Exponentially Fast Distributed Coordination for Nonsmooth Convex Optimization

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Problem. Given a network of n agents whose objective is to cooperatively solve a convex optimization problem

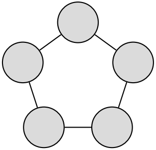

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Figure: A network of agents that seek to solve a convex program

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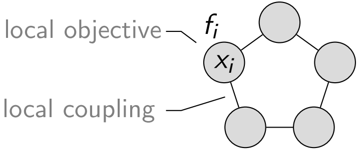

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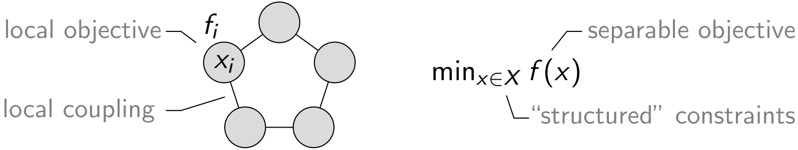

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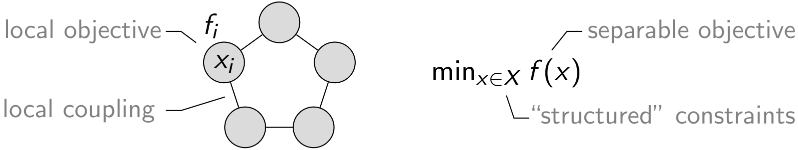

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Objective. Develop continuous-time algorithms that

- are asymptotically correct
- are providing performance guarantees*
- are amenable to distributed implementation

1. Distributed Convex Optimization

2. Continuous-Time Distributed Convex Optimization
   - Problem Statement
   - Saddle-Point Dynamics
   - Saddle-Point-Like Dynamics

3. Performance Characterization
   - Distributed Optimization under Inequality Constraints
   - Distributed Optimization under Equality Constraints

4. Conclusions

Problem. Consider the convex optimization problem

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Idea. Saddle-point dynamics to solve (P)

Augmented Lagrangian. Let $\kappa > 0$ and $L_\kappa : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ be

$$L_\kappa(x, \lambda) = f(x) + \frac{1}{2}\|h(x)\|^2 + \langle \lambda, h(x) \rangle + \kappa \langle \mathbb{1}_m, [g(x)]^+ \rangle$$

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Saddle Points. ((P) convex $\wedge$ strong Slater assumption) $\Rightarrow$

$$\forall x^* \ \exists (\lambda^*, \mu^*) \ : \ L_\kappa(x^*, \lambda) \leq L_\kappa(x^*, \lambda^*) \leq L_\kappa(x, \lambda^*), \ \forall (x, \lambda)$$

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Lemma (Saddle-point theorem)**.** Let $L_\kappa : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ and let $(x^*, \lambda^*) \in \mathrm{sp}(L_\kappa)$ with $\kappa > \|\mu^*\|_\infty$ for some dual solution μ^* of (P). Then, x^* is a solution of (P).

Saddle-point dynamics.

$$\begin{cases} \dot{x}(t) + \partial_x L_\kappa(x(t), \lambda(t)) \overset{a.e.}{\ni} 0_n, & x(t_0) \in \mathbb{R}^n \\ \dot{\lambda}(t) - \partial_\lambda L_\kappa(x(t), \lambda(t)) \overset{a.e.}{\ni} 0_p, & \lambda(t_0) \in \mathbb{R}^p \end{cases} \tag{SPD}$$

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Interpretation.

- gradient-descent dynamics in the primal variable
- gradient-ascent dynamics in the dual variable

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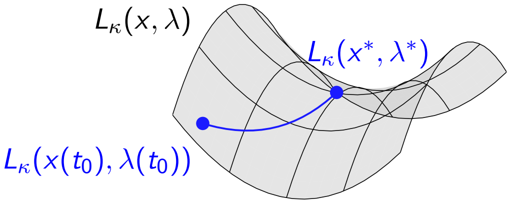

Theorem (Asymptotic convergence). Let $\kappa > \|\mu^*\|_\infty$ for some dual solution μ^* of (P). Let ∂f be (strictly) monotone. Then,

$$\lim_{t \to +\infty} \operatorname{dist}\big((x(t), \lambda(t)), \operatorname{sp}(L_\kappa)\big) = 0.$$

