SIMPLE MECHANICAL CONTROL SYSTEMS WITH CONSTRAINTS AND SYMMETRY

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Abstract. We develop tools for studying the control of underactuated mechanical systems that evolve on a configuration space with a principal fiber bundle structure. Taking the viewpoint of affine connection control systems, we derive reduced formulations of the Levi-Civita and the nonholonomic affine connections, along with the symmetric product, in the presence of symmetries and nonholonomic constraints. We note that there are naturally two kinds of connections to be considered here, affine and principal connections, leading to what we term a "connection within a connection". These results are then used to describe controllability tests that are specialized to simple, underactuated mechanical systems on principal fiber bundles, including the notion of fiber configuration controllability. We present examples of the use of these tools in studying the planar rigid body with a variable direction (vectored) thruster and the snakeboard robot.

Key words. nonlinear control, reduction, configuration controllability, symmetric product

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1. Introduction. In the area of control for mechanical systems, there is a newly emerging body of work that utilizes the special Lagrangian structure of such systems to help focus the control analysis [5, 6, 8, 12, 20, 21]. This perspective, in which the dynamics of simple mechanical systems is interpreted using an affine connection, has led to new insights into both control and motion planning for a number of underactuated mechanical systems. In this paper, we study the effect of symmetries and constraints on the tools that are used in studying affine connection control systems, namely the *affine connection* and the *symmetric product*.

In studying the controllability of a mechanical system, classical tools from nonlinear control theory [27] suggest that one compute the closure by the Lie bracket of all the control inputs and the drift vector field. When the control inputs enter in as forcing terms for second-order ODE's, such as is the case with forces or torques, this procedure requires the system to be transformed into a first-order form. The drawback of this, however, is that the conversion requires that one treat the velocities as a part of the state, and more importantly that the intrinsic structure of the mechanical system as a second-order Lagrangian system is covered up. However, work by Lewis and Murray [21] has shown that a proper geometric interpretation of simple mechanical systems can be achieved through the use of the affine connection formalism and the symmetric product that derives from it.

Bullo, Leonard, and Lewis later applied these results to underactuated Lagrangian systems evolving on a Lie group [6, 7]. They took advantage of the special Lie group

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structure to derive algorithms for generating the control inputs that lead to motion along the directions generated through the operation of the symmetric product. This work serves as a starting point for this paper, in which we explore a generalization of these ideas to systems that evolve on principal fiber bundles, which are locally the product of a Lie group and a general smooth manifold.

We also note that there has been extensive work in the area of understanding the role of symmetries in mechanical systems (e.g., see [2, 4, 9, 10, 16, 23, 24, 25] and references therein). We focus on one aspect of such systems, where internal shape variables play an important role in determining the motion of a system along a Lie group. Lagrangian reduction provides powerful tools for analyzing mechanical systems on fiber bundles. Generally (as is the case for our examples), the Lie group describes the position and orientation of the system, while the remaining variables constitute an internal shape space. Some examples of the shape variables that result are the thruster angle of a blimp [35, 36], the leg angle of a robot leg [21], or the wheel direction angles of a snakeboard [30, 31]. Through a local trivialization, we can use the internal symmetries of the system to decouple the dynamics into two parts, vertical and horizontal, and connect them with a mechanical connection (or constraints) [4, 31]. Likewise, we can apply the same technique to the computation of covariant derivatives by finding the vertical and horizontal parts, and then use the Lie bracket and symmetric product to take advantage of the geometric structure of the system, leading to simplified tests of configuration accessibility and controllability. When nonholonomic constraints are present, the situation is further complicated, though it was shown by Lewis [20] that one can use a *nonholonomic affine connection* that directly extends the controllability results.

Our motivation for studying this class of systems comes from robotics, where it has been noted that *robotic locomotion systems* possess this structure – the dynamics evolve on a product bundle between a Lie group and a general "shape" manifold [15, 28, 31]. This leads us to consider mechanical systems on principal fiber bundles, in which the motion of the system is generated through a complex interaction of thrusts/forces and internal changes in the shape or configuration of the robot. There is an extensive literature studying such systems, including kinematic versions [15], dynamic systems that evolve purely on Lie groups [6], and dynamic systems with nonholonomic constraints [28, 30]. An important quantity for such robotic systems that is highlighted here is the notion of *fiber controllability*, introduced by Kelly and Murray [15] for driftless, kinematic systems. The notion of fiber controllability stems from the fact that for many robotic systems of this form, one only cares that the robot be able to control its position and orientation, without regard to the configuration of its internal shape. Thus, the emphasis is on understanding whether a system is controllable only along the fiber (position and orientation). We extend this notion to dynamic systems with symmetries living on trivial principal fiber bundles.

The paper is organised as follows. In Section 2 we give some background on simple mechanical control systems and the role of symmetries. In Section 3 we study the reduced version of the Levi-Civita affine connection, and hence the symmetric product, for principal fiber bundles. We present the computations in terms of local forms of the quantities that arise, including the *mechanical connection* and the *locked inertia tensor*, since these allow for a reduced and compact representation. We follow up this derivation in Section 4 with a parallel formulation of the nonholonomic affine connection that arises when constraints are present. In Section 5, we describe how these results extend previous notions of configuration controllability to fiber bundles, and introduce a new concept of fiber controllability. In Section 6, we demonstrate the use of these tools in two motivating examples: the underactuated rigid body (or planar blimp) and the snakeboard. Finally, Section 7 is devoted to some concluding remarks.

2. Background on Simple Mechanical Control Systems. In this section we describe the geometric framework utilized in the study of mechanical control systems. We follow [20, 21] in the exposition of affine connection control systems. The reader is referred to [1, 17] for more details on notions such as principal bundles or affine connections.

2.1. Affine Connection Control Systems. Let Q be a n-dimensional manifold. We denote by TQ the tangent bundle of Q, by $\mathfrak{X}(Q)$ the set of vector fields on Q and by $C^\infty(Q)$ the set of smooth functions on Q. A **simple mechanical control system** is defined by a tuple $(Q, \mathcal{G}, V, \mathcal{F})$, where Q is the manifold of configurations of the system, $\mathcal{G}$ is a Riemannian metric on Q (the kinetic energy metric of the system), $V \in C^\infty(Q)$ is the potential function and $\mathcal{F} = \{F^1, \dots, F^m\}$ is a set of m linearly independent 1-forms on Q, which physically correspond to forces or torques.

The dynamics of simple mechanical control systems is classically described by the **forced Euler-Lagrange's equations**

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = \sum_{i=1}^{m} u_i(t) F^i \,, \tag{2.1}$$

where $L(q, \dot q) = \frac{1}{2}\mathcal{G}(\dot q, \dot q) - V(q)$ is the **Lagrangian** of the system.

Alternatively, one can express the control equations 2.1 using the natural affine connection associated to the metric $\mathcal{G}$, the Levi-Civita connection. An **affine connection** [17] is defined as an assignment

$$\begin{array}{rccc} \nabla : & \mathfrak{X}(Q) \times \mathfrak{X}(Q) & \longrightarrow & \mathfrak{X}(Q) \\ & (X, Y) & \longmapsto & \nabla_X Y \end{array}$$

which is $\mathbb{R}$-bilinear and satisfies $\nabla_{fX} Y = f \nabla_X Y$ and $\nabla_X (fY) = f \nabla_X Y + X(f) Y$, for any X, $Y \in \mathfrak{X}(Q)$, $f \in C^\infty(Q)$. In local coordinates,

$$\nabla_X Y = \left(\frac{\partial Y^a}{\partial q^b} X^b + \Gamma^a_{bc} X^b Y^c\right) \frac{\partial}{\partial q^a} \,,$$

where $\Gamma^a_{bc}(q)$ are the **Christoffel symbols** of the affine connection defined by

$$\nabla_{\frac{\partial}{\partial q^b}} \frac{\partial}{\partial q^c} = \Gamma^a_{bc} \frac{\partial}{\partial q^a} \,. \tag{2.2}$$

For simple mechanical control systems, the **Levi-Civita connection** $\nabla^{\mathcal{G}}$ associated to the metric $\mathcal{G}$ is determined by the formula

$$\begin{aligned} 2\,\mathcal{G}(Z, \nabla_X Y) &= X(\mathcal{G}(Z, Y)) + Y(\mathcal{G}(Z, X)) - Z(\mathcal{G}(Y, X)) \\ &\quad + \mathcal{G}(X, [Z, Y]) + \mathcal{G}(Y, [Z, X]) - \mathcal{G}(Z, [Y, X]) \,, \quad X, Y, Z \in \mathfrak{X}(Q) \,. \end{aligned} \tag{2.3}$$

One can compute the Christoffel symbols of $\nabla^{\mathcal{G}}$ to be

$$\Gamma^a_{bc} = \frac{1}{2} \mathcal{G}^{ad} \left(\frac{\partial \mathcal{G}_{db}}{\partial q^c} + \frac{\partial \mathcal{G}_{dc}}{\partial q^b} - \frac{\partial \mathcal{G}_{bc}}{\partial q^d}\right) \,,$$

where $(\mathcal{G}^{ad})$ denotes the inverse matrix of $(\mathcal{G}_{da} = \mathcal{G}(\frac{\partial}{\partial q^d}, \frac{\partial}{\partial q^a}))$. Instead of the input forces $F^1, \ldots, F^m$, we shall make use of the vector fields $Y_1, \ldots, Y_m$, defined as $Y_i = \sharp_{\mathcal{G}}(F^i)$, where $\sharp_{\mathcal{G}} = \flat_{\mathcal{G}}^{-1}$ and $\flat_{\mathcal{G}} : TQ \longrightarrow T^*Q$ is the musical isomorphism given by $\flat_{\mathcal{G}}(X)(Y) = \mathcal{G}(X,Y)$. In local coordinates, we have that $Y_i^a = \mathcal{G}^{ab} F_b^i$, for each $1 \leq i \leq m$. Roughly speaking, this corresponds to considering the effect of the controls on "accelerations" rather than on forces. The control equations 2.1 for the mechanical system may then be recasted as

$$\nabla^{\mathcal{G}}_{\dot{c}(t)} \dot{c}(t) = -\operatorname{grad} V + \sum_{i=1}^{m} u^i(t) Y_i(c(t)) , \tag{2.4}$$

where $\operatorname{grad} V = \sharp_{\mathcal{G}}(dV)$. Observe that we can use a general affine connection in eq. 2.4 instead of the Levi-Civita connection without changing the structure of the equation. This is particularly interesting, since nonholonomic mechanical control systems also give rise to equations of the form of eq. 2.4, as we review in the following [20]. This observation is actually very powerful, since controllability analyses based on a general affine connection (cf. Section 5) are valid for both unconstrained and constrained control systems.

A **constrained mechanical control system** $(Q, \mathcal{G}, V, \mathcal{F}, \mathcal{D})$ is a simple mechanical control system $(Q, \mathcal{G}, V, \mathcal{F})$ subject to the constraints given by the $(n-l)$-dimensional (nonholonomic) distribution $\mathcal{D}$ on Q. In a local description, $\mathcal{D}$ can be defined by the vanishing of l independent constraint functions $\omega_j(q)\dot{q}$, $1 \leq j \leq l$. The application of Lagrange-d'Alembert's principle leads to the constrained equations of motion

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \sum_{i=1}^{m} u_i(t) F^i + \sum_{j=1}^{l} \lambda^j \omega_j , \tag{2.5}$$

which, together with the constraint equations $\omega_j(q)\dot{q} = 0$, describe the dynamics of the nonholonomic system. Here, the λ^j are the Lagrange multipliers. The term $\sum_{j=1}^{l} \lambda^j \omega_j$ represents the "reaction force" due to the constraints.

