Nonholonomic Integrators

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Abstract. We introduce a discretization of the Lagrange-d'Alembert principle for Lagrangian systems with nonholonomic constraints, which allows us to construct numerical integrators that approximate the continuous flow. We study the geometric invariance properties of the discrete flow which provide an explanation for the good performance of the proposed method. This is tested on two examples: a nonholonomic particle with a quadratic potential and a mobile robot with fixed orientation.

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1. Introduction

Discrete mechanics has become a field of intensive research activity in the last years. The increasing interest in the subject is mainly due to its dual character. On the one hand, discrete mechanics allows for the construction of integration schemes, the so-called mechanical integrators, that turn out to be numerically competitive in many situations. On the other hand, many of the geometric properties of mechanical systems in the continuous case admit an appropriate counterpart in the discrete setting, which makes it a rich area to be explored. Both aspects of discrete mechanics mutually interact, since the geometric properties of the discrete model play a key role in the explanation of the good behaviour of the integrators derived from it in a number of situations.

Mechanical integrators preserve some of the invariants of the mechanical system, such as energy, momentum or the symplectic form. It is well known that if the energy and the momentum map include all integrals belonging to a certain class (see [11]), then one cannot create constant time step integrators that are simultaneously symplectic, energy preserving and momentum preserving, unless they integrate the equations *exactly* up to a time reparametrization. (Recently, it has been shown that the construction of energy-symplectic-momentum integrators is indeed possible if one allows varying time steps [15]). This justifies the focus on mechanical integrators that are either symplectic-momentum or energy-momentum preserving (although other types may also be considered, such as methods preserving reversing symmetries).

Based on certain applications, such as molecular dynamics simulation, the necessity of treating holonomic constraints in discrete mechanics has also been discussed in the literature. For example, the popular Verlet algorithm for unconstrained mechanical systems was adapted to handle holonomic constraints, resulting in the Shake algorithm [30] and the Rattle algorithm [1] (see [21] for a discussion of the symplectic character of these methods). The case of general Hamiltonian systems (i.e. not necessarily mechanical) subject to holonomic constraints has also been studied [14, 31, 32]. A different approach, based on the Dirac theory of constraints to find unconstrained formulations in which the constraints appear as invariants, may be found in [20]. Energy-momentum integrators derived from discrete directional derivatives and discrete versions of Hamiltonian mechanics have also been recently adapted to deal with holonomic constraints [12, 13].

Variational integrators are symplectic-momentum mechanical integrators derived from a discretization of Hamilton's principle [2, 28, 34, 35]. This discrete variational principle leads to the obtention of the discrete Euler-Lagrange (DEL) equations. Different discrete Lagrangians result in different variational integrators, including the Verlet algorithm and the whole family of Newmark algoritms [16] used in structural mechanics. Variational integrators handle constraints in a simple and efficient manner by using Lagrange multipliers [36]. It is worth mentioning that, when treated variationally, holonomic constraints do not affect the symplectic or conservative nature of the algorithms, while other techniques can run into trouble in this regard [20].

In this paper, we address the problem of constructing integrators for mechanical systems with nonholonomic constraints. This problem has been stated in a number of recent papers [8, 36], including the presentation of open problems in symplectic integration given in [27]. In nonholonomic Lagrangian mechanics, the symplectic form constructed from the Lagrangian is no longer preserved as in the unconstrained case. Moreover, in the case of a nonholonomic system with symmetry, the momentum map is not conserved in general, due to the presence of the constraint force. However, one can consider a nonholonomic momentum map along the symmetry directions compatible with the constraints, and verify that its evolution along the integral curves of the constrained system is given by the nonholonomic momentum equation [4, 7, 29]. On the other hand, at least in the case of linear (or, more generally, homogeneus) constraints, the energy is still a conservation law for the system. Consequently, two of the three cornerstones on which the construction of mechanical integrators for unconstrained systems relies (i.e. preservation of symplectic structure and momentum) are lacking in the nonholonomic case.

Our starting point to develop integrators in the presence of nonholonomic constraints is the introduction of a discrete version of the Lagrange-d'Alembert principle. This follows the idea that, by respecting the geometric structure of nonholonomic systems, one can create integrators capturing the essential features of this kind of systems. Indeed, we show that the nonholonomic integrators derived from this discrete principle preserve the structure of the evolution of the symplectic form along the trajectories of the system. We also prove that, for nonholonomic systems with symmetry, the nonholonomic integrators give rise to a discrete version of the nonholonomic momentum equation. Moreover, in the presence of horizontal symmetries, the discrete flow exactly preserves the associated momenta. We also treat the case where no nonholonomic momentum map exists, due to the absence of symmetry directions fulfilling the constraints. This situation, known in the literature as the vertical or purely kinematic case [4, 7, 10, 18], allows one to reduce the continuous flow to that of an unconstrained system with a nonconservative force. We show that the nonholonomic integrator also passes to the discrete reduced space, yielding a generalized variational integrator in the sense of [16]. In case the nonconservative force vanishes, we prove that the reduced nonholonomic integator is indeed a variational integrator.

The paper is organised as follows. In Section 2, we give a brief review of variational integrators. This serves as a motivation to introduce the discrete Lagrange-d'Alembert principle in Section 3, from which we derive the discrete Lagrange-d'Alembert (DLA) equations. In case the constraints are holonomic, we show that the DLA algorithm is just a variational integrator. Section 4 deals with the construction of integrators, by way of appropriate discretizations of the Lagrangian and the constraints. Generalizing a theorem presented in [36], we prove that if the configuration manifold Q can be embedded into a linear space V, then the DLA algorithm on Q is equivalent to the DLA algorithm on V subject to the nonholonomic constraints *plus* the holonomic ones defining Q as a subspace of V. As expected, the equations resulting from the nonholonomic integrator inherit some of the geometric characteristics of the continuous system. This is investigated in Section 5, where we study the invariance properties related to the nonholonomic momentum equation and to the kinematic case, providing complementary insights into the geometric structure of discrete nonholonomic mechanics. Finally, in Section 6, we present some numerical tests in the examples of a nonholonomic particle with a quadratic potential and a mobile robot with fixed orientation, illustrating the good performance of the method when compared to the standard 4^{th} order Runge-Kutta.

2. Variational Integrators

Mechanical integrators based on the Veselov discretization technique [28, 34, 35] have been studied intensively in the last years and are by now well known [5, 15, 16, 26, 36]. We briefly review here the main ideas of this approach.

Let Q be a n-dimensional configuration manifold and $L_d : Q \times Q \longrightarrow \mathbb{R}$ a smooth map playing the role of a discrete Lagrangian. The action sum is the map $S : Q^{N+1} \longrightarrow \mathbb{R}$ defined by

$$S = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1}) \,, \tag{1}$$

where $q_k \in Q$ for $k \in \{0, 1, \ldots, N\}$ and k is the discrete time. The discrete variational principle states that the evolution equations extremize the action sum, given fixed end points q_0, q_N. This leads to the discrete Euler-Lagrange (DEL) equations:

$$D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) = 0 \,. \tag{2}$$

Under appropriate regularity assumptions on the discrete Lagrangian L_d, the DEL equations define a map $\Phi : Q \times Q \longrightarrow Q \times Q$, $\Phi(q_{k-1}, q_k) = (q_k, q_{k+1})$ which describes the discrete time evolution of the system.

Now, define the fiber derivative or discrete Legendre transform corresponding to L_d by

$$\begin{array}{rccl} \mathcal{F}L_d : & Q \times Q & \longrightarrow & T^*Q \\ & (q, q') & \longmapsto & (q', D_2 L_d(q, q')) \,, \end{array}$$

and the 2-form Ω_{L_d} on $Q \times Q$ by pulling back the canonical 2-form $\Omega_Q = -d\Theta_Q$ from T^*Q,

$$\Omega_{L_d} = \mathcal{F}L_d^*(\Omega_Q) \,.$$

The alternative discrete fiber derivative $\tilde{\mathcal{F}}L_d(q, q') = (q, -D_1 L_d(q, q'))$ may also be used and the results obtained will be essentially unchanged. A fundamental fact is that the algorithm Φ exactly preserves the symplectic form Ω_{L_d}, that is, $\Phi^*\Omega_{L_d} = \Omega_{L_d}$ (see [36]). If we further assume that the discrete Lagrangian is invariant under the action of a Lie group G on Q, one can prove that the associated discrete momentum map, $J_d : Q \times Q \longrightarrow \mathfrak{g}^*$ (where $\mathfrak{g}^*$ denotes the dual of the Lie algebra $\mathfrak{g}$ of G) defined by

$$\langle J_d(q, q'), \xi \rangle = \langle D_2 L_d(q, q'), \xi_Q(q') \rangle \,,$$

is exactly preserved by the algorithm Φ [36]. Here, ξ_Q denotes the fundamental vector field corresponding to the element $\xi \in \mathfrak{g}$. Moreover, when regarding the discrete mechanical model as an approximation to a continuous system, one can verify that the constant value of the discrete momentum map approaches the value of its continuous counterpart, as the time step decreases.

Consequently, variational integrators are symplectic-momentum integrators.

3. Discrete Lagrange-d'Alembert Principle

In this section, we propose a discrete version of the Lagrange-d'Alembert principle for nonholonomic systems. Before doing so, we first recall the general picture in the continuous case.

Consider a distribution $\mathcal{D}$ on the configuration space Q, describing some kinematic constraints imposed on a Lagrangian system. We say that a curve $q(t)$ in Q satisfies the constraints if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t. The dynamics of the nonholonomic system is determined by a Lagrangian $L : TQ \longrightarrow \mathbb{R}$ through the application of the Lagrange-d'Alembert principle, which states that a curve $q(t)$ is an admissible motion of the system if

$$\delta \mathcal{J} = \delta \int_a^b L(q(t), \dot{q}(t)) dt = 0 \,,$$

for all variations such that $\delta q(t) \in \mathcal{D}_{q(t)}$, $a \leq t \leq b$, and if it satisfies the constraints. It is worth noting that the Lagrange-d'Alembert principle is not variational, since we impose the constraints on the curve *after* extremizing the functional $\mathcal{J}$. The inverse procedure, that is, imposing the constraints *before* extremizing $\mathcal{J}$, results in a different set of equations (this time truly variational) termed vakonomic.

Some straightforward manipulations show that the principle holds precisely when

$$-\delta L = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0 \,,$$

for all the variations $\delta q \in \mathcal{D}_{q(t)}$. If $\{\omega^a = \omega_i^a dq^i\}_{a=1}^m$ is a set of m independent 1-forms defining the annihilator $\mathcal{D}^o$ of $\mathcal{D}$, we arrive at the equations describing the nonholonomic dynamics

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \lambda_a \omega_i^a \,, \tag{3}$$

$$\omega_i^a \dot{q}^i = 0 \,, \tag{4}$$

where λ_a, $a \in \{1, \ldots, m\}$, is a set of Lagrange multipliers. The right-hand side of equation (3) represents the force of constraint.

If we introduce coordinates $q^i = (r^\alpha, s^a)$ on Q, where $\alpha \in \{1, \ldots, n-m\}$, in terms of which ω^a takes the form

$$\omega^a(q) = ds^a + A_\alpha^a(r, s) dr^\alpha \,,$$

then the Lagrange multipliers are exactly given by

$$\lambda_a = \frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a}\,,$$

and the constraint force reads

$$F = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{s}^a} - \frac{\partial L}{\partial s^a}\right)\omega^a\,.$$

Now, we turn to the discrete version of nonholonomic mechanics. Consider as before a discrete Lagrangian $L_d : Q \times Q \longrightarrow \mathbb{R}$ and the associated action sum

$$S = \sum_{k=0}^{N-1} L_d(q_k, q_{k+1})\,, \tag{5}$$

where $q_k \in Q$ and $k \in \{0, 1, \ldots, N\}$ is the discrete time. In the unconstrained discrete mechanics case (cf. Section 2), we have seen that one extremizes the action sum with respect to all possible sequences of $N-1$ points, given fixed end points q_0, q_N. This means that at each point $q \in Q$, the allowed variations are given by the whole tangent space T_qQ. However, in the nonholonomic case, we must restrict the allowed variations. These are exactly given by the distribution $\mathcal{D}$. In addition, we will consider a discrete constraint space $\mathcal{D}_d \subset Q \times Q$ with the same dimension as $\mathcal{D}$ and such that $(q, q) \in \mathcal{D}_d$ for all $q \in Q$. This discrete constraint space imposes constraints on the solution sequence $\{q_k\}$, namely, $(q_k, q_{k+1}) \in \mathcal{D}_d$. Later, when regarding the discrete principle as an approximation of the continuous one, we shall impose more conditions on the selection of $\mathcal{D}_d$ in order to obtain a consistent discretization of the continuous equations of motion.

