Optimal Control for Nonholonomic Systems with Symmetry

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Abstract

We study the optimal control problem for nonholonomic systems with symmetry. This work is motivated by the idea of taking advantage of the geometric structure exhibited by the optimal equations to integrate them.

1 Introduction

As is well known, the application of tools from modern differential geometry in the fields of mechanics and control theory has meant a great advance in these research areas in the last years. The study of the geometrical formulation of nonholonomic equations of motion has led to a better comprehension of locomotion, controllability, motion planning and trajectory tracking (see [7] for an excellent overview).

In this spirit, we address here the optimal control problem for nonholonomic systems with symmetry. We assume that the shape variables are directly controlled as a part of the internal shape of the system. Making use of the nonholonomic equations as derived in [2, 8], we pose the optimal control problem within a geometrical setting which allows us to interpret it as a presymplectic system. Then, the application of a constraint algorithm provides a natural enviroment where the optimal solutions "live" and, in particular, enables us to find that the optimal equations for a range of systems as those falling into the kinematic case, are in fact Hamiltonian. This makes possible the simplification of the equations by implementing a symplectic reduction.

2 Nonholonomic Systems with Symmetry

In this section, we review the formulation of the constrained dynamics and make use of the reduction techniques as developed in [2, 8]. Let Q be the configuration space of our system. Assume the existence of a Lagrangian function $L : TQ \longrightarrow \mathbb{R}$ of the form kinetic minus potential energy governing the dynamics. The interaction of the system with its enviroment is modeled by a distribution $\mathcal{D}$ on Q, which establishes the allowed velocities. Assume also that our system exhibits symmetries: there exists a Lie group G acting on Q and leaving both L and $\mathcal{D}$ invariant. This geometric picture is common to a wide variety of locomotion and robotic systems [5, 8, 10].

For unconstrained systems, Noether's theorem [2] states that the invariance of the Lagrangian gives rise to a momentum conservation law. If p denotes the body momentum, then the conservation law takes the form $g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \mathbb{I}(r)^{-1}p$, where $\mathbb{A}(r)$ is the local form of the mechanical connection and $\mathbb{I}(r)$ is the locked inertia tensor. $\mathbb{A}$ plays a central role since it determines the motion of the system as a combination of momentum p and internal shape changes $(r, \dot{r})$.

In the nonholonomic case, the situation is more involved. The conservation laws must be modified to take into account the effect of the constraint forces. In general, the momentum is no longer conserved, but may vary depending on the internal shape. The equations of motion are

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \mathbb{I}(r)^{-1}p\,, \tag{1}$$

$$\dot{p} = \frac{1}{2}\dot{r}^T\sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T\sigma_{pr}(r)\dot{r} + \frac{1}{2}p^T\sigma_{pp}(r)p, \tag{2}$$

$$M(r)\ddot{r} = -C(r,\dot{r}) + N(r,\dot{r},p) + \tau\,. \tag{3}$$

In this formulation, a nonholonomic momentum p is defined along the kinematic symmetry directions, with an associated governing equation called the nonholonomic momentum equation [2]. Here $\mathbb{A}$ is the nonholonomic connection which plays a similar role as in the unconstrained case. The variable τ represents the control forces applied to the system, which are assumed to affect only the shape variables.

In the principal kinematic case (when constraints $\mathcal{D}$ and symmetries do not intersect) this set of equations reduces to

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r}\,, \tag{4}$$

$$M(r)\ddot{r} = -C(r,\dot{r}) + \tau\,. \tag{5}$$

A rich number of systems fall into this category: the wheeled mobile robot, the N-trailer system, inchworm locomotion, motion at low Reynold's number, etc.

3 A geometric formulation for the optimal control problem

We describe a geometric framework suitable to formulate optimal control problems. In particular, we will make full use of it to write intrinsically the optimal control problem for nonholonomic systems with symmetry.

Roughly speaking, an optimal control problem consists of a cost function C and some constraints that the system must satisfy. We want to steer the system between two given points, q_0, q_1, while extremizing the funtional

$$\mathcal{J} = \int_0^1 C dt$$

among all twice differentiable curves $c(t)$ joining $c(0) = q_0$ and $c(1) = q_1$, and satisfying the constraints. These elements, the functional and the constraints, are precisely the ingredients on which is based vakonomic dynamics [3]. Vakonomic dynamics consists of the optimization of C under the given constraints.