The second order equation 2.5 can alternatively be written as

$$\left\{ \begin{array}{l} \nabla^{\mathcal{G}}_{\dot{c}(t)} \dot{c}(t) = \lambda(t) - \operatorname{grad} V + \sum_{i=1}^{m} u_i(t) Y_i(c(t)) \\ \dot{c}(t) \in \mathcal{D}_{c(t)} , \end{array} \right. \tag{2.6}$$

where now λ is seen as a section of $\mathcal{D}^\perp$, the $\mathcal{G}$-orthogonal complement to $\mathcal{D}$, along the curve c. Letting $\mathcal{P} : TQ \longrightarrow \mathcal{D}$, $\mathcal{Q} : TQ \longrightarrow \mathcal{D}^\perp$ denote the complementary $\mathcal{G}$-orthogonal projectors, we can define an affine connection

$$\overline{\nabla}_X Y = \nabla^{\mathcal{G}}_X Y + (\nabla^{\mathcal{G}}_X \mathcal{Q})(Y) = \mathcal{P}(\nabla^{\mathcal{G}}_X Y) + \nabla^{\mathcal{G}}_X (\mathcal{Q}(Y)) ,$$

such that the nonholonomic control equations 2.6 can be rewritten as

$$\overline{\nabla}_{\dot{c}(t)} \dot{c}(t) = -\mathcal{P}(\operatorname{grad} V) + \sum_{i=1}^{m} u_i(t) \mathcal{P}(Y_i(c(t))) , \tag{2.7}$$

and where we select the initial velocity in $\mathcal{D}$ (cf. [20] for details). Observe that the inputs Y_i act on the system only through their $\mathcal{D}$-component. Indeed, the Lagrange multiplier $\lambda \in \mathcal{D}^\perp$ absorbs their $\mathcal{D}^\perp$-components. The connection $\overline{\nabla}$ is called the

nonholonomic affine connection [3, 19, 20, 33]. Note that equations 2.4 and 2.7 have the same structure.

It can be easily deduced from its definition that $\overline{\nabla}$ restricts to $\mathcal{D}$, that is,

$$\overline{\nabla}_X Y = \mathcal{P}(\nabla^{\mathcal{G}}_X Y) \in \mathcal{D}\,, \text{ for all } Y \in \mathcal{D}\,,\ X \in \mathfrak{X}(Q)\,.$$

This kind of affine connection, which restricts to a given distribution, has been studied in [19]. In particular, such a behaviour implies that the distribution $\mathcal{D}$ is **geodesically invariant**, that is, for every geodesic $c(t)$ of $\overline{\nabla}$ starting from a point in $\mathcal{D}$, $\dot{c}(0) \in \mathcal{D}_{c(0)}$, we have that $\dot{c}(t) \in \mathcal{D}_{c(t)}$.

As we shall see later, a key tool in the controllability analysis and description of mechanical control systems is the **symmetric product** $\langle \cdot : \cdot \rangle$ associated to an affine connection ∇ (see [13, 21, 34]). Given X, $Y \in \mathfrak{X}(Q)$, define

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X\,.$$

The symmetric product characterizes geodesically invariant distributions. Indeed, one can prove that $\mathcal{D}$ is geodesically invariant for the nonholonomic connection if and only if $\langle X : Y \rangle \in \mathcal{D}$, $\forall X$, $Y \in \mathcal{D}$ (see [19]). Recently, Bullo [5] has shown that the evolution of mechanical control systems when starting from rest can be described by a series involving repeated symmetric products of the input vector fields, extending the possibilities of use of the symmetric product to the design of motion control algorithms.

2.2. Principal fiber bundles. The notion of principal fiber bundle is present in many locomotion and robotic systems, since they commonly exhibit translational and rotational symmetries. Examining the configuration space Q, one can observe that there exists a splitting $Q = G \times M$ between variables describing the position and orientation of the robot, i.e., the **pose** coordinates $g \in G$, and variables describing the internal shape of the system, the **shape** coordinates $r \in M$. This exactly corresponds to the case of a trivial principal fiber bundle, decomposed into **fiber** space, G, and **base** space, M, respectively.

Geometrically, this situation is described as follows. Assume there is a Lie group G acting on Q

$$\begin{array}{rccl} \Phi : & G \times Q & \longrightarrow & Q \\ & (g,q) & \longmapsto & \Phi(g,q) = \Phi_g(q) = gq\,. \end{array}$$

The orbit through a point q is $\mathrm{Orb}_G(q) = \{gq \mid g \in G\}$. We denote by $\mathfrak{g}$ the Lie algebra of G. For any element $\xi \in \mathfrak{g}$, let ξ_Q denote the corresponding infinitesimal generator of the group action on Q. Then,

$$T_q(\mathrm{Orb}_G(q)) = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}\,.$$

If the action Φ is free and proper, we can endow the quotient space $Q/G \cong M$ with a manifold structure such that the canonical projection $\pi : Q \longrightarrow M$ is a surjective submersion. Then, we have that $Q(M, G, \pi)$ is a principal bundle with bundle space Q, base space M, structure group G and projection π. Note that the kernel of $\pi_*(= T\pi)$ consists of the **vertical** tangent vectors, i.e., the vectors tangent to the orbits of G in Q. We denote the bundle of vertical vectors by $\mathcal{V}$, with $\mathcal{V}_q = T_q(\mathrm{Orb}_G(q))$, $q \in Q$.

Throughout the paper, we will usually deal with general principal fiber bundles, unless otherwise stated. Locally, one can always trivialize Q and work with $Q \supset$

$\pi^{-1}(U) \equiv G \times U$, where $U \subset M$ is an open subset of M. In the bundle coordinates (g, r), the projection reads $\pi(g, r) = r$ and the Lagrangian L can be written as

$$L(q, \dot{q}) = \frac{1}{2}(\dot{g}^T \, \dot{r}^T) \, \mathcal{G} \begin{pmatrix} \dot{g} \\ \dot{r} \end{pmatrix} - V(g, r)\,,$$

where we note the abuse of notation resulting from changing between $\dot{g}$ as an argument in TG and as a vector (the same stands for $\mathcal{G}$ seen as a bilinear form or as a matrix). In the remainder of the paper, we will often make use of the same notation for coordinate-free and matrix formulas. The precise meaning should be clear from the context.

A **principal connection** on $Q(M, G, \pi)$ can be defined as a G-invariant distribution $\mathcal{H}$ on Q satisfying $T_qQ = \mathcal{H}_q \oplus \mathcal{V}_q$, $\forall q \in Q$. The subspace $\mathcal{H}_q$ of T_qQ is called the **horizontal subspace** at q determined by the connection.

Alternatively, a principal connection can be characterized by a $\mathfrak{g}$-valued 1-form $\mathcal{A}$ on Q satisfying the following conditions

(i) $\mathcal{A}(\xi_Q(q)) = \xi$ for all $\xi \in \mathfrak{g}$,

(ii) $\mathcal{A}((\Phi_g)_* X) = \mathrm{Ad}_g(\mathcal{A}(X))$ for all $X \in TQ$.

The horizontal subspace at q is then given by $\mathcal{H}_q = \{v_q \in T_qQ \mid \mathcal{A}(v_q) = 0\}$. In coordinates, using (i) and (ii) we can write

$$\begin{aligned}\mathcal{A}(g, r, \dot{g}, \dot{r}) &= \mathcal{A}(g(e, r, \xi, \dot{r})) = Ad_g\mathcal{A}(e, r, \xi, \dot{r}) \\ &= Ad_g(\mathcal{A}(e, r, \xi, 0) + \mathcal{A}(e, r, 0, \dot{r})) = Ad_g(\xi + A(r)\dot{r})\,.\end{aligned}$$

Note that A depends only on the shape variables. It is called the **local form of the connection** $\mathcal{A}$.

Given a principal connection, we have that every vector $v \in T_qQ$ can be uniquely written as $v = v^{hor} + v^{ver}$, with $v^{hor} \in \mathcal{H}_q$ and $v^{ver} = \mathcal{A}(v)_Q(q) \in \mathcal{V}_q$. The curvature $\mathcal{B}$ of the principal connection $\mathcal{A}$ is a $\mathfrak{g}$-valued 2-form on Q defined as follows: for each $q \in Q$ and $u, v \in T_qQ$

$$\mathcal{B}(u, v) = d\mathcal{A}(u^{hor}, v^{hor}) = -\mathcal{A}([u^{hor}, v^{hor}])\,.$$

The curvature measures the lack of integrability of the horizontal distribution and plays a fundamental role in the theory of geometric phases (see [17] for a comprehensive treatment). In a local representation, the curvature can be written

$$\mathcal{B}((g\xi, v), (g\eta, w)) = (B(r)(v, w)) = B^a_{\alpha\beta} v^\alpha w^\beta Ad_g e_a\,,$$

where $\{e_a\}_{a=1}^k$ is a basis of the Lie algebra $\mathfrak{g}$ and

$$B^a_{\alpha\beta} = \frac{\partial A^a_\alpha}{\partial r^\beta} - \frac{\partial A^a_\beta}{\partial r^\alpha} + c^a_{bc} A^b_\alpha A^c_\beta\,.$$

The c^a_{bc} are the **structure constants** of the Lie algebra defined by $[e_b, e_c] = c^a_{bc} e_a$.

An additional derivative operator related to a principal connection will appear in the derivations below. Let κ be a $\bigotimes_\nu \mathfrak{g}^*$-valued function on Q, $\kappa : Q \longrightarrow \bigotimes_\nu \mathfrak{g}^*$. Define then the **derivative of** κ **along** $\mathcal{A}$, $D\kappa : TQ \longrightarrow \bigotimes_\nu \mathfrak{g}^*$, by

$$D\kappa(\dot{q})(\xi_1, \ldots, \xi_\nu) = d\kappa(\dot{q})(\xi_1, \ldots, \xi_\nu) + \sum_{k=1}^{\nu} \kappa(q)(\xi_1, \ldots, ad_{\mathcal{A}\dot{q}}\xi_k, \ldots, \xi_\nu)\,.$$

If the mapping κ is G-equivariant, $\kappa(g,r) = Ad^*_{g^{-1}}\kappa_{\mathrm{loc}}(r)$, where $\kappa_{\mathrm{loc}}(r) = \kappa(e,r)$, meaning

$$\kappa(g,r)(\xi_1,\ldots,\xi_\nu) = \kappa_{\mathrm{loc}}(r)(Ad_{g^{-1}}\xi_1,\ldots,Ad_{g^{-1}}\xi_\nu)\,,$$

then one can see that

$$D\kappa(\dot g,\dot r) = Ad^*_{g^{-1}}D\kappa_{\mathrm{loc}}(\dot r)\,.$$

In bundle coordinates, $D\kappa_{\mathrm{loc}}(\dot r)(\xi_1,\ldots,\xi_\nu) = (D\kappa_{\mathrm{loc}})_{\alpha a_1\ldots a_\nu}\dot r^\alpha \xi_1^{a_1}\ldots\xi_\nu^{a_\nu}$, where

$$(D\kappa_{\mathrm{loc}})_{\alpha a_1\ldots a_\nu} = \frac{\partial(\kappa_{\mathrm{loc}})_{a_1\ldots a_\nu}}{\partial r^\alpha} + \sum_{k=1}^{\nu}(\kappa_{\mathrm{loc}})_{a_1\ldots d_k\ldots a_\nu}A^e_\alpha c^{d_k}_{ea_k}$$

2.3. Systems with symmetry. In the reduction of unconstrained mechanical systems with symmetry, it naturally arises a principal connection, called the **mechanical** connection A^{mech}. Assume that the control system $(Q,\mathcal{G},V,\mathcal{F})$ is **invariant** under the action of a Lie group G, that is, $\Phi_g^*\mathcal{G} = \mathcal{G}$, $\Phi_g^*V = V$ and $\Phi_g^*F^i = F^i$, for $1 \le i \le m$ and all $g \in G$ (note that it may happen that a particular element of the control system be invariant under the action of a larger Lie group H, $G \subseteq H$, but we are only considering Lie groups which leave invariant *all* the components of the problem). The horizontal subspace of the mechanical connection is then given by the orthogonal complement of the vertical bundle $\mathcal{V}$ with respect to the kinetic energy metric $\mathcal{G}$, $\mathcal{H} = \mathcal{V}^\perp$. An explicit formula for its associated 1-form is the following. Define the **locked inertia tensor** at configuration $q \in Q$, $\mathcal{I}(q) : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ by

$$\langle \mathcal{I}(q)\xi,\eta\rangle = \mathcal{G}(\xi_Q(q),\eta_Q(q))\,.$$

In local coordinates, this can be expressed as $\mathcal{I}(r,g) = Ad^*_{g^{-1}}I(r)Ad_{g^{-1}}$. $I(r)$, the **local form of** $\mathcal{I}$, has the interpretation of the inertia of the system when frozen at shape r. If we further defined the **momentum map** $J : TQ \longrightarrow \mathfrak{g}^*$ by $\langle J(\dot q),\xi\rangle = \langle \frac{\partial L}{\partial \dot q}(\dot q),\xi_Q(q)\rangle$, then the mechanical connection is just $A^{mech}(\dot q) = I(q)^{-1}J(\dot q)$.

