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An Introduction to Mechanics with Symmetry

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Abstract. Symmetries are known to be an important instrument to reduce and integrate the equations of motion in Classical Mechanics through Noether theorems which provide conserved quantities. In this paper, some different types of infinitesimal symmetries are reviewed, from the almost classical results for unconstrained systems to the more recent research in nonholonomic mechanics. The case of vakonomic dynamics with some applications to optimal control theory is also discussed.

1. Introduction

Symmetry has played an important role in the study of differential equations and mechanics since the early times of Newton, Euler, Lagrange, Jacobi, Hamilton and others. This interaction has continued to be present in the work of Noether, Poincaré, Routh, Appell, Cartan, etc. More recently, the geometrical description of mechanical systems has allowed important new developments like the symplectic reduction procedure by Meyer [59] and Marsden and Weinstein [56]. General references are the following books: Abraham and Marsden [1], Binz *et al* [5], Liberman and Marle [47], Marsden [52], Marsden and Ratiu [54], and Olver [62]. An excellent and recent review paper is [23].

The main interest in the identification of symmetries for mechanical systems is that they permit to reduce the number of degrees of freedom, i.e. to pass to a reduced phase space of less dimension. This procedure facilitates in principle the integration of the equations of motion. Even more, the existence of conserved quantities leads to development of new numerical integrators preserving them.

The presence of symmetries and associated conserved quantities is not only an interesting matter for mechanics, but, they play a crucial role in economics (for instance, in economical growth theories [68]). In [42] it is proved that some conservation laws in economics can be obtained by a direct application of standard results in mechanics.

In this survey we will only consider some particular aspects of the theory, and by no means this should be understood as a full view of the state of the art. Indeed, our main interest is towards the classification of infinitesimal symmetries for lagrangian systems, which may be found in the seminal papers by Prince [65, 66]. Noether, Lie or Cartan symmetries have also been discussed in more general contexts: time-dependent lagrangian systems [66], singular lagrangian systems [13, 21, 27, 29, 40] and higher-order lagrangian systems [39, 45] (see [20, 30, 44, 55, 50, 51, 70] for general treatments). Generalizations of the concept of infinitesimal symmetries have been widely studied by Sarlet *et al* (see [67] for instance). An analysis of symmetries in hamiltonian mechanics with important applications to the study of integrable hamiltonian systems was developed by Bolsinov and Fomenko [10, 11, 12].

Special attention is devoted to mechanical systems subjected to nonholonomic constraints. These systems frequently exhibit symmetries and the issue has been extensively studied in the classical books of Mechanics such as the ones by Appell [3], Painlevé [63], Pars [64], Whittaker [74] and others. However, up to our knowledge, the first modern treatment is due to Koiller [33] (see also [73]), who considered the particular case of Čaplygin systems. Koiller' s work has produced a renewed interest in the subject, and together with the work by Bloch *et al* [9] can be considered as the introduction of the modern geometrical methods in nonholonomic mechanics. Special attention deserves the fundamental book by Neimark and Fuffaev [60]. Classical results as the Routh method, Appell equations or Maggi method have been reinterpreted in this new language. The use of symmetries to obtain constants of the motion is really a complicated matter in the presence of nonholonomic constraints, since a symmetry for the lagrangian does not produce automatically a conserved quantity. The theory requires the introduction of a convenient nonholonomic momentum map [9, 15, 16, 17, 24]. Here we put the emphasis on our own approach to nonholonomic mechanics, (inspired on a preliminar work by Cariñena and Rañada [22]), based on finding explicitly the constrained vector field through a convenient decomposition of the tangent bundle of the phase space. Alternative approaches together with the corresponding reduction procedures can be found in [4, 35, 48].

The last part of the survey is devoted to the so-called vakonomic dynamics, a terminology coined by Kozlov [2]. Vakonomic dynamics is obtained from a lagrangian function subjected to nonholonomic constraints, but instead of using the d'Alembert principle, one uses a variational principle, that is, one looks for the extremals among all the curves satisfying the constraints. This does not provide the correct equations of motion for nonholonomic systems (except if the constraints are integrable) but interesting problems of optimal control theory fit nicely in the theory [6, 7, 8, 46]. Indeed, an optimal control problem can be interpreted as a vakonomic system where the lagrangian is the cost function, and the constraints are just the control equation [58]. The study of symmetries for such a systems were initiated in [37, 58], and the results by van der Schaft *et al* [61, 71] can be recovered in a very natural way.

The paper is structured as follows. Section 2 is devoted to the discussion of symmetries in the general context of hamiltonian systems. The main goal is the symplectic reduction procedure when the action of a Lie group of symmetries possesses an equivariant momentum map. In Section 3 we apply the results of the precedent section to the case of mechanical systems. A classification of infinitesimal symmetries is also given in both settings, lagrangian and hamiltonian. Section 4 deals with hamiltonian systems subjected to constraints. Here, the constrained momentum map is introduced and a momentum equation is presented. The results are particularized in the next section for mechanical systems, where Čaplygin systems deserve special arrention. The case of singular lagrangian systems is briefly analyzed in Section 6. Finally, Section 7 discusses the so-called vakonomic dynamics; in addition, an application to optimal control theory is given.

2. Symmetries in hamiltonian systems

Let (M, ω, H) be a hamiltonian system, that is, the geometry is provided by a symplectic manifold M with symplectic form ω, and the dynamics is derived from a hamiltonian function H on M.

As is well known, the symplectic form ω defines two vector bundle isomorphisms $\flat : TM \longrightarrow T^*M$ by $\flat(X) = i_X\omega$, and its inverse $\sharp = \flat^{-1}$. These mappings are usually called the *musical isomorphisms*.

Let us recall that given two functions f, g on M, their Poisson bracket is defined as

$$\{f, g\} = \omega(X_f, X_g)$$

where X_f denotes the hamiltonian vector field defined by $X_f = \sharp(df)$, i.e. $i_{X_f}\omega = df$.

The dynamics of the hamiltonian system (M, ω, H) is then given by the hamiltonian vector field X_H. Darboux theorem ensures the existence of canonical coordinates (q^A, p_A) on M such that $\omega = dq^A \wedge dp_A$. Then we have

$$X_H = \frac{\partial H}{\partial p_A}\frac{\partial}{\partial q^A} - \frac{\partial H}{\partial q^A}\frac{\partial}{\partial p_A}.$$

Therefore, the integral curves $(q^A(t), p_A(t))$ of X_H satisfy the Hamilton equations

$$\frac{dq^A}{dt} = \frac{\partial H}{\partial p_A} \quad , \quad \frac{dp_A}{dt} = -\frac{\partial H}{\partial q^A}. \tag{1}$$

A *conserved quantity* (also called *first integral* or *constant of the motion*) of the hamiltonian system (M, ω, H) is a function f on M which remains constant along the solutions of the Hamilton equations (1), that is, $X_H(f) = 0$.

A *symmetry* of the hamiltonian system (M, ω, H) is a symplectomorphism ϕ (i.e. a diffeomorphism $\phi : M \longrightarrow M$ such that $\phi^*\omega = \omega$) preserving the hamiltonian function, that is, $H \circ \phi = H$. This means that the hamiltonian vector field X_H is ϕ-related with itself and, therefore, a symmetry preserves the Hamilton equations.

A vector field X on M is an *infinitesimal symmetry* of the hamiltonian system (M, ω, H) if its flow consists of symmetries, or, equivalently: $\mathcal{L}_X\omega = 0$ and $X(H) = 0$.

Since $\mathcal{L}_X\omega = 0$, we deduce that X is a local hamiltonian vector field, i.e. $i_X\omega$ is a closed 1-form. Assume that X is a global hamiltonian vector field, say $X = X_f$ for some function f on M. Then $X_f(H) = -X_H(f) = -\{f, H\} = 0$, which implies that f is a conserved quantity.

Moreover, we have the following.

Theorem 2.1. *A function f on M is a conserved quantity if and only if X_f preserves the hamiltonian H.*

Remark 2.2. If the symplectic form is exact, say $\omega = -d\theta$, then the condition $i_X\omega = df$ is equivalent to the following one:

$$\mathcal{L}_X\theta = -df + d\langle\theta, X\rangle$$

(see the next section).

An important case is that given by a Lie group acting symplectically on the phase space of a hamiltonian system.

Let $\Phi : G \times M \longrightarrow M$ be a symplectic action of a Lie group G on a symplectic manifold (M, ω), i.e. the mapping $\Phi_g : M \longrightarrow M$ defined by $\Phi_g(x) = \Phi(g, x) = gx$, is a symplectomorphism, for all $g \in G$. For any element ξ in the Lie algebra $\mathfrak{g}$ of G, we denote by ξ_M the fundamental vector field which generates the flow $\Phi_{\exp(t\xi)}$.

Definition 2.3. A momentum map for this symplectic action is a mapping $J : M \longrightarrow \mathfrak{g}^*$ such that the fundamental vector field ξ_M generated by an element $\xi \in \mathfrak{g}$ through the action Φ is a hamiltonian vector field. That is,

$$i_{\xi_M}\omega = d(\widehat{J\xi})$$

where the function $\widehat{J\xi}$ is defined by $\widehat{J\xi}(x) = \langle J(x), \xi\rangle$, for all $x \in M$. The momentum map J is said to be equivariant if

$$J \circ \Phi_g = Ad^*_{g^{-1}} \circ J$$

for all $g \in G$, where $Ad^*_{g^{-1}}$ denotes the coadjoint representation of G.