In [3], the following intrinsic formulation for vakonomic dynamics has been developed. Consider the Whitney sum $T^*Q \oplus TQ$ and its canonical projections

$$pr_1 : T^*Q \oplus TQ \longrightarrow T^*Q\,, \quad pr_2 : T^*Q \oplus TQ \longrightarrow TQ\,.$$

Assume that the constraints are given by a submanifold M of TQ, locally defined by the equations $\dot q^\alpha = \Psi^\alpha(q^A, \dot q^a)$, $1 \le \alpha \le m$, $m+1 \le a \le n$ and $1 \le A \le n$. Let us take the submanifold $W_0 = pr_2^{-1}(M)$. Denote $W_0 = T^*Q \times_Q M$ and $\pi_1 = pr_{1|W_0}$, $\pi_2 = pr_{2|W_0}$. Now, define on $T^*Q \times_Q M$ the presymplectic 2-form $\omega = \pi_1^*\omega_Q$, where ω_Q is the canonical symplectic form on T^*Q. Define also the function $H_{W_0} = \langle \pi_1, \pi_2\rangle - \pi_2^*\tilde C$, with $\tilde C : M \to \mathbb{R}$ the restriction of C to M.

Now, the dynamics of the vakonomic system is determined by studying the solutions of the equation

$$i_X\omega = dH_{W_0}\,. \tag{6}$$

Being the system $(T^*Q \times_Q M, \omega, H_{W_0})$ presymplectic, we apply to it the Gotay-Nester's constraint algorithm (see [4]). First we consider the points W_1 of $T^*Q \times_Q M$ where (6) has a solution. This first constraint submanifold is determined by

$$W_1 = \{x \in T^*Q \times_Q M : dH_{W_0}(x)(V) = 0,\ \forall V \in \ker\omega(x)\}\,.$$

which is locally characterized by the vanishing of the constraints

$$\varphi_a = \lambda_a + \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial \dot q^a} - \frac{\partial \tilde C}{\partial \dot q^a} = 0\,,$$

or, equivalently, $\lambda_a = \frac{\partial \tilde C}{\partial \dot q^a} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial \dot q^a}$, $m+1 \le a \le n$. The equations of motion along W_1 can be written as

$$\left\{\begin{array}{rcl} \dot q^\alpha &=& \Psi^\alpha(q^A, \dot q^a)\,, \\ \dot\lambda_\alpha &=& \dfrac{\partial \tilde C}{\partial q^\alpha} - \lambda_\beta \dfrac{\partial \Psi^\beta}{\partial q^\alpha}\,, \\ \dfrac{d}{dt}\left(\dfrac{\partial \tilde C}{\partial \dot q^a} - \lambda_\alpha \dfrac{\partial \Psi^\alpha}{\partial \dot q^a}\right) &=& \dfrac{\partial \tilde C}{\partial q^a} - \lambda_\beta \dfrac{\partial \Psi^\beta}{\partial q^a}\,. \end{array}\right. \tag{7}$$

Nevertheless, the solutions on W_1 may not be tangent to W_1. In such a case, we have to restrict W_1 to the submanifold W_2 where these solutions are tangent to W_1. Proceeding further, we obtain a sequence of submanifolds

$$\cdots \hookrightarrow W_s \hookrightarrow \cdots \hookrightarrow W_2 \hookrightarrow W_1 \hookrightarrow W_0 = T^*Q \times_Q M\,.$$

If this constraint algorithm stabilizes, then we will have obtained a final constraint submanifold $W_f = W_s$ on which a vector field X exists such that $(i_X\omega = dH_{W_0})_{|W_f}$.

4 The Optimal Control Problem for Nonholonomic Systems

We assume in the following that the shape space is fully controllable, that is, the curve $r(t) \in B$ can be specified arbitrarily using a suitable control force τ. Then equation (4) above can be rewritten as $\ddot r = u$.

Given a cost function C which is a positive definite quadratic function of $\dot r$, we formulate the optimal control problem as follows [6]: given q_0, $q_1 \in Q$, find the curves $r(t) \in M$ which steer the system from q_0 to q_1 while minimizing the total cost $\int_0^1 C(\dot r)dt$, where $r = \pi(q)$, subject to the constraints (2) and the momentum equation (3).

In studying the optimal control problem, it is appropriate to treat p as a set of independent variables and the momentum equation as an additional set of constraints, as considered in [6]. Therefore, our configuration space will be $\tilde Q = Q \times \mathbb{R}^k$, where k is the number of momentum directions. The cost function C can naturally be extended to $T\tilde Q$. The constraint submanifold $M \subset T\tilde Q$ is determined by the annihilation of

$$\begin{array}{rcl} g^{-1}\dot g &=& -\mathbb{A}(r)\dot r + \mathbb{I}(r)^{-1}p\,, \\ \dot p &=& \dfrac{1}{2}\dot r^T\sigma_{\dot r\dot r}(r)\dot r + p^T\sigma_{pr}(r)\dot r + \dfrac{1}{2}p^T\sigma_{pp}(r)p\,. \end{array}$$

The dimension of M is $2n + k - \dim G$. Note that the constraints are nonlinear in general, due to the presence of the matrix $\sigma_{\dot r\dot r}(r)$. This term plays a fundamental role in controllability results for nonholonomic systems [10].