The invariance of the metric and the potential function implies also that $L(g,r,\dot g,\dot r) = L(e,r,g^{-1}\dot g,\dot r) = \ell(r,\dot r,\xi)$, where $\xi = g^{-1}\dot g$. The function $\ell : TQ/G \longrightarrow \mathbb{R}$ is given by

$$\ell(r,\dot r,\xi) = \frac{1}{2}(\xi^T\,\dot r^T)\,\hat{\mathcal{G}}\begin{pmatrix}\xi\\ \dot r\end{pmatrix} - V(r)\,,$$

where $\hat{\mathcal{G}}$ stands for the reduced metric [28]

$$\hat{\mathcal{G}} = \begin{pmatrix} I(r) & I(r)A(r)\\ A(r)^T I(r) & m(r)\end{pmatrix}\,. \tag{2.8}$$

Here, A denotes the local form of the mechanical connection. This reduced metric is block diagonalized if we write it in terms of the shape variables $(r,\dot r)$ and the **locked body angular velocity**, $\Omega = \xi + A(r)\dot r$. Indeed, one can see that $\hat{\mathcal{G}}$ takes the form

$$\tilde{\mathcal{G}} = \begin{pmatrix} I(r) & 0\\ 0 & m(r) - A^T(r)I(r)A(r)\end{pmatrix} = \begin{pmatrix} I(r) & 0\\ 0 & \Delta(r)\end{pmatrix}\,.$$

We will see below that the terms I and Δ plays a central role in deriving a local expression for the Levi-Civita affine connection.

The study of nonholonomic systems with symmetry has by now many contributions, starting from the work by Koiller on the kinematic case [16] and going through the use of the Hamiltonian formalism [2], Lagrangian reduction [4], the geometry of the tangent bundle [9, 10] or Poisson methods [23], among others. We review here some of the results found in [4, 28] for such systems which will be relevant for establishing later the decomposition for the nonholonomic affine connection.

Assume that the constrained mechanical control system is **invariant** under the action of a Lie group G, meaning that both $(Q, \mathcal{G}, V, \mathcal{F})$ and the constraint distribution $\mathcal{D}$ are invariant. Assume further that $\mathcal{D} + \mathcal{V} = TQ$ (the so-called *dimension assumption* [4]). We are interested in knowing which symmetry directions (i.e. tangent to the action of the Lie group) are compatible with the constraints. Consequently, we consider the intersection $S_q = \mathcal{V}_q \cap \mathcal{D}_q$ at each $q \in Q$. Since $S \subset \mathcal{V}$, we can consider a bundle $\mathfrak{g}^{\mathcal{D}} \longrightarrow Q$ whose fiber is given by $\mathfrak{g}^q = \{\xi \in \mathfrak{g} : \xi_Q(q) \in S_q\}$. The nonholonomic momentum map is then defined as

$$\begin{array}{rccccc} J^{nh}: & TQ & \longrightarrow & \mathfrak{g}^{\mathcal{D}^*} & & \\ & (q,\dot q) & \longmapsto & J^{nh}(q,\dot q): & \mathfrak{g}^q & \rightarrow & \mathbb{R} \\ & & & & \xi^q & \longmapsto & \langle \frac{\partial L}{\partial \dot q}(\dot q), \xi^q_Q(q)\rangle\,. \end{array}$$

This momentum map can be used to "augment" the constraints and provide a principal connection on $Q \longrightarrow Q/G$, the so-called nonholonomic principal connection [4]. The horizontal subspace at $q \in Q$ of this connection is given by the orthogonal complement of S in the constraint distribution, $\mathcal{H}_q = S_q^{\perp} \cap \mathcal{D}_q$.

Alternatively, let $\{e_1(r), \dots, e_s(r), e_{s+1}(r), \dots, e_k(r)\} \in \mathfrak{g}$ be a basis of $\mathfrak{g}$ such that the first s elements span $\mathfrak{g}^{(r,e)}$ and both set of generators are orthogonal in the kinetic energy metric restricted to $\mathcal{V}$. Denote by $\frac{\partial e_i}{\partial r^\alpha} = \sum_{a=1}^k \gamma^a_{i\alpha} e_a$, a notation which will be useful later. Define the momentum

$$p_i = \langle \frac{\partial \ell}{\partial \xi}, e_i(r)\rangle\,,\ 1 \le i \le s\,.$$

Now consider the map

$$\begin{array}{rccc} A^{sym}: & T_qQ & \longrightarrow & S_q \\ & (q,\dot q) & \longmapsto & (\tilde{\mathcal{I}}^{-1}(q) J^{nh}(q,\dot q))_Q\,, \end{array}$$

where $\tilde{\mathcal{I}}(q) : \mathfrak{g}^{\mathcal{D}} \longrightarrow \mathfrak{g}^{\mathcal{D}^*}$ is the locked inertia tensor relative to $\mathfrak{g}^{\mathcal{D}}$. Notice that A^{sym} maps S onto itself. Additionally, let $A^{kin} : T_qQ \longrightarrow S_q^{\perp}$ be the orthogonal projection relative to the kinetic energy metric. The constraints plus the momentum can be written

$$A^{kin}(q)\dot q = 0\,, \quad A^{sym}(q)\dot q = (\tilde{\mathcal{I}}^{-1}(q)p)_Q\,.$$

The nonholonomic connection 1-form is then given by

$$A^{nh} = A^{kin} + A^{sym}\,.$$

It is an instructive exercise to verify that A^{nh} satisfies indeed conditions (i) and (ii) defining a principal connection (cf. Section 2.2). This principal connection plays a fundamental role in the reduction of nonholonomic systems with symmetry [4].

3. Decomposition of the Levi-Civita connection under symmetry. Given a mechanical control system with symmetry, it seems reasonable that the controllability tests can be simplified by taking into account the symmetry properties of the problem. In order to do that, we will obtain decompositions of the Levi-Civita connection and the nonholonomic affine connection according to the principal fiber bundle structure of the configuration space Q. This will be the subject of the following two sections.

Let $(Q, \mathcal{G}, V, \mathcal{F})$ be a simple mechanical control system invariant under the action of a Lie group G. The following simple lemma [14] will be helpful.

LEMMA 3.1. *The Levi-Civita connection associated to a left-invariant metric H on the Lie group G is given by*

$$\nabla^H_{g\xi} g\eta = \frac{1}{2} g\left([\xi, \eta] - \sharp_H \left(ad^*_\xi \flat_H \eta + ad^*_\eta \flat_H \xi\right)\right) ,$$

where $g\xi$ stands for $(L_g)_* \xi$ and so on. Consequently, the symmetric product associated to ∇^H takes the form

$$\langle g\xi : g\eta \rangle_H = -g \, \sharp_H \left(ad^*_\xi \flat_H \eta + ad^*_\eta \flat_H \xi\right) .$$

Now, we come to the main result of this section, where we derive the properties of the "connection within a connection". Emphasis is placed on the role of I, A and Δ in determining $\nabla^{\mathcal{G}}$.

PROPOSITION 3.2. *Given G-invariant vector fields on Q, $X = (g\xi, v)$ and $Y = (g\eta, w)$, with $\xi(r)$, $\eta(r) \in \mathfrak{g}$ and v, $w \in TM$, the covariant derivative of Y along X can be expressed as*

$$\nabla^{\mathcal{G}}_X Y = g\left\{ \begin{pmatrix} \nabla^I_\Omega \Psi \\ \nabla^\Delta_v w \end{pmatrix} - \frac{1}{2} \begin{pmatrix} I^{-1}\mathbb{L} \\ \Delta^{-1}\mathbb{S} \end{pmatrix} \right\} , \tag{3.1}$$

where

$$\begin{aligned} \mathbb{L} &= -D(I\Omega)(\cdot, w) - D(I\Psi)(\cdot, v) + I([\Omega, \Psi] - [\xi, \eta] + \xi_r w - \eta_r v - A[v, w]) \\ &\quad + 2\, I(A(\nabla^{\mathcal{G}}_X Y)_M) \in \mathfrak{g}^* , \\ \mathbb{S} &= I(\Omega, B(w, \cdot)) + I(\Psi, B(v, \cdot)) + DI(\cdot)(\Omega, \Psi) \in T^*M , \end{aligned}$$

and $\Omega = \xi + Av$, $\Psi = \eta + Aw$, $\xi_r \equiv \frac{\partial \xi}{\partial r}$, $\eta_r \equiv \frac{\partial \eta}{\partial r}$

Proof. As we have recalled above, the Levi-Civita connection can be characterized as the unique affine connection verifying eq. 2.3. Let Z be a G-invariant vector field, $Z = (g\mu, u)$. The invariance of the metric implies

$$\mathcal{G}(Z, Y) = \hat{\mathcal{G}}\left((\mu, u), (\eta, w)\right) = \tilde{\mathcal{G}}\left((\Theta, u), (\Psi, w)\right) ,$$

where $\Theta = \mu + Aw$. The first three terms in eq. 2.3 can be expanded in a similar way,

$$X(\mathcal{G}(Z, Y)) = X(\Theta^T I \Psi + u^T \Delta w) = v(\Theta^T I \Psi) + v(u^T \Delta w) .$$

For the remaining ones, we have that

$$\begin{aligned} \mathcal{G}(X, [Z, Y]) &= \tilde{\mathcal{G}}\left((\Omega, v), ([\mu, \eta] + \eta_r u - w\mu_r + A[u, w], [u, w])\right) \\ &= \Omega^T I[\mu, \eta] + \Omega^T I(\eta_r u - w\mu_r + A[u, w]) + v^T \Delta[u, w] . \end{aligned}$$

As a result, eq. 2.3 can be written as $2\,\mathcal{G}(Z,\nabla^{\mathcal{G}}_X Y) = \langle(\delta,\gamma),(\mu,u)\rangle$, where $\delta=\delta_1+\delta_2$, $\gamma=\gamma_1+\gamma_2$ and

$$\begin{aligned}
\delta_1 &= \xi^T I[\cdot,\eta] + \eta^T I[\cdot,\xi] - \cdot^T I[\eta,\xi] \\
\delta_2 &= \cdot^T v(I\Psi) + \cdot^T w(I\Omega) + (Av)^T I[\cdot,\eta] + (Aw)^T I[\cdot,\xi] - \cdot^T I(\xi_r w - \eta_r v + A[w,v]) \\
\gamma_1 &= v(\cdot^T \Delta w) + w(\cdot^T \Delta v) - \cdot(w^T \Delta v) + v^T \Delta[\cdot,w] + w^T\Delta[\cdot,v] - \cdot^T \Delta[w,v] \\
\gamma_2 &= v((A\cdot)^T I\Psi) + w((A\cdot)^T I\Omega) - \cdot(\Psi^T I\Omega) + \Omega^T I(\eta_r\cdot + A[\cdot,w]) \\
&\quad + \Psi^T I(\xi_r\cdot + A[\cdot,v]) - (A\cdot)^T I([\eta,\xi] + \xi_r w - \eta_r v + A[w,v])\,.
\end{aligned}$$