Consequently, to develop the discrete nonholonomic mechanics, one needs three ingredients: a discrete Lagrangian L_d, a constraint distribution $\mathcal{D}$ on Q and a discrete constraint space $\mathcal{D}_d$. Notice that the discrete mechanics can also be seen within this framework, where $\mathcal{D} = TQ$ and $\mathcal{D}_d = Q \times Q$.

Then, we define the discrete Lagrange-d'Alembert principle to be the extremization of (5) among the sequence of points (q_k) with given fixed end points q_0 and q_N, where the variations must satisfy $\delta q_k \in \mathcal{D}_{q_k}$ and $(q_k, q_{k+1}) \in \mathcal{D}_d$, for all $k \in \{0, \ldots, N-1\}$. This leads to the set of equations

$$\left(D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k)\right)_i \delta q_k^i = 0\,,\ 1 \le k \le N-1\,,$$

where $\delta q_k \in \mathcal{D}_{q_k}$, along with $(q_k, q_{k+1}) \in \mathcal{D}_d$. If $\omega_d^a : Q \times Q \to \mathbb{R}$, $a \in \{1, \ldots, m\}$, are functions whose annihilation defines $\mathcal{D}_d$, what we have got is the following **discrete Lagrange-d'Alembert** (DLA) algorithm

$$\left\{\begin{array}{rcl} D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) & = & \lambda_a \omega^a(q_k) \\ \omega_d^a(q_k, q_{k+1}) & = & 0\,. \end{array}\right. \tag{6}$$

Notice that the discrete Lagrange-d'Alembert principle is not truly variational, as the continuous principle. Alternatively, we will refer to the DLA algorithm (6) as a **nonholonomic integrator**, by analogy with the unconstrained case. Note also that, under appropriate regularity assumptions, the implicit function theorem ensures us that we have obtained a well-defined algorithm $\Phi : Q\times Q \longrightarrow Q\times Q$, $\Phi(q_{k-1},q_k) = (q_k,q_{k+1})$. In fact, this is guaranteed if the matrix

$$\begin{pmatrix} D_1D_2L_d(q_k,q_{k+1}) & \omega^a(q_k) \\ D_2\omega^a_d(q_k,q_{k+1}) & 0 \end{pmatrix} \tag{7}$$

is invertible for each (q_k,q_{k+1}) in a neighbourhood of the diagonal of $Q\times Q$.

Remark 3.1 Assume we are given a continuous nonholonomic problem with data $L : TQ \longrightarrow \mathbb{R}$ and $\mathcal{D} \subset TQ$. In the following section, we shall discuss some types of discretizations of this problem. To guarantee that the DLA algorithm approximates the continuous flow within a desired order of accuracy, one should select the discrete Lagrangian $L_d : Q\times Q \longrightarrow \mathbb{R}$ and the discrete constraint space $\mathcal{D}_d$ in a consistent way. This essentially means that if $\omega^1,\dots,\omega^m$ are 1-forms on Q whose annihilation locally define the constraint distribution $\mathcal{D}$, one performs the same type of discretization of both the Lagrangian $L : TQ \longrightarrow \mathbb{R}$ and the 1-forms (interpreted as functions linear in the velocities, $\omega^a : TQ \longrightarrow \mathbb{R}$). For instance, if L_d is constructed by means of a discretization mapping $\Psi : Q\times Q \longrightarrow TQ$ defined on a neighbourhood of the diagonal of $Q\times Q$, that is, $L_d = L\circ\Psi$, then $\mathcal{D}_d$ must locally be defined by the annihilation of the functions $\omega^a_d = \omega^a\circ\Psi$. Stated otherwise, $\mathcal{D}_d$ should be such that $\Psi(\mathcal{D}_d) = \mathcal{D}$.

Remark 3.2 Consider the continuous nonholonomic problem given by L and $\mathcal{D}$, and let L_d and $\mathcal{D}_d$ be appropriate discrete versions of them. Then, if the matrix

$$\begin{pmatrix} D_1D_2L_d(q_k,q_k) & \omega^a(q_k) \\ D_2\omega^a_d(q_k,q_k) & 0 \end{pmatrix}$$

is invertible for each $q_k\in Q$, a sufficiently small stepsize h guarantees that the matrix (7) is also nonsingular and hence the DLA algorithm is solvable for q_{k+1}.

Remark 3.3 (The holonomic case) Let us examine the nonholonomic integrator when the constraints are holonomic, that is, the case when the distribution $\mathcal{D}$ is integrable. Assume that there exists a function $g : Q \longrightarrow \mathbb{R}^l$ whose level surfaces are precisely the integral manifolds of $\mathcal{D}$, i.e. for each $r\in\mathbb{R}^l$, $N_r = g^{-1}(r)$ is a submanifold of Q such that $T_qN_r = \mathcal{D}_q$ for all $q\in N_r$. Then, we can consider as constraint space the following subspace of $Q\times Q$,

$$\mathcal{D}_d = \cup_{r\in\mathbb{R}^l} N_r\times N_r\,.$$

Observe that if we take $q_0\in N_0$, then $(q_0,q_1)\in\mathcal{D}_d$ is equivalent to $q_1\in N_0$. We then find that the nonholonomic integrator for an initial pair q_0, $q_1\in N_0$ becomes

$$\left\{ \begin{array}{rcl} D_1L_d(q_k,q_{k+1}) + D_2L_d(q_{k-1},q_k) & = & \lambda_a Dg^a(q_k) \\ g(q_{k+1}) & = & 0\,, \end{array} \right. \tag{8}$$

where $g^a : Q \longrightarrow \mathbb{R}$ denotes the a component of g. Notice that (8) is just a variational integrator [36]. It is known that for an appropriate discrete Lagrangian, one recovers the Shake algorithm [21, 30], written in terms of position variables. The Shake algorithm is very useful in molecular dynamics simulation.

4. Construction of integrators

In the unconstrained case [36], there are mainly two ways of constructing mechanical integrators, depending on whether Q is seen as a manifold in its own right (the "intrinsic" point of view) or as being embedded in a larger space (the "extrinsic" point of view). Assume that we have a continuous nonholonomic problem given by $L : TQ \longrightarrow \mathbb{R}$ and $\mathcal{D} \subset TQ$.

When adopting the intrinsic point of view, one makes use of coordinate charts on Q to construct the discrete Lagrangian. Let $\varphi : U \subset Q \longrightarrow \mathbb{R}^n$ be a local chart whose coordinate domain U contains q_k and asume that $q_{k+1} \in U$ (a condition guaranteed by a sufficiently small timestep h). A choice of discrete Lagrangian is the following

$$L_d^\alpha(q_k, q_{k+1}) = L\left(\varphi^{-1}((1-\alpha)\varphi(q_k) + \alpha\varphi(q_{k+1})), (\varphi^{-1})_*\left(\frac{\varphi(q_{k+1}) - \varphi(q_k)}{h}\right)\right), \tag{9}$$

where $0 \leq \alpha \leq 1$ is an interpolation parameter and the differential $(\varphi^{-1})_*$ is taken at the point $x = (1-\alpha)\varphi(q_k) + \alpha\varphi(q_{k+1})$. Of course there are other possible choices of discretizations, as for instance,

$$\begin{aligned} L_d^{sym,\alpha}(q_k, q_{k+1}) = {} & \frac{1}{2}L\left(\varphi^{-1}((1-\alpha)\varphi(q_k) + \alpha\varphi(q_{k+1})), (\varphi^{-1})_*\left(\frac{\varphi(q_{k+1}) - \varphi(q_k)}{h}\right)\right) \\ & + \frac{1}{2}L\left(\varphi^{-1}(\alpha\varphi(q_k) + (1-\alpha)\varphi(q_{k+1})), (\varphi^{-1})_*\left(\frac{\varphi(q_{k+1}) - \varphi(q_k)}{h}\right)\right). \end{aligned} \tag{10}$$

In the unconstrained case, the choice (10) always yields second order accurate numerical methods, whereas in general this is only guaranteed for the discretization (9) if $\alpha = \frac{1}{2}$ (although for natural Lagrangians of the form $L = \frac{1}{2}\dot{q}M\dot{q} - V(q)$, (9) also gives second order numerical methods [16]).

This approach is called the Generalized Coordinate Formulation.

Remark 4.1 In general, this viewpoint is necessarily local, since the discretizations are only valid in the coordinate domain U of φ. If we choose an atlas of charts covering the whole manifold Q, we cannot guarantee that the construction of the discrete Lagrangian L_d will coincide on the chart overlaps. There are certain cases, however, in which this is indeed possible. For example, if we can find an atlas $\{(U_s, \varphi_s)\}$ such that for any two overlapping charts, φ_{s_1} and φ_{s_2}, the local diffeomorphism $\varphi_{s_1 s_2} = \varphi_{s_1} \circ \varphi_{s_2}^{-1}$ verifies $\varphi_{s_1 s_2}((1-\alpha)x + \alpha y) = (1-\alpha)\varphi_{s_1 s_2}(x) + \alpha\varphi_{s_1 s_2}(y)$, for any x, $y \in \varphi_{s_2}(U_{s_2})$ and $(\varphi_{s_1 s_2})_* = id$, then it is easy to see that one can "paste" the local constructions (9) (respectively (10)) to have a well-defined discrete Lagrangian on a neighbourhood of the diagonal of $Q \times Q$. [A simple example of this situation is given by the manifold $\mathbb{S}^1$, with the local charts $\varphi_1(z_1, z_2) = \arcsin(z_2/z_1) \in (0, 2\pi)$ and $\varphi_2(z_1, z_2) = \arcsin(z_2/z_1) \in (-\pi, \pi)$].

Remark 4.2 Another way of constructing a well-defined discrete Lagrangian on a neighbourhood of the diagonal of $Q \times Q$ is the following. Assume that there exist a $q_0 \in Q$ and a differentiable mapping $\Upsilon : Q \longrightarrow Diff(Q)$ such that $\Upsilon(q)(q) = q_0$, for all $q \in Q$. In this case, we can define L_d for each (q, q') according to either (9) or (10) by means of $\varphi = \varphi_0 \circ \Upsilon(q)$, where φ_0 is a local chart whose coordinate domain contains q_0. It is important to note that in this construction the mapping $\varphi_0 \circ \Upsilon(q)$ *varies* with the pair (q, q'). This is the case for instance of finite dimensional Lie groups $Q = G$, where one can take $q_0 = e$, the identity element, $\Phi(g) = L_{g^{-1}}$ for each $g \in G$ and $\varphi_0 = \exp_e^{-1}$ (see [26]). We shall make use of this construction in Section 5.3.3.

The extrinsic point of view assumes that Q is embedded in some linear space V and that we have a Lagrangian $\mathcal{L} : TV \longrightarrow \mathbb{R}$ such that $\mathcal{L}_{|TQ} = L$. In addition, it is assumed that there exists a vector valued constraint function $g : V \longrightarrow \mathbb{R}^l$, such that $g^{-1}(0) = Q \subset V$, with 0 a regular value of g. According to (9) and (10), we can define the following discrete Lagrangians on $V \times V$

$$\mathcal{L}_d^\alpha(v_k, v_{k+1}) = \mathcal{L}\left((1-\alpha)v_k + \alpha v_{k+1}, \frac{v_{k+1} - v_k}{h}\right), \tag{11}$$

and

$$\begin{aligned} \mathcal{L}_d^{sym,\alpha}(v_k, v_{k+1}) = & \frac{1}{2}\mathcal{L}\left((1-\alpha)v_k + \alpha v_{k+1}, \frac{v_{k+1} - v_k}{h}\right) \\ & + \frac{1}{2}\mathcal{L}\left(\alpha v_k + (1-\alpha)v_{k+1}, \frac{v_{k+1} - v_k}{h}\right). \end{aligned} \tag{12}$$

In this way, we can sum and subtract points in Q because we are regarding them as vectors in V by means of the natural inclusion $j : Q \hookrightarrow V$. Of course, we must ensure that the points obtained by the algorithm all belong to Q. Then, the solution sequence (v_k) will extremize the action sum $\mathbb{S} = \sum_{k=0}^{N-1} \mathcal{L}_d(v_k, v_{k+1})$ subject to the holonomic constraints imposed by g. This leads to the discrete equations

$$\left\{ \begin{array}{rcl} D_1\mathcal{L}_d(v_k, v_{k+1}) + D_2\mathcal{L}_d(v_{k-1}, v_k) & = & \lambda_l Dg^l(v_k) \\ g(v_{k+1}) & = & 0\,. \end{array} \right.$$

This approach is called the Constrained Coordinate Formulation.

Both formulations are shown to be equivalent in the domain of definition of the local chart φ selected in the Generalized Coordinate Formulation (see [36]), whereby the following identification is understood: $L_d(q_k, q_{k+1}) = \mathcal{L}_d(j(q_k), j(q_{k+1}))$, which is valid for choices of the chart (U, φ) in the definition of L_d such that the map $J = j \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \longrightarrow V$ is linear. Notice that this assumption is not at all restrictive, since j is an injective immersion and such a chart (U, φ) can always be chosen.

