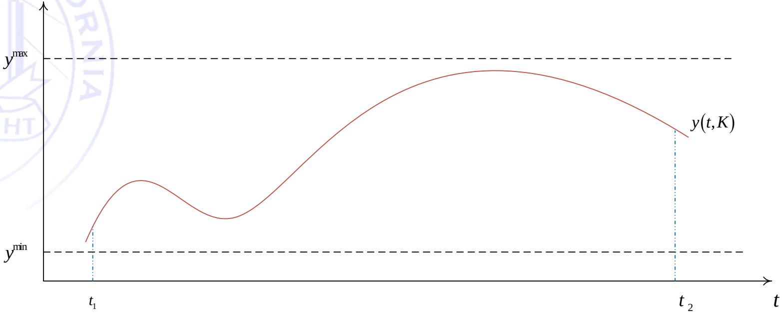
$$\Sigma \triangleq \left\{ K \in \mathbb{R}^s \mid y^{\min} \leqslant y(t, K) \leqslant y^{\max},\ \forall t \in [t_1, t_2] \right\}$$

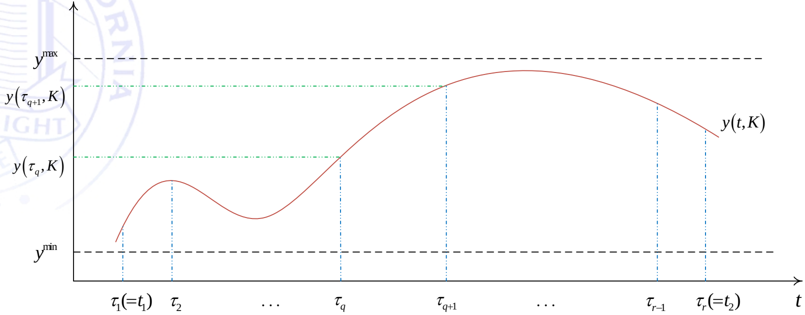
Strategy: Construct **inner approximation** Σ_I & **outer approximation** Σ_O

$$\Sigma_I \subseteq \Sigma \subseteq \Sigma_O$$

Outer Approximation

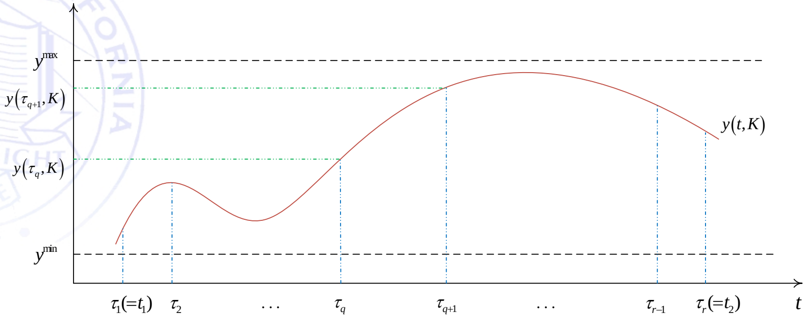


Outer Approximation



$t_1 = \tau_1 < \tau_2 < \cdots < \tau_r = t_2$: sampling points

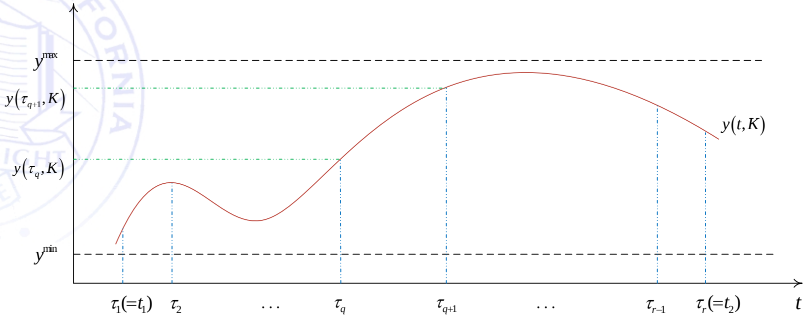
Outer Approximation



$t_1 = \tau_1 < \tau_2 < \cdots < \tau_r = t_2$: sampling points

$$y^{\min} \leqslant y(t, K) \leqslant y^{\max},\ \forall t \in [t_1, t_2] \Rightarrow y^{\min} \leqslant y(\tau_q, K) \leqslant y^{\max},\ \forall q \in [1, r]_{\mathbb{N}}$$