On the other hand, we have that

$$2\,\mathcal{G}(Z,\nabla^{\mathcal{G}}_X Y) = 2\,(\mu^T,u^T)\begin{pmatrix} I & IA \\ A^T I & m\end{pmatrix}\begin{pmatrix}(\nabla^{\mathcal{G}}_X Y)_{\mathfrak{g}} \\ (\nabla^{\mathcal{G}}_X Y)_M\end{pmatrix}$$

As both expansions for $\mathcal{G}(Z,\nabla^{\mathcal{G}}_X Y)$ are valid for any Z, we can conclude that

$$\begin{aligned}
2\begin{pmatrix}(\nabla^{\mathcal{G}}_X Y)_{\mathfrak{g}} \\ (\nabla^{\mathcal{G}}_X Y)_M\end{pmatrix} &= \begin{pmatrix} I & IA \\ A^T I & m\end{pmatrix}^{-1}\begin{pmatrix}\delta\\ \gamma\end{pmatrix} \\
&= \begin{pmatrix} I^{-1}+A\Delta^{-1}A^T & -A\Delta^{-1} \\ -\Delta^{-1}A^T & \Delta^{-1}\end{pmatrix}\begin{pmatrix}\delta\\ \gamma\end{pmatrix}.
\end{aligned}\tag{3.2}$$

Noting that $\delta_1 = 2I\nabla^I_\xi \eta$ (see Lemma 3.1) and $\gamma_1 = 2\Delta\nabla^\Delta_v w$, we can further develop the right-hand side of eq. 3.2 as

$$\begin{aligned}
&\begin{pmatrix} I^{-1} & 0 \\ 0 & \Delta^{-1}\end{pmatrix}\begin{pmatrix}\delta\\ \gamma\end{pmatrix} + \begin{pmatrix} A\Delta^{-1}A^T & -A\Delta^{-1} \\ -\Delta^{-1}A^T & 0\end{pmatrix}\begin{pmatrix}\delta\\ \gamma\end{pmatrix} = \\
&2\begin{pmatrix}\nabla^I_\xi\eta \\ \nabla^\Delta_v w\end{pmatrix} + \begin{pmatrix} I^{-1} & 0 \\ 0 & \Delta^{-1}\end{pmatrix}\left\{\begin{pmatrix}\delta_2\\ \gamma_2\end{pmatrix} + \begin{pmatrix} IA\Delta^{-1}A^T & -IA\Delta^{-1} \\ -A^T & 0\end{pmatrix}\begin{pmatrix}\delta\\ \gamma\end{pmatrix}\right\}.
\end{aligned}$$

In this way, we get

$$\begin{pmatrix}(\nabla^{\mathcal{G}}_X Y)_{\mathfrak{g}} \\ (\nabla^{\mathcal{G}}_X Y)_M\end{pmatrix} = \begin{pmatrix}\nabla^I_\xi\eta \\ \nabla^\Delta_v w\end{pmatrix} - \frac{1}{2}\begin{pmatrix} I^{-1} & 0 \\ 0 & \Delta^{-1}\end{pmatrix}\begin{pmatrix}\mathbb{L}'\\ \mathbb{S}\end{pmatrix},$$

where $\mathbb{L}' = -\delta_2 - IA\Delta^{-1}\mathbb{S} + IA\Delta^{-1}\gamma_1$ and $\mathbb{S} = A^T\delta - \gamma_2$. To complete the proof, we only have to identify these terms in a more geometrical manner, which we do in the following.

We begin with $\mathbb{S}$. Noting that

$$A^b_\beta \frac{\partial v^\beta}{\partial r^\alpha} + \frac{\partial \xi^b}{\partial r^\alpha} - \frac{\partial \Omega^b}{\partial r^\alpha} = -\frac{\partial A^b_\beta}{\partial r^\alpha} v^\beta\,,$$

we can rewrite γ_2 as

$$\begin{aligned}
\gamma_2 = {}& v^\beta A^b_\alpha \frac{\partial (I\Psi)_b}{\partial r^\beta} + w^\beta A^b_\alpha \frac{\partial (I\Omega)_b}{\partial r^\beta} - \Psi^b \frac{\partial I_{ba}}{\partial r^\alpha}\Omega^a \\
&+(I\Psi)_b\left\{\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha}\right\} v^\beta + (I\Omega)_b\left\{\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha}\right\} w^\beta \\
&-A^b_\alpha I_{ba} c^a_{de}\eta^d\xi^e - A^b_\alpha I_{ba}\left\{\frac{\partial \xi^a}{\partial r^\beta}w^\beta - \frac{\partial \eta^a}{\partial r^\beta}v^\beta + A^a_\beta [w,v]^\beta\right\}.
\end{aligned}$$

Substituting into the expression for $\mathbb{S}$, one obtains after some computations

$$
\begin{aligned}
-\mathbb{S} &= (I\Psi)_b \left\{ \frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^c_\beta E^b_{\alpha c} \right\} v^\beta + (I\Omega)_b \left\{ \frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^c_\beta E^b_{\alpha c} \right\} w^\beta \\
&\quad -\Psi^b \frac{\partial I_{ba}}{\partial r^\alpha}\Omega^a - \Omega^d I_{db} E^b_{\alpha e}\Psi^e - \Psi^d I_{db} E^b_{\alpha e}\Omega^e \\
&= -I(\Psi, B(v,\cdot)) - I(\Omega, B(w,\cdot)) - DI(\cdot)(\Omega,\Psi)\,,
\end{aligned}
$$

where $E^b_{\alpha c} = c^b_{dc} A^d_\alpha$. Now we turn our attention to $\mathbb{L}'$. Note that

$$
\begin{aligned}
\mathbb{L}' &= IA\Delta^{-1}(\gamma_1 - \mathbb{S}) - \delta_2 \\
&= 2IA(\nabla^\Delta_v w - \frac{1}{2}\Delta^{-1}\mathbb{S}) - \delta_2 = 2IA(\nabla^{\mathcal{G}}_X Y)_M - \delta_2\,.
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
\delta_2 &= v^\alpha \frac{\partial (I\Psi)_a}{\partial r^\alpha} + w^\alpha \frac{\partial (I\Omega)_a}{\partial r^\alpha} + A^d_\alpha v^\alpha I_{db} c^b_{ae}\eta^e + A^d_\alpha w^\alpha I_{db} c^b_{ae}\xi^e \\
&\quad -I_{ba}\left\{ \frac{\partial \xi^b}{\partial r^\beta} w^\beta - \frac{\partial \eta^b}{\partial r^\beta} v^\beta + A^b_\beta [w,v]^\beta \right\}\,.
\end{aligned}
$$

Adding and substracting $(I\Psi)_b E^b_{\alpha a} v^\alpha$ and $(I\Omega)_b E^b_{\alpha a} w^\alpha$ and re-grouping, we obtain

$$
\begin{aligned}
\delta_2 &= D(I\Psi)(\cdot, v) + D(I\Omega)(\cdot, w) + 2I\nabla^I_\Omega \Psi - 2I\nabla^I_\xi \eta \\
&\quad -I(\cdot, [\Omega,\Psi]) + I(\cdot, [\xi,\eta]) - I(\cdot, \xi_r w - \eta_r v + A[w,v])\,.
\end{aligned}
$$

Finally, we can write

$$
(\nabla^{\mathcal{G}}_X Y)_{\mathfrak{g}} = \nabla^I_\xi \eta - \frac{1}{2} I^{-1}\mathbb{L}' = \nabla^I_\Omega \Psi - \frac{1}{2} I^{-1}\mathbb{L}\,,
$$

where $\mathbb{L}$ is as above. $\square$

As a consequence of this proposition, we have the following interesting result.

Corollary 3.3. *The symmetric product associated to the Levi-Civita connection* $\nabla^{\mathcal{G}}$ *of two G-invariant vector fields,* $X = (g\xi, v)$ *and* $Y = (g\eta, w)$ *is given by*

$$
\langle X : Y \rangle_{\mathcal{G}} = g\left\{ \begin{pmatrix} \langle \Omega : \Psi \rangle_I \\ \langle v : w \rangle_\Delta \end{pmatrix} - \begin{pmatrix} I^{-1}\mathbb{L}^s \\ \Delta^{-1}\mathbb{S} \end{pmatrix} \right\}\,, \tag{3.3}
$$

where

$$
\begin{aligned}
\mathbb{L}^s &= -D(I\Omega)(\cdot, w) - D(I\Psi)(\cdot, v) + IA\left(\langle v : w\rangle_\Delta - \Delta^{-1}\mathbb{S}\right) \in \mathfrak{g}^* \\
\mathbb{S} &= I(\Omega, B(w,\cdot)) + I(\Psi, B(v,\cdot)) + DI(\cdot)(\Omega,\Psi) \in T^*M\,,
\end{aligned}
$$

and $\langle \cdot : \cdot \rangle_I$, $\langle \cdot : \cdot \rangle_\Delta$ *denote the symmetric products defined by the Levi-Civita connections* ∇^I *and* ∇^Δ*, respectively.*

We shall return to these results in Section 6 in computing the symmetric product in specific examples.

4. Decomposition of the nonholonomic affine connection under symmetry. Let $(Q, \mathcal{G}, V, \mathcal{F}, \mathcal{D})$ be a constrained mechanical control system invariant under the action of a Lie group G. As expected, the invariance of the Levi-Civita connection and the nonholonomic distribution $\mathcal{D}$ can be combined to find a decomposition of the nonholonomic affine connection similar to that of Proposition 3.2.

First, notice that if $\mathcal{D}$ is generated by a basis of G-invariant vector fields X_i, $1 \leq i \leq n-l$, the projector $\mathcal{P} : TQ \longrightarrow \mathcal{D}$ with respect to the orthogonal decomposition $TQ = \mathcal{D} \oplus \mathcal{D}^\perp$ is given by

$$\mathcal{P}(Z) = \sum_{i,j} C^{ij} \mathcal{G}(X_i, Z) X_j \, , \; Z \in \mathfrak{X}(Q) \, ,$$

where (C^{ij}) is the inverse matrix of $(C_{ij} = \mathcal{G}(X_i, X_j))$. A geometrically revealing choice of generators of $\mathcal{D}$ making use of the exposition in Section 2.3 is the following. Recall that the nonholonomic principal connection A^{nh} induces a decomposition of the tangent bundle, $TQ = \mathcal{H} \oplus \mathcal{V}$. This in particular implies that

$$\mathcal{D} = \mathcal{H} \oplus S \, .$$

On the one hand, we know that $S_{(r,e)} = \operatorname{span}\{e_1(r)_Q, \ldots, e_s(r)_Q\}$. Furthermore, the generators of $\mathcal{H}_{(r,e)}$ are of the form $(-\mathbb{A}\dot{r}, \dot{r})$, where $\mathbb{A}$ denotes the local form of A^{nh}. Hence, we have that

$$\mathcal{D}_{(r,g)} = g \mathcal{D}_{(r,e)} = g \operatorname{span} \left\{ (-\mathbb{A}\dot{r}, \dot{r}) \, , (e_i, 0) \right\} \, .$$

For these vector fields we compute

$$\begin{aligned}
\mathcal{G}\left(g\left(e_i, 0\right), g\left(e_j, 0\right)\right) &= e_i^T I e_j = e_i^T \tilde{I} e_j \\
\mathcal{G}\left(g\left(-\mathbb{A}(\dot{r}), \dot{r}\right), g\left(e_j, 0\right)\right) &= -(\mathbb{A}\dot{r})^T I e_j + (A\dot{r})^T I e_j = (\tilde{A}\dot{r})^T I e_j = 0 \\
\mathcal{G}\left(g\left(-\mathbb{A}(\dot{r}), \dot{r}\right), g\left(-\mathbb{A}(\dot{r}), \dot{r}\right)\right) &= (\mathbb{A}\dot{r})^T I \mathbb{A}\dot{r} - (\mathbb{A}\dot{r})^T I A\dot{r} - (A\dot{r})^T I \mathbb{A}\dot{r} + \dot{r}^T m \dot{r} \\
&= \dot{r}^T (m + \mathbb{A}^T I \mathbb{A} - \mathbb{A}^T I A - A^T I \mathbb{A}) \dot{r} = \dot{r}^T \tilde{\Delta} \dot{r} \, ,
\end{aligned}$$

where $\tilde{A} = A - \mathbb{A}$, $\tilde{\Delta} = m - A^T I A + \tilde{A}^T I \tilde{A}$ and we have used the fact that $\tilde{A}\dot{r} \in S^\perp$. Hence, we can write the matrix C as

$$C = \begin{pmatrix} \tilde{I} & 0 \\ 0 & \tilde{\Delta} \end{pmatrix} \, .$$

Now, we are in a position to prove the following result.