In the nonholonomic case, we can construct an appropriate adaptation of both formulations. In the Generalized Coordinate Formulation, we introduce $\mathcal{D}_d$ as follows. Take a local basis of 1-forms of the annihilator of the constraint distribution $\mathcal{D}$, $\{\omega^1, \ldots, \omega^m\} \in \mathcal{D}^o$. These 1-forms can be intrepreted as functions linear in the velocities, locally defined on TQ. Then, we discretize them according to the previous discretizations of the Lagrangian, that is, we take either

$$\omega_d^a(q_k, q_{k+1}) = \omega^a\left(\varphi^{-1}((1-\alpha)\varphi(q_k) + \alpha\varphi(q_{k+1})), (\varphi^{-1})_*\left(\frac{\varphi(q_{k+1}) - \varphi(q_k)}{h}\right)\right), \tag{13}$$

or

$$\begin{aligned}\omega_d^a(q_k,q_{k+1}) \quad &= \frac{1}{2}\omega^a\left(\varphi^{-1}((1-\alpha)\varphi(q_k)+\alpha\varphi(q_{k+1})),(\varphi^{-1})_*\left(\frac{\varphi(q_{k+1})-\varphi(q_k)}{h}\right)\right) \quad (14)\\ &+\frac{1}{2}\omega^a\left(\varphi^{-1}((1-\alpha)\varphi(q_k)+\alpha\varphi(q_{k+1})),(\varphi^{-1})_*\left(\frac{\varphi(q_{k+1})-\varphi(q_k)}{h}\right)\right).\end{aligned}$$

In this way we obtain the functions $\omega_d^a : Q\times Q \longrightarrow \mathbb{R}$ whose annihilation defines $\mathcal{D}_d \subset Q\times Q$. As in the unconstrained case, it is not hard to prove that the discretization (10) together with(14) yields second order accurate approximations to the continuous flow, whereas this is only guaranteed for the discretization (9), (13) if $\alpha=\frac{1}{2}$.

In the Constrained Coordinate Formulation, we assume that there exist local 1-forms on V defining $\mathcal{D}^o$, $\{\tilde\omega^1,\dots,\tilde\omega^m\}$ such that $\tilde\omega^a(q)_{|T_qQ}=\omega^a(q)$ for $q\in Q$. Then, we discretize them according to

$$\tilde\omega_d^a(v_k,v_{k+1})=\tilde\omega^a\left((1-\alpha)v_k+\alpha v_{k+1},\frac{v_{k+1}-v_k}{h}\right), \tag{15}$$

and

$$\begin{aligned}\tilde\omega_d^a(v_k,v_{k+1}) &= \frac{1}{2}\tilde\omega^a\left((1-\alpha)v_k+\alpha v_{k+1},\frac{v_{k+1}-v_k}{h}\right)\\ &\quad+\frac{1}{2}\tilde\omega^a\left(\alpha v_k+(1-\alpha)v_{k+1},\frac{v_{k+1}-v_k}{h}\right).\end{aligned} \tag{16}$$

Observe that we can identify $\omega_d^a(q_k,q_{k+1})=\tilde\omega_d^a(j(q_k),j(q_{k+1}))$ in the same way as we have done for the discrete Lagrangians. Then, the discrete Lagrange-d'Alembert principle with the holonomic constraints g and the nonholonomic constraints $\tilde\omega^1,\dots,\tilde\omega^m$ leads us to the equations

$$\left\{\begin{array}{rcl} D_1\mathcal{L}_d(v_k,v_{k+1})+D_2\mathcal{L}_d(v_{k-1},v_k) &=& \lambda_l Dg^l(v_k)+\mu_a\tilde\omega^a(v_k)\\ g(v_{k+1}) &=& 0\\ \tilde\omega_d^a(v_k,v_{k+1}) &=& 0\,. \end{array}\right. \tag{17}$$

The following theorem, analogous to the one presented in [36], ensures that both formulations (6) and (17) are indeed equivalent in the same sense as before, as one might expect. We prove it for the discretizations (9), (13). The proof for the symmetric discretizations (10), (14) is analogous.

Theorem 4.3 *Let $\varphi : U\subset Q \longrightarrow \mathbb{R}^n$ be a local chart of Q such that $J=j\circ\varphi^{-1}$ is linear. Identify U with $\varphi(U)$ and $j_{|U}$ with $J_{|U}$ through φ. Let q_{k-1}, q_k be two initial points in the coordinate chart and let $v_{k-1}=J(q_{k-1})$, $v_k=J(q_k)$. Then, the Generalized Coordinate Formulation (6) has a solution $(q_{k+1},\mu_a^{(k)})$ if and only if the Constrained Coordinate Formulation (17) has a solution $(v_{k+1},\lambda_l^{(k)},\bar\mu_a^{(k)})$. Indeed, $v_{k+1}=J(q_{k+1})$ and $\bar\mu_a^{(k)}=\mu_a^{(k)}$.*

Proof: To establish the equivalence, we first expand equations (6) and (17) in terms of $\mathcal{L}$ and the 1-forms $\{\tilde\omega^a\}_{a=1}^m$. Let $(v,\dot v)$ denote the canonical coordinates of TV.

Equations (6) become, when written in matrix form,

$$\left\{\begin{array}{rcl} D^TJ(q_k)\left\{\dfrac{1}{h}\left[\dfrac{\partial\mathcal{L}}{\partial\dot v}(a_k,b_k)-\dfrac{\partial\mathcal{L}}{\partial\dot v}(a_{k+1},b_{k+1})\right]\right. & + & (1-\alpha)\dfrac{\partial\mathcal{L}}{\partial v}(a_{k+1},b_{k+1}) \\ & + & \left.\alpha\dfrac{\partial\mathcal{L}}{\partial v}(a_k,b_k)+\mu_a^{(k)}\tilde\omega^a(J(q_k))\right\}=0 \\ \tilde\omega_d^a(J(q_k),J(q_{k+1})) & = & \tilde\omega^a(a_{k+1},b_{k+1})=0\,, \end{array}\right. \tag{18}$$

where $a_k=\alpha J(q_k)+(1-\alpha)J(q_{k-1})$ and $b_k=\dfrac{1}{h}(J(q_k)-J(q_{k-1}))$. Note that we are using the identifications $\omega_d^a=(\tilde\omega_d^a)_{|Q\times Q}$, $(\mathcal{L}_d^\alpha)_{|Q\times Q}=L_d^\alpha$ and $\tilde\omega^a_{|TQ}=\omega^a$. If $\tilde\omega^a=\tilde\omega^a_l dv^l$ and $\omega^a=\omega^a_i dq^i$, we have that

$$\omega_i^a(q_k)dq^i=J^*(\tilde\omega_l^a dv^l)(q_k)=\frac{\partial J^l}{\partial q^i}(q_k)\tilde\omega_l^a(J(q_k))dq^i\,,$$

which can be written in a more compact way as $\omega^a(q_k)=D^TJ(q_k)\tilde\omega^a(J(q_k))$. Here and in the following the superscript T refers to the transpose of a matrix.

On the other hand, equation (17) can be written as

$$\left\{\begin{array}{rcl} \dfrac{1}{h}\left[\dfrac{\partial\mathcal{L}}{\partial\dot v}\left(v_{k-1+\alpha},\dfrac{v_k-v_{k-1}}{h}\right)\right. & - & \left.\dfrac{\partial\mathcal{L}}{\partial\dot v}\left(v_{k+\alpha},\dfrac{v_{k+1}-v_k}{h}\right)\right]+(1-\alpha)\dfrac{\partial\mathcal{L}}{\partial v}\left(v_{k+\alpha},\dfrac{v_{k+1}-v_k}{h}\right) \\ & + & \alpha\dfrac{\partial\mathcal{L}}{\partial v}\left(v_{k-1+\alpha},\dfrac{v_k-v_{k-1}}{h}\right)+\bar\mu_a^{(k)}\tilde\omega^a(v_k)=\lambda_l^{(k)}Dg^l(v_k) \\ g(v_{k+1}) & = & 0 \\ \tilde\omega_d^a(v_k,v_{k+1}) & = & \tilde\omega_l^a(v_{k+\alpha})\left(\dfrac{v_{k+1}-v_k}{h}\right)^l=0\,, \end{array}\right. \tag{19}$$

where the shorthand notation $v_{k+\alpha}=(1-\alpha)v_k+\alpha v_{k+1}$ is used. Now, assume that $(v_{k+1},\lambda_l^{(k)},\bar\mu_a^{(k)})$ is a solution of (19) with $v_k=J(q_k)$ and $v_{k-1}=J(q_{k-1})$. The fact that $g(v_{k+1})=0$ implies that v_{k+1} belongs to the image of J. Let $q_{k+1}=J^{-1}(v_{k+1})$. Multiplying the first equation of (19) by $D^TJ(q_k)$ and making the corresponding substitutions, one obtains for the pair $(q_{k+1},\bar\mu_a^{(k)})$ just the first equation of (18), since the term $D^TJ(q_k)D^Tg(v_k)$ cancels due to $g\circ J=0$.

Conversely, if $(q_{k+1},\mu_a^{(k)})$ is a solution of (18), then one can find Lagrange multipliers $\lambda_l^{(k)}$, such that $(v_{k+1}=J(q_{k+1}),\lambda_l^{(k)},\mu_a^{(k)})$ is a solution of (19) as follows. The second and the third equation of (19) are automatically satisfied because of $v_{k+1}\in Q$ and taking into account the second equation of (18). Moreover, as $DJ(q_k)$ and $Dg(q_k)$ are assumed to have full rank, we have that $T_{v_k}V=\mathcal{R}(DJ(q_k))\oplus\mathcal{N}(D^TJ(q_k))$, where $\mathcal{R}(DJ(q_k))$ and $\mathcal{N}(D^TJ(q_k))$ refer to the range and the kernel, respectively, of the operator under consideration. Since $\mathcal{R}(D^Tg(q_k))\subset\mathcal{N}(D^TJ(q_k))$ and $\dim\mathcal{R}(D^Tg(q_k))=\dim\mathcal{N}(D^TJ(q_k))$, we can write $T_{v_k}V=\mathcal{R}(DJ(q_k))\oplus\mathcal{R}(D^Tg(q_k))$. Now, the left-hand side of the first equation of (19) can be decomposed into a part belonging to $\mathcal{R}(DJ(q_k))$ and a part belonging to $\mathcal{R}(D^Tg(q_k))$. But the part in $\mathcal{R}(DJ(q_k))$ is zero, because of the first equation of (18). Consequently, the entire expression belongs to $\mathcal{R}(D^Tg(q_k))$, and thus there exist some $\lambda_l^{(k)}$ such that

$$\frac{1}{h}\left[\frac{\partial\mathcal{L}}{\partial\dot v}\left(v_{k-1+\alpha},\frac{v_k-v_{k-1}}{h}\right)-\frac{\partial\mathcal{L}}{\partial\dot v}\left(v_{k+\alpha},\frac{v_{k+1}-v_k}{h}\right)\right]+(1-\alpha)\frac{\partial\mathcal{L}}{\partial v}\left(v_{k+\alpha},\frac{v_{k+1}-v_k}{h}\right)$$

$$+ \alpha \frac{\partial \mathcal{L}}{\partial v}\left(v_{k-1+\alpha}, \frac{v_k - v_{k-1}}{h}\right) + \bar{\mu}_a^{(k)} \tilde{\omega}^a(v_k) = \lambda_l^{(k)} Dg^l(v_k)\,,$$

which is precisely the first equation of (19). $\boxed{QED}$

The relevance of Theorem 4.3 becomes apparent when handling concrete examples. Generally, it is easier to treat the nonholonomic integrator following the Constrained Coordinate Formulation, since points in Q can be treated as points in some $\mathbb{R}^s$, and this is a definite advantage for the numerical implementation. On the other hand, the geometric study of the properties of discrete nonholonomic mechanics is carried out from the "intrinsic" point of view, so things are proved for the Generalized Coordinate Formulation.

5. Geometric invariance properties

In the unconstrained case, one can study discrete mechanics by itself, starting from a given discrete Lagrangian L_d and investigating the geometric properties that the discrete flow enjoys, such as the preservation of the symplectic form or of the momentum in the presence of symmetry. Furthermore, when one regards a discrete mechanical system as an approximation of a continuous one, it turns out that the symplectic-momentum nature of the variational integrators makes the difference in capturing the essential features of Lagrangian systems.

In the following, we provide some geometric arguments for the good performance of the DLA algorithm when compared to other standard higher order numerical methods, such as the 4^{th} order Runge-Kutta, as will be shown in Section 6. Of course, a more thorough error analysis would be of interest, but here we focus our attention on the invariance properties that the discrete nonholonomic mechanics possesses, as a sign of its appropriateness for approximating the continuous counterpart.

