Chapter 1

Vector-valued quadratic forms in control theory

Francesco Bullo
Coordinated Science Laboratory
University of Illinois
Urbana-Champaign, IL 61801
United States
bullo@uiuc.edu

Jorge Cortés
Systems, Signals and Control
University of Twente
Enschede, 7500 AE
The Netherlands
j.cortesmonforte@utwente.nl

Andrew D. Lewis
Mathematics & Statistics
Queen's University
Kingston, ON K7L 3N6
Canada
andrew@mast.queensu.ca

Sonia Martínez
Institute of Mathematics & Physics
High Council of Scientific Research
Madrid, 28006
Spain
s.martinez@imaff.cfmac.csic.es

1.1 Problem statement and historical remarks

For finite dimensional $\mathbb{R}$-vector spaces U and V we consider a symmetric bilinear map $B\colon U \times U \to V$. This then defines a quadratic map $Q_B\colon U \to V$ by $Q_B(u) = B(u,u)$. Corresponding to each $\lambda \in V^*$ is a $\mathbb{R}$-valued quadratic form λQ_B on U defined by $\lambda Q_B(u) = \lambda \cdot Q_B(u)$. B is *definite* if there exists $\lambda \in V^*$ so that λQ_B is positive-definite. B is *indefinite* if for each $\lambda \in V^* \setminus \operatorname{ann}(\operatorname{image}(Q_B))$, λQ_B is neither positive nor negative-semidefinite, where ann denotes the annihilator.

Given a symmetric bilinear map $B\colon U \times U \to V$, the problems we consider are as follows.

1. *Find necessary and sufficient conditions characterizing when Q_B is surjective.*

2. *If Q_B is surjective and $v \in V$, design an algorithm to find a point $u \in Q_B^{-1}(v)$.*

3. *Find necessary and sufficient conditions to determine when B is indefinite.*

From the computational point of view, the first two questions are the more interesting ones. Both can be shown to be NP-complete, whereas the third one can be recast as a semidefinite programming problem.[1] Actually, our main interest is in a geometric characterization of these problems. Section 1.4 below constitutes an initial attempt to unveil the essential geometry behind these questions. By understanding the geometry of the problem properly, one may be lead to simple characterizations like the one presented in Proposition 3, which turn out to be checkable in polynomial time for certain clases of quadratic mappings.

Before we comment on how our problem impinges on control theory, let us provide some historical context for it as a purely mathematical one. The classification of $\mathbb{R}$-valued quadratic forms is well understood. However, for quadratic maps taking values in vector spaces of dimension two or higher, the classification problem becomes more difficult. The theory can be thought of as beginning with the work of Kronecker, who obtained a finite classification for pairs of symmetric matrices. For three or more symmetric matrices, that the classification problem has an uncountable number of equivalence classes for a given dimension of the domain follows from the work of Kac [12]. For quadratic forms, in a series of papers Dines (see [8] and references cited therein) investigated conditions when a finite collection of $\mathbb{R}$-valued quadratic maps were simultaneously positive-definite. The study of vector-valued quadratic maps is ongoing. A recent paper is [13], to which we refer for other references.

1.2 Control theoretic motivation

Interestingly and perhaps not obviously, vector-valued quadratic forms come up in a variety of places in control theory. We list a few of these here.

Optimal control: Agračhev [2] explicitly realises second-order conditions for optimality in terms of vector-valued quadratic maps. The geometric approach leads naturally to the consideration of vector-valued quadratic maps, and here the necessary conditions involve definiteness of these maps. Agračhev and Gamkrelidze [1, 3] look at the map $\lambda \mapsto \lambda Q_B$ from V^* into the set of vector-valued quadratic maps. Since λQ_B is a $\mathbb{R}$-valued quadratic form, one can talk about its *index* and *rank* (the number of -1's and nonzero terms, respectively, along the diagonal when the form is diagonalised). In [1, 3] the topology of the surfaces of constant index of the map $\lambda \mapsto \lambda Q_B$ is investigated.

[1] We thank an anonymous referee for these observations.