Let us describe now how the dynamics enters into the picture.

Take a hamiltonian function $H : M \longrightarrow \mathbb{R}$ which is invariant by G. A direct computation shows that $X_H(J) = 0$, which proves that the functions $\widehat{J\xi}$ are conserved quantities.

Assume that $\mu \in \mathfrak{g}^*$ is a regular value of J, that is, the differential mapping of J is surjective at every point in $J^{-1}(\mu)$. This condition ensures that $J^{-1}(\mu)$ is a submanifold of M. If the action is supposed to be equivariant, then it restricts to an action $G_\mu \times J^{-1}(\mu) \longrightarrow J^{-1}(\mu)$, where $G_\mu = \{g \in G | Ad^*_{g^{-1}}(\mu) = \mu\}$ is the isotropy group of μ with respect to the coadjoint representation. If this action is in addition free and proper, we obtain a principal bundle $\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu = J^{-1}(\mu)/G_\mu$ with structure group G_μ. Moreover, if $i_\mu : J^{-1}(\mu) \longrightarrow M$ denotes the natural inclusion, then it can be proven that there exists a unique symplectic form ω_μ on M_μ such that $\pi_\mu^*\omega_\mu = i_\mu^*\omega$. (M_μ, ω_μ) is called the *reduced symplectic manifold*, and we have that $\dim M_\mu = \dim M - \dim G - \dim G_\mu$.

Suppose now that the hamiltonian H is G-invariant, then the restriction of X_H to $J^{-1}(\mu)$ will be tangent to $J^{-1}(\mu)$, and, since it is G_μ-invariant, it projects onto a vector field X_μ on M_μ. On the other hand, the restriction of H to $J^{-1}(\mu)$ also projects onto a function H_μ on M_μ. If X_{H_μ} denotes the hamiltonian vector field of H_μ with respect to the reduced symplectic form, we have $X_{H_\mu} = X_\mu$; X_{H_μ} is the *reduced dynamics*.

Remark 2.4. If $0 \in \mathfrak{g}^*$ is a regular value, then $G_\mu = G$ and $M_0 = J^{-1}(0)/G$. In this case, $\dim M_0 = \dim M - 2\dim G$.

After the integration of the reduced dynamics, the next step is the reconstruction of the original dynamics. This process involves two steps: a horizontal lifting with respect to some connection in the principal bundle $\pi_\mu : J^{-1}(\mu) \longrightarrow M_\mu$, and an integration on the Lie algebra $\mathfrak{g}_\mu$ of the isotropy group. We refer to [1, 53, 54] for details.

3. Symmetries in hamiltonian and lagrangian mechanics

3.1. Hamiltonian mechanics.

The phase space for mechanics is the cotangent bundle T^*Q of the configuration manifold Q, with canonical projection $\pi_Q : T^*Q \longrightarrow Q$. Here, T^*Q is equipped with its canonical symplectic form ω_Q. Let us recall that $\omega_Q = -d\lambda_Q$, where λ_Q is the *Liouville 1-form* on T^*Q defined by

$$\lambda_Q(\alpha_q)(X_{\alpha_q}) = \langle \alpha_q, T\pi_Q(\alpha_q)(X_{\alpha_q})\rangle$$

for all $\alpha_q \in T^*Q$, and for all $X_{\alpha_q} \in T_{\alpha_q}(T^*Q)$.

In bundle coordinates (q^A, p_A), these objects are expressed as

$$\lambda_Q = p_A dq^A \ , \ \omega_Q = dq^A \wedge dp_A.$$

Take a hamiltonian function H on T^*Q. We could consider only general infinitesimal symmetries of the hamiltonian system (T^*Q, ω_Q, H), but the bundle structure of T^*Q allows us also to consider point symmetries.

Definition 3.1. A vector field X on Q is a point symmetry for the hamiltonian system defined by H if the complete lift (which will be defined below) X^{C^*} of X to T^*Q is an infinitesimal symmetry of (T^*Q, ω_Q, H).

Denote by ιX the evaluation function of X, that is, $\iota X(q^A, p_A) = X^A p_A$, where $X = X^A \dfrac{\partial}{\partial q^A}$. Since

$$X^{C^*} = X^A \frac{\partial}{\partial q^A} - p_B \frac{\partial X^B}{\partial q^A}\frac{\partial}{\partial p_A}$$

we deduce that X^{C^*} is just the hamiltonian vector field for ιX, say $X^{C^*} = X_{\iota X}$. Therefore, if X is a point symmetry then ιX is a conserved quantity.

Let $\Phi : G \times Q \longrightarrow Q$ be an action of a Lie group G on Q. The lifted action $G \times T^*Q \longrightarrow T^*Q$ is given by $\tilde{\Phi}_g = (\Phi_g^{-1})^*$, for all $g \in G$.

Proposition 3.2. *The lifted action $\tilde{\Phi}$ is symplectic with respect to the canonical symplectic form ω_Q. In addition, it admits an equivariant momentum map $J : T^*Q \longrightarrow \mathfrak{g}^*$ defined by $\langle J(\alpha_q), \xi\rangle = \langle \alpha_q, (\xi_Q)(q)\rangle$, for all $\alpha_q \in T^*Q$ and for all $\xi \in \mathfrak{g}$.*

Remark 3.3. Notice that the result holds for any exact symplectic form $\omega = -d\theta$ provided that the potential 1-form θ is invariant with respect to the action of the Lie group.

Take now a regular value $\mu \in \mathfrak{g}^*$ of J, and apply the reduction procedure described in the former section to the present case. On the one hand, it is obtained a reduced symplectic manifold $(T^*Q)_\mu$ with reduced symplectic form ω_μ, and on the other hand, we have a principal bundle $\bar{\pi}_\mu : Q \longrightarrow Q_\mu = Q/G_\mu$ (assuming that the action of G on Q is free and proper). Let us briefly investigate the relations between $((T^*Q)_\mu, \omega_\mu)$ and $(T^*Q_\mu, \omega_{Q_\mu})$.

Consider γ a connection in the principal bundle $\bar{\pi}_\mu : Q \longrightarrow Q_\mu$. Denote by $\mu' \in \mathfrak{g}_\mu^*$ the restriction of $\mu : \mathfrak{g} \longrightarrow \mathbb{R}$ to the subalgebra $\mathfrak{g}_\mu$. If $\mathrm{curv}(\gamma)$ denotes the curvature 2-form of γ, we obtain a real 2-form $\langle \mu', \mathrm{curv}(\gamma)\rangle$ on Q which projects onto a 2-form on Q_μ; its pull-back to T^*Q_μ is denoted by B. A direct computation shows that $\omega_{T^*Q_\mu} - B$ is still a symplectic form on T^*Q_μ.

The following result was proved in [36] (see also [1, 53]).

Theorem 3.4. *$((T^*Q)_\mu, \omega_\mu)$ is symplectically embedded in $(T^*Q_\mu, \omega_{T^*Q_\mu} - B)$. The image is a vector subbundle over Q_μ. This embedding is one-to-one if and only if $\mathfrak{g} = \mathfrak{g}_\mu$.*

The above embedding is just the so-called *shifting trick*. In many mechanical systems, there is a nice method to choose an appropiate connection γ (the mechanical connection) which facilitates the task of the reconstruction (see [53]).

An interesting remark is that the above procedure describes a particle moving in an (in general nonabelian) Yang-Mills field. The reduced phase space $(T^*Q)_\mu$ is described in more detail by a fibre bundle over $T^*(Q/G)$ with typical fibre the coadjoint orbit $G\dot{\mu}$ (see [32] for details).

3.2. Lagrangian mechanics.

The lagrangian description of a mechanical system involves a lagrangian function $L = L(q^A, \dot{q}^A)$, which is defined as $L = T - V$, where T is the kinetic energy and V denotes the potential energy. Here, q^A are the generalized coordinates, and $\dot{q}^A$ the generalized velocities, where A ranges from 1 to n.

The Hamilton principle states that the motion from the point q_0 to another point q_1 is done along the extremals of the functional

$$\mathcal{J}(c(t)) = \int_0^1 L(c(t), \dot{c}(t))dt\,,$$

defined by L on the set of twice piecewise differentiable curves $c(t)$ joining $c(0) = q_0$ and $c(1) = q_1$.

Now, the use of the fundamental lemma of the Calculus of Variations implies that the equations of motion are just the so-called Euler-Lagrange equations for L:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = 0 \quad , 1 \leq A \leq n. \tag{2}$$

The coordinates (q^A) live on the configuration manifold Q (usually a Riemannian manifold) and the lagrangian is then a function $L : TQ \longrightarrow \mathbb{R}$ defined on the tangent bundle TQ of Q with canonical projection $\tau_Q : TQ \longrightarrow Q$. TQ is the space of velocities with bundle coordinates $(q^A, \dot{q}^A)$.

