Controllability of mechanical systems with constraints and symmetry

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Abstract

We develop tools within the affine connection formalism for the control of underactuated mechanical systems evolving on a principal fiber bundle. We present reduced formulations of the Levi-Civita and the nonholonomic affine connections in the presence of symmetries and nonholonomic constraints. Specialized controllability tests are developed, and the notion of fiber configuration controllability is introduced. The results are illustrated in a planar blimp.

1 Introduction

In the area of control of mechanical systems, there is a newly emerging body of work that utilizes their special Lagrangian structure to help focus the analysis [3, 4, 5, 7, 11, 12, 14]. This perspective, where the dynamics is interpreted using affine connections, has led to new insights into both control and motion planning for a number of underactuated mechanical systems. In this paper, we study the effect of symmetries and constraints on the affine connection and the symmetric product.

There has been extensive work in the area of understanding the role of symmetries in mechanical systems (e.g., see [2, 6, 13] and references therein). Lagrangian reduction provides powerful tools for analyzing mechanical systems on principal fiber bundles. Generally, the Lie group describes the position and orientation, while the remaining variables constitute an internal shape space. The symmetries can be used to decouple the dynamics into two parts, vertical and horizontal, and connect them with a principal connection [2, 10, 16]. Likewise, we can apply the same technique to the covariant derivatives, and then use the Lie bracket and symmetric product to obtain simplified tests of controllability. Our motivation for studying this class of systems comes from robotics, where it has been noted that *locomotion systems* possess this structure [9, 16]. An important notion here is that of *fiber controllability*, introduced in [9] for driftless, kinematic systems. This concept stems from the fact that for many robotic systems, one only cares that the robot be able to control its position and orientation, without regards to the configuration of its internal shape. Thus, the emphasis is put on whether a system is controllable along the fiber. We extend this notion to dynamic systems living on trivial principal fiber bundles.

The paper is organized as follows. In Section 2 we give some background on mechanical control systems and symmetries. In Section 3 we study the reduced version of the Levi-Civita and the nonholonomic affine connection for principal fiber bundles. In Section 4, we describe how these results extend previous notions of configuration controllability, and introduce a new concept of fiber controllability. We also demonstrate the use of these tools in the example of the planar blimp.

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2 Affine connections and mechanical control systems

Here we describe the geometric framework utilized in the study of mechanical control systems [11, 12]. A *simple mechanical control system* is defined by a tuple $(Q, \mathcal{G}, V, \mathcal{F})$, where Q is the configuration manifold, $\mathcal{G}$ is a Riemannian (kinetic energy) metric on Q, $V \in C^\infty(Q)$ is the potential function and $\mathcal{F} = \{F^1, \dots, F^m\}$ is a set of m linearly independent 1-forms on Q. The *forced Euler-Lagrange's equations* describe the dynamics of the system and can be expressed using the Levi-Civita connection (see [10] for a comprehensive treatment on affine connections) as

$$\nabla^{\mathcal{G}}_{\dot{c}(t)} \dot{c}(t) = -\mathrm{grad}\, V + \sum_{i=1}^{m} u^i(t) Y_i(c(t)) \,, \tag{2.1}$$

where $\mathrm{grad}\, V = \sharp_{\mathcal{G}}(dV)$ and $Y_i = \sharp_{\mathcal{G}}(F^i)$, with $\sharp_{\mathcal{G}} = \flat_{\mathcal{G}}^{-1}$ and $\flat_{\mathcal{G}}(X)(Y) = \mathcal{G}(X, Y)$.