As we have mentioned above, in nonholonomic mechanics the symplectic form is not preserved by the flow of the system, so one can not expect the discrete version to preserve it. However, we will show in Section 5.1 that the discrete flow preserves the structure of the evolution of the symplectic form along the trajectories of the system. This property generalizes the symplectic character of variational integrators systems and, in fact, one precisely recovers the preservation of the symplectic form in the absence of constraints.

Moreover, under the action of a Lie group G on the configuration manifold Q, leaving invariant the Lagrangian $L : TQ \longrightarrow \mathbb{R}$ and the constraints $\mathcal{D} \subset TQ$, the associated momentum $J : TQ \longrightarrow \mathfrak{g}^*$ in general will not be conserved either. The development of the reduction theory of nonholonomic Lagrangian systems with symmetry has drawn on a careful examination of the compatibility of the symmetry directions and the constraints, which is encoded in the intersection $\mathcal{V} \cap \mathcal{D}$. Koiller [18] started with the so-called vertical or purely kinematic case, $\mathcal{V} \cap \mathcal{D} = 0$, and subsequent works [4, 7, 10, 29] have treated the horizontal, $\mathcal{V} \subset \mathcal{D}$, and the general cases $0 \subseteq \mathcal{V} \cap \mathcal{D} \subseteq \mathcal{V}$ (other relevant contributions make use, among others, of the Hamiltonian formalism [3] or of Poisson methods [19, 25], among others). An important geometric object in this reduction theory is the so-called nonholonomic momentum map, which corresponds to the usual momentum map restricted to the symmetry directions compatible with the constraints. This momentum map can be used to "augment" the constraints and provide a principal connection on $Q \longrightarrow Q/G$, the nonholonomic principal connection, a fact with important applications, for instance, to the control of nonholonomic systems [29]. In addition, one can measure the evolution of this mapping along the integral curves of the Lagrange-d'Alembert equations. This "measurement" constitutes the nonholonomic momentum equation [4]. We shall show in Section 5.2 that the DLA algorithm satisfies a discrete version of the nonholonomic momentum equation. In addition, in the presence of horizontal symmetries, we shall show that the associated momenta are actually conservation laws.

In the vertical or purely kinematic case, there are no symmetry directions lying in the constraint distribution and one does not have any nonholonomic momentum. Nonholonomic systems of Chaplygin type form the most representative class of systems falling into this category. In Section 5.3, we will discuss how, for Chaplygin systems, the DLA algorithm passes to the reduced space Q/G and yields a variational integrator in the sense of [16]. In some cases (in agreement with the continuous counterpart), this reduced formulation exactly yields a standard variational integrator.

5.1. The symplectic form

In this section, we investigate the behaviour of the DLA algorithm with respect to the discrete symplectic form Ω_{L_d} defined in Section 2. In doing so, we first recall the properties of the continuous flow in this regard, and then show that the discrete algorithm follows the same pattern.

The nonholonomic equations of motion (3) can be written in a coordinate free form [22] in a symplectic context. To do so, we need to introduce some geometric objects. In terms of the tangent bundle coordinates $(q^A, \dot{q}^A)$, let us denote by $\Delta = \dot{q}^A \frac{\partial}{\partial \dot{q}^A}$ the dilation or Liouville vector field on TQ (see [24]) and by $S = dq^A \otimes \frac{\partial}{\partial \dot{q}^A}$ the canonical vertical endomorphism (see [23]). The action of S on a 1-form will be denoted by S^*. Then we can define the Poincaré-Cartan 1-form and 2-form, corresponding to a given Lagrangian L, by $\Theta_L = S^* dL$ and $\Omega_L = -d\Theta_L$, respectively. We further have that $E_L = \Delta L - L$ represents the energy function of the system. If the Lagrangian L is regular, which will always be tacitly assumed in the sequel, Ω_L is symplectic. The equations of motion for the nonholonomic system are then given by

$$\left\{ \begin{array}{l} (i_X \Omega_L - dE_L)_{|\mathcal{D}} \in S^*((T\mathcal{D})^o) \, , \\ X_{|\mathcal{D}} \in T\mathcal{D} \, . \end{array} \right. \tag{20}$$

The integral curves of the dynamical vector field X satisfy precisely the nonholonomic equations (3).

From (20), we can write $i_X \Omega_L = dE_L + \beta$, with $\beta \in S^*((T\mathcal{D})^o)$. This implies that the evolution of the symplectic form along the trajectories of the system is given by

$$\mathcal{L}_X \Omega_L = i_X d\Omega_L + d i_X \Omega_L = d\beta \, , \tag{21}$$

where $\mathcal{L}$ denotes the Lie derivative.

The DLA algorithm also preserves this structure for the evolution of the discrete symplectic form Ω_{L_d}. Indeed we have that

$$\begin{aligned}\Phi^*\Omega_{L_d} &= -\Phi^*\mathcal{F}L_d^* d\Theta_Q = -d(\mathcal{F}L_d\circ\Phi)^*\Theta_Q = -d(\tilde{\mathcal{F}}L_d\circ\Phi)^*\Theta_Q\\ &= -d(\mathcal{F}L_d\Theta_Q - \beta_d)\,,\end{aligned}$$

where $\beta_d \in \mathcal{D}^o$ and in the last equality we have used the definition of the discrete principle (6). Finally, we get

$$\Phi^*\Omega_{L_d} = \Omega_{L_d} + d\beta_d\,, \tag{22}$$

which is the discrete version of Eq. (21). Note that in the absence of constraints, we precisely recover the conservation of the discrete symplectic form.

5.2. The momentum

In nonholonomic mechanics, the momentum associated to a symmetry group G of the system in general is not a conserved quantity. Instead, one considers a nonholonomic momentum map J^{nh}, which is the usual one restricted to the symmetry directions compatible with the constraints, and derives a momentum equation describing the evolution of J^{nh}. What we develop in the following is a discrete version of the nonholonomic momentum map and we show that the nonholonomic integrator (6) fulfills a discrete version of the momentum equation.

Let us briefly recall how the theory is developed in the continuous picture [4, 7]. Consider a Lie group G acting on the configuration manifold Q, such that the Lagrangian $L : TQ \longrightarrow \mathbb{R}$ and the constraints $\mathcal{D} \subset TQ$ are G-invariant. For each $q \in Q$, the following subspace of the Lie algebra of G,

$$\mathfrak{g}_q = \{\xi \in \mathfrak{g} \,/\, \xi_Q(q) \in \mathcal{D}_q\}\,,$$

is introduced, where ξ_Q denotes the fundamental vector field associated to the element $\xi \in \mathfrak{g}$. Denote by $\mathfrak{g}^{\mathcal{D}}$ the disjoint union of all such subspaces, $\mathfrak{g}^{\mathcal{D}} = \cup_{q\in Q}\mathfrak{g}_q$. Then, we have a generalized bundle $\mathfrak{g}^{\mathcal{D}} \longrightarrow Q$ which captures at each point $q \in Q$ the symmetry directions which lie in the constraint distribution. Define then

$$\begin{array}{rccclll} J^{nh}: & TQ & \longrightarrow & \mathfrak{g}^{\mathcal{D}*} & & & \\ & v_q & \longmapsto & J^{nh}(v_q): & \mathfrak{g}_q & \longrightarrow & \mathbb{R} \\ & & & & \xi & \longmapsto & \langle J^{nh}(v_q),\xi\rangle = \langle \mathcal{F}L(v_q), \xi_Q(q)\rangle\,. \end{array}$$

Note in passing that the nonholonomic momentum map coincides with the usual momentum map along the bundle $\mathfrak{g}^{\mathcal{D}}$.

Assume we have a smooth section $\tilde{\xi}$ of the bundle $\mathfrak{g}^{\mathcal{D}} \longrightarrow Q$ and consider the function $J^{nh}_{\tilde{\xi}} : TQ \longrightarrow \mathbb{R}$ given by $J^{nh}_{\tilde{\xi}} = \langle J^{nh}, \tilde{\xi}\rangle$. By means of the Lagrange-d'Alembert principle one can now prove the following

Theorem 5.1 *[4] Any solution $q(t)$ of the Lagrange-d'Alembert equations for the nonholonomic system must satisfy the momentum equation*

$$\frac{d}{dt}\left(J^{nh}_{\tilde{\xi}}\right) = \left\langle \mathcal{F}L(\dot{q}(t)), \left(\frac{d}{dt}\tilde{\xi}(q(t))\right)_Q \right\rangle = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}\tilde{\xi}(q(t))\right]^i_Q . \tag{23}$$

In some cases, it can happen that an element ξ of the Lie algebra belongs to $\mathfrak{g}_q$, for all $q \in Q$. This then defines a constant section $\tilde{\xi}$ of the bundle $\mathfrak{g}^{\mathcal{D}}$ and ξ is called a **horizontal symmetry**. As a consequence of the momentum equation (23), we get

Corollary 5.2 *If ξ is a horizontal symmetry, then $J^{nh}_{\tilde{\xi}}$ is a conservation law.*

Next, we investigate these issues for the discrete Lagrange-d'Alembert principle developed in Section 3. First, given the discrete Lagrangian $L_d : Q \times Q \longrightarrow \mathbb{R}$, define the discrete momentum map by

$$\begin{array}{rccccc} J^{nh}_d : & Q \times Q & \longrightarrow & \mathfrak{g}^{\mathcal{D}*} & & \\ & (q_{k-1}, q_k) & \longmapsto & J^{nh}_d(q_{k-1}, q_k) : \ \mathfrak{g}_q & \longrightarrow & \mathbb{R} \\ & & & \xi & \longmapsto & \langle D_2 L_d(q_{k-1}, q_k), \xi_Q(q_k)\rangle . \end{array}$$

Take, as in the continuous case, a smooth section $\tilde{\xi}$ of the bundle $\mathfrak{g}^{\mathcal{D}}$ and consider the function $(J^{nh}_d)_{\tilde{\xi}}$ on $Q \times Q$. Then, one finds that the nonholonomic integrator fullfils the following discrete version of the momentum equation.

Theorem 5.3 *The flow $(q_{k-1}, q_k) \longmapsto (q_k, q_{k+1})$ of the discrete Lagrange-d'Alembert equations verifies*

$$(J^{nh}_d)_{\tilde{\xi}}(q_k, q_{k+1}) - (J^{nh}_d)_{\tilde{\xi}}(q_{k-1}, q_k) = D_2 L_d(q_k, q_{k+1})\left(\tilde{\xi}(q_{k+1}) - \tilde{\xi}(q_k)\right)_Q (q_{k+1}) . \tag{24}$$

Proof: The invariance of the discrete Lagrangian L_d implies that

$$L(\exp(s\tilde{\xi}(q_k))q_k, \exp(s\tilde{\xi}(q_k))q_{k+1}) = L(q_k, q_{k+1}) .$$

Differentiating with respect to s and setting $s = 0$ yields

$$D_1 L_d(q_k, q_{k+1})\left(\tilde{\xi}(q_k)\right)_Q (q_k) + D_2 L_d(q_k, q_{k+1})\left(\tilde{\xi}(q_k)\right)_Q (q_{k+1}) = 0 . \tag{25}$$

On the other hand, the discretization of the Lagrange-d'Alembert principle (6) implies that

$$D_1 L_d(q_k, q_{k+1})\left(\tilde{\xi}(q_k)\right)_Q (q_k) + D_2 L_d(q_{k-1}, q_k)\left(\tilde{\xi}(q_k)\right)_Q (q_k) = 0 . \tag{26}$$

Subtracting equation (25) from equation (26), we find that

$$D_2 L_d(q_k, q_{k+1})\left(\tilde{\xi}(q_k)\right)_Q (q_{k+1}) = D_2 L_d(q_{k-1}, q_k)\left(\tilde{\xi}(q_k)\right)_Q (q_k) . \tag{27}$$

Finally, the result follows from (27) since

$$\begin{aligned} (J^{nh}_d)_{\tilde{\xi}}(q_k, q_{k+1}) &- (J^{nh}_d)_{\tilde{\xi}}(q_{k-1}, q_k) = \\ &= D_2 L_d(q_k, q_{k+1})\left(\tilde{\xi}(q_{k+1})\right)_Q (q_{k+1}) - D_2 L_d(q_{k-1}, q_k)\left(\tilde{\xi}(q_k)\right)_Q (q_k) \\ &= D_2 L_d(q_k, q_{k+1})\left(\tilde{\xi}(q_{k+1})\right)_Q (q_{k+1}) - D_2 L_d(q_k, q_{k+1})\left(\tilde{\xi}(q_k)\right)_Q (q_{k+1}) . \end{aligned}$$

$\boxed{QED}$

In the presence of horizontal symmetries we find that the algorithm (6) exactly preserves the associated components of the momentum.