There are two canonical geometric objects on TQ:

- the *vertical endomorphism* $S = dq^A \otimes \dfrac{\partial}{\partial \dot{q}^A}$;
- and the *Liouville vector field* $\Delta = \dot{q}^A \dfrac{\partial}{\partial \dot{q}^A}$.

Using L we can construct the *Poincaré-Cartan* 2-form

$$\omega_L = -d\theta_L \ , \theta_L = S^*(dL).$$

In bundle coordinates we have

$$\omega_L = dq^A \wedge d\hat{p}_A$$

where $\hat{p}_A = \dfrac{\partial L}{\partial \dot{q}^A}$, $A = 1, \ldots, n$ are the *generalized momenta*.

A fact that is immediately observed is that ω_L is symplectic if and only if the lagrangian L is *regular*, i.e. the Hessian matrix $\left(\dfrac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}\right)$ is nonsingular. This will be the case in Classical Mechanics, since $L = T - \tau_Q^* V$, where T is the kinetic energy of a Riemannian metric on Q, and $V : Q \longrightarrow \mathbb{R}$. From now on, we will assume the regularity of L, or, in order words, that (TQ, ω_L) is a symplectic manifold. The musical isomorphisms will be denoted by $\flat_L : T(TQ) \longrightarrow T^*(TQ)$ and $\sharp_L = (\flat_L)^{-1} : T^*(TQ) \longrightarrow T(TQ)$.

Consider the *energy function* $E_L = \Delta(L) - L$, which in local coordinates reads as $E_L = \dot{q}^A \hat{p}_A - L$, and let Γ_L be the hamiltonian vector field for E_L with respect to ω_L, that is, we have

$$i_{\Gamma_L} \omega_L = dE_L. \tag{3}$$

Γ_L is called the *Euler-Lagrange vector field*, and enjoys the following properties:

- Γ_L is a SODE (Second order differential equation), i.e. $S(\Gamma_L) = \Delta$.
- The solutions of Γ_L, that is, the projections onto Q of the integral curves of Γ_L are the solutions of the Euler-Lagrange equations.

Next, we shall study symmetries for a lagrangian system on the configuration manifold Q with a regular lagrangian function $L : TQ \longrightarrow \mathbb{R}$.

First of all, let us recall that a conserved quantity will be a function $f : TQ \longrightarrow \mathbb{R}$ which remains constant along the solutions of equations (3), in other words, $\Gamma_L(f) = 0$.

We can introduce the following definition.

Definition 3.5. 1. A symmetry of L is a diffeomorphism $\Phi : Q \longrightarrow Q$ such that $L \circ T\Phi = L$.

2. An infinitesimal symmetry of L is a vector field X on Q whose flow consists of symmetries of L, or, equivalently, $X^C(L) = 0$.

Here X^C denotes the *complete* or *tangent* lift of X to the tangent bundle TQ. If the local expression of X is $X = X^A \dfrac{\partial}{\partial q^A}$, then we obtain

$$X^C = X^A \frac{\partial}{\partial q^A} + \dot{q}^B \frac{\partial X^A}{\partial q^B} \frac{\partial}{\partial \dot{q}^A}.$$

Also, the *vertical lift* of X to TQ is the vector field

$$X^V = X^A \frac{\partial}{\partial \dot{q}^A}.$$

Of course, we have $S(X^C) = X^V$.

Theorem 3.6. *(Noether theorem) If X is an infinitesimal symmetry of L then $X^V(L)$ is a conserved quantity.*

Proof. One can follow the discussion in [44], although a simpler proof can be derived as follows.

First, notice that X^C leaves invariant E_L, θ_L and ω_L. Therefore, $\mathcal{L}_{X^C}\theta_L = 0$, which implies

$$i_{X^C}\omega_L = d\langle\theta_L, X^C\rangle.$$

Thus, X^C is a hamiltonian vector field for the function $\langle\theta_L, X^C\rangle = X^V(L)$. Finally, the result follows from Theorem 2.1. ■

Theorem 3.6 can be generalized considering more general types of infinitesimal symmetries.

Definition 3.7.

1. A Noether symmetry of the lagrangian system with lagrangian L is a vector field X on Q such that $\mathcal{L}_{X^C}\theta_L = dF$, for some function F on TQ and, in addition, $X^C(E_L) = 0$.
2. A Cartan symmetry of the lagrangian system with lagrangian L is a vector field $\tilde{X}$ on TQ such that $\mathcal{L}_{\tilde{X}}\theta_L = dF$, for some function F on TQ and, in addition, $\tilde{X}(E_L) = 0$.

Remark 3.8.

1. Notice that if X is a Noether symmetry for L then the associated function F is constant along the fibers so that $F = f^V$, for some function $f : Q \longrightarrow \mathbb{R}$. Here f^V denotes the vertical lift of f to TQ, that is, $f^V = \tau_Q^* f$.
2. Alternatively, a Noether symmetry can be defined as a vector field X on Q such that $X^C(L) = F^C$. This condition is dynamically interpreted saying that $X^C(L)$ is a total derivative. Here F^C denotes the complete lift of F to TQ, that is, $F^C(X_q) = \langle dF(q), X_q\rangle$, for all $X_q \in T_qQ$. In local coordinates we have $F^C = \dot{q}^A \dfrac{\partial F}{\partial q^A}$.
3. Every infinitesimal symmetry of L is a Noether symmetry.
4. The complete lift of a Noether symmetry is a Cartan symmetry.

Proposition 3.9. *Let X (resp. $\tilde{X}$) a Noether (resp. Cartan) symmetry for L. Then, the function $X^V(L) - F$ (resp. $\theta_L(\tilde{X}) - F$) is a conserved quantity.*

Remark 3.10. According to Theorem 2.1 all the conserved quantities are determined by Cartan symmetries.

Next, let $\Phi : G \times Q \longrightarrow Q$ be an action of a Lie group G on the configuration manifold Q. This action can be lifted in a canonical way to an action $\tilde{\Phi} : G \times TQ \longrightarrow TQ$ by $\tilde{\Phi}_g = T\Phi_g$, for all $g \in G$.

¿From now on, we shall assume that L is G-invariant, i.e. $L \circ T\Phi_g = L$, for all $g \in G$. Then, E_L, θ_L and ω_L are also G-invariant and, a fortiori Γ_L is so too. Under these hypotheses, we know that there is an equivariant momentum map

$$J : TQ \longrightarrow \mathfrak{g}^*$$

defined by

$$\langle J(X_q), \xi \rangle = \langle \theta_L(X_q), \xi_{TQ}(X_q) \rangle$$

for all $X_q \in TQ$ and for all $\xi \in \mathfrak{g}$.

If $\mu \in \mathfrak{g}^*$ is a regular value of J, we can develop the symplectic reduction procedure to get a reduced symplectic manifold $(TQ)_\mu$ with reduced symplectic form ω_μ. In [53] it was proved a similar result to Theorem 3.4. Indeed, the only difference is that in order to perform the shifting trick one needs to invoke the existence of an appropiate vector field (we refer to [53] for a detailed discussion).

The lagrangian and hamiltonian descriptions of mechanics are related by means of the *Legendre transformation* $FL : TQ \longrightarrow T^*Q$ which is defined as follows:

$$FL(X_q)(Y_q) = \langle \theta_L(X_q), \tilde{Y}_q \rangle$$

where $\tilde{Y}_q \in T_{X_q}(TQ)$ projects onto Y_q. FL is a fibred morphism over Q. In bundle coordinates, we have

$$FL(q^A, \dot{q}^A) = (q^A, \hat{p}_A). \tag{4}$$

A direct computation shows that

$$FL^*\lambda_Q = \theta_L \ , \ FL^*\omega_Q = \omega_L .$$

¿From (4) we deduce that L is regular if and only if FL is a local diffeomorphism. Since we have assumed that $L = T - \tau_Q^* V$, where T is the kinetic energy of a Riemannian metric $\langle \, , \, \rangle$ on Q with local components g_{AB}, we conclude that FL becomes $FL(q^A, \dot{q}^A) = (q^A, g_{AB}\dot{q}^B)$, that is, a linear mapping lowering indexes, and then is a vector bundle isomorphism over Q.

The hamiltonian energy on T^*Q is defined as $H = E_L \circ FL$, and using the canonical symplectic form we obtain the hamiltonian vector field X_H. The relation between both dynamics is a consequence of the fact that Γ_L and X_H are FL-related, which allows us to transform the Euler-Lagrange equations into the Hamiltonian ones.

Accordingly, the infinitesimal symmetries on both sides are related.

Proposition 3.11. *1. Let X be a vector field on Q. Then X is an infinitesimal symmetry for L if and only if it is a point symmetry for the hamiltonian system.*

2. Let $\tilde{X}$ be a vector field on TQ. Then $\tilde{Y}$ is a Cartan symmetry for L if and only if $FL(\tilde{Y})$ is an infinitesimal symmetry for the hamiltonian system (T^*Q, ω_Q, H).

Remark 3.12. More classes of infinitesimal symmetries were introduced in [65, 66], even for time-dependent lagrangian systems. A careful discussion comparing all these classes of symmetries can be found in [39, 44].

4. Symmetries in constrained hamiltonian systems

The results of this section are mainly contained in [16, 17].