Local controllability: The use of vector-valued quadratic forms arises from the attempt to arrive at feedback-invariant conditions for controllability. Basto-Gonçalves [6] gives a second-order sufficient condition for local controllability, one of whose hypotheses is that a certain vector-valued quadratic map be indefinite (although the condition is not stated in this way). This condition is somewhat refined in [11], and a necessary condition for local controllability is also given. Included in the hypotheses of the latter is the condition that a certain vector-valued quadratic map be definite.

Control design via power series methods and singular inversion: Numerous control design problems can be tackled using power series and inversion methods. The early references [5, 9] show how to solve the optimal regulator problem and the recent work in [7] proposes local steering algorithms. These strong results apply to linearly controllable systems, and no general methods are yet available under only second-order sufficient controllability conditions. While for linearly controllable systems the classic inverse function theorem suffices, the key requirement for second-order controllable systems is the ability to check surjectivity and compute an inverse function for certain vector-valued quadratic forms.

Dynamic feedback linearisation: In [14] Sluis gives a necessary condition for the dynamic feedback linearisation of a system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n, \ u \in \mathbb{R}^m.$$

The condition is that for each $x \in \mathbb{R}^n$, the set $D_x = \{f(x,u) \in T_x\mathbb{R}^n \mid \ u \in \mathbb{R}^m\}$ admits a *ruling*, that is, a foliation of D_x by lines. Some manipulations with differential forms turns this necessary condition into one involving a symmetric bilinear map B. The condition, it turns out, is that $Q_B^{-1}(0) \neq \{0\}$. This is shown by Agračhev [1] to generically imply that Q_B is surjective.

1.3 Known results

Let us state a few results along the lines of our problem statement that are known to the authors. The first is readily shown to be true (see [11] for the proof). If X is a topological space with subsets $A \subset S \subset X$, we denote by $\mathrm{int}_S(A)$ the interior of A relative to the induced topology on S. If $S \subset V$, $\mathrm{aff}(S)$ and $\mathrm{conv}(S)$ denote, respectively, the affine hull and the convex hull of S.

Proposition 1 *Let $B\colon U \times U \to V$ be a symmetric bilinear map with U and V finite-dimensional. The following statements hold:*

(i) B is indefinite if and only if $0 \in \mathrm{int}_{\mathrm{aff}(\mathrm{image}(Q_B))}(\mathrm{conv}(\mathrm{image}(Q_B)))$;

(ii) B is definite if and only if there exists a hyperplane $P \subset V$ so that $\mathrm{image}(Q_B) \cap P = \{0\}$ and so that $\mathrm{image}(Q_B)$ lies on one side of P;

(iii) if Q_B is surjective then B is indefinite.

The converse of (iii) is false. The quadratic map from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by $Q_B(x, y, z) = (xy, xz, yz)$ may be shown to be indefinite but not surjective.

Agračhev and Sarychev [4] prove the following result. We denote by $\operatorname{ind}(Q)$ the index of a quadratic map $Q\colon U \to \mathbb{R}$ on a vector space U.

Proposition 2 *Let $B\colon U \times U \to V$ be a symmetric bilinear map with V finite-dimensional. If $\operatorname{ind}(\lambda Q_B) \geq \dim(V)$ for any $\lambda \in V^* \setminus \{0\}$ then Q_B is surjective.*

This sufficient condition for surjectivity is not necessary. The quadratic map from $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $Q_B(x, y) = (x^2 - y^2, xy)$ is surjective, but does not satisfy the hypotheses of Proposition 2.

1.4 Problem simplification

One of the difficulties with studying vector-valued quadratic maps is that they are somewhat difficult to get ones hands on. However, it turns out to be possible to simplify their study by a reduction to a rather concrete problem. Here we describe this process, only sketching the details of how to go from a given symmetric bilinear map $B\colon U \times U \to V$ to the reformulated end problem. We first simplify the problem by imposing an inner product on U and choosing an orthonormal basis so that we may take $U = \mathbb{R}^n$.