A *constrained mechanical control system* $(Q, \mathcal{G}, V, \mathcal{F}, \mathcal{D})$ is a simple mechanical control system $(Q, \mathcal{G}, V, \mathcal{F})$ subject to the constraints given by the $(n-l)$-dimensional distribution $\mathcal{D}$ on Q. Letting $\mathcal{P} : TQ \longrightarrow \mathcal{D}$, $\mathcal{Q} : TQ \longrightarrow \mathcal{D}^\perp$ denote the complementary $\mathcal{G}$-orthogonal projectors, we can define the *nonholonomic affine connection* [1, 11, 17] by $\overline{\nabla}_X Y = \nabla^{\mathcal{G}}_X Y + (\nabla^{\mathcal{G}}_X \mathcal{Q})(Y) = \mathcal{P}(\nabla^{\mathcal{G}}_X Y) + \nabla^{\mathcal{G}}_X(\mathcal{Q}(Y))$, such that the nonholonomic control equations can be rewritten as

$$\overline{\nabla}_{\dot{c}(t)} \dot{c}(t) = -\mathcal{P}(\mathrm{grad}\, V) + \sum_{i=1}^{m} u_i(t) \mathcal{P}(Y_i(c(t))) \,, \tag{2.2}$$

where $\dot{c}(0) \in \mathcal{D}$. Note that equations (2.1) and (2.2) have the same structure. For the rest of the paper we will take $V \equiv 0$. The absence of the potential makes the picture considerably more clear while capturing the essential aspects of the analysis. On the other hand, a potential function can be incorporated to the controllability analysis along the lines of [12].

A key tool in the controllability analysis of mechanical control systems is the *symmetric product* $\langle \cdot : \cdot \rangle$ associated to an affine connection ∇. Given X, $Y \in \mathfrak{X}(Q)$, define $\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$.

2.1 Mechanical systems with symmetry

Assume that a Lie group G acts on Q, $\Phi : G \times Q \to Q$, $(g, q) \mapsto \Phi(g, q) = \Phi_g(q)$. If Φ is free and proper, the quotient space $Q/G \cong M$ is a manifold and the projection $\pi : Q \to M$ is a surjective submersion. Then, $Q(M, G, \pi)$ is a principal fiber bundle [10]. The bundle of vertical vectors is denoted by $\mathcal{V}_q = T_q(\mathrm{Orb}_G(q))$, $q \in Q$. Throughout the paper, we treat trivial bundles, $Q = G \times M$. This is a situation often found in robotic locomotion [16], where a splitting exists between variables describing position/orientation, $g \in G$, and variables describing the internal shape, $r \in M$.

A *principal connection* on $Q(M, G, \pi)$ is a G-invariant distribution $\mathcal{H}$ on Q such that $T_q Q = \mathcal{H}_q \oplus \mathcal{V}_q$, $\forall q \in Q$. The subspace $\mathcal{H}_q$ of $T_q Q$ is called the *horizontal subspace* at q. Alternatively, a principal connection can be characterized by a $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on Q ($\mathfrak{g}$ is the Lie algebra of G) satisfying: (i) $\mathcal{A}(\xi_Q(q)) = \xi$, for all $\xi \in \mathfrak{g}$; (ii) $\mathcal{A}((\Phi_g)_* X) = \mathrm{Ad}_g(\mathcal{A}(X))$, for all $X \in TQ$. The horizontal subspace at q is then given by $\mathcal{H}_q = \{v_q \in T_q Q \mid \mathcal{A}(v_q) = 0\}$. Using (i) and (ii),

$$\mathcal{A}(g, r, \dot{g}, \dot{r}) = \mathcal{A}(g(e, r, \xi, \dot{r})) = Ad_g \mathcal{A}(e, r, \xi, \dot{r}) = Ad_g(\xi + A(r)\dot{r}) \,,$$

where $A(r)$ is called the *local form* of $\mathcal{A}$. For reasons of space, we refer the reader to [8, 10] for other important concepts such as the curvature and the derivative along the connection.

Assume that a mechanical control system $(Q, \mathcal{G}, \mathcal{F})$ is *invariant* under a Lie group G, that is, $\Phi_g^* \mathcal{G} = \mathcal{G}$ and $\Phi_g^* F^i = F^i$, for $1 \leq i \leq m$ and all $g \in G$. In the reduction of mechanical systems, the *mechanical connection* A^{mech} plays a prominent role. One has $A^{mech}(\dot{q}) = \mathcal{I}(q)^{-1} J(\dot{q})$, where $\mathcal{I}$ is the *locked inertia tensor* and J is the natural *momentum map*. The invariance of the metric implies that it can be represented in the space $(r, \dot{r}, \xi) \in TQ/G$, $\xi = g^{-1}\dot{g}$, by the reduced metric

$$\hat{\mathcal{G}} = \begin{pmatrix} I(r) & I(r)A(r) \\ A(r)^T I(r) & m(r) \end{pmatrix} .$$