Consider a symplectic manifold (M, ω), a hamiltonian function $H : M \longrightarrow \mathbb{R}$, a submanifold N of M, and a distribution F on M along N, i.e. a vector subbundle F of $TM_{|_N}$. N will represent the *constraint submanifold*, and the annihilator F^o of F will be the *bundle of constraint forces*.

Set up the equations

$$(i_X\omega - dH)_{|_N} \in F^o \ , \ X \in TN \tag{5}$$

and let us try to find their solutions.

Denote by $F^\perp$ the symplectic complement of F in TM with respect to ω. Notice that $\flat(F^\perp) = F^o$.

Assume that $\operatorname{rank} F = \dim N$, and, in addition $TN \cap F^\perp = 0$. Then we have

$$TM_{|_N} = TN \oplus F^\perp$$

Proposition 4.1. *Under the above conditions, equations (5) have a unique solution X_{const}.*

Indeed, if we denote by

$$\mathcal{P} : TM_{|_N} \longrightarrow TN \ , \mathcal{Q} : TM_{|_N} \longrightarrow F^\perp$$

the complementary projectors associated to the above decomposition, we have

$$X_{const} = \mathcal{P}(X_H).$$

Take a local basis $\{\mu_i \mid i = 1, \ldots, m\}$ of F^o, and a family of independent constraint functions $\{\Phi_i | i = 1, \ldots, m\}$ defining N. If $Z_i = \sharp(\mu_i)$ we have $X_{const} = X_H + \lambda^i Z_i$. The Lagrange multipliers λ^i are determined using the tangency condition, i.e.

$$0 = X_{const}(\Phi_i) = X_H(\Phi_i) + \lambda^j Z_j(\Phi_i)$$

which implies

$$\lambda^i = -\mathcal{C}^{ij} X_H(\Phi_j)$$

where $(\mathcal{C}^{ij})$ is the inverse matrix of $(Z_i(\Phi_j))$. (Notice that the matrix $(Z_i(\Phi_j))$ is non-singular since $TM_{|_N} = TN \oplus F^\perp$).

Next, we will consider symmetries of such a constrained hamiltonian system.

Definition 4.2. An infinitesimal symmetry of (M, ω, H, N, F) is a vector field Y such that

- $\mathcal{L}_Y\omega = 0$, $Y(H) = 0$;
- Y is tangent to N;
- Y preserves F, i.e. $\mathcal{L}_Y U \in F$, for all section U of F;
- $Y \in F$.

Proposition 4.3. *Let Y be an infinitesimal symmetry of (M,ω,H,N,F). If Y is globally hamiltonian, i.e. $Y = X_f$ for some function f on M, then f is a conserved quantity.*

Proof. Let $X_{const} = X_H + \lambda^i Z_i$ be the solution of equations (5). Since $Y = X_f$ we have that

$$X_{const}(f) = X_H(f) + \lambda^i Z_i(f)$$

vanishes, since $X_H(f) = -X_f(H) = 0$ and $Z_i(f) = -\mu_i(X_f) = 0$. ■

Next, let $G \times M \longrightarrow M$ be a symplectic action of a Lie group G on (M,ω) such that H, N and F are G-invariant. We suppose that the action is free and proper in such a way that we have a well-defined principal G-bundle $\rho : M \longrightarrow \bar{M} = M/G$. Of course, the restriction of the action to N is still free and proper, and we obtain a new principal G-bundle $\rho_N : N \longrightarrow \bar{N} = N/G$. (Obviously, $\bar{N}$ is a submanifold of $\bar{M}$). Since H is G-invariant, it defines a function $\bar{H}$ on $\bar{M}$. Also, X_{const} is G-invariant. (Notice that if $X_{const} \in F$ then $X_{const}(H) = 0$, that is, H is a conserved quantity, but this property does not hold in general).

Denote by $\mathcal{V}$ the subbundle of TM whose fibers are the tangent spaces to the G-orbits, that is, $\mathcal{V}_x = T_x(Gx)$ for all $x \in M$. In other words, we have $\mathcal{V} = \ker T\rho$. Of course, the restriction of $\mathcal{V}$ to N is the vector subbundle $\mathcal{V}_{|N} = \ker T\rho_N$. It should be noticed that the elements in $\mathcal{V}$ do not automatically satisfy the fourth requirement in Definition 4.2 so that we are obliged to consider those symmetries compatible with the "constraint forces" represented by the vector bundle F^o.

Define now a vector subbundle U of $TN_{|N}$ as the symplectic complement of $\mathcal{V} \cap F$ into $F \cap TN$. A direct computation shows that

$$U = (F \cap TN) \cap (\mathcal{V} \cap F)^{\perp}.$$

U is again G-invariant, and it projects onto a vector subbundle $\bar{U}$ of $T\bar{M}_{|\bar{N}}$. If we denote by ω_U the restriction of ω to U, then it projects onto a 2-form $\omega_{\bar{U}}$ on $\bar{U}$. Similarly, the restriction of dH to U, denoted by $d_U H$ pushed down to a 1-form on $\bar{U}$ which is just the restriction $d_{\bar{U}}\bar{H}$ of $d\bar{H}$ to $\bar{U}$.

The following result was proved in [17], and generalizes a result by Bates and Śniatycki [4].

Proposition 4.4. *Assume that $X_{const} \in F$. Then the projection $\bar{X}$ of X_{const} onto $\bar{N}$ is a section of $\bar{U}$ satisfying the equation*

$$i_{\bar{X}}\omega_{\bar{U}} = d_{\bar{U}}\bar{H}. \tag{6}$$

Assume that our symplectic form ω is exact, i.e. $\omega = -d\theta$, such that θ is G-invariant. As we have shown in Section 2, in this case there is a well-defined momentum map $J : M \longrightarrow \mathfrak{g}^*$. However, this momentum map is not adequate for our purposes: indeed, it does not provide conserved quantities in general.

Therefore, we have to "correct" J in order to get a momentum map which takes into account the constraints.

Firstly, for each $x \in N$ we put

$$\mathfrak{g}^x = \{\xi \in \mathfrak{g} | \xi_N(x) \in F_x\}.$$

(Here ξ_N denotes the fundamental vector field generated by the action on N, and, obviously, it is the restriction of ξ_M to N.)

Next, we take the following generalized vector bundles over N:

$$\mathfrak{g}^F = \cup_{x \in N} \mathfrak{g}^x.$$

Finally, we define the constrained momentum map $J^c : M \longrightarrow (\mathfrak{g}^F)^*$, by

$$\langle J^c(x), \xi \rangle = \langle J(x), \xi \rangle$$

for all $\xi \in \mathfrak{g}^x$ and for all $x \in M$, where $(\mathfrak{g}^F)^*$ denotes the dual vector bundle of $\mathfrak{g}^F$ (defined fiber by fiber).

Consider now a smooth section $\bar{\xi}$ of the vector bundle $\mathfrak{g}^F$; it defines a function $\widehat{J^c\bar{\xi}}$ on N according to $\widehat{J^c\bar{\xi}}(x) = \langle J^c(x), \bar{\xi}(x) \rangle$. In addition, we can construct a vector field Ξ on N as

$$\Xi(x) = (\bar{\xi}(x))_N(x)$$

for all $x \in N$. The following result was proved in [17], extending the work in [9] for nonholonomic systems.

Proposition 4.5. *(The momentum equation) We have*

$$X_{const}(\widehat{J^c\bar{\xi}}) = (\mathcal{L}_\Xi \theta)(X_{const}).$$

According to the relative position between the constraints and the reaction forces, we can consider the following classification of constrained systems.

1. The *purely kinematic* case: $\mathcal{V}_x \cap F_x = \{0\}$ and $T_xN = \mathcal{V}_x + (F_x \cap T_xN)$, for all $x \in N$.
2. The case of *horizontal symmetries*: $\mathcal{V}_x \subset F_x$, for all $x \in N$.
3. The *general* case: $\{0\} \subsetneq \mathcal{V}_x \cap F_x \subsetneq \mathcal{V}_x$, for all $x \in N$.

4.1. The purely kinematic case.

In this case we will have

$$T_xN = U_x \oplus \mathcal{V}_x,$$

for all $x \in N$, since $U_x = \mathcal{V}_x \cap T_xN$. As U is G-invariant we have obtained a principal connection Γ in the principal bundle $\rho_N : N \longrightarrow \bar{N}$ with horizontal distribution U. Denote by $\mathbf{h}$ the horizontal projector of Γ and by $R = \frac{1}{2}[\mathbf{h}, \mathbf{h}]$ its curvature. If we assume that $X_{const} \in F$ (and hence $X_{const} \in U$), and taking into account that $\bar{U} = T\bar{N}$ we conclude that equation (6) reads now as

$$i_{\bar{X}} \bar{\omega} = d\bar{H}$$

since $\bar{\omega} = \omega_{\bar{U}}$ is a genuine 2-form on $\bar{N}$, and $d_{\bar{U}}\bar{H} = d\bar{H}$.