Outer Approximation



$t_1 = \tau_1 < \tau_2 < \cdots < \tau_r = t_2$: sampling points

$$y^{\min} \leqslant y(t, K) \leqslant y^{\max}, \ \forall t \in [t_1, t_2] \Rightarrow y^{\min} \leqslant y(\tau_q, K) \leqslant y^{\max}, \ \forall q \in [1, r]_{\mathbb{N}}$$

Outer approximation

Define $\Sigma_O \triangleq \left\{K \mid y^{\min} \leqslant y(\tau_q, K) \leqslant y^{\max}, \ \forall q \in [1, r]_{\mathbb{N}}\right\}$, then $\Sigma \subseteq \Sigma_O$

Outer Approximation

Outer approximation

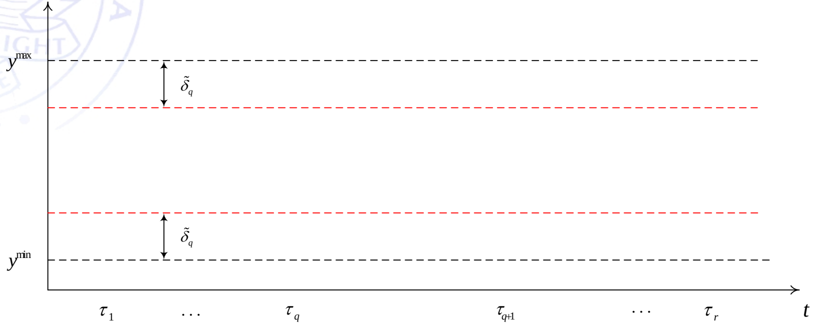
Define $\Sigma_O \triangleq \{K \mid y^{\min} \leqslant y(\tau_q, K) \leqslant y^{\max},\ \forall q \in [1, r]_{\mathbb{N}}\}$, then $\Sigma \subseteq \Sigma_O$

Note:

1. If $\dot{y}(t, K)$ is bounded, and $\forall q \in [1, r-1]_{\mathbb{N}}$, $(\tau_{q+1} - \tau_q) \to 0^+$, then $\Sigma_O \to \Sigma$
2. #constraints in Σ_O is r

Inner Approximation

Focus on $[\tau_q, \tau_{q+1}]$



Inner Approximation

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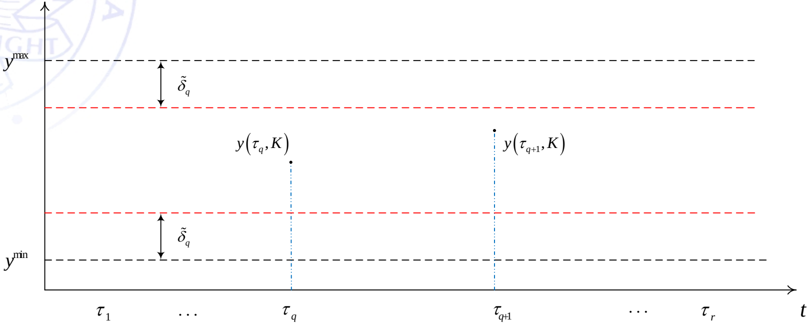

Inner Approximation

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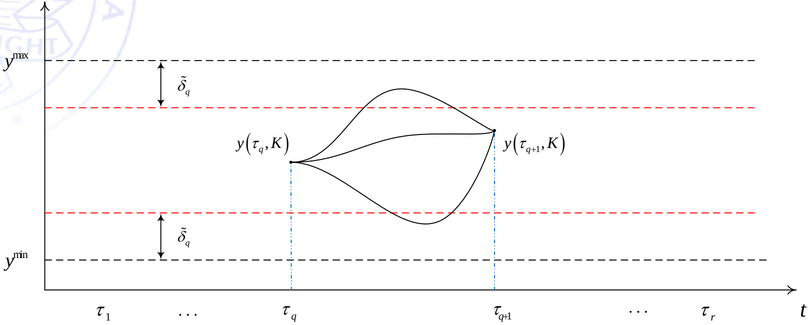
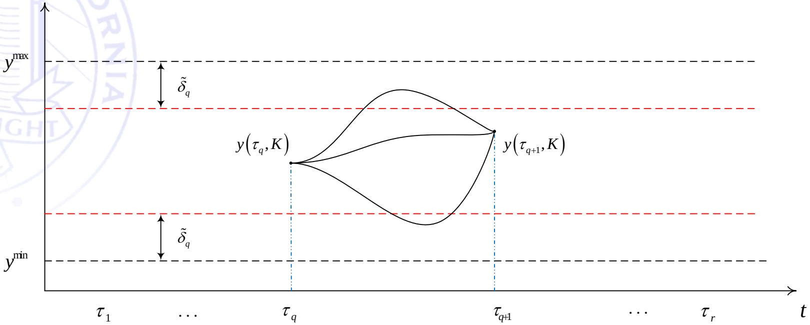