Here, I and A denote the local form of the inertia tensor and the mechanical connection, respectively. This reduced metric is block diagonalized in terms of the shape variables $(r, \dot{r})$ and the *locked body angular velocity*, $\Omega = \xi + A(r)\dot{r}$ as $\tilde{\mathcal{G}} = \operatorname{diag}(I(r), \Delta(r))$, with $\Delta(r) = m(r) - A^T(r)I(r)A(r)$.

Assume that a constrained mechanical control system is *invariant* under a Lie group G, meaning that both $(Q, \mathcal{G}, \mathcal{F})$ and the constraint distribution $\mathcal{D}$ are invariant. Assume that $\mathcal{D} + \mathcal{V} = TQ$. Define the intersection $S = \mathcal{V} \cap \mathcal{D}$. The horizontal subspace at $q \in Q$ of the *nonholonomic principal connection* [2, 15] is given by $\mathcal{H}_q = S_q^{\perp} \cap \mathcal{D}_q$. Let $\mathfrak{g}^{\mathcal{D}} \to Q$ be a bundle whose fiber is given by $\mathfrak{g}^q = \{\xi \in \mathfrak{g} : \xi_Q(q) \in S_q\}$. The nonholonomic momentum map is defined as $J^{nh} : TQ \to \mathfrak{g}^{\mathcal{D}*}$, $J^{nh}(q, \dot{q})(\xi^q) = \langle \frac{\partial L}{\partial \dot{q}}(\dot{q}), \xi_Q^q(q) \rangle$. Consider the map $A^{sym} : T_qQ \to S_q$, $(q, \dot{q}) \mapsto (\tilde{\mathcal{I}}^{-1}(q) J^{nh}(q, \dot{q}))_Q$, where $\tilde{\mathcal{I}}$ is the *locked inertia tensor relative to* $\mathfrak{g}^{\mathcal{D}}$. Additionally, let $A^{kin} : T_qQ \to S_q^{\perp}$ be the orthogonal projection. The nonholonomic connection 1-form is $A^{nh} = A^{kin} + A^{sym}$. Useful for the latter derivations will be fixing a basis $\{e_1(r), \ldots, e_s(r), e_{s+1}(r), \ldots, e_k(r)\}$ of $\mathfrak{g}$ such that the first s elements span $\mathfrak{g}^{(r,e)}$ and both set of generators are orthogonal. We will denote $\frac{\partial e_i}{\partial r^\alpha} = \sum_{a=1}^{k} \gamma_{i\alpha}^a e_a$.

3 Levi-Civita and nonholonomic affine connections under symmetry

This section presents decompositions of the Levi-Civita and the nonholonomic affine connections according to the principal fiber bundle structure of the configuration space Q. These decompositions will enable us to simplify the controllability tests for mechanical control systems with symmetry.

Proposition 3.1. *Given G-invariant vector fields on Q, $X = (g\xi, v)$ and $Y = (g\eta, w)$, with $\xi(r)$, $\eta(r) \in \mathfrak{g}$ and v, $w \in TM$, the covariant derivative of Y along X can be expressed as*

$$\nabla_X^{\mathcal{G}} Y = g \left\{ \begin{pmatrix} \nabla_\Omega^I \Psi \\ \nabla_v^\Delta w \end{pmatrix} - \frac{1}{2} \begin{pmatrix} I^{-1}\mathsf{L} \\ \Delta^{-1}\mathsf{S} \end{pmatrix} \right\} , \tag{3.3}$$

where

$$\mathsf{L} = -D(I\Omega)(\cdot, w) - D(I\Psi)(\cdot, v) + I([\Omega, \Psi] - [\xi, \eta] + \xi_r w - \eta_r v - A[v, w]) + 2\, I(A(\nabla_X^{\mathcal{G}} Y)_M) \in \mathfrak{g}^* ,$$
$$\mathsf{S} = I(\Omega, B(w, \cdot)) + I(\Psi, B(v, \cdot)) + DI(\cdot)(\Omega, \Psi) \in T^*M ,$$

and D, B denote, respectively, the local forms of the derivative along and the curvature of the mechanical connection and $\Omega = \xi + Av$, $\Psi = \eta + Aw$, $\xi_r \equiv \frac{\partial \xi}{\partial r}$, $\eta_r \equiv \frac{\partial \eta}{\partial r}$.