If $\omega = -d\theta$, we can go further into the reduction by defining a 1-form θ_c on N as follows:

$$\theta_c = i_{X_{const}}(\mathbf{h}^* d\theta' - d\mathbf{h}^*\theta')$$

where θ' is the restriction of θ to N. The 1-forms $\mathbf{h}^*\theta'$ and θ_c projects onto $\bar{\theta}$ and $\bar{\theta}_c$ respectively. We have the following.

Proposition 4.6. *The projection $\bar{X}$ satisfies the reduced equation*

$$i_{\bar{X}}(-d\bar{\theta}) = d\bar{H} + \bar{\theta}_c.$$

Remark 4.7. Notice that the momentum map does not play any role in this case. However, one can "project" the constrained system eliminating all the symmetries.

4.2. The case of horizontal symmetries.

Assume as above that $\omega = -d\theta$ and that θ is G-invariant. In this case, the constrained momentum map is just the usual one, say $J^c = J$. Therefore, any element $\xi \in \mathfrak{g}$ provides a horizontal symmetry, that is, the function $\widehat{J\xi}$ is a conserved quantity:

$$X_{const}(\widehat{J\xi}) = 0.$$

For this reason, such a vector field ξ_N is called a *horizontal symmetry*.

If $\mu \in \mathfrak{g}^*$ is a regular value, $N' = N \cap J^{-1}(\mu)$ is a clean intersection, and, in addition, G_μ acts freely and properly on it, then we obtain a submanifold $N_\mu = (N \cap J^{-1}(\mu))/G_\mu$. We also have a distribution F' on M along N by taking $F'_{x'} = T_{x'}(J^{-1}(\mu)) \cap F_{x'}$, for all $x' \in N \cap J^{-1}(\mu)$. F' is G_μ-invariant and projects onto a subbundle F_μ of TM_μ along N_μ. We deduce the following.

Proposition 4.8. *X_{const} induces a vector field X_μ on N_μ such that*

$$(i_{X_\mu}\omega_\mu - dH_\mu)_{|N_\mu} \in F_\mu^o\,,\ X_\mu \in TN_\mu$$

4.3. The general case.

The general case is really complicated and requires a careful analysis of the constrained momentum mapping. For more details we refer to [9, 24].

5. Non-holonomic constraints

In this section we particularize some of the results of the precedent one (see [41] for more details).

Consider a mechanical system given by a lagrangian function $L : TQ \longrightarrow \mathbb{R}$ and subjected to nonholonomic constraints given by a submanifold N of TQ. Locally, N is defined as the zero set of a family of independent constraint functions $\Phi_i(q^A, \dot{q}^A) = 0$, $i = 1, \ldots, m \leq n = \dim Q$. We assume that $\tau_Q(N) = Q$ which means that all the configurations are available.

The constraints are said to be linear if N is linear, that is, if N is the total space of a vector subbundle of TQ, or, equivalently, it defines a distribution on the configuration manifold Q. Otherwise, they are called nonlinear. A particular and interesting case occurs when the constraints are affine in the velocities. For simplicity, we will assume along this section that the constraints are linear. Therefore the submanifold N can be locally described by a family of independent linear constraints of the form

$$\Phi_i(q^A, \dot{q}^A) = \mu_{iA}(q)\dot{q}^A.$$

The bundle of constraint forces is now $F^o = S^*(TN^o)$, where TN^o denotes the annihilator of the tangent bundle TN of N. The assumption $\operatorname{rank} F = \dim N$ is obviously fullfilled, and, in addition, we always have $TN \cap F^\perp = 0$. To prove the last equality, take a local basis $\{S^*(d\Phi_i) \mid i = 1, \dots, m\}$ of $S^*(TN^o)$, and put $Z_i = \sharp_L(d\Phi_i)$. If $Z = a^i Z_i$ is tangent to N, then $a^i Z_i(\Phi_j) = 0$. If we denote $\mathcal{C}_{ij} = Z_i(\Phi_j)$ we have

$$C_{ij} = -g^{AB}\mu_{iA}\mu_{jB}$$

which shows that the matrix $(\mathcal{C}_{ij})$ is nonsingular. (Here (g^{AB}) are the components of the inverse matrix of (g_{AB}) corresponding to the Riemannian metric $\langle\, , \rangle$ on Q definig the kinetic energy.) Thus, $a^i = 0$, for all i is the only solution of the above system, and therefore we have a Whitney sum

$$T(TQ)_{|_N} = TN \oplus F^\perp.$$

The solution of the constrained dynamics is now denoted by Γ_{const}, and, as we know, it is the solution of the equations

$$(i_X\omega_L - dE_L)_{|_N} \in S^*(TN^o)\ ,\ X \in TN \tag{7}$$

As before, if we denote by

$$\mathcal{P} : T(TQ)_{|_N} \longrightarrow TN\ ,\ \mathcal{Q} : T(TQ)_{|_N} \longrightarrow F^\perp$$

the complementary projectors associated to the above decomposition, we have

$$\Gamma_{const} = \mathcal{P}(\Gamma_L).$$

Assume now that a Lie group acts on Q preserving L and the constraint submanifold N. Therefore, the action also preserves the bundle of reaction forces $S^*(TN^o)$.

Introduce the following generalized vector bundle over Q:

$$\mathfrak{g}^N = \cup_{q\in Q}\mathfrak{g}^q$$

where

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} | \xi_Q(q) \in N_q\},$$

N_q denoting the vector subspace defined by the distribution N. In [9, 17] the reduction procedure is studied using the nonholonomic momentum map

$$J^{nh} : TQ \longrightarrow (\mathfrak{g}^N)^*$$

defined by $\langle J^{nh}(X_q), \xi\rangle = \langle \theta_L(X_q), \xi_{TQ}(X_q)\rangle$, for all $X_q \in TQ$ and $\xi \in \mathfrak{g}^q$.

We shall discuss in the following the relation between the vector bundles $\mathfrak{g}^N \longrightarrow Q$ and $\mathfrak{g}^F \longrightarrow N$. Since the constraints are linear, a direct computation shows that

$$\mathfrak{g}^q = \mathfrak{g}^{X_q},$$

for all $X_q \in N \cap T_q Q$. Notice that this does not hold for arbitrary constraints (see [24]).

A global section $\bar{\xi}$ of $\mathfrak{g}^N \longrightarrow Q$ induces a vector field Ξ on Q as follows:

$$\Xi(q) = ((\bar{\xi}(q))_Q)(q),$$

for all $q \in Q$.

The momentum equation reads now as [15]

$$\Gamma_{const}(\widehat{J^{nh}\bar{\xi}}) = \Xi^C(L),$$

with the obvious notations.

The classification introduced in the above section is applied to the present case with some particularities that are carefully analyzed in [9, 24]. In addition, in [24] a reduction procedure using directly the momentum map is given.

To end this section, we shall discuss a particular and important case that fits nicely in the purely kinematic case.

Consider a nonholonomic lagrangian system given by the following data:

- A regular lagrangian $L : TQ \longrightarrow \mathbb{R}$;
- $\sigma : Q \longrightarrow Q/G$ is a principal bundle with structure group G;
- The constraints are given by the horizontal distribution H of a principal connection in $\sigma : Q \longrightarrow Q/G$. γ.

Such a system is called a *Čaplygin mechanical system* [33]. However, Čaplygin only studied abelian principal bundles, and it was Voronec who extended the study for arbitrary Lie groups (see [60]).

First of all, one can project L onto a new lagrangian function $L^* : T(Q/G) \longrightarrow \mathbb{R}$ by

$$L^*(Y) = L(Y_q^H),$$

for $Y \in T_{\sigma(q)}Y$, where Y_q^H denotes the horizontal lift of Y at q with respect to the connection γ.

The system fullfills the conditions of the purely kinematic case. Moreover, we have $\bar{N} = T(Q/G)$, and Proposition 4.6 implies that the projection $\bar{\Gamma}$ of the constrained dynamics satisfies the equation

$$i_{\bar{X}}\omega_{L^*} = dE_{L^*} + \bar{\theta}_c.$$

In addition the 1-form $\bar{\theta}_c$ satisfies the condition $i_{\Gamma^*}\bar{\theta}_c = 0$, for any SODE Γ^* on $T(Q/G)$ which says that it is a gyroscopic term.

Remark 5.1. A further study on Čaplygin systems can be found in [14], where the existence of invariant measures of the reduced system is investigated.

Remark 5.2. Since our lagrangian system is hyperregular, i.e. the Legendre transformation $FL : TQ \longrightarrow T^*Q$ is a diffeomorphism, then the results on TQ can be carried out word by word in the cotangent bundle. This will not happen with singular lagrangians.

Remark 5.3. The case of nonlinear constraints is more involved. First of all, there does not exist as unanimous convention about the principle to adopt to derive the equations of motion [49]. Furthermore, it is really difficult to find nice examples. We refer to [43, 69] for details.

Remark 5.4. The usual Poisson bracket induced from ω_L is not enough to obtain the evolution of an observable in the presence of nonholonomic constraints. A seminal work is due to van der Schaft and Maschke [72] who modified the canonical bracket to obtain the so-called nonholonomic bracket. A full understanding of the role played by this nonholonomic bracket (showing its relationship with the Dirac bracket) was given in [19, 18] (see also [35]). The nonholonomic bracket can also be used to reduce the equations of motion.