PROPOSITION 4.1. *Given G-invariant vector fields, $X = (g\xi, v) \in TQ$, $Y = (g\eta, w) \in \mathcal{D}$ on Q, with $\xi(r)$, $\eta(r) \in \mathfrak{g}$ and v, $w \in TM$ the nonholonomic affine connection $\overline{\nabla}$ can be expressed as*

$$\overline{\nabla}_X Y = g \left\{ \begin{pmatrix} A^{sym}(\nabla^I_{\bar{\Omega}} \bar{\Psi}) \\ \nabla^{\tilde{\Delta}}_v w \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \tilde{I}^{-1}\tilde{\mathbb{L}} + 2\mathbb{A}(\overline{\nabla}_X Y)_M \\ \tilde{\Delta}^{-1}\tilde{\mathbb{S}} \end{pmatrix} \right\} \, , \tag{4.1}$$

where

$$\begin{aligned}
\tilde{\mathbb{L}} &= -\mathbb{D}(I\bar{\Omega})(\cdot, w) - \mathbb{D}(I\bar{\Psi})(\cdot, v) + I(\tilde{A}v, \gamma.w - [\cdot, \eta]) + I(\tilde{A}w, \gamma.v - [\cdot, \xi]) \\
&\quad + I([\bar{\Omega}, \bar{\Psi}] - [\xi, \eta] + \xi_r w - \eta_r v - \mathbb{A}[v, w]) \in \mathfrak{g}^{\mathcal{D}^*} \, , \\
\tilde{\mathbb{S}} &= I(\bar{\Psi}, B(v, \cdot)) + I(\bar{\Omega}, B(w, \cdot)) + I(\tilde{A}w, \mathbb{B}(v, \cdot)) + I(\tilde{A}v, \mathbb{B}(w, \cdot)) \\
&\quad - D(I\bar{\Psi})(\tilde{A}\cdot, v) - D(I\bar{\Omega})(\tilde{A}\cdot, w) + \mathbb{D}I(\cdot)(\bar{\Omega} + \tilde{A}v, \bar{\Psi} + \tilde{A}w) - \mathbb{D}I(\cdot)(\tilde{A}v, \tilde{A}w) \\
&\quad - I([\xi, \eta], \tilde{A}\cdot) - I(\eta_r v - \xi_r v, \tilde{A}\cdot) - I(\mathbb{A}[v, w], \tilde{A}\cdot) \in T^*M \, ,
\end{aligned}$$

and $\mathbb{D}$, $\mathbb{B}$ denote, respectively, the local forms of the derivative along and the curvature of the nonholonomic connection A^{nh} and $\bar{\Omega} = \xi + \mathbb{A}v$, $\bar{\Psi} = \eta + \mathbb{A}w$.

Proof. Since $Y \in \mathcal{D}$, $\overline{\nabla}_X Y = \mathcal{P}(\nabla^{\mathcal{G}}_X Y) = \sum C^{ij}\mathcal{G}(X_i, \nabla^{\mathcal{G}}_X Y)X_j$. We first compute

$$\mathcal{G}\left(g\left(e_i, 0\right), \nabla^{\mathcal{G}}_X Y\right) = e_i^T I \left\{(\nabla^{\mathcal{G}}_X Y)_{\mathfrak{g}} + A(\nabla^{\mathcal{G}}_X Y)_M\right\} \tag{4.2}$$

$$\mathcal{G}\left(g\left(-\mathbb{A}\dot{r}, \dot{r}\right), \nabla^{\mathcal{G}}_X Y\right) = (\tilde{A}\dot{r})^T I(\nabla^{\mathcal{G}}_X Y)_{\mathfrak{g}} + \dot{r}^T(m - A^T I \mathbb{A})(\nabla^{\mathcal{G}}_X Y)_M \tag{4.3}$$

$$= (\tilde{A}\dot{r})^T I \left\{(\nabla^{\mathcal{G}}_X Y)_{\mathfrak{g}} + A(\nabla^{\mathcal{G}}_X Y)_M\right\} + \dot{r}^T \Delta(\nabla^{\mathcal{G}}_X Y)_M$$

Let us denote $(\nabla^{\mathcal{G}}_X Y)_{\mathfrak{g}} + A(\nabla^{\mathcal{G}}_X Y)_M = \widetilde{\nabla^{\mathcal{G}}_X Y}$ for brevity. In terms of $\bar{\Omega}$, $\bar{\Psi}$ and using Proposition 3.2 it can be expanded as

$$\begin{aligned}\widetilde{\nabla^{\mathcal{G}}_X Y} = \nabla^I_{\bar{\Omega}}\bar{\Psi} - \frac{1}{2}I^{-1}\Big\{&-\mathbb{D}(I\bar{\Omega})(\cdot, w) - \mathbb{D}(I\tilde{A}v)(\cdot, w) - \mathbb{D}(I\bar{\Psi})(\cdot, v) \\ &- \mathbb{D}(I\tilde{A}w)(\cdot, v) - I(\tilde{A}w, [\cdot, \bar{\Omega}]) - I(\tilde{A}v, [\cdot, \bar{\Psi}]) \\ &+ I([\bar{\Omega}, \bar{\Psi}] - [\xi, \eta] + \xi_r w - \eta_r v - A[v, w], \cdot)\Big\}\end{aligned}$$

Before plugging this expression into eq. 4.2, notice that

$$-\mathbb{D}(I\tilde{A}v)(e_i, w) - I(\tilde{A}v, [e_i, \bar{\Psi}]) = I(\tilde{A}v, \gamma_i w - [e_i, \eta]),$$

where we have used the fact that $e_i \in S$ and $\tilde{A}v \in S^{\perp}$. After substituting, we find that eq. 4.2 can be expressed as $\mathcal{G}\left(g\left(e_i, 0\right), \nabla^{\mathcal{G}}_X Y\right) = \langle I(\nabla^I_{\bar{\Omega}}\bar{\Psi}, \cdot) - \frac{1}{2}\tilde{\mathbb{L}}, e_i\rangle$, where

$$\begin{aligned}\tilde{\mathbb{L}} = &-\mathbb{D}(I\bar{\Omega})(\cdot, w) - \mathbb{D}(I\bar{\Psi})(\cdot, v) + I(\tilde{A}v, \gamma.w - [\cdot, \eta]) + I(\tilde{A}w, \gamma.v - [\cdot, \xi]) \\ &+ I([\bar{\Omega}, \bar{\Psi}] - [\xi, \eta], \cdot) - I(\eta_r v - \xi_r w + \mathbb{A}[v, w], \cdot)\end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned}\Delta(\nabla^{\mathcal{G}}_X Y)_M &= \Delta\nabla^{\Delta}_v w - \frac{1}{2}\mathbb{S} \\ &= \tilde{\Delta}\nabla^{\tilde{\Delta}}_v w - D\nabla^D_v w - \frac{1}{2}\mathbb{S} = \tilde{\Delta}\nabla^{\tilde{\Delta}}_v w - \left(D\nabla^D_v w + \frac{1}{2}\mathbb{S}\right),\end{aligned}$$

where $D = \tilde{A}^T I \tilde{A}$ and $D\nabla^D_v w$ is a shorthand notation to denote the expression eq. 2.3 for the symmetric tensor D. Then, we can rewrite (4.3) as

$$\dot{r}^T\left(\tilde{\Delta}\nabla^{\tilde{\Delta}}_v w - \left(D\nabla^D_v w + \frac{1}{2}\mathbb{S} - \tilde{A}^T I\widetilde{\nabla^{\mathcal{G}}_X Y}\right)\right).$$

Therefore $\overline{\nabla}_X Y$ becomes

$$\overline{\nabla}_X Y = \mathcal{P}(\nabla^{\mathcal{G}}_X Y) = \begin{pmatrix} g(\overline{\nabla}_X Y)_{\mathfrak{g}} \\ (\overline{\nabla}_X Y)_M \end{pmatrix},$$

with

$$\begin{aligned}(\overline{\nabla}_X Y)_{\mathfrak{g}} &= \tilde{I}^{-1}\left\{I(\nabla^I_{\bar{\Omega}}\bar{\Psi}, \cdot) - \frac{1}{2}\tilde{\mathbb{L}}\right\} - \mathbb{A}(\overline{\nabla}_X Y)_M \\ &= A^{sym}(\nabla^I_{\bar{\Omega}}\bar{\Psi}) - \frac{1}{2}\tilde{I}^{-1}\tilde{\mathbb{L}} - \mathbb{A}(\overline{\nabla}_X Y)_M, \\ (\overline{\nabla}_X Y)_M &= \nabla^{\tilde{\Delta}}_v w - \tilde{\Delta}^{-1}\left(D\nabla^D_v w + \frac{1}{2}\mathbb{S} - \tilde{A}^T I\widetilde{\nabla^{\mathcal{G}}_X Y}\right),\end{aligned}$$

where we have used the fact that $A^{sym}(\zeta) \equiv A^{sym}(\zeta_Q(e,r)) = \tilde{I}^{-1}I(\zeta)$, for $\zeta \in \mathfrak{g}$. To end the proof, let us write explicitely the terms in $(\overline{\nabla}_X Y)_M$. Adding and substracting terms in the expression for $2\,D\nabla^D_v w$, we can find that

$$\begin{aligned}2\,D\nabla^D_v w &= v^\beta \frac{\partial \tilde{A}^a_\alpha}{\partial r^\beta} I_{ab}\tilde{A}^b_\gamma w^\gamma + D(I\tilde{A}w)(\tilde{A}\cdot, v) - I(\tilde{A}v, [Aw, \tilde{A}\cdot]) \\ &\quad + w^\beta \frac{\partial \tilde{A}^a_\alpha}{\partial r^\beta} I_{ab}\tilde{A}^b_\gamma v^\gamma + D(I\tilde{A}v)(\tilde{A}\cdot, w) - I(\tilde{A}w, [Av, \tilde{A}\cdot]) \\ &\quad - w^\beta v^\gamma \frac{\partial \tilde{A}^a_\beta I_{ab}\tilde{A}^b_\gamma}{\partial r^\alpha} + \tilde{A}^T I\tilde{A}[v,w] \\ &= D(I\tilde{A}w)(\tilde{A}\cdot, v) + D(I\tilde{A}v)(\tilde{A}\cdot, w) + I(\tilde{A}w)B(\cdot, v) + I(\tilde{A}w)\mathbb{B}(v,\cdot) \\ &\quad + I(\tilde{A}v)B(\cdot, w) + I(\tilde{A}v)\mathbb{B}(w,\cdot) - \mathbb{D}I(\tilde{A}v, \tilde{A}w) + \tilde{A}^T I\tilde{A}[v,w]\,.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathbb{S} &= I(\Omega, B(w,\cdot)) + I(\Psi, B(v,\cdot)) + DI(\cdot)(\Omega,\Psi) \\ &= I(\Omega, B(w,\cdot)) + I(\Psi, B(v,\cdot)) + \mathbb{D}I(\cdot)(\Omega,\Psi) + I(\Omega, [\tilde{A}\cdot, \Psi]) + I(\Psi, [\tilde{A}\cdot, \Omega])\,,\end{aligned}$$

and the term $\tilde{A}^T I\widetilde{\nabla^{\mathcal{G}}_X Y}$ can be written as

$$\begin{aligned}-\tilde{A}^T I\widetilde{\nabla^{\mathcal{G}}_X Y} &= -\tilde{A}^T I\nabla^I_\Omega \Psi + \frac{1}{2}\Big(-D(I\Psi)(\tilde{A}\cdot, v) - D(I\Omega)(\tilde{A}\cdot, w) \\ &\quad + I([\Omega,\Psi] - [\xi,\eta], \tilde{A}\cdot) - I(\eta_r v - \xi_r v, \tilde{A}\cdot) - \tilde{A}^T IA[v,w]\Big)\,.\end{aligned}$$