Now, we are going to address the optimal control problem within the framework we have exposed above. First, notice that locally we can identify $T^*\tilde Q$ with $T^*(Q/G) \times T^*G \times T^*\mathbb{R}^k$. We further trivialize T^*G by left translations and identify it with $G \times \mathfrak{g}^*$, by $\lambda_g = (g, \lambda)$, where $\lambda = L_g^*\lambda_g$. The Hamiltonian then reads as

$$\begin{array}{rcl} H &=& \lambda_a\dot r^a + \lambda_\beta(-\mathcal{A}_a^\beta\dot r^a + I^{\beta i}p_i) + \lambda_i\left(\dfrac{1}{2}\sigma_{iab}\dot r^a\dot r^b\right. \\ && \left. +\sigma_{ia}^j p_j\dot r^a + \dfrac{1}{2}\sigma_i^{jl}p_jp_l\right) - \dfrac{1}{2}C_{ab}\dot r^a\dot r^b\,, \end{array}$$

where $1 \le \beta \le \dim G$, $1 \le i,j,l \le k$ and $1 \le a,b \le \dim Q/G$. The first constraint submanifold, W_1, is determined by the equations

$$\varphi_a = \lambda_a - \lambda_\beta\mathcal{A}_a^\beta + \lambda_i(\sigma_{iab}\dot r^b + \sigma_{ia}^j p_j) - C_{ab}\dot r^b = 0\,.$$

Equations (7) on W_1 take the form

$$\begin{array}{rcl} (g^{-1}\dot g)^\beta &=& -\mathcal{A}_a^\beta\dot r^a + I^{\beta i}p_i \\ \dot p_i &=& \dfrac{1}{2}\sigma_{iab}\dot r^a\dot r^b + \sigma_{ia}^j p_j\dot r^a + \dfrac{1}{2}\sigma_i^{jl}p_jp_l \\ \dot\lambda_\beta &=& c_{\gamma\beta}^\delta\lambda_\delta(-A_a^\gamma\dot r^a + I^{\gamma j}p_j) \\ \dot\lambda_i &=& -\lambda_\beta I^{\beta i} - \lambda_j(\sigma_{ja}^i\dot r^a + \sigma_j^{il}p_l) \end{array}$$

$$\begin{array}{l} \dfrac{d}{dt}\left(C_{ab}\dot r^b + \lambda_\beta\mathcal{A}_a^\beta - \lambda_i(\sigma_{iab}\dot r^b + \sigma_{ia}^j p_j)\right) \\ \quad = \lambda_\beta\left(\dfrac{\partial\mathcal{A}_b^\beta}{\partial r^a}\dot r^b - \dfrac{\partial I^{\beta j}}{\partial r^a}p_j\right) \\ \quad -\lambda_i\left(\dfrac{1}{2}\dfrac{\partial\sigma_{ibc}}{\partial r^a}\dot r^b\dot r^c + \dfrac{\partial\sigma_{ib}^j}{\partial r^a}p_j\dot r^b + \dfrac{1}{2}\dfrac{\partial\sigma_i^{jl}}{\partial r^a}p_jp_l\right)\,, \end{array} \tag{8}$$

where $c_{\gamma\beta}^\delta$ are the structure constants of the Lie algebra $\mathfrak{g}$. These equations are precisely the ones obtained in [6] through reduced Lagrangian optimization. However, there is still more things to do. Indeed, we have not proven that the algorithm stops at W_1. The matrix given by $\bar{\mathcal{C}}_{ab} = C_{ab} - \lambda_i \sigma_{iab}$ is not invertible in general in all the points of W_1. Consequently the constraint algorithm must continue. In the principal kinematic case we have no nonholonomic momentum p and so $\tilde{Q} = Q$. Equations (8) are greatly simplified [9] and we get

$$\left\{ \begin{array}{rcl} (g^{-1}\dot{g})^\beta & = & -\mathcal{A}_a^\beta \dot{r}^a \\ \dot{\lambda}_\beta & = & -\lambda_\delta c_{\gamma\beta}^\delta \mathcal{A}_a^\gamma \dot{r}^a \\ C_{ab}\ddot{r}^b & = & -\lambda_\beta \mathcal{B}_{ac}^\beta \dot{r}^c , \end{array} \right. \tag{9}$$

where $\mathcal{B}_{ac}^\beta = \dfrac{\partial \mathcal{A}_a^\beta}{\partial r^c} - \dfrac{\partial \mathcal{A}_c^\beta}{\partial r^a} - c_{\alpha\gamma}^\beta \mathcal{A}_c^\alpha \mathcal{A}_a^\gamma$ are the components of the local form of the curvature of the connection $\mathbb{A}$. The matrix $\bar{\mathcal{C}} = (C_{ab})$ is invertible everywhere, and therefore these equations correspond to the flow of the Hamiltonian system (W_1, w_{W_1}, H_{W_1}).