6. Singular lagrangian systems

The formalism to treat singular lagrangian systems is due to Dirac (see [26]). The study of these systems is motivated by problems appearing in Field theories (theory of monopoles, the relativistic string, interactions between relativistic particles, electromagnetism). The goal of Dirac was to develop a quantization method for such singular systems. An intrinsic geometric formulation and generalization of Dirac approach is provided by the so-called presymplectic constraint algorithm, developed by Gotay and Nester [31].

The lagrangian L is singular or degenerate if the Hessian matrix $\left(\frac{\partial^2 L}{\partial \dot{q}^A \partial \dot{q}^B}\right)$ is singular. In such a case the equations of motion

$$i_X \omega_L = dE_L \tag{8}$$

do not have a solution in general, and if such a solution exists, it will not be unique. However, for quantization purposes, it is more convenient to work on the symplectic ambient provided by the Hamiltonian formulation. For a singular lagrangian system, the Legendre transformation $FL : TQ \to T^*Q$ is not a diffeomorphism. Yet, the regularity condition can be conveniently weakened: we will assume that FL is almost-regular, i.e. $FL(TQ) = M_1$ is a submanifold of T^*Q and FL is a submersion onto M_1 with connected fibers. In such a case, $h_1 : M_1 \to \mathbb{R}$ given by $h_1 \circ FL_{|M_1} = E_L$ is well defined. If we denote by ω_1 the restriction of the canonical symplectic form ω_Q to M_1, then the equation

$$i_X \omega_1 = dh_1 \tag{9}$$

is the hamiltonian counterpart of equation (8).

Equation (9) will not have in general a global well defined solution on M_1. Applying the presymplectic algorithm [31] a sequence of submanifolds

$$\ldots \hookrightarrow M_k \hookrightarrow \ldots \hookrightarrow M_3 \hookrightarrow M_2 \hookrightarrow M_1 .$$

is generated. If the algorithm stabilizes at some step k and a final constraint submanifold $M_k = M_f$ exists, then there is at least a solution on M_f of the following equation

$$(i_X\omega_1 = dh_1)_{|M_f} . \tag{10}$$

Taking $H_1 : T^*Q \to \mathbb{R}$ an arbitrary extension of the hamiltonian $h_1 : M_1 \to \mathbb{R}$ then Equation (10) is equivalent to

$$(i_X\omega_Q = dH_1)_{|M_f} \in (TM_1^o)_{|M_f} \tag{11}$$

(See [17]). Therefore, the study of the symmetries of singular lagrangian systems fits nicely in the case of constrained hamiltonian systems previously analyzed in Section 4, by considering the hamiltonian system $(M = T^*Q, \omega = \omega_Q, H = H_1)$ with constraint submanifold $N = M_f$ and $F = (TM_1)_{|M_f}$ as the bundle of constraint forces.

Remark 6.1. An extensive study of symmetries for singular lagrangian systems can be found in [40], where the symmetries are identified with respect to the whole family of solutions, not only with respect to a particular one.

7. Symmetries in vakonomic dynamics

Unlike what happens in nonholonomic mechanics, in vakonomic mechanics the equations of motion for systems in the presence of nonholonomic constraints are obtained through the application of a variational principle.

The starting point is a n-dimensional configuration manifold Q, a $(2n-m)$-dimensional constraint submanifold N of TQ, locally defined by the independent equations $\phi_\alpha = 0$, $1 \leq \alpha \leq m$, and a Lagrangian $L : TQ \longrightarrow \mathbb{R}$. If (q^A) are coordinates in Q with $(q^A, \dot{q}^A)$ the induced coordinates in TQ, then we write $L = L(q^A, \dot{q}^A)$. In general, N will be a subbundle of TQ over Q. In the discussion that follows, we will treat the case of a subbundle of TQ, $N \equiv D$, defined by a distribution D on Q, or the case of an affine subbundle N modelled on the vector subbundle D of TQ with an additional vector field γ on Q.

Now, according to the theory of the calculus of variations, we extremize the functional

$$\mathcal{J}(c(t)) = \int_0^1 L(c(t), \dot{c}(t))dt ,$$

defined by L on the set $\tilde{\mathcal{C}}^2(q_0, q_1)$ of twice piecewise differentiable curves $c(t)$ joining $c(0) = q_0$ and $c(1) = q_1$, and satisfying the constraints $\dot{c}(t) \in N_{c(t)}$, $\forall t$.

Using the Lagrange Multipliers Theorem in an infinite dimensional context, one can see that c is an admissible motion if and only if there exist m functions

$\{\lambda^\alpha : [0,1] \longrightarrow \mathbb{R}; \alpha = 1, \dots m\}$ such that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^A}\right) - \frac{\partial L}{\partial q^A} = -\lambda^\alpha \left(\frac{d}{dt}\left(\frac{\partial \phi_\alpha}{\partial \dot{q}^A}\right) - \frac{\partial \phi_\alpha}{\partial q^A}\right) - \frac{d\lambda^\alpha}{dt}\frac{\partial \phi_\alpha}{\partial \dot{q}^A}\,,\; 1 \leq A \leq n\,, \tag{12}$$

and $\phi_\alpha(q^A, \dot{q}^A) = 0,\; 1 \leq \alpha \leq m$ (see [2, 6, 46]). ¿From (12) we deduce that a curve $c = (q^A(t))$ in $\tilde{\mathcal{C}}^2(q_0, q_1)$ is a solution of the vakonomic equations if and only if there exist local functions $\lambda^1, \dots, \lambda^m$ on $\mathbb{R}$ such that $\bar{c}(t) = (q^A(t), \lambda^\alpha(t))$ is an extremal for the extended Lagrangian

$$\mathcal{L} : T(Q \times \mathbb{R}^m) \longrightarrow \mathbb{R}\,,\;\; \mathcal{L} = L + \lambda^\alpha \phi_\alpha\,,$$

i.e. it satisfies the Euler-Lagrange equations

$$\begin{cases} \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{q}^A}\right) - \dfrac{\partial \mathcal{L}}{\partial q^A} = 0\,,\; 1 \leq A \leq n\,, \\[2ex] \dfrac{d}{dt}\left(\dfrac{\partial \mathcal{L}}{\partial \dot{\lambda}^\alpha}\right) - \dfrac{\partial \mathcal{L}}{\partial \lambda^\alpha} \equiv \phi_\alpha(q^A, \dot{q}^A) = 0\,,\; 1 \leq \alpha \leq m\,, \end{cases}$$

(see [2, 46, 57] for details).

¿From the extended Lagrangian $\mathcal{L}$ we can construct the system $(TP, \omega_{\mathcal{L}}, dE_{\mathcal{L}})$, where $\omega_{\mathcal{L}} = -d\theta_{\mathcal{L}}$ is the Poincaré-Cartan 2-form, $\theta_{\mathcal{L}} = S^*(d\mathcal{L})$ is the Poincaré-Cartan 1-form, and $S = \dfrac{\partial}{\partial \dot{q}^A} \otimes dq^A + \dfrac{\partial}{\partial \dot{\lambda}^\alpha} \otimes d\lambda^\alpha$ is the vertical endomorphism on TP. $E_{\mathcal{L}} = \Delta\mathcal{L} - \mathcal{L}$ is the energy associated with $\mathcal{L}$, where $\Delta = \dot{q}^A \dfrac{\partial}{\partial \dot{q}^A} + \dot{\lambda}^\alpha \dfrac{\partial}{\partial \dot{\lambda}^\alpha}$ is the Liouville vector field on TP. We will assume that $(TP, \omega_{\mathcal{L}}, dE_{\mathcal{L}})$ is presymplectic, i.e. $\omega_{\mathcal{L}}$ has constant rank.

Within this geometrical framework we can pose the equation

$$i_\Gamma \omega_{\mathcal{L}} = dE_{\mathcal{L}}\,, \tag{13}$$

which codifies the vakonomic equations (12). Γ will be a second order differential equation to be found on TP whose integral curves $(q^A(t), \lambda^\alpha(t))$ are the vakonomic solutions $(q^A(t))$ together with the corresponding Lagrange multipliers $(\lambda^\alpha(t))$.

Equation (13) will not have in general a global well defined solution on TP. Applying the Gotay-Nester algorithm [31] for presymplectic systems, one can generate a sequence of submanifolds

$$\dots \hookrightarrow P_k \hookrightarrow \dots \hookrightarrow P_3 \hookrightarrow P_2 \hookrightarrow P_1\,.$$

In the most favourable case, the algorithm will stabilize at some step k and a final constraint submanifold $P_k = P_f$ will exist where there is a well defined vector field $\Gamma \in TP_f$ such that

$$(i_\Gamma \omega_{\mathcal{L}} = dE_{\mathcal{L}})_{|P_f}\,. \tag{14}$$

Remark 7.1. In [25] an alternative geometric description of vakonomic dynamics in the extended phase space $T^*Q \times_Q M$ was described. This formulation was used to compare the solutions of vakonomic dynamics with the solutions of nonholonomic mechanics for nonholonomic Lagrangian systems (see also [28, 38, 46]).