Inner Approximation

Focus on $[\tau_q, \tau_{q+1}]$



Suppose $\exists\ \infty > \tilde{d}_q \geqslant \max\limits_{K, t \in [\tau_q, \tau_{q+1}]} \{|\dot{y}(t, K)|\}$. Let $\tilde{\delta}_q \triangleq \tilde{d}_q(\tau_{q+1} - \tau_q)/2$

If $y^{\min} + \tilde{\delta}_q \leqslant y(\tau_q, K),\ y(\tau_{q+1}, K) \leqslant y^{\max} - \tilde{\delta}_q$, then

$$y^{\min} \leqslant y(t, K) \leqslant y^{\max},\ \forall t \in [\tau_q, \tau_{q+1}]$$

Inner Approximation

Let q go through $1, 2, \ldots r-1 \Rightarrow$

Inner approximation

Define

$$\Sigma_I \triangleq \left\{ K \mid y^{\min} + \tilde{\delta}_q \leqslant y(\tau_q, K),\; y(\tau_{q+1}, K) \leqslant y^{\max} - \tilde{\delta}_q,\; \forall q \in [1, r-1]_{\mathbb{N}} \right\},$$

then $\Sigma_I \subseteq \Sigma$

Note:

1. If $\forall q \in [1, r-1]$, $(\tau_{q+1} - \tau_q) \to 0^+$, then $\Sigma_I \to \Sigma$
2. #constraints in Σ_I is $2(r-1)$

Back to the Vector Case

$$\Psi = \bigcap_{i=1,2,\dots n+m} \left\{ K \in \mathbb{R}^s \mid x_i^{\min} \leqslant [S(t)]_i + [V(t)]_i K \leqslant x_i^{\max},\ \forall t \in [t_1, t_2] \right\}$$
$$\triangleq \bigcap_{i=1,2,\dots n+m} \Psi_i$$

1. Associate each Ψ_i sampling points $t_1 = \tau_1^i, \tau_2^i, \dots, \tau_{r(i)}^i = t_2$
2. Obtain $\Psi_{i,O}$ and $\Psi_{i,I}$ s.t. $\Psi_{i,I} \subseteq \Psi_i \subseteq \Psi_{i,O}$
3. Define

$$\Psi_O \triangleq \bigcap_{i=1,2,\dots n+m} \Psi_{O,i},\ \Psi_I \triangleq \bigcap_{i=1,2,\dots n+m} \Psi_{I,i}$$

$\Rightarrow \Psi_I \subseteq \Psi \subseteq \Psi_O$

4. If $(\tau_{q+1}^i - \tau_q^i) \to 0^+$ for every $q \in [1, r(i)-1]_{\mathbb{N}}$ and every $i \in [1, m+n]_{\mathbb{N}}$, then $\Psi_I \to \Psi$ and $\Psi_O \to \Psi$

Outline

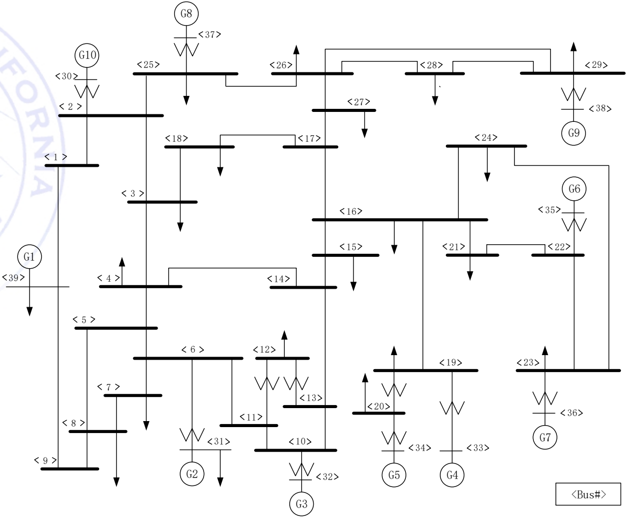


Figure: IEEE 39-bus power network.