On the other hand, we note that for systems on abelian Lie groups, equations (8) are more simple: the Lagrange multipliers λ_α are constant along the optimal curves [9]. In the nonabelian case, we can make use of the geometry of vakonomic dynamics and obtain some constants of the motion for the equations, as we show in the following section.

5 Conserved quantities

We can further exploit the symmetry of the nonholonomic system to gain more insight into equations (8). Indeed, the action of G on Q, Φ, can naturally be extended to $T^*Q \oplus TQ$, just by taking $\Psi_g = \Phi_{g^{-1}}^* \oplus \Phi_{g*}$. A direct computation shows that Ψ leaves the presymplectic system $(T^*Q \times_Q M, \omega, H)$ invariant.

In a local trivialization, the action Ψ can be expressed as $\Psi_h(r, g, p, \lambda_r, \lambda, \lambda_p, \dot{r}) = (r, hg, p, \lambda_r, \lambda, \lambda_p, \dot{r})$. Associated to it, we have a natural vakonomic momentum map given by

$$\begin{array}{rccc} J: & T^*Q \times_Q M & \longrightarrow & \mathfrak{g}^* \\ & (r, g, p, \lambda_r, \lambda, \lambda_p, \dot{r}) & \longmapsto & CoAd_g \lambda . \end{array}$$

Noether's theorem in this context implies that the vakonomic momentum is conserved. Therefore, we have obtained $\dim G$ constants of the motion for equations (8).

Now, we consider again the kinematic case. By means of the conserved momentum J, we can implement a symplectic reduction on the Hamiltonian system $(W_1, \omega_{W_1}, H_{W_1})$, which simplifies the optimal equations. Indeed, Ψ_1 is a Hamiltonian action on W_1 with associated $CoAd$-equivariant momentum map $J_1 \equiv J_{|W_1}$. We will consider the inverse image of the coadjoint orbit $\Theta_\mu \subset \mathfrak{g}^*$, $J_1^{-1}(\Theta_\mu) = \{(r, g, \dot{r}, \lambda) : \lambda \in \Theta_\mu\}$ and quotient by the whole Lie group G. To do that, we choose local coordinates $(\nu_1, \ldots, \nu_{w(\mu)})$ in Θ_μ. For each $\lambda \in \mathfrak{g}^*$ which belongs to Θ_μ, we have that $\lambda_\alpha = \lambda_\alpha(\nu)$ and viceversa. We take local coordinates $(r^a, g^\alpha, \dot{r}^a, \nu_\eta)$, $1 \le \eta \le w(\mu)$, in $J_1^{-1}(\Theta_\mu)$ and $(r^a, \dot{r}^a, \nu_\eta)$ in $J_1^{-1}(\Theta_\mu)/G$. The reduced optimal control equations are then

$$\left\{ \begin{array}{rcl} \dot{\nu}_\eta & = & \dfrac{\partial \nu_\eta}{\partial \lambda_\beta} \dot{\lambda}_\beta = -\dfrac{\partial \nu_\eta}{\partial \lambda_\beta} \lambda_\delta(\nu) c_{\gamma\beta}^\delta \mathcal{A}_a^\gamma \dot{r}^a \\ C_{ab}\ddot{r}^b & = & -\lambda_\beta(\nu) \mathcal{B}_{ac}^\beta \dot{r}^c . \end{array} \right. \tag{10}$$

Observe that, like equations (9), these ones are Hamiltonian too. Notice also that integrating equations (9) with initial condition $(r(0), g(0), \dot{r}(0), \lambda(0))$ is equivalent to integrating equations (10) with initial condition $(r(0), \dot{r}(0), \nu(0))$, once we have fixed the value $\mu = \lambda(0)$.

6 Conclusions

We have shown how the geometric formulation of the optimal control problem for nonholonomic systems with symmetry leads naturally to a better understanding of the structure of the optimal equations. In the kinematic case, we have been able to reduce the number of equations.

This approach provides interesting points for future research. In particular, the application of symplectic numerical methods [11] for the optimal equations in the kinematic case must be explored. On the other hand, in the mixed dynamic and kinematic case, the algorithm must be completed.

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