7.1. Symmetries.

Here, we study the general symmetries of a vakonomic system (L, N) on TQ and their relationship with the symmetries of $\mathcal{L}$, an extended Lagrangian of the form $\mathcal{L} = L + \lambda^\alpha \phi_\alpha$, where $\{\phi_\alpha \,|\, 1 \leq \alpha \leq m\}$ is a global basis of functions defining the submanifold of constraints N.

We will consider that N is an affine subbundle of TQ modelled on the vector subbundle $D \subseteq TQ$, $\dim N = \dim D = 2n - m$, with an additional vector field $\gamma : Q \longrightarrow TQ$. We say that a vector X_q is in M_q if and only if $X_q - \gamma_q \in D_q$.

In the sequel, $\pi_1 : Q \times \mathbb{R}^m \longrightarrow Q$ and $\pi_2 : Q \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ will denote the projections onto each factor of $Q \times \mathbb{R}^m$.

Definition 7.2. ([2]) A vakonomic symmetry for (L, N) will be a diffeomorphism $s : Q \longrightarrow Q$ such that Ts leaves N and $L_{|N}$ invariant, i.e. $Ts(N) = N$ and $(L \circ Ts)_{|N} = L_{|N}$.

In this way, we assure that the constrained variational problem is preserved by s and so will be its solutions. The condition $(L \circ Ts)_{|N} = L_{|N}$ is equivalent to say that there exist m local functions $\{\lambda_0^\alpha : TQ \longrightarrow \mathbb{R} \,|\, 1 \leq \alpha \leq m\}$ such that $L \circ Ts - L = \lambda_0^\alpha \phi_\alpha$, while the condition $Ts(N) = N$ means that the transformation $\{\phi_\alpha \circ Ts = \bar{\phi}_\alpha \,|\, 1 \leq \alpha \leq m\}$, gives rise to new independent constraint functions defining N.

In fact, if $D \subseteq TQ$ is the distribution modelling N we have

1. Since $\gamma \in N$, then, $Ts(q)(\gamma_q) \in N_{s(q)}$, or, equivalently, $Ts(q)(\gamma_q) - \gamma_{s(q)} \in D_q$.

2. Let X_q be a vector in D_q. Then, $X_q + \gamma_q \in N_q$ and $Ts(X_q) + Ts(\gamma_q) \in N_{s(q)}$. But again, this means $Ts(X_q) + Ts(\gamma_q) - \gamma_{s(q)} \in D_{s(q)}$. By (i) we deduce that $Ts(X_q) \in D_{s(q)}$.

That is, D is invariant by Ts and so is D^o. Thus, a basis $\{\omega_\alpha\}_{\alpha=1}^m$ of D^o is transformed into a new one $T^*s(\omega_\alpha) = \bar{\omega}_\alpha$. Then, there exists a non-singular matrix-valued function on Q, $\Lambda_\alpha^\beta(s) : Q \longrightarrow GL(m, \mathbb{R})$ such that $\bar{\omega}_\alpha = \Lambda_\alpha^\beta(s) \omega_\beta$.

When $L \circ Ts = L$, we can extend the diffeomorphism s to $P = Q \times \mathbb{R}^m$ as

$$\begin{array}{rccc} \bar{s} : & P & \longrightarrow & P \\ & (q^A, \lambda^\alpha) & \longmapsto & (s^A(q), \bar{\Lambda}_\beta^\alpha(s)(q)\lambda^\beta)\,, \end{array}$$

so that $T\bar{s}$ leaves $\mathcal{L}$ invariant. This systematic procedure allows us to translate all the vakonomic symmetries s into symmetries $\bar{s}$ of the singular Lagrangian $\mathcal{L}$ and viceversa, we can recover them just by projecting $\bar{s}$ to Q.

Definition 7.3. A vakonomic infinitesimal symmetry (from now on VIS) for (L, N) is a vector field X on Q such that its complete lift X^C to TQ is tangent to N and satisfies $X^C(L)_{|N} = X^C_{|N}(L_{|N}) = 0$.

In other words, X is a VIS if and only if its flow, $\{s_t : Q \longrightarrow Q\}$, consists of vakonomic symmetries for (L, N). For simplicity, we will consider those X such that $X^C(L) = 0$. Then, from a VIS X on Q, one can obtain an infinitesimal symmetry of $\mathcal{L}$, a vector field $\bar{X}$ on P. Indeed, since $X^C(L) = 0$ and $X^C(\phi_\alpha)_{|_N} = 0$, $\forall 1 \leq \alpha \leq m$, the flow of X, $\{s_t\}$ verifies for all $-\epsilon < t < \epsilon$,

$$L \circ Ts_t = L\,, \quad \phi_\alpha \circ Ts_t = \bar{\phi}_{\alpha t} = \Lambda^\alpha_\beta(t)\phi_\alpha\,.$$

We can then define the one-parameter group

$$\begin{array}{rccc} \bar{s}_t : & P & \longrightarrow & P \\ & (q,\lambda) & \longmapsto & (s_t(q), \Lambda^\alpha_\beta(-t)(q)\lambda^\beta)\,, \end{array}$$

and take the vector field whose flow is given by $\{\bar{s}_t\}$ (its infinitesimal generator), $\bar{X} \equiv X + Y_{\mathcal{L}}$, where

$$Y_{\mathcal{L}} = \left(\frac{d}{dt}_{|t=0}\Lambda^\alpha_\beta(-t)(q)\right)\lambda^\beta\frac{\partial}{\partial\lambda^\alpha}\,.$$

Since $\mathcal{L} \circ T\bar{s}_t = \mathcal{L}$, for all $-\epsilon < t < \epsilon$, it is immediate that $\bar{X}^C(\mathcal{L}) = 0$.

Conversely, given an infinitesimal symmetry of $\mathcal{L}$, $\bar{X} = X^A(q)\dfrac{\partial}{\partial q^A}$ $+f^\alpha_\beta(q)\lambda^\beta\dfrac{\partial}{\partial\lambda^\alpha}$, we have

$$\bar{X}^C(\mathcal{L}) = \bar{X}^c(L) + \lambda^\alpha(f^\beta_\alpha\phi_\beta + \bar{X}^c(\phi_\alpha)) = 0\,.$$

Since this is valid for every λ^α, we obtain

$$\bar{X}^c(L) = 0\,, \quad \bar{X}^c(\phi_\alpha) = -f^\beta_\alpha\phi_\beta\,.$$

That is, $\bar{X}$ projects onto a vector field on Q, $X = X^A(q)\dfrac{\partial}{\partial q^A}$, which is a VIS for (L,N). For this reason, we will focus our attention on infinitesimal symmetries of $\mathcal{L}$ given by $\bar{X} = X + \lambda^\beta f^\alpha_\beta(q)\dfrac{\partial}{\partial\lambda^\alpha}$, where (f^α_β) is a matrix-valued function on Q, $(f^\alpha_\beta) : Q \longrightarrow \mathfrak{gl}(m,\mathbb{R})$. We will call to this type of symmetry a VIS for (L,N) on P.

Definition 7.4. A vakonomic Noether symmetry (VNS) for (L,N) will be a vector field X on Q such that $X^C_{|_N}$ is tangent to N and $X^C(L)_{|_N} = F^C_{|_N}$ for some associated function $F : Q \longrightarrow \mathbb{R}$.

Observe that, although the flow of a Noether symmetry preserves N, it does not consists of vakonomic symmetries in the sense of Definition 7.2. In case $X^C(L) = F^C$ on the whole of TQ, the above-defined extension $\bar{X} = X + Y_{\mathcal{L}}$ gives rise to a Noether symmetry of $\mathcal{L}$; that is, $\bar{X}^C(\mathcal{L}) = \pi_1^*(F^C) \equiv F^C$.

Conversely, if $\bar{X} = X^A(q)\dfrac{\partial}{\partial q^A} + f^\alpha_\beta(q)\lambda^\beta\dfrac{\partial}{\partial\lambda^\alpha} = X + f^\alpha_\beta(q)\lambda^\beta\dfrac{\partial}{\partial\lambda^\alpha}$ is a Noether symmetry for $\mathcal{L}$, say,

$$\bar{X}^C(\mathcal{L}) = \bar{X}^C(L) + \lambda^\alpha(f^\beta_\alpha\phi_\beta + \bar{X}^C(\phi_\alpha)) = \bar{F}^C\,, \tag{15}$$

for some $\bar{F} : P \longrightarrow \mathbb{R}$. Since $\dfrac{\partial\bar{F}}{\partial\lambda^\alpha} = \dfrac{\partial\bar{X}^C(\mathcal{L})}{\partial\dot{\lambda}^\alpha} = 0$, equating $(\lambda^\alpha) = (0)$, we have $\bar{X}^C(L) = \bar{F}^C$, and being (15) valid for all λ^α, we also have $\bar{X}^C(\phi_\alpha) = -f^\beta_\alpha\phi_\beta$.

Thus, $\bar{F}$ must be the pullback of a function $F : Q \longrightarrow \mathbb{R}$, and $\bar{X}$ projects to X, a VNS for (L, N) with associated function F. These type of symmetries $\bar{X}$ will be referred as VNS for (L, N) on P.