Proposition 3.2. *Given G-invariant vector fields, $X = (g\xi, v) \in TQ$, $Y = (g\eta, w) \in \mathcal{D}$ on Q, with $\xi(r)$, $\eta(r) \in \mathfrak{g}$ and v, $w \in TM$ the nonholonomic affine connection $\overline{\nabla}$ can be expressed as*

$$\overline{\nabla}_X Y = g \left\{ \begin{pmatrix} A^{sym}(\nabla_{\bar{\Omega}}^{I} \bar{\Psi}) \\ \nabla_v^{\tilde{\Delta}} w \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \tilde{I}^{-1}\tilde{\mathsf{L}} + 2\mathsf{A}(\overline{\nabla}_X Y)_M \\ \tilde{\Delta}^{-1}\tilde{\mathsf{S}} \end{pmatrix} \right\} , \tag{3.4}$$

where

$$\tilde{\mathsf{L}} = -\mathsf{D}(I\bar{\Omega})(\cdot, w) - \mathsf{D}(I\bar{\Psi})(\cdot, v) + I(\tilde{A}v, \gamma.w - [\cdot, \eta]) + I(\tilde{A}w, \gamma.v - [\cdot, \xi]) + I([\bar{\Omega}, \bar{\Psi}] - [\xi, \eta] + \xi_r w - \eta_r v - \mathsf{A}[v, w]) \in \mathfrak{g}^{\mathcal{D}*},$$

$$\tilde{\mathsf{S}} = I(\bar{\Psi}, B(v, \cdot)) + I(\bar{\Omega}, B(w, \cdot)) + I(\tilde{A}w, \mathsf{B}(v, \cdot)) + I(\tilde{A}v, \mathsf{B}(w, \cdot)) - D(I\bar{\Psi})(\tilde{A}\cdot, v) - D(I\bar{\Omega})(\tilde{A}\cdot, w) + \mathsf{D}I(\cdot)(\bar{\Omega} + \tilde{A}v, \bar{\Psi} + \tilde{A}w) - \mathsf{D}I(\cdot)(\tilde{A}v, \tilde{A}w) - I([\xi, \eta], \tilde{A}\cdot) - I(\eta_r v - \xi_r v, \tilde{A}\cdot) - I(\mathsf{A}[v, w], \tilde{A}\cdot) \in T^*M,$$

and D, B *denote, respectively, the local forms of the derivative along and the curvature of the nonholonomic connection* A^{nh} *and* $\bar{\Omega} = \xi + \mathsf{A}v$, $\bar{\Psi} = \eta + \mathsf{A}w$.

The corresponding decompositions of the associated symmetric products can be obtained from these propositions. We refer to [8] for the proofs and the explicit expressions.

4 Controllability analysis

Here we refine the controllability analysis of [12] for mechanical control systems evolving on principal fiber bundles. The notions of local configuration accessibility (LCA) and controllability (STLCC), along with that of good/bad symmetric products and degree are taken from [12]. Let

$$\nabla_{\dot{c}(t)}\dot{c}(t) = \sum_{i=1}^{m} u_i(t)Y_i(c(t)), \tag{4.5}$$

where ∇ can be either the Levi-Civita or the nonholonomic affine connection. Assume that the control system (4.5) is invariant under the action of a Lie group G. Denote by $\mathfrak{B} = \{B_1, \ldots, B_m\}$ the representants of the input vector fields $\mathfrak{Y} = \{Y_1, \ldots, Y_m\}$ at $\mathfrak{g} \times TM$, i.e. $Y_i(r, g) = gB_i(r, e) = g(\xi_i(r), v_i)$, $1 \le i \le m$. Due to the invariance of the system we have that $\langle Y_i : Y_j\rangle = \langle gB_i : gB_j\rangle \equiv g\langle B_i : B_j\rangle$ for all $1 \le i, j \le m$. Note also that the Lie brackets $[Y_i, Y_j]$ can be written as