Corollary 5.4 *If ξ is a horizontal symmetry, then $(J_d^{nh})_\xi$ is conserved by the nonholonomic integrator.*

5.3. Chaplygin systems

Consider the situation in which the action of the Lie group G has no symmetry direction lying in $\mathcal{D}$, i.e. $\mathcal{V} \cap \mathcal{D} = 0$, where $\mathcal{V}$ denotes the vertical bundle of the projection $\pi : Q \longrightarrow Q/G$. Under the common assumption that $\mathcal{D}_q + \mathcal{V}_q = T_qQ$ for all $q \in Q$ (dimension assumption), one has indeed a splitting of the tangent bundle at each point $q \in Q$, $T_qQ = \mathcal{D}_q \oplus \mathcal{V}_q$. The distribution $\mathcal{D}$ being G-invariant, this situation corresponds precisely to the notion of a principal connection on the principal fiber bundle $\pi : Q \longrightarrow Q/G$. Such systems are known in the literature as generalized Chaplygin systems [6, 18].

It is known that one of the peculiarities of nonholonomic Chaplygin systems is that, after reduction by the Lie group G, they take on the form of an unconstrained system, subject to an "external" force of a special type. In the following, we briefly review these facts for the sake of clarity of the exposition.

5.3.1. Reduction in the continuous case In the Chaplygin case, reduction can be achieved as follows. Consider the lifted action of the Lie group G on TQ and denote by $\rho : TQ \longrightarrow \overline{TQ}$ the associated projection. Define by F the subbundle of TTQ along $\mathcal{D}$ whose annihilator is given by $S^*((T\mathcal{D})^o)$. Then, $\mathcal{V} \cap \mathcal{D} = 0$ implies that $F \cap \mathcal{V}_\rho = 0$ and therefore

$$T\mathcal{D} = (F \cap T\mathcal{D}) \oplus \mathcal{V}_\rho \,.$$

This means that on the principal bundle $\rho_{|\mathcal{D}} : \mathcal{D} \longrightarrow \overline{\mathcal{D}} \equiv T(Q/G)$ we have another principal connection. Denote by $h : T\mathcal{D} \longrightarrow F \cap T\mathcal{D}$ and $v : T\mathcal{D} \longrightarrow \mathcal{V}_\rho$ the horizontal and vertical projectors, respectively. Then, we consider the 1-form

$$\alpha = i_X(h^* di^* \Theta_L - dh^* i^* \Theta_L) \,,$$

where $i : \mathcal{D} \hookrightarrow TQ$ is the canonical inclusion.

On the other hand, the Lagrangian L induces a Lagrangian $L^* : T(Q/G) \longrightarrow \mathbb{R}$ by

$$L^*(\bar{q}, v_{\bar{q}}) = L(q, v^h_{\bar{q}}) \,,$$

where $\pi(q) = \bar{q}$ and $v^h_{\bar{q}}$ denotes the unique vector in $\mathcal{D}_q$ such that $\pi_*(v^h_{\bar{q}}) = v_{\bar{q}}$. This function is well-defined because of the G-invariance of L. Then, one can prove that the solution X of (20) is ρ-projectable and that the projected dynamics $\overline{X} = \rho_*(X)$ satisfies

$$i_{\overline{X}} \omega_{L^*} = dE_{L^*} + \overline{\alpha} \,, \tag{28}$$

where $\overline{\alpha}$ is the projection of the 1-form α. Moreover, one can show that the contraction of $\overline{\alpha}$ with $\overline{X}$ vanishes, that is, $i_{\overline{X}}\overline{\alpha} = 0$. Hence, the nonconservative force represented by $\overline{\alpha}$ is of "gyroscopic" type. This implies in particular that the energy E_{L^*} is a conserved quantity of the reduced dynamics.

Alternatively, a principal connection can be characterized by a $\mathfrak{g}$-valued 1-form A on Q (the connection 1-form) satisfying $A(\xi_Q(q)) = \xi$ for all $\xi \in \mathfrak{g}$ and $A(\phi_{g_*}X) = Ad_g(A(X))$ for all $X \in TQ$. The constraint distribution is precisely given by the horizontal space $\mathcal{D}_q = \{v_q \in T_qQ : A(v_q) = 0\}$. The principal bundle structure implies that the configuration manifold Q can be locally seen as $Q/G \times G$. In the sequel, we will not make a notational distinction between (r, g) considered as a point on the product manifold and considered as the corresponding adapted coordinates. In each case, the precise meaning should be clear from the context.

In terms of adapted bundle coordinates, the G-action on Q reads $\phi_h(r, g) = (r, hg)$ and the projection $\pi : Q \longrightarrow Q/G$ is given by $\pi(r, g) = r$. The coordinate expression for the connection 1-form then reads $A(r, g)(\dot{r}, \dot{g}) = Ad_g(g^{-1}\dot{g} + A(r, e)\dot{r})$, so that the constraint 1-forms become $\omega = g^{-1}dg + A(r, e)dr$. If we fix a basis $\{e_1, \dots, e_m\}$ of the Lie algebra $\mathfrak{g}$, then we can write

$$\omega = g^{-1}dg + A^b_\beta(r)dr^\beta e_b \,. \tag{29}$$

The G-invariance of the Lagrangian yields $L(r, g, \dot{r}, \dot{g}) = L(r, e, \dot{r}, g^{-1}\dot{g})$. Denote then by $\ell(r, \dot{r}, \xi)$ the projection of L onto TQ/G. It follows that the reduced Lagrangian L^* on $T(Q/G)$ is given by $L^*(r, \dot{r}) = \ell(r, \dot{r}, -A^b_\beta(r)\dot{r}^\beta e_b)$. The gyroscopic 1-form can be locally written as

$$\alpha = -\left(\frac{\partial \ell}{\partial \xi^a}\right)^* \left(\frac{\partial A^a_\beta}{\partial r^\gamma} - \frac{\partial A^a_\gamma}{\partial r^\beta} + c^a_{bc}A^b_\beta A^c_\gamma\right) \dot{r}^\gamma dr^\beta \,,$$

where the $*$ on the right-hand side indicates that, after computing the derivative of ℓ with respect to ξ^b, one replaces the ξ^a everywhere by $-A^a_\beta(r)\dot{r}^\beta$. The constants c^b_{ac} appearing in the last term on the right-hand side are the structure constants of $\mathfrak{g}$ with respect to the chosen basis, i.e. $[e_b, e_c] = c^a_{bc}e_a$. Note in passing that the expressions $\dfrac{\partial A^b_\beta}{\partial r^\gamma} - \dfrac{\partial A^b_\gamma}{\partial r^\beta} + c^b_{ac}A^a_\beta A^c_\gamma$ are the coefficients of the curvature of the principal connection A in local form. Consequently, the integral curves of the projected solution $\bar{X}$ verify the equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}^\beta}\right) - \frac{\partial L}{\partial r^\beta} = -\alpha_\beta \,,$$

where $\beta \in \{1, \dots, n-m\}$.

5.3.2. Reduction of the discrete principle Next, we examine the possibility of passing the discrete nonholonomic principle to the reduced space Q/G. Consider a discrete Lagrangian $L_d : Q \times Q \longrightarrow \mathbb{R}$ and a discrete space $\mathcal{D}_d$, described by the annihilation of some constraint functions $\omega^a_d : Q \times Q \longrightarrow \mathbb{R}$, $a \in \{1, \dots, m\}$. Assume that both the Lagrangian and the constraints are G-invariant under the diagonal action of the Lie group on the manifold $Q\times Q$. The DLA algorithm then becomes

$$\left\{\begin{array}{rcl} D_1L_d(r_k,g_k,r_{k+1},g_{k+1})+D_2L_d(r_{k-1},g_{k-1},r_k,g_k) & = & \lambda_a\omega^a(r_k,g_k)\\ \omega^a_d(r_k,g_k,r_{k+1},g_{k+1}) & = & 0\,, \end{array}\right. \tag{30}$$

where it should be recalled that D_1 denotes the derivative with respect to $q_k=(r_k,g_k)$. These equations can be rewritten, using the expression (29) for the constraint 1-forms, in the following form

$$\left\{\begin{array}{l} \dfrac{\partial L_d}{\partial r^\beta_k}(r_k,g_k,r_{k+1},g_{k+1})+\dfrac{\partial L_d}{\partial r^\beta_k}(r_{k-1},g_{k-1},r_k,g_k)=\\ \qquad =\left(\dfrac{\partial L_d}{\partial g_k}(r_k,g_k,r_{k+1},g_{k+1})+\dfrac{\partial L_d}{\partial g_k}(r_{k-1},g_{k-1},r_k,g_k)\right)A^b_\beta(r_k)L_{g_k*}e_b\\ \omega^a_d(r_k,g_k,r_{k+1},g_{k+1})=0\,, \end{array}\right. \tag{31}$$

where L_g denotes the left multiplication by g in G. It must be noted that in the right-hand side of the first equation, a shorthand notation is used to denote the natural pairing between tangent vectors and covectors on G.

Observe that $\mathcal{D}_d$ can be locally identified with $Q/G\times Q/G\times G$ via the assignement

$$(r_k,g_k,r_{k+1},g_{k+1})\in\mathcal{D}_d\longmapsto(r_k,g_k,r_{k+1})\,,$$

since g_{k+1} is uniquely determined by the equations $\omega^a_d(r_k,g_k,r_{k+1},g_{k+1})=0$, $a\in\{1,\ldots,m\}$. In addition, the G-invariance of the constraint functions implies that $g_{k+1}(r_k,g_k,r_{k+1})=g_k\cdot g_{k+1}(r_k,e,r_{k+1})$.

Let us consider the restriction of $L_d:Q\times Q\longrightarrow\mathbb{R}$ to $\mathcal{D}_d$, $L^c_d:\mathcal{D}_d\longrightarrow\mathbb{R}$. The G-invariance of L_d and $\mathcal{D}_d$ implies the G-invariance of L^c_d. Define a discrete Lagrangian L^*_d on the reduced manifold as

$$\begin{array}{rccc} L^*_d: & Q/G\times Q/G & \longrightarrow & \mathbb{R}\\ & (r_k,r_{k+1}) & \longmapsto & L^c_d(r_k,e,r_{k+1})\,. \end{array}$$

Now, we shall write the DLA algorithm (31) in terms of the constrained discrete Lagrangian L^c_d and then examine the possibility of passing the equations to Q/G, in terms of the reduced discrete Lagrangian L^*_d. First, we have that

$$\begin{aligned} \frac{\partial L^c_d}{\partial r^\beta_k} &= \frac{\partial L_d}{\partial r^\beta_k}+\frac{\partial L_d}{\partial g_{k+1}}\frac{\partial g_{k+1}}{\partial r^\beta_k}\,,\\ \frac{\partial L^c_d}{\partial r^\beta_{k+1}} &= \frac{\partial L_d}{\partial r^\beta_{k+1}}+\frac{\partial L_d}{\partial g_{k+1}}\frac{\partial g_{k+1}}{\partial r^\beta_{k+1}}\,. \end{aligned}$$

Secondly, we also have

$$0=\frac{\partial L^c_d}{\partial g_k}=\frac{\partial L_d}{\partial g_k}+\frac{\partial L_d}{\partial g_{k+1}}\frac{\partial g_{k+1}}{\partial g_k}=\frac{\partial L_d}{\partial g_k}+R^*_{g_{k+1}}\frac{\partial L_d}{\partial g_{k+1}}\,,$$

where R_g denotes the right multiplication in the Lie group by the element $g\in G$.