Summing up these terms, we get the expression for $\tilde{\mathbb{S}}$ stated in the proposition. ◻

Corollary 4.2. *The symmetric product associated to $\overline{\nabla}$ of two G-invariant vector fields, $X = (g\xi, v) \in \mathcal{D}$ and $Y = (g\eta, w) \in \mathcal{D}$ is given by*

$$\langle X : Y\rangle = g\left\{\begin{pmatrix} A^{sym}(\langle\bar{\Omega} : \bar{\Psi}\rangle_I) \\ \langle v : w\rangle_{\tilde{\Delta}}\end{pmatrix} - \begin{pmatrix} \tilde{I}^{-1}\tilde{\mathbb{L}}^s + \mathbb{A}\left(\langle v : w\rangle_{\tilde{\Delta}} - \tilde{\Delta}^{-1}\tilde{\mathbb{S}}^s\right) \\ \tilde{\Delta}^{-1}\tilde{\mathbb{S}}^s\end{pmatrix}\right\}, \tag{4.4}$$

where

$$\begin{aligned}\tilde{\mathbb{L}}^s &= -\mathbb{D}(I\bar{\Omega})(\cdot, w) - \mathbb{D}(I\bar{\Psi})(\cdot, v) + I(\tilde{A}v, \gamma.w - [\cdot,\eta]) + I(\tilde{A}w, \gamma.v - [\cdot,\xi]) \in \mathfrak{g}^{\mathcal{D}*} \\ \tilde{\mathbb{S}}^s &= I(\bar{\Psi}, B(v,\cdot)) + I(\bar{\Omega}, B(w,\cdot)) + I(\tilde{A}w, \mathbb{B}(v,\cdot)) + I(\tilde{A}v, \mathbb{B}(w,\cdot)) - D(I\bar{\Psi})(\tilde{A}\cdot, v) \\ &\quad - D(I\bar{\Omega})(\tilde{A}\cdot, w) + \mathbb{D}I(\cdot)(\bar{\Omega} + \tilde{A}v, \bar{\Psi} + \tilde{A}w) - \mathbb{D}I(\cdot)(\tilde{A}v, \tilde{A}w) \in T^*M\,.\end{aligned}$$

and $\langle\cdot : \cdot\rangle_{\tilde{\Delta}}$ denotes the symmetric product defined by the Levi-Civita connection $\nabla^{\tilde{\Delta}}$.

5. Controllability analysis. The point in the approach of Lewis and Murray to simple mechanical control systems is precisely to know what is happening to configurations, rather than to states, since in many of these systems configurations may be controlled, but not configurations and velocities at the same time. The basic question they pose is "what is the set of configurations that are attainable from a given configuration starting from rest?"

Consider the control equation

$$\nabla_{\dot{c}(t)}\dot{c}(t) = \sum_{i=1}^m u_i(t)Y_i(c(t))\,, \tag{5.1}$$

where the affine connection ∇ can be either the Levi-Civita affine connection associated to a kinetic energy metric or the nonholonomic affine connection for a constrained system (recall that in the latter case we select $\dot{c}(0) \in \mathcal{D}$ and Y_i denotes the projection by $\mathcal{P}$ to $\mathcal{D}$ of the i^{th} input vector field). Notice that we are considering now that $V \equiv 0$. The absence of the potential makes the picture considerably more clear while capturing the essential aspects of the analysis. On the other hand, a potential function could be incorporated to the controllability tests along the lines of [21].

Take $q_0 \in Q$ and let $U \subset Q$ be a neighbourhood of q_0. Define

$$\begin{aligned}\mathcal{R}_Q^U(q_0, T) = \{q \in Q \,|\, &\text{there exists a solution } (c, u) \text{ of (5.1) such that} \\ &\dot{c}(0) = 0_{q_0}, c(t) \in U \text{ for } t \in [0, T] \text{ and } \dot{c}(T) \in T_q Q\}\end{aligned}$$

and denote by $\mathcal{R}_Q^U(q_0, \leq T) = \cup_{0 \leq t \leq T} \mathcal{R}_Q^U(q_0, t)$.

We shall focus our attention on the following notions of accessibility and controllability [21].

Definition 5.1. *The system (5.1) is* **locally configuration accessible (LCA) at** $q_0 \in Q$ *if there exists* $T > 0$ *such that* $\mathcal{R}_Q^U(q_0, \leq t)$ *contains a non-empty open set of* Q*, for all neighbourhoods* U *of* q_0 *and all* $0 \leq t \leq T$*. If this holds for any* $q_0 \in Q$ *then the system is called* **locally configuration accessible**.

Definition 5.2. *The system (5.1) is* **small-time locally configuration controllable (STLCC) at** $q_0 \in Q$ *if there exists* $T > 0$ *such that* $\mathcal{R}_Q^U(q_0, \leq t)$ *contains a non-empty open set of* Q *to which* q_0 *belongs, for all neighbourhoods* U *of* q_0 *and all* $0 \leq t \leq T$*. If this holds for any* $q_0 \in Q$ *then the system is called* **small-time locally configuration controllable**.

Given the input vector fields $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$, let us denote by $\overline{Sym}(\mathcal{Y})$ the distribution obtained by closing the set $\mathcal{Y}$ under the symmetric product and by $\overline{Lie}(\mathcal{Y})$ the involutive closure of $\mathcal{Y}$. With these ingredients, one can prove

Theorem 5.3. *([21]) The control system (5.1) is locally configuration accessible at* q *if* $\overline{Lie}(\overline{Sym}(\mathcal{Y}))_q = T_q Q$.

If P is a symmetric product of vector fields in $\mathcal{Y}$, we let $\gamma_i(P)$ denote the number of occurrences of Y_i in P. The **degree** of P will be $\gamma_1(P) + \cdots + \gamma_m(P)$. We say that P is **bad** if $\gamma_i(P)$ is even for each $1 \leq i \leq m$. Otherwise, we say that P is **good**. The following theorem gives sufficient conditions for STLCC.

Theorem 5.4. *Suppose that the system (5.1) is LCA at* q *and that every bad symmetric product* P *at* q *in* $\mathcal{Y}$ *can be written as a linear combination of good symmetric products at* q *of lower degree than* P*. Then it is STLCC at* q*.*

Remark 5.5. This theorem was proved in [21], as an application to mechanical systems of previous work by Sussmann [32] on general control systems with drift. There has been some effort in trying to obtain sufficient *and* necessary conditions for configuration controllability. A conjecture that remains open is that the system (5.1) is STLCC at q if and only if there exists a basis of vector fields generating the input distribution which satisfies the sufficient conditions of the theorem. Lewis [18] proved the validity of the conjecture for the one-input case. Recently, Cortés and Martínez [11] have proved that it is also valid for underactuated systems by one control.

The exposed controllability analysis can be further refined for mechanical control systems with symmetry, taking into account the results of the previous sections. Assume that the control system (5.1) is invariant under the action of a Lie group G. Let us denote by $\mathfrak{B} = \{B_1, \ldots, B_m\}$ the representants of the input vector fields

$\mathcal{Y} = \{Y_1, \dots, Y_m\}$ at $\mathfrak{g} \times TM$, that is,

$$Y_i(r,g) = gB_i(r,e) = g\begin{pmatrix} \xi_i(r) \\ v_i \end{pmatrix}, \; 1 \le i \le m\,.$$

Due to the invariance of the system we have that $\langle Y_i : Y_j\rangle = \langle gB_i : gB_j\rangle \equiv g\langle B_i : B_j\rangle$ for all $1 \le i, j \le m$. The explicit expression in bundle coordinates for this symmetric product is given by Corollaries 3.3 and 4.2. Note also that the Lie brackets $[Y_i, Y_j]$ can be written as

$$[Y_i, Y_j] \equiv g[B_i, B_j] = g\begin{pmatrix} [\xi_i, \xi_j]_{\mathfrak{g}} + \frac{\partial \xi_j}{\partial r} v_i - \frac{\partial \xi_i}{\partial r} v_j \\ [v_i, v_j]_M \end{pmatrix}$$

As a result, we have the following version of the former results.

Theorem 5.6. *Let the control system (5.1) be invariant under the action of a Lie group G.*

(i) The system is LCA at $q = (r,g) \in Orb_G(r,e)$ if $\overline{Lie}(\overline{Sym}(\mathfrak{B}))_{(r,e)} = \mathfrak{g} \times T_rM$.

(ii) Suppose that the system is LCA at (r,e) and that every bad symmetric product P at (r,e) in $\mathfrak{B}$ can be written as a linear combination of good symmetric products at (r,e) of lower degree than P. Then (5.1) is STLCC at $q \in Orb_G(r,e)$.

These simplified tests of the accessibility and controllability properties of mechanical control systems under symmetry are indeed quite useful in practical examples, since they remove completely the dependence on the Lie group elements $g \in G$ from the computations. In examples such as the blimp, the underwater vehicle, the snakeboard or the roller racer, where symmetry plays an important role, this property may be a definitive advantage.

An additional important simplification from the computational point of view stems from the fact that for many dynamic robotic locomotion systems, the set of inputs at disposal consists of the full tangent bundle of the shape space M. This essentially corresponds to the observation that the system can adjust its shape as desired. For such problems, we can state the following result.

Theorem 5.7. *Let the control system* (5.1) *be invariant under the action of a Lie group G. Additionally assume that the system is fully actuated in the shape space, i.e. the set of input forces consist of $F^1 = dr^1, \dots, F^m = dr^m$, where m now also denotes the dimension of M. Then, the locked body angular velocities of the input vector fields all vanish, $\Omega_i = 0$, $1 \le i \le m$. Moreover, in the presence of nonholonomic constraints, the projections of the input vector fields to $\mathcal{D}$ also have $\bar{\Omega}_i = 0$, $1 \le i \le m$.*

Proof. It is not difficult to verify that the input vector fields are of the form $\left(-gA\Delta^{-1}\dot{r}, \Delta^{-1}\dot{r}\right)$. Then, $\Omega_i = 0$ follows. On the other hand, their projections to the constraint distribution $\mathcal{D}$ can be written as $(-g\mathbb{A}\tilde{\Delta}^{-1}\dot{r}, \tilde{\Delta}^{-1}\dot{r})$, which implies that $\bar{\Omega}_i = 0$. ☐

As a consequence of Theorem 5.7, the necessary calculations in the controllability analysis of the successive symmetric products involving the input vector fields (cf. Corollary 3.3) or their projections to $\mathcal{D}$ (cf. Corollary 4.2) are further simplified. In fact, for two vector fields $X = (g\xi, v)$ and $Y = g(\eta, w)$ having vanishing associated locked body angular velocities $\Omega = 0$, $\Psi = 0$, we have by Corollary 3.3 that

$$\langle X : Y\rangle_{\mathcal{G}} = g\begin{pmatrix} -A\,\langle v : w\rangle_{\Delta} \\ \langle v : w\rangle_{\Delta} \end{pmatrix},$$

which also has vanishing locked body angular velocity. On the other hand, for two vector fields $X = (g\xi, v) \in \mathcal{D}$ and $Y = g(\eta, w) \in \mathcal{D}$ having $\bar{\Omega} = 0$, $\bar{\Psi} = 0$ respectively, we have by Corollary 4.2 that

$$\langle X : Y\rangle = g \begin{pmatrix} -\tilde{I}^{-1}\tilde{\mathbb{L}}^s - \mathbb{A}\left(\langle v : w\rangle_{\tilde{\Delta}} - \tilde{\Delta}^{-1}\tilde{\mathbb{S}}^s\right) \\ \langle v : w\rangle_{\tilde{\Delta}} - \tilde{\Delta}^{-1}\tilde{\mathbb{S}}^s \end{pmatrix},$$

with $\tilde{\mathbb{L}}^s = I(\tilde{A}v, \gamma.w - [\cdot, \eta]) + I(\tilde{A}w, \gamma.v - [\cdot, \xi])$ and $\tilde{\mathbb{S}}^s = I(\tilde{A}w, \mathbb{B}(v, \cdot)) + I(\tilde{A}v, \mathbb{B}(w, \cdot))$.