$$[Y_i, Y_j] \equiv g[B_i, B_j] = g \begin{pmatrix} [\xi_i, \xi_j]_{\mathfrak{g}} + \frac{\partial \xi_j}{\partial r}v_i - \frac{\partial \xi_i}{\partial r}v_j \\ [v_i, v_j]_M \end{pmatrix}$$

Theorem 4.1. *Let the system* (4.5) *be invariant under the action of a Lie group* G. *Then it is*

- *LCA at* $q = (r, g) \in Orb_G(r, e)$ *if* $\overline{Lie}(\overline{Sym}(\mathfrak{B}))_{(r,e)} = \mathfrak{g} \times T_rM$.
- *STLCC at* $q \in Orb_G(r, e)$ *if it is LCA at* (r, e) *and every bad symmetric product* P *at* (r, e) *in* $\mathfrak{B}$ *can be written as a linear combination of good symmetric products at* (r, e) *of lower degree.*

where $\overline{Sym}(\mathfrak{B})$ *and* $\overline{Lie}(\mathfrak{B})$ *denote, respectively, the symmetric and involutive closures of* $\mathfrak{B}$.

These tests remove from the computations the dependence on G. Furthermore, the decompositions of the symmetric products can be used to compute the terms $\langle B_i : B_j\rangle$. An additional simplification stems from the fact that many dynamic locomotion systems have the full tangent bundle of M as the set of inputs, which corresponds to the observation that the system can adjust its shape as desired. Then, one can prove [8] that the input vector fields have $\Omega_i = 0$, $1 \le i \le m$. Moreover, when constraints are present, their projections to $\mathcal{D}$ also have $\bar{\Omega}_i = 0$, $1 \le i \le m$.

Another interesting aspect for this kind of mechanical control systems is the adaptation of the concept of *weak controllability* for kinematic systems defined in [9]. This notion means controllability in the fiber, without regards to the intermediate or final positions of the shape variables. Let V^τ denote any subset of Q such that $\tau(V^\tau)$ is an open subset of G, where $\tau : Q \to G$ denotes the natural projection. Take $q_0 \in Q$ and let $U \subset Q$ be a neighborhood of q_0. Define

$$\mathcal{R}_Q^U(q_0, T) = \{q \in Q \mid \exists\ (c, u) \text{ solving (4.5) with } \dot{c}(0) = 0_{q_0}, c(t) \in U \text{ for } t \in [0, T] \text{ and } \dot{c}(T) \in T_q Q\}$$

and denote by $\mathcal{R}_Q^U(q_0, \leq T) = \cup_{0 \leq t \leq T} \mathcal{R}_Q^U(q_0, t)$.

Definition 4.1. *The system* (4.5) *is* locally fiber configuration accessible (LFCA) (resp. small-time locally fiber configuration controllable (STLFCC)) at $q_0 \in Q$ *if there exists* $T > 0$ *such that* $\mathcal{R}_Q^U(q_0, \leq t)$ *contains a non-empty subset* V^τ *of* Q *(resp.* $\mathcal{R}_Q^U(q_0, \leq t)$ *contains a non-empty subset* V^τ *of* Q *and* $g_0 \in \tau(V^\tau)$*), for all neighborhoods* U *of* q_0 *and all* $0 \leq t \leq T$.