In view of this, we see that the nonholonomic integrator can be expressed in the following way

$$D_1L^c_d(r_k,g_k,r_{k+1})+D_2L^c_d(r_{k-1},g_{k-1},r_k)=F^-(q_k,q_{k+1})+F^+(q_{k-1},q_k)\,,$$

where

$$
\begin{aligned}
F^-(q_k, q_{k+1}) \quad &= \frac{\partial L_d}{\partial g_{k+1}} \frac{\partial g_{k+1}}{\partial r_k^\beta} + \frac{\partial L_d}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1}) A_\beta^b(r_k) L_{g_k *} e_b \\
&= \left(\frac{\partial L_d}{\partial g_{k+1}} \frac{\partial g_{k+1}}{\partial r_k^\beta}(r_k, r_{k+1}) + \frac{\partial L_d}{\partial g_k} A_\beta^b(r_k) \right) L_{g_k *} e_b \,, \\
F^+(q_{k-1}, q_k) \quad &= \frac{\partial L_d}{\partial g_k} \frac{\partial g_k}{\partial r_k^\beta} + \frac{\partial L_d}{\partial g_k}(r_{k-1}, g_{k-1}, r_k, g_k) A_\beta^b(r_k) L_{g_k *} e_b \\
&= \frac{\partial L_d}{\partial g_k} \frac{\partial g_k}{\partial r_k^\beta}(r_{k-1}, r_k) L_{g_{k-1} *} e_b - \frac{\partial L_d}{\partial g_{k-1}} A_\beta^b(r_k) L_{g_{k-1} *} Ad_{g_k(r_{k-1}, r_k)} e_b \,.
\end{aligned}
$$

Note that both discrete forces, F^- and F^+, are G-invariant. This can be seen as follows. As L_d is G-invariant, we have that $L_d(r_k, g_k, r_{k+1}, g_{k+1}) = L_d(r_k, e, r_{k+1}, g_k^{-1} g_{k+1}) = \ell_d(r_k, r_{k+1}, f_{k,k+1})$, where we use the shorthand notation $f_{k,k+1} = g_k^{-1} g_{k+1}$. From here, one can derive that

$$
\begin{aligned}
\frac{\partial L_d}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1}) \quad &= -\, L^*_{g_k^{-1}} R^*_{f_{k,k+1}} \frac{\partial \ell_d}{\partial f_{k,k+1}} \\
\frac{\partial L_d}{\partial g_{k+1}}(r_k, g_k, r_{k+1}, g_{k+1}) &= L^*_{g_k^{-1}} \frac{\partial \ell_d}{\partial f_{k,k+1}} \,.
\end{aligned}
$$

Moreover, if $(r_k, g_k, r_{k+1}, g_{k+1}) \in \mathcal{D}_d$, then $f_{k,k+1} = g_k^{-1} g_{k+1} = g_{k+1}(r_k, r_{k+1})$. Therefore, substituting in the expressions for the discrete forces, one verifies that

$$
\begin{aligned}
F^-(q_k, q_{k+1}) = &\frac{\partial \ell_d}{\partial f_{k,k+1}}(r_k, r_{k+1}, f_{k,k+1}) \frac{\partial g_{k+1}}{\partial r_k^\beta}(r_k, r_{k+1}) \\
&-\, R^*_{f_{k,k+1}} \frac{\partial \ell_d}{\partial f_{k,k+1}}(r_k, r_{k+1}, f_{k,k+1}) A_\beta^b(r_k) e_b \,, \\
F^+(q_{k-1}, q_k) = &\frac{\partial \ell_d}{\partial f_{k-1,k}} \frac{\partial g_k}{\partial r_k^\beta}(r_{k-1}, r_k) \\
&+\, L_{g_k(r_{k-1}, r_k)} \frac{\partial \ell_d}{\partial f_{k-1,k}}(r_{k-1}, r_k, f_{k-1,k}) A_\beta^b(r_k) e_b \,.
\end{aligned}
$$

Therefore, we can write a well-defined algorithm on Q/G of the form

$$
D_1 L_d^*(r_k, r_{k+1}) + D_2 L_d^*(r_{k-1}, r_k) = F^-(r_k, r_{k+1}) + F^+(r_{k-1}, r_k) \,. \tag{32}
$$

Equation (32) belongs to the type of discretization generalizing variational integrators for systems with external forces developed in [16]:

$$
\delta \sum L_d(q_k, q_{k+1}) + \sum (F_d^-(q_k, q_{k+1}) \delta q_k + F_d^+(q_k, q_{k+1}) \delta q_{k+1}) = 0 \,, \tag{33}
$$

where F_d^-, F_d^+ are the left and right discrete friction forces. Equation (33) defines an integrator $(q_{k-1}, q_k) \longmapsto (q_k, q_{k+1})$ given implicitly by the forced discrete Euler-Lagrange equations

$$
D_1 L_d(q_k, q_{k+1}) + D_2 L_d(q_{k-1}, q_k) + F_d^-(q_k, q_{k+1}) + F_d^+(q_{k-1}, q_k) = 0 \,. \tag{34}
$$

We summarize the above discussion in the following result.

Theorem 5.5 *Consider a discrete nonholonomic problem with data $L_d : Q \times Q \longrightarrow \mathbb{R}$, a distribution $\mathcal{D}$ on Q and a discrete constraint space $\mathcal{D}_d$. Let G be a Lie group acting freely and properly on Q, leaving $\mathcal{D}$ invariant and such that $T_qQ = \mathcal{D}_q \oplus \mathcal{V}_q$, for all $q \in Q$, where $\mathcal{V}$ denotes the vertical bundle of the G-action. Assume further that both L_d and $\mathcal{D}_d$ are invariant under the diagonal action of a Lie group G on the manifold $Q \times Q$. Then, the DLA algorithm (31) passes to the reduced space Q/G, yielding a generalized variational integrator in the sense of [16]. We call the algorithm on Q/G the reduced discrete Lagrange-d'Alembert algorithm (RDLA).*

So far, we have obtained that the DLA algorithm respects the structure of the evolution of the symplectic form along the flow of the system (cf. Eq (22)) and that, in the presence of symmetries, it satisfies a discrete version of the nonholonomic momentum equation. In addition, we have been able to establish in the two extremal cases (horizontal and vertical) Corollary 5.4 and Theorem 5.5, respectively. These results are important, both from a geometrical and from a numerical perspective. On the one hand, they show interesting interactions between the discrete unconstrained and nonholonomic mechanics, similar to those occuring in the continuous case. On the other hand, when regarding the discrete version of mechanics as an approximation of the continuous one, they provide good arguments to consider the proposed DLA algorithm (6) as an appropriate (in a symplectic-momentum sense) discretization of the continuous flow.

It is worth noting, though, that when regarding the discrete nonholonomic mechanics as an approximation of the continuous one, one cannot expect the diagram

$$\begin{tikzcd} \mathrm{LA} \arrow[r] \arrow[d] & \mathrm{RLA} \\ \mathrm{DLA} \arrow[r] & \mathrm{RDLA} \end{tikzcd}$$

to be commutative in general, because the two horizontal arrows symbolize processes that are of a different mathematical nature (discrete and continuous, respectively). For instance, there exist some special situations in which the reduced Chaplygin system admits a Hamiltonian description, that is, the gyroscopic force F vanishes [6]. But in general, the RDLA will not be a standard variational integrator. In the following section we show that, under strong assumptions on the linearity of the geometric operations involved, the midpoint RDLA algorithm [which corresponds to take $\alpha = 1/2$ in Eq. (9) (or in Eq. (10), since in this case both discretizations coincide)] yields indeed a variational integrator, i.e. the diagram is commutative. The hypothesis on the linearity are justified by the fact that the diagram involves both discrete and continuous systems. This result provides an additional reason (since we already know that this type of discretization always guarantees a second order accurate numerical approximation to the continuous flow) to consider the midpoint rule as a reliable integrator.

5.3.3. The midpoint RDLA algorithm Consider a nonholonomic Chaplygin system, with the following data: a principal G-bundle $\pi : Q \longrightarrow M = Q/G$, associated to a free and proper action Ψ of G on Q, a Lagrangian $L : TQ \longrightarrow \mathbb{R}$ which is G-invariant with respect to the lifted action on TQ, and linear nonholonomic constraints determined by the horizontal distribution $\mathcal{D}$ of a principal connection γ on π. In this section, we will focus our attention on the midpoint RDLA algorithm.

For each $q = (r_k, g_k) \in Q$, take a product chart $\varphi = \varphi_1 \times \varphi_2$, given by a chart φ_1 in Q/G and a chart φ_2 in G. For the latter, we take (see [26]): $\varphi_2 = \exp^{-1} \circ L_{g_k^{-1}}$, which is defined in a neighbourhod of g_k, where $\exp : \mathfrak{g} \longrightarrow G$ is the exponential mapping. Denote by

$$\eta = \frac{1}{2}\varphi_2(g_k) + \frac{1}{2}\varphi_2(g_{k+1}) = \frac{1}{2}\log(g_k^{-1}g_{k+1})\,,$$
$$\zeta = \frac{\varphi_2(g_{k+1}) - \varphi_2(g_k)}{h} = \frac{\log(g_k^{-1}g_{k+1})}{h}\,.$$

We assume that Q/G is itself a linear space, so that we can always take the identity chart $\varphi_1 = id_{Q/G}$. With this type of charts, we can construct the discrete Lagrangian and the discrete constraint distribution as explained in Remark 4.2.

The discrete Lagrangian then reads

$$L_d^{\frac{1}{2}}(r_k, g_k, r_{k+1}, g_{k+1}) = L\left(r_{k+\frac{1}{2}}, \varphi_2^{-1}(\eta), \frac{r_{k+1} - r_k}{h}, (\varphi_2^{-1})_*(\zeta)\right)\,,$$

and the discrete nonholonomic constraints

$$\zeta + A_\beta^b(r_{k+\frac{1}{2}})\left(\frac{r_{k+1} - r_k}{h}\right)^\beta e_b = 0\,.$$

As before, the shorthand notation $r_{k+\frac{1}{2}} = \frac{1}{2}r_k + \frac{1}{2}r_{k+1}$ is understood. The above discretizations of the Lagrangian and of the constraints are G-invariant under the diagonal action of the Lie group on the manifold $Q \times Q$.

Here, we will make a different identification between $\mathcal{D}_d$ and $Q/G \times Q/G \times G$, taking into account the specific structure of the constraint functions. More precisely, we identify $\mathcal{D}_d$ with $Q/G \times Q/G \times G$ via the assignment

$$(r_k, g_k, r_{k+1}, g_{k+1}) \in \mathcal{D}_d \longmapsto (r_k, r_{k+1}, \hat{g})\,,$$

where

$$\hat{g} = \varphi_2^{-1}(\eta) = L_{g_k}\exp(\frac{1}{2}h\zeta) = g_k\exp(-\frac{1}{2}A(r_{k+\frac{1}{2}})(r_{k+1} - r_k))\,.$$

The inverse mapping $(r_k, r_{k+1}, \hat{g}) \longmapsto (r_k, g_k, r_{k+1}, g_{k+1}) \in \mathcal{D}_d$ is given by

$$g_{k+1} = \hat{g}\exp(\frac{1}{2}h\zeta)\,, \quad g_k = \hat{g}\exp(-\frac{1}{2}h\zeta)\,. \tag{35}$$

Consider the restriction of $L_d^{\frac{1}{2}} : Q \times Q \longrightarrow \mathbb{R}$ to $\mathcal{D}_d$, $L_d^c : \mathcal{D}_d \longrightarrow \mathbb{R}$. Define, as before, the discrete Lagrangian L_d^* on the reduced manifold as

$$\begin{array}{rccl} L_d^* : & Q/G \times Q/G & \longrightarrow & \mathbb{R} \\ & (r_k, r_{k+1}) & \longmapsto & L_d^c(r_k, r_{k+1}, e)\,. \end{array}$$

Then, we have that

$$\frac{\partial L_d^c}{\partial r_k^\beta} = \frac{\partial L_d^{\frac{1}{2}}}{\partial r_k^\beta} + \frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}\frac{\partial g_k}{\partial r_k^\beta} + \frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k+1}}\frac{\partial g_{k+1}}{\partial r_k^\beta},$$

where, from (35),

$$\begin{aligned} \frac{\partial g_k}{\partial r_k^\beta} &= -\frac{1}{2}\left(A_\beta^b(r_{k+\frac{1}{2}}) - \frac{1}{2}\frac{\partial A_\gamma^b}{\partial r^\beta}(r_{k+\frac{1}{2}})(r_{k+1}^\gamma - r_k^\gamma)\right) L_{g_k *}e_b\,, \\ \frac{\partial g_{k+1}}{\partial r_k^\beta} &= \frac{1}{2}\left(A_\beta^b(r_{k+\frac{1}{2}}) - \frac{1}{2}\frac{\partial A_\gamma^b}{\partial r^\beta}(r_{k+\frac{1}{2}})(r_{k+1}^\gamma - r_k^\gamma)\right) L_{g_{k+1} *}e_b\,. \end{aligned}$$

Analogously, we see that

$$\frac{\partial L_d^c}{\partial r_{k+1}^\beta} = \frac{\partial L_d^{\frac{1}{2}}}{\partial r_{k+1}^\beta} + \frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}\frac{\partial g_k}{\partial r_{k+1}^\beta} + \frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k+1}}\frac{\partial g_{k+1}}{\partial r_{k+1}^\beta},$$

where

$$\begin{aligned} \frac{\partial g_k}{\partial r_{k+1}^\beta} &= -\frac{1}{2}\left(-A_\beta^b(r_{k+\frac{1}{2}}) - \frac{1}{2}\frac{\partial A_\gamma^b}{\partial r^\beta}(r_{k+\frac{1}{2}})(r_{k+1}^\gamma - r_k^\gamma)\right) L_{g_k *}e_b\,, \\ \frac{\partial g_{k+1}}{\partial r_{k+1}^\beta} &= \frac{1}{2}\left(-A_\beta^b(r_{k+\frac{1}{2}}) - \frac{1}{2}\frac{\partial A_\gamma^b}{\partial r^\beta}(r_{k+\frac{1}{2}})(r_{k+1}^\gamma - r_k^\gamma)\right) L_{g_{k+1} *}e_b\,. \end{aligned}$$