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Idea. Two strategies:

1. Let ∂f be monotone. Let $(x^*, \lambda^*) \in \mathrm{sp}(L_\kappa)$ and define

$$V_1(x, \lambda) = \frac{1}{2}\|x - x^*\|^2 + \frac{1}{2}\|\lambda - \lambda^*\|^2$$

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2. Let ∂f be strictly monotone. Let $(\tilde{x}, \tilde{\lambda}) \in \operatorname{sp}(L_\kappa)$ and define

$$V_2(x, \lambda) = L_\kappa(x, \lambda) - L_\kappa(\tilde{x}, \tilde{\lambda}) + \min_{(x^*, \lambda^*) \in \operatorname{sp}(L_\kappa)} \left(\frac{1}{2}\|x - x^*\|^2 + \frac{1}{2}\|\lambda - \lambda^*\|^2\right)$$

$\rightsquigarrow$ V_2 is a Liapunov function

Let $G = \{x \in \mathbb{R}^n \mid g(x) \leq 0_m\}$ and define $F : G \times \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ by

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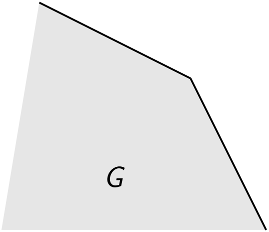

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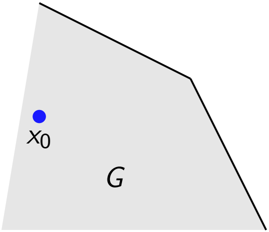

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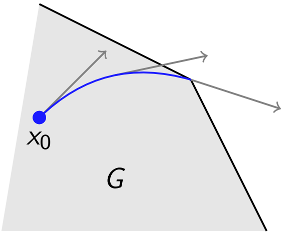

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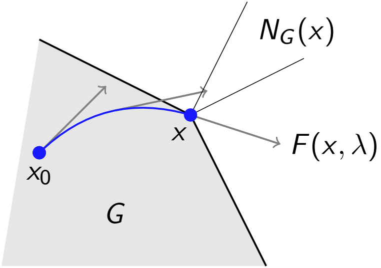

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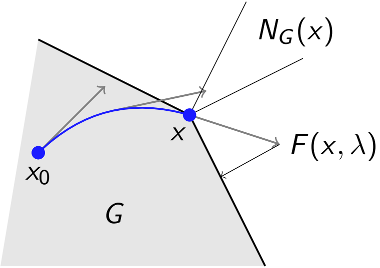

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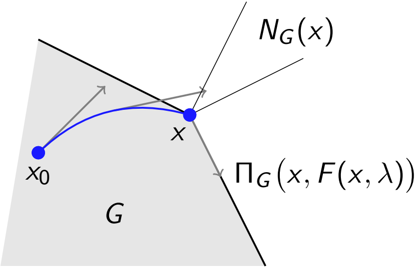

Theorem (Relationship of solutions). Let $(x, \lambda) : [t_0, +\infty) \to G \times \mathbb{R}^p$ be any solution of (SPLD) starting from $(x_0, \lambda_0) \in G \times \mathbb{R}^p$. Then, there exists $\kappa > 0$ such that it is also a solution of (SPD).

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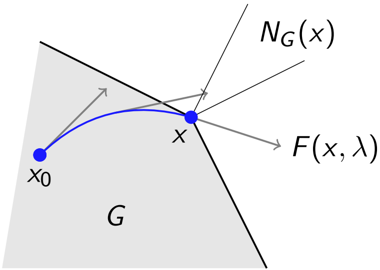

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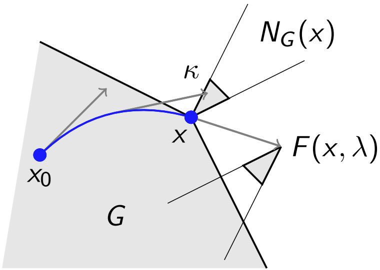