Theorem 4.2. *Let the system* (4.5) *be invariant under* G. *Then it is*

- *LFCA at* $q = (r, g)$ *if* $\tau_* \overline{Lie}(\overline{Sym}(\mathfrak{B}))_{(r,e)} = \mathfrak{g}$.
- *STLFCC at* q *if it is LFCA at* q *and the projection through* τ *of every bad symmetric product* P *at* q *in* $\mathfrak{B}$, $\tau_* P$, *can be written as a linear combination of projections through* τ *of good symmetric products at* q *of lower degree.*

4.1 An example: the planar blimp

Consider a rigid body moving in $SE(2)$ with a thruster to adjust its pose [8]. The control inputs are the thruster force F^1 and a torque F^2 that actuates its orientation with respect to the body axis $\{X^b, Y^b\}$. The configuration of the blimp is determined by a tuple $(x, y, \theta, \gamma) \in SE(2) \times \mathbb{S}^1$, where (x, y) is the position of the center of mass, θ is the orientation with respect to the fixed basis $\{X^f, Y^f\}$ and γ is the orientation of the thrust with respect to the body basis $\{X^b, Y^b\}$. The Riemannian metric is $\mathcal{G} = m(dx \otimes dx + dy \otimes dy) + (J_1 + J_2) d\theta \otimes d\theta + J_2 d\gamma \otimes d\gamma + J_2(d\theta \otimes d\gamma + d\gamma \otimes d\theta)$, where m denotes the mass of the blimp, J_1 is its moment of inertia and J_2 is the inertia of the thruster. The input forces are given by $F^1 = \cos(\theta + \gamma) dx + \sin(\theta + \gamma) dy - h \sin\gamma d\theta$, $F^2 = d\gamma$.

This mechanical control system is invariant under the left multiplication of the Lie group $G = SE(2)$. The reduced representation of the input vector fields at $\mathfrak{g} \times TM$ is given by

$$B_1 = \frac{1}{m} \cos\gamma \frac{\partial}{\partial x} + \frac{1}{m} \sin\gamma \frac{\partial}{\partial y} - \frac{h}{J_1} \sin\gamma \frac{\partial}{\partial \theta} + \frac{h}{J_1} \sin\gamma \frac{\partial}{\partial \gamma}\,, \quad B_2 = -\frac{1}{J_1} \frac{\partial}{\partial \theta} + \frac{J_1 + J_2}{J_1 J_2} \frac{\partial}{\partial \gamma}\,.$$

Resorting to the results presented in Section 3, we can compute the following symmetric brackets

$$\langle B_1 : B_1 \rangle_{\mathcal{G}} = \frac{h^2}{J_1^2} \sin(2\gamma)\,(0, 0, -1, 1)\ , \quad \langle B_2 : \langle B_1 : B_1 \rangle_{\mathcal{G}} \rangle_{\mathcal{G}} = 2\frac{h^2}{J_1^2} \frac{J_1 + J_2}{J_1 J_2} \cos(2\gamma)\,(0, 0, -1, 1)\ ,$$
$$\langle B_2 : B_2 \rangle_{\mathcal{G}} = 0\,, \quad \langle B_1 : B_2 \rangle_{\mathcal{G}} = \left(-\frac{1}{m J_2} \sin\gamma, \frac{1}{m J_2} \cos\gamma, -\frac{h(J_1 + J_2)}{J_1^2 J_2} \cos\gamma, \frac{h(J_1 + J_2)}{J_1^2 J_2} \cos\gamma \right)\,.$$

Note that $\{B_1, B_2, \langle B_1 : B_2 \rangle_{\mathcal{G}}, \langle B_1 : B_1 \rangle_{\mathcal{G}}, \langle B_2 : \langle B_1 : B_1 \rangle_{\mathcal{G}} \rangle_{\mathcal{G}}\}$ span $\mathfrak{g} \times TM$ at every (e, r) and hence the system is LCA. However, the bad bracket $\langle B_1 : B_1 \rangle_{\mathcal{G}}$ is not in general a combination of the lower order good brackets B_1 and B_2. Therefore, we can not conclude STLCC. In any case, at $\gamma = 0$, $\langle B_1 : B_1 \rangle_{\mathcal{G}}(e, 0) = 0$ and hence the system is STLCC at $(g, 0)$, for all $g \in G$. However, for fiber configuration controllability, we see that $\tau_* \langle B_1 : B_1 \rangle_{\mathcal{G}} \in \mathrm{span}\{\tau_* B_2\}$, which implies STLFCC.

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