We let $\mathrm{Sym}_n(\mathbb{R})$ denote the set of symmetric $n \times n$ matrices with entries in $\mathbb{R}$. On $\mathrm{Sym}_n(\mathbb{R})$ we use the canonical inner product

$$\langle \boldsymbol{A}, \boldsymbol{B} \rangle = \operatorname{tr}(\boldsymbol{AB}).$$

We consider the map $\pi\colon \mathbb{R}^n \to \mathrm{Sym}_n(\mathbb{R})$ defined by $\pi(\boldsymbol{x}) = \boldsymbol{x}\boldsymbol{x}^t$, where ^{t} denotes transpose. Thus the image of π is the set of symmetric matrices of rank at most one. If we identify $\mathrm{Sym}_n(\mathbb{R}) \simeq \mathbb{R}^n \otimes \mathbb{R}^n$, then $\pi(\boldsymbol{x}) = \boldsymbol{x} \otimes \boldsymbol{x}$. Let K_n be the image of π and note that it is a cone of dimension n in $\mathrm{Sym}_n(\mathbb{R})$ having a singularity only at its vertex at the origin. Furthermore, K_n may be shown to be a subset of the hypercone in $\mathrm{Sym}_n(\mathbb{R})$ defined by those matrices $\boldsymbol{A}$ in $\mathrm{Sym}_n(\mathbb{R})$ forming angle $\arccos(\frac{1}{n})$ with the identity matrix. Thus the ray from the origin in $\mathrm{Sym}_n(\mathbb{R})$ through the identity matrix is an axis for the cone K_N. In algebraic geometry, the image of K_n under the projectivisation of $\mathrm{Sym}_n(\mathbb{R})$ is known as the *Veronese surface* [10], and as such is well-studied, although perhaps not along lines that bear directly on the problems of interest in this article.

We now let $B\colon \mathbb{R}^n \times \mathbb{R}^n \to V$ be a symmetric bilinear map with V finite-dimensional. Using the universal mapping property of the tensor product, B induces a *linear* map $\tilde{B}\colon \mathrm{Sym}_n(\mathbb{R}) \simeq \mathbb{R}^n \otimes \mathbb{R}^n \to V$ with the property that $\tilde{B} \circ \pi = B$. The dual of this map gives an injective linear map $\tilde{B}^*\colon V^* \to \mathrm{Sym}_n(\mathbb{R})$ (here we assume that the image of B spans V). By an appropriate choice of inner product on V one can render the embedding $\tilde{B}^*$ an isometric embedding of V in $\mathrm{Sym}_n(\mathbb{R})$. Let us denote by L_B the image of V under this isometric embedding. One may then show that with these identifications, the image of Q_B in V is the orthogonal projection of K_n onto the subspace L_B. Thus we reduce the problem to one of orthogonal projection of a canonical object, K_n, onto a subspace in $\mathrm{Sym}_n(\mathbb{R})$! To simplify things further, we decompose L_B into a component along the identity matrix in $\mathrm{Sym}_n(\mathbb{R})$ and a component orthogonal to the identity matrix. However, the matrices orthogonal to the identity are readily seen to simply be the traceless $n \times n$ symmetric matrices. Using our picture of K_n as a subset of a hypercone having as an axis the ray through the identity matrix, we see that questions of surjectivity, indefiniteness, and definiteness of B impact only on the projection of K_n onto that component of L_B orthogonal to the identity matrix.

The following summarises the above discussion.

The problem of studying the image of a vector-valued quadratic form can be reduced to studying the orthogonal projection of $K_n \subset \mathrm{Sym}_n(\mathbb{R})$*, the unprojectivised Veronese surface, onto a subspace of the space of traceless symmetric matrices.*

This is, we think, a beautiful interpretation of the study of vector-valued quadratic mappings, and will surely be a useful formulation of the problem. For example, with it one easily proves the following result.

Proposition 3 *If* $\dim(U) = \dim(V) = 2$ *with* $B\colon U \times U \to V$ *a symmetric bilinear map, then* Q_B *is surjective if and only if* B *is indefinite.*

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