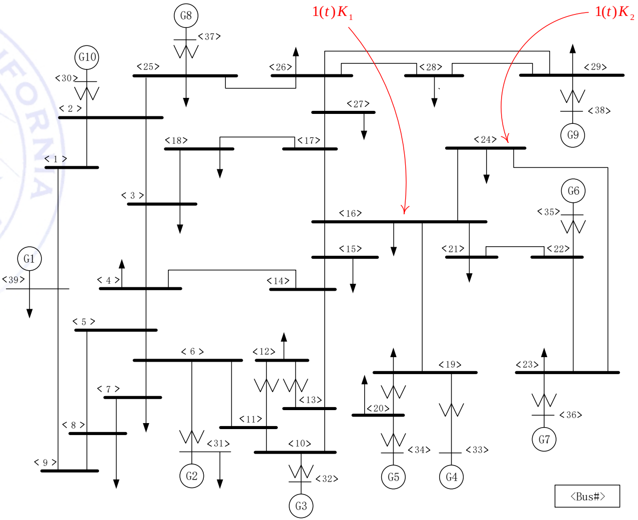


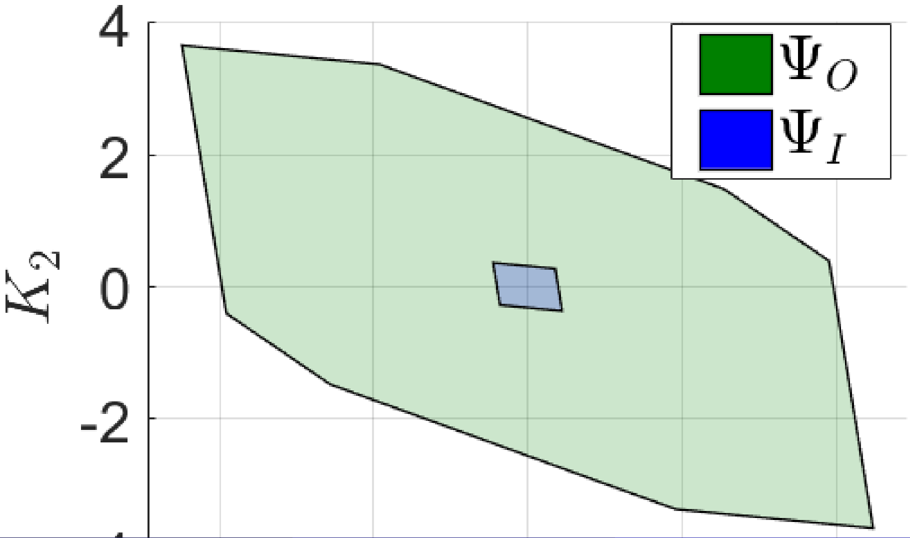
Figure: IEEE 39-bus power network.

$$\Psi \triangleq \left\{ K \mid \Omega^{\min} \leqslant \Omega(t, K) \leqslant \Omega^{\max},\ F^{\min} \leqslant Y_b \Lambda(t, K) \leqslant F^{\max},\ \forall t \in [t_1, t_2] \right\}$$

$$\begin{aligned}
K &= \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}, \\
t_0 &= 0\text{s}, \\
t_1 &= 3\text{s}, \\
\Omega^{\min} &= -0.5\text{Hz} \times \mathbf{1}_{39}, \\
\Omega^{\max} &= 0.5\text{Hz} \times \mathbf{1}_{39}, \\
F^{\min} &= -10\text{unit} \times \mathbf{1}_{46}, \\
F^{\max} &= 10\text{unit} \times \mathbf{1}_{46}, \\
\tau^i &= (0s, 0.02s, 0.04s, ..., 2.98s, 3s),\ \forall i = 1, 2, \ldots 39
\end{aligned}$$