Secondly, we also have

$$0 = \frac{\partial L_d^c}{\partial \hat g} = \frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}\frac{\partial g_k}{\partial \hat g} + \frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k+1}}\frac{\partial g_{k+1}}{\partial \hat g} = R^*_{\exp(-\frac{1}{2}h\zeta)}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k} + R^*_{\exp(\frac{1}{2}h\zeta)}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k+1}}\,. \tag{36}$$

Now, we expand the term $\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1})$ on the right-hand side of the first equation in the DLA algorithm (31) as

$$\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1}) = \frac{1}{2}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1}) + \frac{1}{2}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1})\,,$$

and then make use of (36) to get the expression

$$\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1}) = \frac{1}{2}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1}) - \frac{1}{2}R^*_{\exp(h\zeta)}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k+1}}(r_k, g_k, r_{k+1}, g_{k+1})\,.$$

Analogously, we find for the other term

$$\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_{k-1}, g_{k-1}, r_k, g_k) = \frac{1}{2}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_{k-1}, g_{k-1}, r_k, g_k) - \frac{1}{2}R^*_{\exp(-h\zeta)}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k-1}}(r_{k-1}, g_{k-1}, r_k, g_k)\,.$$

Then, the discrete forces in the RDLA algorithm take the form

$$\begin{aligned} F^-(q_k, q_{k+1}) &= \frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}\frac{\partial g_k}{\partial r_k^\beta} + \frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k+1}}\frac{\partial g_{k+1}}{\partial r_k^\beta} + \frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_k, g_k, r_{k+1}, g_{k+1})A_\beta^b(r_k)L_{g_k *}e_b \\ &= \frac{1}{2}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}\left(-A_\beta^b(r_{k+\frac{1}{2}}) + \frac{1}{2}\frac{\partial A_\gamma^b}{\partial r^\beta}(r_{k+\frac{1}{2}})(r_{k+1}^\gamma - r_k^\gamma) + A_\beta^b(r_k)\right) L_{g_k *}e_b \end{aligned}$$

$$
\begin{aligned}
&+\ \frac{1}{2}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k+1}}\left(\left(A^b_\beta(r_{k+\frac{1}{2}})-\frac{1}{2}\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)\right)L_{g_{k+1}*}e_b\right.\\
&-\ \left.A^b_\beta(r_k)L_{g_{k+1}}Ad_{\exp(-h\zeta)}e_b\right)\ ,
\end{aligned}
$$

$$
\begin{aligned}
F^+(q_{k-1},q_k)\quad &= \frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k-1}}\frac{\partial g_{k-1}}{\partial r^\beta_k}+\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}\frac{\partial g_k}{\partial r^\beta_k}+\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}(r_{k-1},g_{k-1},r_k,g_k)A^b_\beta(r_k)L_{g_k*}e_b\\
&= \frac{1}{2}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_{k-1}}\left(\left(A^b_\beta(r_{k-\frac{1}{2}})+\frac{1}{2}\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}})(r^\gamma_k-r^\gamma_{k-1})\right)L_{g_{k-1}*}e_b\right.\\
&-\ \left.A^b_\beta(r_k)L_{g_{k-1}}Ad_{\exp(h\zeta)}e_b\right)\\
&+\ \frac{1}{2}\frac{\partial L_d^{\frac{1}{2}}}{\partial g_k}\left(-\left(A^b_\beta(r_{k-\frac{1}{2}})+\frac{1}{2}\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}})(r^\gamma_k-r^\gamma_{k-1})\right)+A^b_\beta(r_k)\right)L_{g_k*}e_b\,.
\end{aligned}
$$

By the linear dependence of $A(r)$ on r, we have that

$$
\begin{aligned}
A^b_\beta(r_k) &= A^b_\beta(r_{k+\frac{1}{2}})-\frac{1}{2}\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)\,,\\
A^b_\beta(r_k) &= A^b_\beta(r_{k-\frac{1}{2}})+\frac{1}{2}\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k-\frac{1}{2}})(r^\gamma_k-r^\gamma_{k-1})\,.
\end{aligned}
$$

Substituting into the expressions for the discrete forces, we get

$$
\begin{aligned}
F^-(q_k,q_{k+1})\quad &= \frac{1}{2}\frac{\partial L_d}{\partial g_k}\left(-\frac{1}{2}\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)+\frac{1}{2}\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)\right)L_{g_k*}e_b\\
&+\ \frac{1}{2}\frac{\partial L_d}{\partial g_{k+1}}\left(\left(A^b_\beta(r_{k+\frac{1}{2}})-\frac{1}{2}\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)\right)L_{g_{k+1}*}e_b\right.\\
&-\ \left.A^b_\beta(r_k)L_{g_{k+1}*}(e_b-[h\zeta,e_b])\right)\\
&= \frac{1}{4}\frac{\partial L_d}{\partial g_k}\left(-\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)+\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)\right)L_{g_k*}e_b\\
&+\ \frac{1}{4}\frac{\partial L_d}{\partial g_{k+1}}\left(\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)-\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k+\frac{1}{2}})(r^\gamma_{k+1}-r^\gamma_k)\right.\\
&-\ \left.2A^a_\beta(r_k)A^c_\gamma(r_{k+\frac{1}{2}})c^b_{ca}(r^\gamma_{k+1}-r^\gamma_k)\right)L_{g_{k+1}*}e_b\,,
\end{aligned}
$$

$$
\begin{aligned}
F^+(q_{k-1},q_k)\quad &= \frac{1}{2}\frac{\partial L_d}{\partial g_{k-1}}\left(\left(A^b_\beta(r_{k-\frac{1}{2}})+\frac{1}{2}\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}})(r^\gamma_k-r^\gamma_{k-1})\right)L_{g_{k-1}*}e_b\right.\\
&-\ \left.A^b_\beta(r_k)L_{g_{k-1}*}(e_b+[h\zeta,e_b])\right)\\
&+\ \frac{1}{2}\frac{\partial L_d}{\partial g_k}\left(-\frac{1}{2}\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}})(r^\gamma_k-r^\gamma_{k-1})+\frac{1}{2}\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k-\frac{1}{2}})(r^\gamma_k-r^\gamma_{k-1})\right)L_{g_k*}e_b\\
&= \frac{1}{4}\frac{\partial L_d}{\partial g_{k-1}}\left(-\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k-\frac{1}{2}})(r^\gamma_k-r^\gamma_{k-1})+\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}})(r^\gamma_k-r^\gamma_{k-1})\right.
\end{aligned}
$$

$$+\ 2A^a_\beta(r_k)A^c_\gamma(r_{k-\frac{1}{2}})c^b_{ca}(r^\gamma_k - r^\gamma_{k-1})\Big)\, L_{g_{k-1}*}e_b$$
$$+\ \frac{1}{4}\frac{\partial L_d}{\partial g_k}\left(-\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}})(r^\gamma_k - r^\gamma_{k-1}) + \frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k-\frac{1}{2}})(r^\gamma_k - r^\gamma_{k-1})\right) L_{g_k*}e_b\,,$$

where the c^b_{ca} are the structure constants of the Lie algebra $\mathfrak{g}$, $[e_c, e_a] = c^b_{ca}e_b$.

From the G-invariance of the continuous Lagrangian, one can derive that

$$\frac{\partial L_d}{\partial g_k} = -\frac{1}{h}L^*_{g_k^{-1}}\frac{\partial \ell}{\partial \xi}\,, \qquad \frac{\partial L_d}{\partial g_{k+1}} = \frac{1}{h}L^*_{g_{k+1}^{-1}}\frac{\partial \ell}{\partial \xi}\,.$$

Therefore, we find that the discrete forces can be rewritten as

$$\begin{aligned}
F^-(q_k, q_{k+1}) &= -\frac{1}{4}\frac{\partial \ell}{\partial \xi}\left(\frac{r^\gamma_{k+1} - r^\gamma_k}{h}\right)\left(-\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k+\frac{1}{2}}) + \frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k+\frac{1}{2}})\right) e_b \\
&\quad + \frac{1}{4}\frac{\partial \ell}{\partial \xi}\left(\frac{r^\gamma_{k+1} - r^\gamma_k}{h}\right)\left(\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k+\frac{1}{2}}) - \frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k+\frac{1}{2}}) - 2A^a_\beta(r_k)A^c_\gamma(r_{k+\frac{1}{2}})c^b_{ca}\right) e_b \\
&= \frac{1}{2}\frac{\partial \ell}{\partial \xi}\left(\frac{r^\gamma_{k+1} - r^\gamma_k}{h}\right)\left(\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k+\frac{1}{2}}) - \frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k+\frac{1}{2}}) - A^a_\beta(r_k)A^c_\gamma(r_{k+\frac{1}{2}})c^b_{ca}\right) e_b\,,
\end{aligned}$$

$$\begin{aligned}
F^+(q_{k-1}, q_k) &= -\frac{1}{4}\frac{\partial \ell}{\partial \xi}\left(\frac{r^\gamma_k - r^\gamma_{k-1}}{h}\right)\left(-\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k-\frac{1}{2}}) + \frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}}) + 2A^a_\beta(r_k)A^c_\gamma(r_{k-\frac{1}{2}})c^b_{ca}\right) e_b \\
&\quad + \frac{1}{4}\frac{\partial \ell}{\partial \xi}\left(\frac{r^\gamma_k - r^\gamma_{k-1}}{h}\right)\left(-\frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}}) + \frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k-\frac{1}{2}})\right) e_b \\
&= \frac{1}{2}\frac{\partial \ell}{\partial \xi}\left(\frac{r^\gamma_k - r^\gamma_{k-1}}{h}\right)\left(\frac{\partial A^b_\beta}{\partial r^\gamma}(r_{k-\frac{1}{2}}) - \frac{\partial A^b_\gamma}{\partial r^\beta}(r_{k-\frac{1}{2}}) - A^a_\beta(r_k)A^c_\gamma(r_{k-\frac{1}{2}})c^b_{ca}\right) e_b\,,
\end{aligned}$$

Note that the sum of both forces $F^-(q_k, q_{k+1}) + F^+(q_{k-1}, q_k)$ is a discretization of the continuous force

$$F = -\frac{\partial \ell}{\partial \xi^a}\dot r^\gamma B^a_{\beta\gamma} = -\frac{\partial \ell}{\partial \xi^a}\dot r^\gamma\left(\frac{\partial A^a_\gamma}{\partial r^\beta} - \frac{\partial A^a_\beta}{\partial r^\gamma} - c^a_{bc}A^c_\beta A^b_\gamma\right)$$

around the point q_k. Now, we are in a position to prove the following

Theorem 5.6 *Consider a nonholonomic Chaplygin system with data: a free and proper action $\Psi : G \times Q \longrightarrow Q$, a G-invariant Lagrangian $L : TQ \longrightarrow \mathbb{R}$ and a G-invariant distribution $\mathcal{D}$ on Q. Assume that Q/G is a linear space, the Lie group G is abelian and the constraints have a linear dependence on the base point. Then, if the reduced continuous Chaplygin system is Hamiltonian, the midpoint RDLA algorithm is a variational integrator.*

Proof: If the Lie group is abelian, then the structural constants vanish, $c^a_{bc} = 0$. Therefore, we can see the discrete forces as

$$\begin{aligned}
F^-(q_{k-1}, q_k) &= F(\frac{r_{k-1} + r_k}{2}, \frac{r_k - r_{k-1}}{h}) \\
F^+(q_k, q_{k+1}) &= F(\frac{r_k + r_{k+1}}{2}, \frac{r_{k+1} - r_k}{h})\,.
\end{aligned}$$

As a consequence, the vanishing of F implies the vanishing of the discrete forces and hence the RDLA algorithm takes the form

$$D_1 L_d^*(r_k, r_{k+1}) + D_2 L_d^*(r_{k-1}, r_k) = 0\,,$$

which is the variational integrator derived from the discrete Lagrangian $L_d^* : Q/G \times Q/G \longrightarrow \mathbb{R}$. $\boxed{QED}$

6. Some Numerical Examples

To illustrate the performance of the algorithms obtained from the nonholonomic integrator, we consider in this section two examples: a nonholonomic particle with a quadratic potential and a mobile robot with fixed orientation.

6.1. Nonholonomic particle

Let a particle of unit mass be moving in space, $Q = \mathbb{R}^3$, with Lagrangian $L : TQ \longrightarrow \mathbb{R}$

$$L = K - V = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \left(x^2 + y^2\right),$$

and subject to the constraint

$$\Phi = \dot{z} - y\dot{x} = 0\,.$$

The constraint distribution is then given by

$$\mathcal{D} = \operatorname{span}\left\{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\right\}\,.$$

Note that this system is a Chaplygin system, as defined above. In fact, consider the Lie group $G = \mathbb{R}$ and its trivial action by translation on Q,

$$\begin{array}{rccc} \Psi : & G \times Q & \longrightarrow & Q \\ & (a, (x, y, z)) & \longmapsto & (x, y, z + a) \end{array}$$

Now, $\mathcal{D}$ is the horizontal subspace of the principal connection $A = (dz - y dx)e$, where e denotes the generator of the Lie algebra.