Finally, let $\Phi : G \times Q \longrightarrow Q$ be a free and proper left action of a Lie group G on the configuration space Q. The group G will be a group of vakonomic symmetries for (L, N), if each Φ_g is a vakonomic symmetry, that is, if the lifted action $T\Phi : G \times TQ \longrightarrow TQ$ satisfies $L_{|N} \circ T\Phi_g = L_{|N}$ and $T\Phi_g(N) = N$, $\forall g \in G$.

We can make use of the procedure described before to extend a symmetry from Q to $P = Q \times \mathbb{R}^m$. Given a fixed Lagrangian, $\mathcal{L} = L + \lambda^\alpha \phi_\alpha$, let us assume that $L \circ T\Phi_g = L$ for all $g \in G$. Then we define the new action

$$\begin{array}{rcl} \Psi : G \times P & \longrightarrow & P \\ (g, (q, \lambda)) & \longmapsto & (\Phi_g(q), \bar{\Lambda}^\alpha_\beta(g)(q)\lambda^\beta) \,. \end{array}$$

It is easy to check that this is indeed a free action and, when G is compact, one can assure that it is also proper.

One is commonly interested in studying the symmetry properties of a dynamical problem because this can yield, via a Noether's theorem for example, information about conservation laws or reduction of the number of degrees of freedom. An example of this situation is the following result.

Proposition 7.5. *(Noether's theorem). Assume that the sequence of submanifolds obtained through the application of the Gotay-Nester algorithm stabilizes at some step $k_f \equiv f$. Let $\bar{X} \in \mathfrak{X}(P)$ be a VNS for (L, N) with associated function $F : P \longrightarrow \mathbb{R}$. Then,*

1. *$\bar{X}^C_{|P_k}$ is tangent to the submanifold P_k $\forall 1 \leq k \leq k_f$*
2. *$(F^v - i_{\bar{X}^c}\theta_{\mathcal{L}})_{|P_f} : P_f \longrightarrow \mathbb{R}$ is a constant of the motion for $\mathfrak{X}^{\omega_{\mathcal{L}}}(P_f)$, where $\mathfrak{X}^{\omega_{\mathcal{L}}}(P_f) = \{X \in \mathfrak{X}(P_f) | i_X \omega_f = (dE_L)_{|P_f}\}$ is the set of solutions.*

In particular, in the case of a Lie group G acting on Q, $\Phi : G \times Q \longrightarrow Q$, freely and properly, we have the following. Let ξ be an element in $\mathfrak{g}$, the Lie algebra of G. Denote by ξ_P (respectively ξ_{P_k}, ξ_Q) the vector field which generates the flow $\Psi_{exp(t\xi)}$ (respectively $(\Psi^T_k)_{exp(t\xi)}$, $\Phi_{exp(t\xi)}$). Then, as a consequence of Proposition 7.5 we have that ξ_P is a VIS for (L, N) on P, ξ_{P_f} is a dynamical symmetry for $\mathfrak{X}^{\omega_{\mathcal{L}}}(P_f)$ and

$$\begin{array}{rrcl} J_f : & P_f & \longrightarrow & \mathfrak{g}^* \\ & x & \longmapsto & J_f(x) : \begin{array}[t]{rcl} \mathfrak{g} & \longrightarrow & \mathbb{R} \\ \xi & \longmapsto & J_f(\xi)(x) = i_{\xi_{P_f}} \theta_{\mathcal{L}}(x) \end{array} \end{array}$$

is a momentum map for the presymplectic system $(P_f, \omega_{P_f}, dE_{\mathcal{L}|P_f})$. We will call it the *vakonomic momentum map* [2, 28]. Therefore, we have that $J_f(\xi) : P_f \longrightarrow \mathbb{R}$, $x \longmapsto J_f(\xi)(x) = J_f(x)(\xi)$ is a constant of the motion.

If $\xi_Q(q) = \xi^A_Q(q) \dfrac{\partial}{\partial q^A}$ and $\xi_P(q, \lambda) = \xi^A_Q(q) \dfrac{\partial}{\partial q^A} + \xi^\alpha_\beta(q) \lambda^\beta \dfrac{\partial}{\partial \lambda^\alpha}$, then, given $x \in P_f$, we have,

$$J_\xi(x) = i_{\xi_{P_k}} \theta_{\mathcal{L}}(x) = \frac{\partial \mathcal{L}}{\partial \dot{q}^A}(x) \xi^A_Q(x) = \left(\frac{\partial L}{\partial \dot{q}^A} + \lambda^\alpha \frac{\partial \phi_\alpha}{\partial \dot{q}^A} \right)(x) \xi^A_Q(x) \,.$$

To end this section, let us mention that the previous discussion can also be developed in the Hamiltonian picture [58].

7.2. A quick look at optimal control problems.

Here, we show how the above exposed formalism can be used in studying some optimal control problems.

A general optimal control problem consists of a set of differential equations

$$\dot{x}^i = f^i(x(t), u(t))\,,\ 1 \leq i \leq n\,, \tag{16}$$

where the x^i denote the states and the u the control variables, and a cost function $L(x,u)$. Given initial and final states x_0, x_f, the objective is to find a C^2-piecewise smooth curve $c(t) = (x(t), u(t))$ such that $x(t_0) = x_0$, $x(t_f) = x_f$, satisfying the control equations (16) and minimizing the functional

$$\mathcal{J}(c) = \int_{t_0}^{t_f} L(x(t), u(t))dt\,.$$

In a global description, one assumes an affine bundle structure $\pi : C \longrightarrow B$, where B is the configuration manifold with local coordinates x^i and C is the bundle of controls, with local coordinates (x^i, u^a).

The ordinary differential equations (16) on B depending on the parameters u can be seen as a vector field Γ along the projection map π, that is, Γ is a smooth map $\Gamma : C \longrightarrow TB$ such that the diagram

$$\begin{tikzcd} C \arrow[rr, "\Gamma"] \arrow[dr, "\pi"'] & & TB \arrow[dl, "\tau_B"] \\ & B & \end{tikzcd}$$

is conmutative. This vector field is locally written as $\Gamma = f^i(x,u)\dfrac{\partial}{\partial x^i}$.

In the following, we show how this kind of problems admit a formulation in terms of vakonomic dynamics. Consider the cost function $L : C \longrightarrow \mathbb{R}$ and its pullback $\tau_C^* L$ to TC. Let us define the set,

$$N = \{v \in TC \,|\, \pi_*(v) = \Gamma(\tau_C(v))\}\,,$$

which is a submanifold of TC. Locally this submanifold is defined by the conditions $\dot{x}^i = f^i(x,u)$, $1 \leq i \leq n$, which are just the differential equations (16). Then, to solve the vakonomic problem with Lagrangian $\tau_N^* L : TC \longrightarrow \mathbb{R}$ and constraint submanifold $N \subset TC$ is equivalent to solve the original general optimal control problem. Moreover, one can make use of the already developed theory of the dynamics of vakonomic systems in the singular Lagrangian framework and of the different types of symmetries associated to such systems, to analyze general control problems.

Remark 7.6. An alternative way of rephrasing the general optimal control problem in terms of a constrained variational problem is considered in [7, 8]. Assuming that equation (16) determines u as a function of $(x, \dot{x})$, one can pose the vakonomic problem with Lagrangian $L = L(x, u(x,\dot{x}))$ and constraints $\dot{x} - f(x, u(x, \dot{x}))$ on TB.

If one performs the Gotay-Nester algorithm with the extended Lagrangian $\mathcal{L} = L + \lambda_i(\dot{x}^i - f^i(x,u))$, one finds that the second constraint submanifold P_2 is the final constraint manifold if and only if the matrix

$$W_{ab} = \frac{\partial^2 L}{\partial u^a \partial u^b} - \lambda_i \frac{\partial^2 f^i}{\partial u^a \partial u^b}$$

is invertible, which is exactly the characterization for the so-called regular optimal control problem.

On the other hand, one can easily state a version of the Noether's theorem for general optimal control problems.

Proposition 7.7. *(Noether's theorem) Consider a regular optimal control problem. Let $X \in \mathfrak{X}(C)$ be a vakonomic Noether symmetry (VNS) for $(\tau_C^* L, N)$ with associated function $F : C \longrightarrow \mathbb{R}$. Then $F^v - i_{\bar{X}^C}\theta_{\mathcal{L}} : P_2 \longrightarrow \mathbb{R}$ is a constant of the motion along any optimal trajectory.*

Locally, if $X = X^i(x,u)\dfrac{\partial}{\partial x^i} + X^a(x,u)\dfrac{\partial}{\partial u^a}$, then $\bar{X} = X + g^i_j(x,u)\lambda_i \dfrac{\partial}{\partial \lambda_j}$, for some $(g^i_j) : C \longrightarrow \mathfrak{gl}(n,\mathbb{R})$, and the constant of the motion reads locally as

$$F^v - i_{\bar{X}^c}\theta_{\mathcal{L}} = F^v - \frac{\partial \mathcal{L}}{\partial \dot{x}^i} X^i - \frac{\partial \mathcal{L}}{\partial \dot{u}^a} X^a = F^v - \sum_{i=1}^{n} \lambda_i X^i \,.$$

This result is a corollary of Proposition 7.5 when applied to general optimal control problems.

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