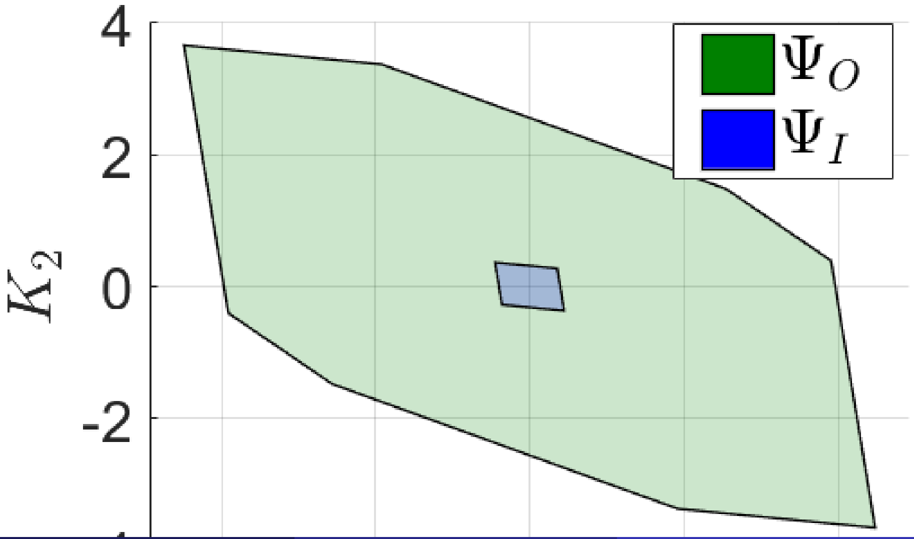
Simulations

[b]0.31



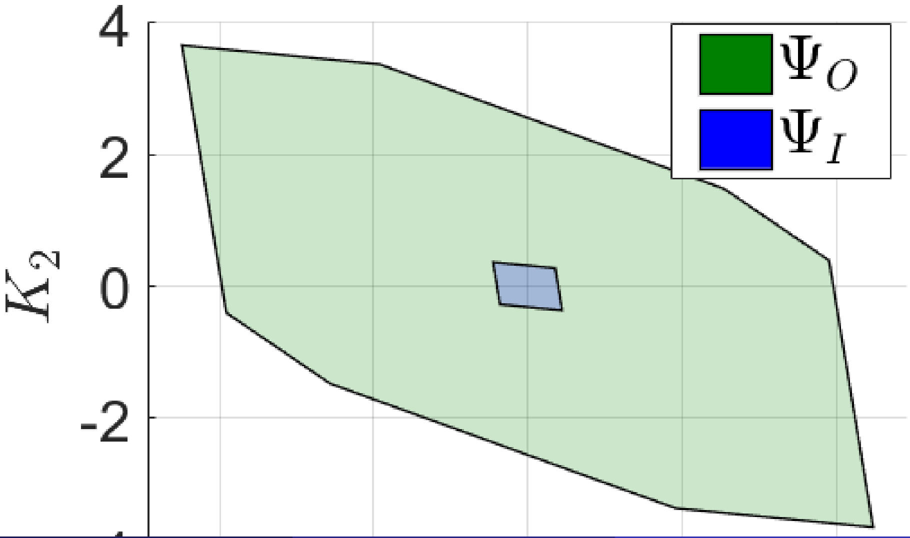
Simulations

[b]0.31

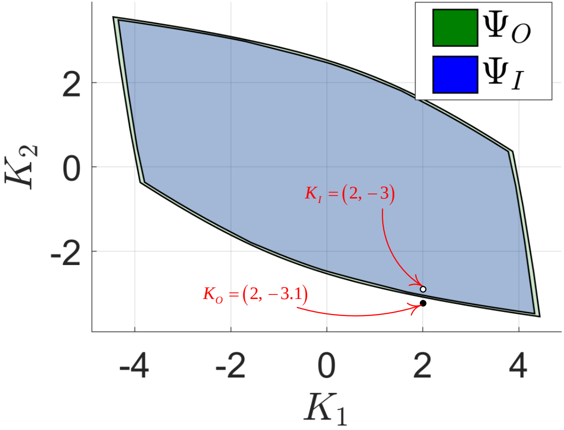


Simulations

[b]0.31



Simulations



$\Rightarrow K_I \in \Psi, K_O \notin \Psi$

Simulations

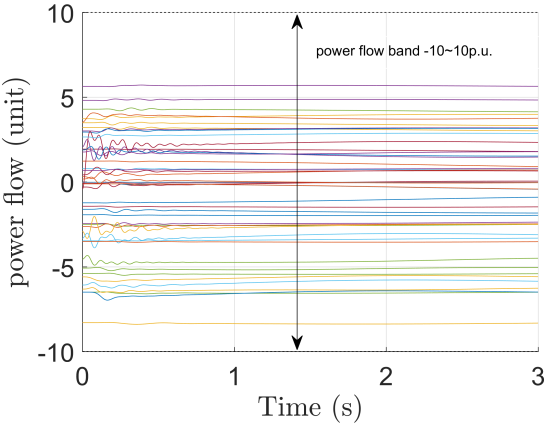


Figure: Flow response w.r.t. K_I.

Simulations

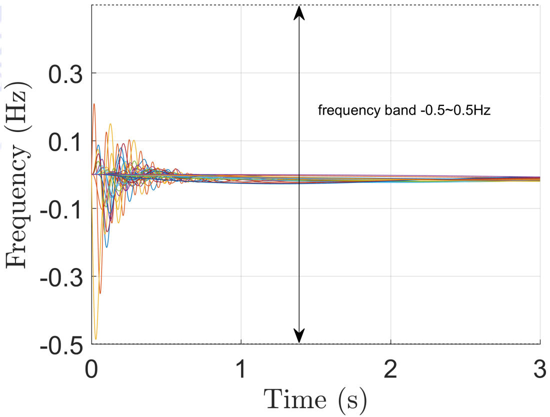


Figure: Frequency response w.r.t. K_I.

Simulations

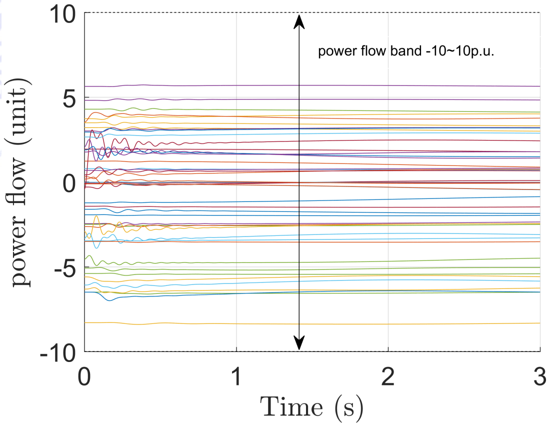


Figure: Flow response w.r.t. K_O.