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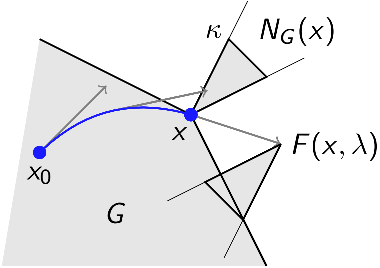

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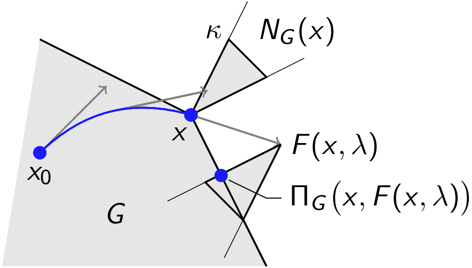

Outline

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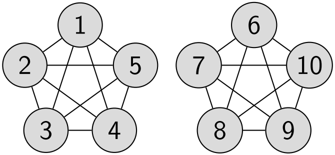
Example. Consider a network of $n = 10$ agents that seek to cooperatively solve the minimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \sum_{i \in \{1,\ldots,n\}} x_i^2/2 + |x_i| \\ \text{subject to} \quad & \|(x_1 - 2, \ldots, x_5 - 2)\|_\infty \leq 1 \\ & \|(x_6 + 2, \ldots, x_n + 2)\|_\infty \leq 1 \end{aligned}$$



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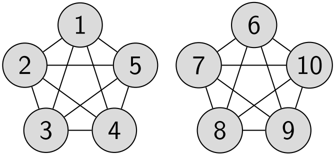

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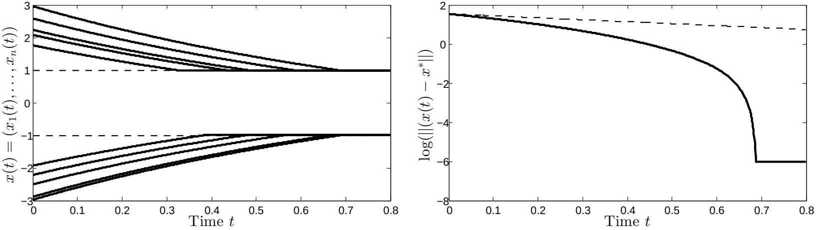

Figure: Execution of the dynamics $\dot{x}(t) - \Pi_G\big(x(t), -\partial f(x(t))\big) \overset{a.e.}{\ni} 0_n$

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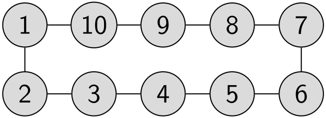
(iii) If $\partial(\nabla f) \succ 0$, then $\mathrm{sp}(L)$ is exponentially stable under (SP).

Example. Consider a network of $n = 10$ agents whose objective is to cooperatively solve the minimization problem

$$\begin{aligned} \underset{x \in \mathbb{R}^n}{\text{minimize}} \quad & \sum_{i \in \{1,\ldots,n\}} x_i^2/2 \\ \text{subject to} \quad & \text{Circ}_n(0,1,1/2)x = \mathbb{1}_n \end{aligned}$$



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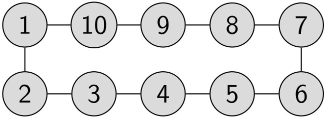

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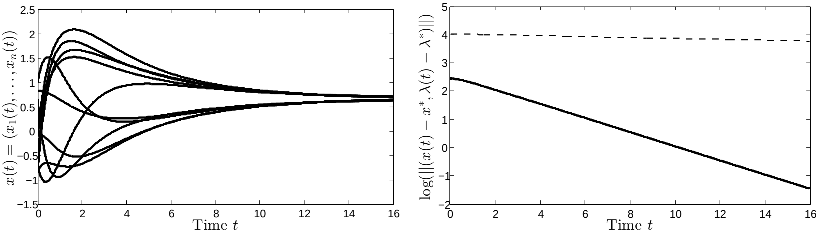

Figure: Execution of the saddle-point dynamics (SP)

Summary.

- continuous-time algorithms for distributed convex optimization
- convergence analysis of saddle-point(-like) dynamics
- identification of a nonsmooth Lyapunov function
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Thank you for your attention!