As can be easily verified, in the present example the energy is a conserved quantity of the nonholonomic system. We are interested here in the extent to which the different integration schemes actually preserve this quantity, as well as the constraint. Energy is commonly used when dealing with symplectic integrators as a fairly reliable indicator [9, 33].

The tested algorithms are the following:

- Nonholonomic integrator: L_d^α and $\omega_{d,\alpha}$ with $\alpha = \dfrac{1}{2}$
- Runge-Kutta: 4^{th} order, time step fixed
- Benchmark: Matlab 5.1 ODE 113 (Predictor-Corrector)

The 4^{th} order Runge-Kutta method is a classical integrator which does not make use of the mechanical nature of the system. To implement it, we have first eliminated the Lagrange multiplier from the nonholonomic equations, so that one gets second order equations in (x, y, z), amenable to integration by RK4.

On the other hand, the nonholonomic integrator has been designed taking into account the special structure of the problem.

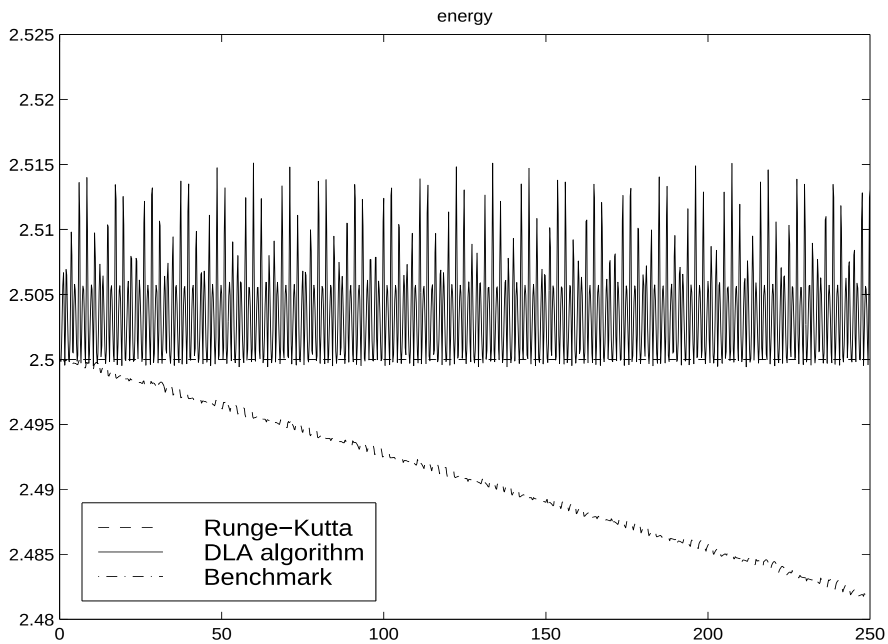

Figure 1. Energy behaviour of integrators for the nonholonomic particle with a quadratic potential. Note the long-time stable behaviour of the nonholonomic integrator, as opposed to classical methods such as Runge Kutta.

The two algorithms are run with the same stepsize, $h = 0.2$ to provide a reasonable comparison between them. The Benchmark algorithm is a high order, multi-step, predictor-corrector method which has been carried out with a very small stepsize. It can be regarded as the true solution for this example.

The results are shown in the figures. In Figure 1, we have plotted the energy behaviour of the integrators for a short time, but the same pattern is observed if we carry out the simulation for arbitrarily long periods of time. It is immediately apparent that the nonholonomic integrator and the Runge-Kutta method have qualitatively different behaviours. We take as a good indication the fluctuating energy behaviour of the nonholonomic integrator, since this property is also observed in symplectic methods.

The extent to which the three algorithms respect the constraints is plotted in Figure 2. Notice that the results from the Benchmark algorithm and the nonholonomic integrator are indistinguishable, whereas the behaviour of the Runge-Kutta technique is much less satisfactory.

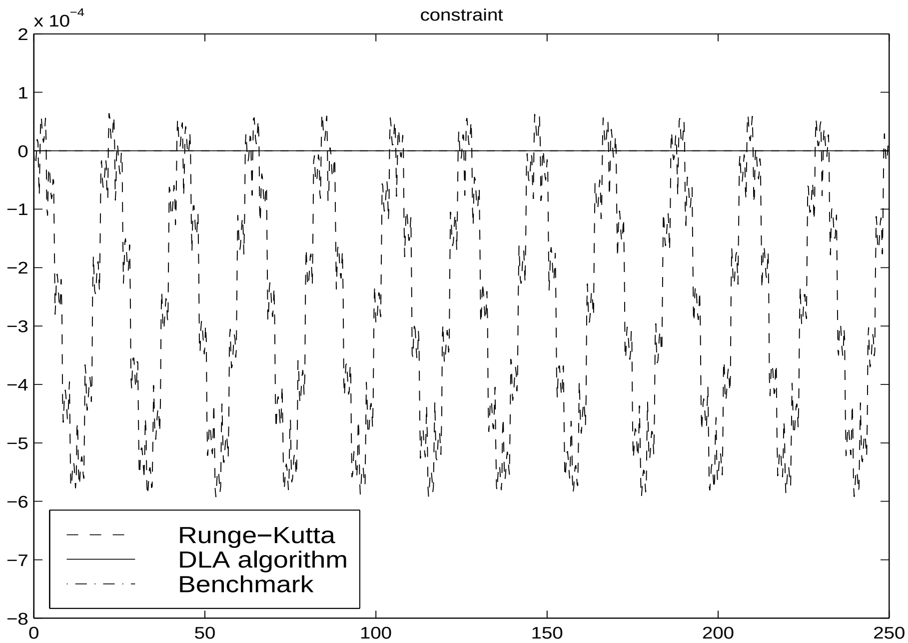

Figure 2. Illustration of the extent to which the tested algorithms respect the constraint. The Runge Kutta technique does not take into account the special nature of nonholonomic systems which explains its bad behaviour in this regard.

6.2. Mobile robot with fixed orientation with a potential

Consider a planar mobile robot with three wheels which roll whithout slipping with a fixed orientation [17]. Let $(x, y) \in \mathbb{R}^2$ denote the position of the center of the body, $\theta \in \mathbb{S}^1$ the orientation angle of the wheels, which are controlled by means of a drive mechanism, and $\psi \in \mathbb{S}^1$ the rotation angle of the wheels. The configuration space for this system is $Q = \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$. The kinetic energy of the robot is given by

$$K = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{3}{2}I_\omega\dot{\psi}^2 \,,$$

where m is the mass of the robot, I its moment of inertia and I_ω the axial moment of inertia of each wheel, respectively. We have introduced an artificial potential $V = 10 \sin\psi$ in order to "force" the behaviour of the numerical methods. The Lagrangian of the system is then $L = K - V$.

The constraints, induced by the conditions of no lateral sliding and rolling without sliding of the wheels, are $\dot{x} - R\cos\theta\dot{\psi} = 0$, $\dot{y} - R\sin\theta\dot{\psi} = 0$, where R is the radius of the wheels.

This system is again a Chaplygin system with Lie group $G = (\mathbb{R}^2, +)$,

$$\begin{array}{rcl} \Phi : G \times Q & \longrightarrow & Q \\ ((a, b), (x, y, \theta, \psi)) & \longmapsto & (x + a, y + b, \theta, \psi) \,, \end{array}$$

and principal connection $A = (x - R\cos\theta d\psi)e_1 + (dy - R\sin\theta d\psi)e_2$. The constraint distribution $\mathcal{D}$ is given by the horizontal subspace of A. As the constraints are linear, the energy is a conserved quantity for the continuous system.

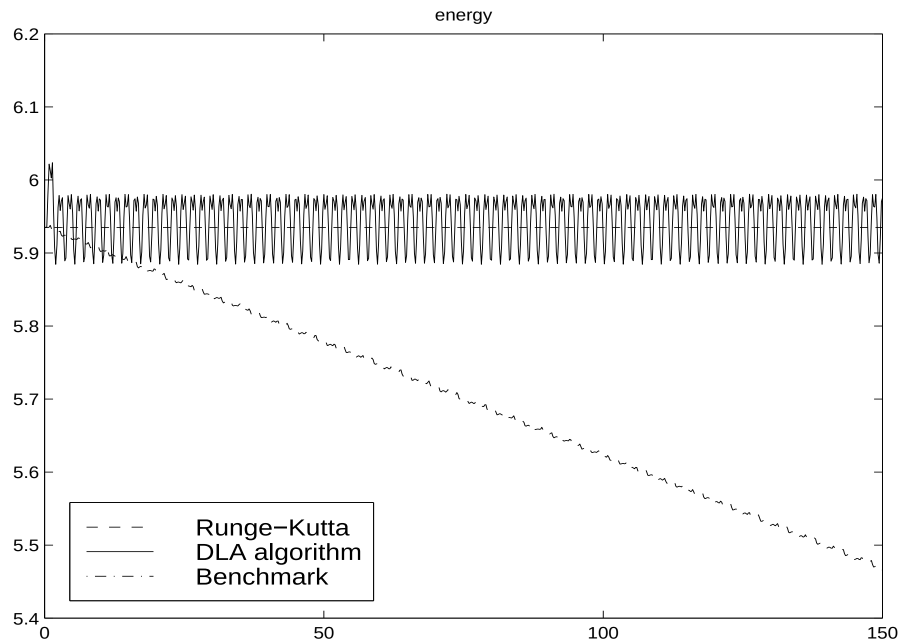

Figure 3. Energy behaviour of integrators for a mobile robot with fixed orientation.

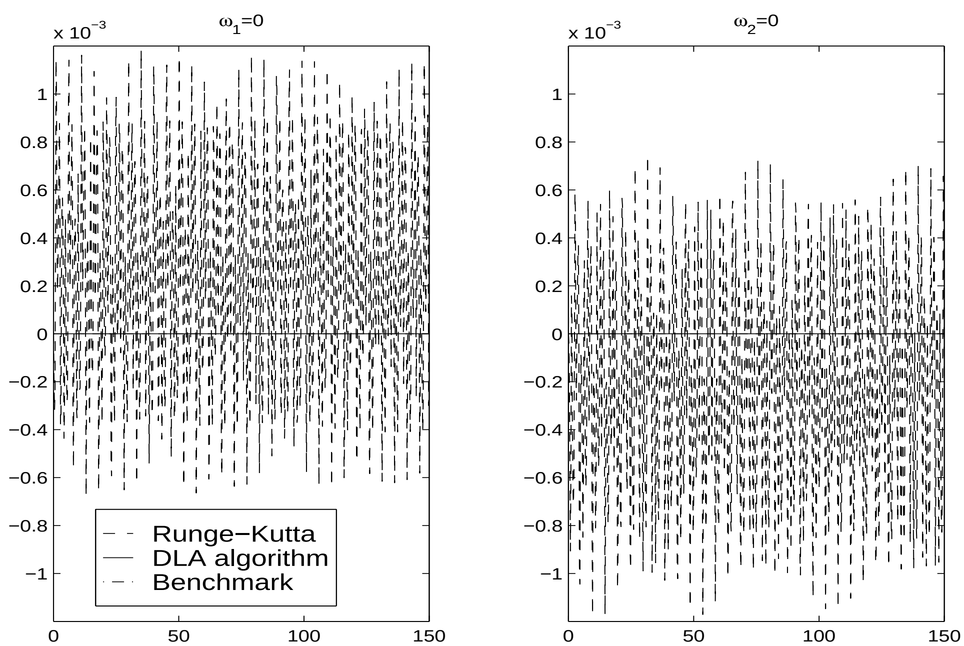

Figure 4. Illustration of the extent to which the tested algorithms respect the constraints $\omega_1 = 0$ and $\omega_2 = 0$. The behaviour of the nonholonomic integrator and the Benchmark algorithm is indistinguishable.

It can be immediately checked that the reduced system on $Q/G = \mathbb{S}^1 \times \mathbb{S}^1$ is the free system determined by the reduced Lagrangian

$$L^* = \frac{1}{2} I \dot{\theta}^2 + \frac{1}{2}(3I_\omega + mR^2)\dot{\psi}^2 - 10 \sin\psi \, .$$

From the discussion in Section 5.3.2, we know that the midpoint DLA scheme can be passed to Q/G and that the RDLA algorithm is indeed variational in the sense defined in [16]. However, the hypotheses of Theorem 5.6 are not fulfilled, and hence we cannot assure that the RDLA is a variational integrator. Nevertheless, the comparison of the DLA algorithm with the 4^{th} order Runge-Kutta method in the approximation of the energy and the constraints turns out to be very satisfactory (see Figures 3 and 4). Again, the 4^{th} order Runge-Kutta method is implemented by elimination of the Lagrange multipliers, while the DLA scheme is implemented using the Newton-Raphson technique. The stepsize employed is $h = 0.2$.

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