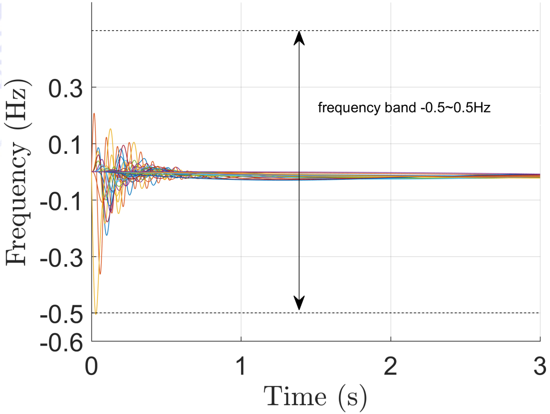


Figure: Frequency response w.r.t. K_O.

Simulations

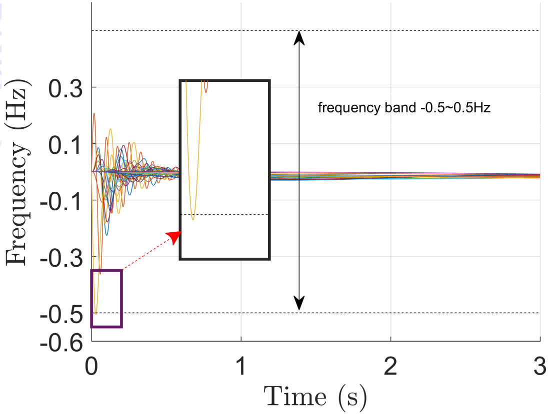


Figure: Frequency response w.r.t. K_O.

Conclusion & Future Work

Conclusion

1. Provided inner and out approximations of the feasibility set.
2. Proved the convergence of the approximations.
3. Developed an algorithm to reduce the approximation gaps w/o adding new sampling points.

Future Work

1. Consider uncertain trajectory form.
2. Extend results to nonlinear swing dynamics.