Notice also that the tests we have obtained here for principal fiber bundles are the natural extension of the results developed in [7] for mechanical control systems on Lie groups. The major difference is that on Lie groups, G-invariance implies that the tests are expressed in $\mathfrak{g}$ in a *purely algebraic* way, whereas on principal fiber bundles we have to take into account the role of the shape space M and therefore *differentiation* is still required.

Another interesting aspect for this kind of mechanical control systems is the adaptation of the concept of **weak controllability** for kinematic systems defined in [15]. This notion essentially means controllability in the fiber, without regards to the intermediate or final positions of the shape variables. This type of controllability is meaningful for locomotion systems, where the group elements correspond to positions and orientation (and therefore are the most interesting variables to control) and one really does not care about the shapes the system is describing. In the following, we discuss it for the second order dynamical problems we are considering.

Assume then that we are dealing with a *trivial* principal fiber bundle, that is, the decomposition $Q = G \times M$ holds globally. Let V^τ denote any subset of Q such that $\tau(V^\tau)$ is an open subset of G, where $\tau : Q \equiv G \times M \longrightarrow G$ denotes the natural projection. Let $q_0 = (r_0, g_0)$ and $U \subset Q$ as before. Then we have

Definition 5.8. *The system (5.1) is* **locally fiber configuration accessible (LFCA) at** $q_0 \in Q$ *if there exists* $T > 0$ *such that* $\mathcal{R}_Q^U(q_0, \leq t)$ *contains a non-empty subset* V^τ *of* Q*, for all neighbourhoods* U *of* q_0 *and all* $0 \leq t \leq T$*. If this holds for any* $q_0 \in Q$ *then the system is called* **locally fiber configuration accessible**.

Definition 5.9. *The system (5.1) is* **small-time locally fiber configuration controllable (STLFCC) at** $q_0 \in Q$ *if there exists* $T > 0$ *such that* $\mathcal{R}_Q^U(q_0, \leq t)$ *contains a non-empty subset* V^τ *of* Q *such that* $g_0 \in \tau(V^\tau)$*, for all neighbourhoods* U *of* q_0 *and all* $0 \leq t \leq T$*. If this holds for any* $q_0 \in Q$ *then the system is called* **small-time locally fiber configuration controllable**.

From the discussion above, one can prove the following:

Theorem 5.10. *Let the mechanical control system (5.1) be invariant under* G*.*

(i) The system is LFCA at $q = (r, g)$ *if* $\tau_* \overline{Lie}(\overline{Sym}(\mathfrak{B}))_{(r,e)} = \mathfrak{g}$.

(ii) Suppose that the system is LFCA at q *and that the projection through* τ *of every bad symmetric product* P *at* q *in* $\mathfrak{B}$*,* $\tau_* P$*, can be written as a linear combination of projections through* τ *of good symmetric products at* q *of lower degree than* P*. Then (5.1) is STLFCC at* q*.*

Proof. Along the zero section of TQ, $q \longmapsto 0_q$, we have that the decomposition $T_{0_q}TQ \equiv T_qQ \oplus T_qQ$ holds, where the first factor corresponds to configurations and the second one to velocities. Then, from [21], we know that the accessibility distribution $\mathcal{C}$ corresponding to the full control system (that is, considering as states both the configurations and the velocities) can be decomposed as $\mathcal{C}_{0_q} = \mathcal{C}_{hor}(q) \oplus \mathcal{C}_{ver}(q)$, with $\mathcal{C}_{hor}(q) = \overline{Lie}(\overline{Sym}(\mathcal{Y}))_q$ and $\mathcal{C}_{ver}(q) = \overline{Sym}(\mathcal{Y})_q$. If $\tau_* \overline{Lie}(\overline{Sym}(\mathfrak{B}))_{(r,e)} = \mathfrak{g}$, we can conclude that $T_gG \subset \mathcal{C}_{hor}(q)$ and hence the system (5.1) is LFCA at q. The other claim follows from the invariance of the system and Sussmann's result in [32]. $\square$

6. Examples.

6.1. The blimp. Consider a rigid body moving in $SE(2)$ with a thruster to adjust its pose (see Figure 6.1). The original motivation for this problem is the blimp system developed by Zhang and Ostrowski [35] restricted to the vertical plane. The control inputs are the thruster force F^1 and a torque F^2 that actuates its orientation with respect to the body axis $\{X^b, Y^b\}$. The acting point of the thruster is assumed to be located along the body's long axis, at a distance h from the center of mass.

The configuration of the blimp is determined by a tuple (x, y, θ, γ), where (x, y) is the position of the center of mass, θ is the orientation of the blimp with respect to the fixed basis $\{X^f, Y^f\}$ and γ denotes the orientation of the thrust with respect to the body basis $\{X^b, Y^b\}$. The configuration manifold is then $Q = SE(2) \times \mathbb{S}^1$.

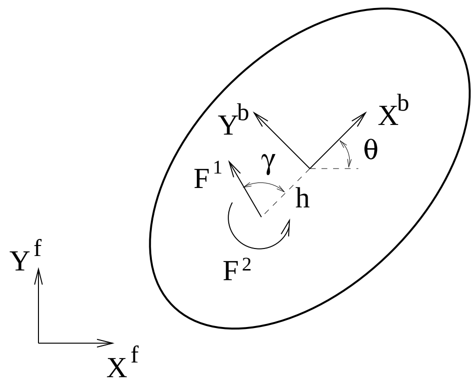

Fig. 6.1. *A planar blimp with rotating thruster*

For simplicity, we assume the thruster is massless. Then, the Riemannian metric of the system is

$$\mathcal{G} = m(dx \otimes dx + dy \otimes dy) + (J_1 + J_2) d\theta \otimes d\theta + J_2 d\gamma \otimes d\gamma + J_2(d\theta \otimes d\gamma + d\gamma \otimes d\theta)\,,$$

where m denotes the mass of the blimp, J_1 is its moment of inertia and J_2 is the inertia of the thruster. The Lagrangian of the system is the kinetic energy associated to this metric, that is

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} J_1 \dot{\theta}^2 + \frac{1}{2} J_2 (\dot{\gamma} + \dot{\theta})^2\,.$$

Finally, the input forces are given by

$$F^1 = \cos(\theta + \gamma) dx + \sin(\theta + \gamma) dy - h \sin\gamma d\theta\,, \quad F^2 = d\gamma\,.$$

The corresponding input vector fields can be computed to be

$$\begin{aligned} Y_1 &= \frac{1}{m} \cos(\theta + \gamma) \frac{\partial}{\partial x} + \frac{1}{m} \sin(\theta + \gamma) \frac{\partial}{\partial y} - \frac{h}{J_1} \sin\gamma \frac{\partial}{\partial \theta} + \frac{h}{J_1} \sin\gamma \frac{\partial}{\partial \gamma}\,, \\ Y_2 &= -\frac{1}{J_1} \frac{\partial}{\partial \theta} + \frac{J_1 + J_2}{J_1 J_2} \frac{\partial}{\partial \gamma}\,. \end{aligned}$$

This simple mechanical control system is invariant under the left multiplication of the Lie group $G = SE(2)$,

$$\begin{array}{rcl} \Phi: \quad G \times Q & \longrightarrow & Q \\ ((a, b, \alpha), (x, y, \theta, \gamma)) & \longmapsto & (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \theta + \alpha, \gamma)\,. \end{array}$$

The reduced representation of the input vector fields at $\mathfrak{g} \times TM$ is given by

$$B_1 = \frac{1}{m}\cos\gamma\frac{\partial}{\partial x} + \frac{1}{m}\sin\gamma\frac{\partial}{\partial y} - \frac{h}{J_1}\sin\gamma\frac{\partial}{\partial\theta} + \frac{h}{J_1}\sin\gamma\frac{\partial}{\partial\gamma}\,,$$
$$B_2 = -\frac{1}{J_1}\frac{\partial}{\partial\theta} + \frac{J_1+J_2}{J_1J_2}\frac{\partial}{\partial\gamma}\,.$$

Let $\{e_x, e_y, e_\theta\}$ be the canonical basis of the Lie algebra $se(2)$. Given the metric $\mathcal{G}$, we can readily identify from its reduced form (2.8) the local form of the mechanical connection and the inertia tensor

$$I = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_1+J_2 \end{pmatrix}\,, \quad A = \begin{pmatrix} 0 \\ 0 \\ \frac{J_2}{J_1+J_2} \end{pmatrix}\,.$$

As the shape space is one-dimensional and B^{mech} is skew-symmetric, we deduce that $B^{mech} = 0$. Some computations yields that DI also vanishes. Consequently, $\mathbb{S} = 0$. In addition,

$$D(I\eta)(\xi, v) = \left\{ \begin{pmatrix} m\frac{\partial\eta^1}{\partial\gamma} \\ m\frac{\partial\eta^2}{\partial\gamma} \\ (J_1+J_2)\frac{\partial\eta^3}{\partial\gamma} \end{pmatrix}^T + \frac{mJ_2}{J_1+J_2}\begin{pmatrix} \eta^2 \\ -\eta^1 \\ 0 \end{pmatrix}^T \right\} \begin{pmatrix} \xi^1 \\ \xi^2 \\ \xi^3 \end{pmatrix} v\,.$$

Note that $\Delta = \left(\frac{J_1J_2}{J_1+J_2}\right)$. Therefore, the Christoffel symbols of ∇^Δ vanish and

$$\langle v : w\rangle_\Delta = \frac{\partial w}{\partial\gamma}v + \frac{\partial v}{\partial\gamma}w\,.$$

Summing up, we conclude that $\mathbb{L}^s$ in eq. 3.3 for $X = (g\xi, v)$, $Y = (g\eta, w)$ is given by

$$\mathbb{L}^s = -\begin{pmatrix} m\left(\frac{\partial\xi^1}{\partial\gamma}w + \frac{\partial\eta^1}{\partial\gamma}v\right) \\ m\left(\frac{\partial\xi^2}{\partial\gamma}w + \frac{\partial\eta^2}{\partial\gamma}v\right) \\ (J_1+J_2)\left(\frac{\partial\xi^3}{\partial\gamma}w + \frac{\partial\eta^3}{\partial\gamma}v\right) \end{pmatrix} - \frac{mJ_2}{J_1+J_2}\begin{pmatrix} \Omega^2w + \Psi^2v \\ -\Omega^1w - \Psi^1v \\ 0 \end{pmatrix}$$

Following Lemma 3.1 we can compute the symmetric product defined by ∇^I

$$\langle\Omega : \Psi\rangle_I = \begin{pmatrix} -\Omega^2\Psi^3 - \Omega^3\Psi^2 \\ \Omega^1\Psi^3 + \Omega^3\Psi^1 \\ 0 \end{pmatrix}\,.$$

With these ingredients, we are now ready to perform the controllability analysis along the lines of Section 5. Consider the following symmetric brackets

$$\langle B_1 : B_1\rangle_{\mathcal{G}} = \tfrac{h^2}{J_1^2}\sin(2\gamma)\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}\,, \qquad \langle B_1 : B_2\rangle_{\mathcal{G}} = \begin{pmatrix} -\frac{1}{mJ_2}\sin\gamma \\ \frac{1}{mJ_2}\cos\gamma \\ -\frac{h(J_1+J_2)}{J_1^2J_2}\cos\gamma \\ \frac{h(J_1+J_2)}{J_1^2J_2}\cos\gamma \end{pmatrix}\,,$$

$$\langle B_2 : B_2\rangle_{\mathcal{G}} = 0\,, \qquad \langle B_2 : \langle B_1 : B_1\rangle_{\mathcal{G}}\rangle_{\mathcal{G}} = 2\tfrac{h^2}{J_1^2}\tfrac{J_1+J_2}{J_1J_2}\cos(2\gamma)\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}\,.$$

Note that $\{B_1, B_2, \langle B_1 : B_2\rangle_{\mathcal{G}}, \langle B_1 : B_1\rangle_{\mathcal{G}}, \langle B_2 : \langle B_1 : B_1\rangle_{\mathcal{G}}\rangle_{\mathcal{G}}\}$ span $\mathfrak{g} \times TM$ at every point (e, r) and hence the system is locally configuration accessible. However, the bad bracket $\langle B_1 : B_1\rangle_{\mathcal{G}}$ is not in general a linear combination of the lower order good brackets B_1 and B_2. Therefore, we can not conclude that the system is STLCC. In any case, at $\gamma = 0$, we have that $\langle B_1 : B_1\rangle_{\mathcal{G}}(e, 0) = 0$ and we can assure that the system is small time locally configuration controllable at $(g, 0)$, for all $g \in G$. However, if we restrict our attention to fiber configuration controllability, we can see that $\tau_*\langle B_1 : B_1\rangle_{\mathcal{G}} \in \mathrm{span}\{\tau_* B_2\}$ and therefore the blimp is STLFCC. Physically, fiber controllability corresponds to the fact that we can use the shape torque to control the orientation angle θ to a desired value, but not θ and γ simultaneously.

6.2. The snakeboard. The Snakeboard [22, 28] is a variant of the skateboard in which the passive wheel assemblies can pivot freely about a vertical axis. By coupling the twisting of the human torso with the appropriate turning of the wheels (where the turning is controlled by the rider's foot movement), the rider can generate a snake-like locomotion pattern without having to kick off the ground.

A simplified model is shown in Figure 6.2. We assume that the front and rear wheel axles move through equal and opposite rotations. This is based on the observations of human snakeboard riders who use roughly the same phase relationship. A momentum wheel rotates about a vertical axis through the center of mass, simulating the motion of a human torso.

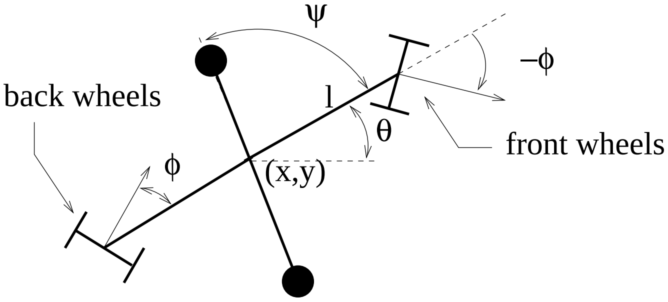

Fig. 6.2. *The snakeboard model*

The position and orientation of the snakeboard is determined by the coordinates of the center of mass (x, y) and its orientation θ. The shape variables are (ψ, ϕ), so the configuration space is $Q = SE(2) \times \mathbb{S}^1 \times \mathbb{S}^1$. The physical parameters for the system are the mass of the board, m; the inertia of the rotor, J_r; the inertia of the wheels about the vertical axes, J_w; and the half-length of the board, l. A key component of the snakeboard is the use of the rotor inertia to drive the body. To keep the rotor and body inertias on similar scales, we make the additional simplifying assumption [4, 30] that the inertias of the system satisfy $J + J_r + 2J_w = ml^2$.

The Riemannian metric of this system is

$$\begin{aligned}\mathcal{G} = m(dx \otimes dx + dy \otimes dy) + (J + J_r + 2J_w)d\theta \otimes d\theta \\ + J_r(d\theta \otimes d\psi + d\psi \otimes d\theta) + J_r d\psi \otimes d\psi + 2J_w d\phi \otimes d\phi\,.\end{aligned}$$

The control torques are assumed to be applied to the rotation of the wheels and the rotor. Hence, we consider

$$F^1 = d\psi\,, \quad F^2 = d\phi\,.$$

Observe that the snakeboard is an example of the type of dynamic locomotion systems we mentioned earlier, since the set of control inputs fully actuate the shape variables, $\mathrm{span}\{F^1, F^2\} = T^*M$. The corresponding input vector fields via the diffeomorphism $\sharp_{\mathcal{G}}$ are

$$Y_1 = -\frac{1}{J+2J_w}\frac{\partial}{\partial\theta} + \frac{ml^2}{J_r(J+2J_w)}\frac{\partial}{\partial\psi}\,, \quad Y_2 = \frac{1}{2J_w}\frac{\partial}{\partial\phi}\,.$$

The assumption that the wheels do not slip in the direction of the wheels axles yields the following two nonholonomic constraints

$$\begin{aligned}
-\sin(\theta+\phi)\dot{x} + \cos(\theta+\phi)\dot{y} - l\cos\phi\,\dot{\theta} &= 0\,,\\
-\sin(\theta-\phi)\dot{x} + \cos(\theta-\phi)\dot{y} + l\cos\phi\,\dot{\theta} &= 0\,.
\end{aligned}$$

A quick set of calculations shows that this constrained mechanical system is invariant under the left multiplication in the Lie group $SE(2)$. The intersection $S = \mathcal{V}\cap\mathcal{D}$ can be seen to be one-dimensional. Moreover, we have that $S_{(e,r)} = e_{1Q}$, where $e_1 = l\cos\phi e_x - \sin\phi e_\theta$. We complete the basis by adding two elements generating $S^\perp_{(e,r)}$

$$e_2 = e_y\,, \quad e_3 = \frac{1}{l}\tan\phi e_x + e_\theta\,.$$

Taking into account the discussion of the preceding sections, we can identify the following elements

$$I = \begin{pmatrix} m & 0 & 0\\ 0 & m & 0\\ 0 & 0 & ml^2 \end{pmatrix}\,, \quad A = \begin{pmatrix} 0 & 0\\ 0 & 0\\ \frac{J_r}{ml^2} & 0 \end{pmatrix}\,, \quad \mathbb{A} = \begin{pmatrix} -\frac{J_r}{2ml}\sin(2\phi) & 0\\ 0 & 0\\ \frac{J_r}{ml^2}\sin^2\phi & 0 \end{pmatrix}\,.$$

Our choice of generators of $\mathcal{D}_{(e,r)}$ following Section 2.3 is then

$$\mathcal{D}_{(r,e)} = \mathrm{span}\left\{\frac{\partial}{\partial\psi} + \frac{J_r}{ml^2}\sin\phi e_1, \frac{\partial}{\partial\phi}, e_1\right\}\,.$$

The projections to $\mathcal{D}$ of the input vector fields under the orthogonal decomposition $TQ = \mathcal{D}\oplus\mathcal{D}^\perp$ are

$$\begin{aligned}
\mathcal{B}_1 = \mathcal{P}(B_1) &= \frac{ml^2}{J_r(ml^2 - J_r\sin^2\phi)}\left(\frac{\partial}{\partial\psi} + \frac{J_r}{ml^2}\sin\phi e_1\right)\,,\\
\mathcal{B}_2 = \mathcal{P}(B_2) &= \frac{1}{2J_w}\frac{\partial}{\partial\phi}\,.
\end{aligned}$$

For the sake of completeness, we have computed the terms $\tilde{\mathbb{L}}^s \in \mathfrak{g}^{\mathcal{D}}$ and $\tilde{\mathbb{S}}^s$ in eq. 4.4 for any G-invariant vector fields $X = (g\xi, v)$ and $Y = (g\eta, w)$, although we already pointed out in Section 5 (cf. Theorem 5.7) that the amount of calculations for the controllability tests can be made quite lighter taking into account the fact that $\bar{\Omega}_i = 0$ for $\mathcal{B}_i$, $i = 1, 2$.

$$\begin{aligned}
\tilde{\mathbb{L}}^s &= -\Bigg\{ml\cos\phi\sum_{\alpha=1}^{2}\left(\frac{\partial\bar{\Psi}^1}{\partial r^\alpha}v^\alpha + \frac{\partial\bar{\Omega}^1}{\partial r^\alpha}w^\alpha\right) - ml^2\sin\phi\sum_{\alpha=1}^{2}\left(\frac{\partial\bar{\Psi}^3}{\partial r^\alpha}v^\alpha + \frac{\partial\bar{\Omega}^3}{\partial r^\alpha}w^\alpha\right)\\
&\quad + J_r\cos\phi(v^2w^1 + w^2v^1) + \frac{J_r}{2l}\sin(2\phi)\sin\phi(w^1\xi^2 + v^1\eta^2)\Bigg\}\,e_1^*\,,\\
\tilde{\mathbb{S}}^s &= -\frac{J_r^2}{2ml^2}\sin(2\phi)\begin{pmatrix} w^1v^2 + w^2v^1\\ -2v^1w^1 \end{pmatrix}\,.
\end{aligned}$$

The controllability analysis yields the following results at the point $\mathbf{0} = (0,0,0,0,0)$

$$\begin{aligned} \langle \mathcal{B}_1 : \mathcal{B}_1 \rangle(\mathbf{0}) &= 0\,, & \langle \mathcal{B}_1 : \mathcal{B}_2 \rangle(\mathbf{0}) &= \tfrac{1}{2J_w ml} e_x\,, \\ \langle \mathcal{B}_2 : \mathcal{B}_2 \rangle(\mathbf{0}) &= 0\,, & [\mathcal{B}_1, \mathcal{B}_2](\mathbf{0}) &= \tfrac{1}{2J_w ml} e_x\,, \\ [\mathcal{B}_2, [\mathcal{B}_1, \mathcal{B}_2]](\mathbf{0}) &= -\tfrac{1}{2J_w^2 ml^2} e_\theta, & [\mathcal{B}_2, [\mathcal{B}_1, [\mathcal{B}_2, [\mathcal{B}_1, \mathcal{B}_2]]]](\mathbf{0}) &= -\tfrac{1}{4J_w^3 m^2 l^3} e_y - \tfrac{1}{2J_w^3 m^2 l^4} e_\theta. \end{aligned}$$

Note that $\{\mathcal{B}_1, \mathcal{B}_2, \langle \mathcal{B}_1 : \mathcal{B}_2 \rangle, [\mathcal{B}_2, [\mathcal{B}_1, \mathcal{B}_2]], [\mathcal{B}_2, [\mathcal{B}_1, [\mathcal{B}_2, [\mathcal{B}_1, \mathcal{B}_2]]]]\}$ span $\mathfrak{g} \times T_{(0,0)}M$, so the system is locally configuration accessible at $(g, 0, 0)$, for all $g \in G$. Moreover, the bad symmetric products $\langle \mathcal{B}_1 : \mathcal{B}_1 \rangle$ and $\langle \mathcal{B}_2 : \mathcal{B}_2 \rangle$ vanish at $\mathbf{0}$ and the remaining ones are either 0 or in $\operatorname{span}\{\mathcal{B}_2(\mathbf{0}), \langle \mathcal{B}_1 : \mathcal{B}_2 \rangle(\mathbf{0})\}$, so we can conclude that the snakeboard is STLCC at $(g, 0, 0)$, for all $g \in G$.

7. Conclusions. We have developed a new set of tools that can be used in the study of simple mechanical systems evolving on principal fiber bundles. These tools have direct application to a large class of problems in robotic locomotion. Using the Lie group symmetries that are associated with an invariant mechanical system on a principal fiber bundle, we have given an explicit formulation of the affine connection in terms of the mechanical and nonholonomic connections, for unconstrained and constrained systems, respectively. This formulation can greatly reduce the amount of computation necessary to derive controllability tests, as was observed during the analysis of the snakeboard system.

We have defined a new notion of fiber configuration controllability, which can be used to focus the analysis on the important components of a locomotion system, namely the fiber variables of position and orientation. The tools developed in this paper were applied to two systems, the planar rigid body and the snakeboard robot.

We are currently working on applying these tools to motion planning for such systems (see [26]). Recent work by Bullo et al. [6] suggests an excellent avenue for applying the affine connection in a motion planning framework. We will also explore connections of these tools to steering for dynamic systems, as for example was done by Ostrowski [29] using the reduced equations for the snakeboard [31].

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