Mechanical control systems on Lie algebroids

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Abstract

This paper considers control systems defined on Lie algebroids. After deriving basic controllability tests for general control systems, we specialize our discussion to the class of mechanical control systems on Lie algebroids. This class of systems includes mechanical systems subject to holonomic and nonholonomic constraints, mechanical systems with symmetry and mechanical systems evolving on semidirect products. We introduce the notions of linear connection, symmetric product and geodesically invariant subbundle on a Lie algebroid. We present appropriate tests for various notions of accessibility and controllability, and analyze the relation between the controllability properties of control systems related by a morphism of Lie algebroids.

1 Introduction

One of the basic problems in control theory is that of deciding the local controllability properties of a given system. Roughly speaking, the local controllability problem consists of finding appropriate conditions guaranteeing that the set of reachable states starting from an initial point is open, i.e., that the system can move locally in *any* direction. Deciding the controllability properties of a system is an *a priori* question that one needs to have addressed before being able to undertake other control problems such as motion planning and trajectory generation. The controllability problem has received a great deal of attention during the last decades (see [13, 28, 31, 32] and references therein). In particular, researchers have undertaken a thorough study of control systems with a rich geometric structure such as mechanical and homogeneous systems, and made use of their special properties to accomplish accurate modeling settings and sharp analysis results. Specific class of control problems include simple mechanical systems [20], systems subject to nonholonomic constraints [2, 5, 19], systems invariant under the action of a Lie group of symmetries [9, 16, 23, 27], systems enjoying special homogeneity properties [8, 15, 34], systems evolving on semidirect products [30], and more.

One of the features which imposes a separate study for each class of systems is the lack of a unified framework. For instance, it is well known that a Lagrangian system invariant under the action of a Lie group of symmetries can be reduced to the quotient space induced by the action, but the reduced dynamics is not the one that corresponds to the reduced Lagrangian. Recent investigations have lead to a unifying geometric framework to overcome this drawback. It is precisely the underlying structure of Lie algebroid on the phase space what allows a unified treatment. This idea was introduced by Weinstein [35] in order to define a Lagrangian formalism which is general enough to account for the various types of systems. A symplectic formalism was later introduced for Lagrangian [25] and Hamiltonian systems [24]. One of the advantages of the Lie algebroid formalism is the possibility of establishing appropriate maps (morphisms) between two systems that respect the structure of the phase space, and allow to relate their respective control properties. The underlying idea (in a similar way to the notion of *abstraction* [29]) is that the property of interest will be easier to decide for one of the systems, and that by means of the morphism one will be able to infer the same knowledge for the other system.

In this paper we study the controllability problem for systems affine in the inputs evolving on a bundle E with an underlying Lie algebroid structure, $\tau : E \to M$. We build on previous work on local controllability of general systems [32] and of mechanical systems [20] to derive tests to check the accessibility and controllability properties of control systems evolving on a Lie algebroid. Throughout the paper, we pay special attention to what we term *mechanical control systems on a Lie algebroid*. This class of systems embraces a variety of different situations that can occur when analyzing mechanical systems, such as the ones mentioned above. Building on the notion of prolongation $\mathcal{T}E \to E$ of the Lie algebroid $E \to M$ introduced in [25], we develop all the necessary differential geometric tools enabling an intrinsic treatment of the second-order dynamics associated with mechanical systems. We focus on the set of reachable points in the base manifold M and in the bundle space E starting from states which belong to the zero section of $E \to M$. We make use of the geometric homogeneity properties of the controlled equations, which turn out to greatly simplify the accessibility and controllability computations. We carefully describe the relation between the controllability properties of control systems that are related by a morphism of Lie algebroids. As a result of the generality of the approach, we are able to present in a unified way previous work on the configuration accessibility and controllability properties of simple mechanical control systems [3, 9, 20, 26] (see also [4] for a comprehensive overview). Regarding systems evolving on semidirect products, the application of the Lie algebroid approach renders novel tests which are valid in slightly more general settings than the ones considered in [30]. We also extend notions such as fiber controllability to what we call *controllability with regards to a manifold* and develop conditions to check this property.

In the course of the preparation of this manuscript, we came across the recent research effort [34]. This reference, which is close in spirit to this work, analyzes the controllability properties of so-called "1-homogeneous control systems" evolving on a vector bundle. However, it deals with vector fields with values in the tangent bundle of the vector bundle, as opposed to deal with the formalism of Lie algebroids and their prolongations. This choice of a higher-dimensional phase space makes necessary to resort to additional geometric tools such as Ehresmann connections in order to describe the structure of the accessibility algebra. We think that the Lie algebroid approach accommodates the same level of generality, while enabling in general a more concise treatment of the controlled dynamics.

The paper is organized as follows. In Section 2 we present some basic facts on Lie algebroids. We also discuss in detail the notion of linear connection on a Lie algebroid, including the generalization of the Levi-Civita connection and the constrained connection. In Section 3, we introduce the prolongation of a Lie algebroid and develop the differential geometry of horizontal sections, homogeneity, SODE sections and geodesically invariant subbundles. In Section 4 we study nonlinear affine control systems whose drift and input vector fields are associated with some sections of a Lie algebroid. This apparent restriction is not such, since most physical systems can be casted into this form. We formulate the conditions for local accessibility and controllability in terms of Lie brackets of sections, and we study the effect of a morphism of Lie algebroids on these properties. In Section 5 we introduce the class of mechanical control systems defined on a Lie algebroid. We show that the notion of affine connection control system can be generalized to the setting of Lie algebroids, thus providing a general framework to study the controllability properties of these systems. We introduce the notions of local base controllability and controllability with regard to a manifold, and we obtain computable sufficient conditions to check them. We also study the effect of morphisms of Lie algebroids in simplifying the controllability analysis. These results are later applied in Section 6 to simple mechanical systems defined on a manifold, simple mechanical systems with symmetry and systems defined on semidirect products and orbits of group actions. Section 7 presents some concluding remarks. We have gathered in an appendix some basic notions for control systems defined on manifolds. A final remark is that the summation convention over repeated indexes is understood throughout the paper.

2 Preliminaries on Lie algebroids

In this section we introduce some known notions and develop new concepts concerning Lie algebroids that are necessary for the further developments. We refer the reader to [10, 22] for thorough studies of Lie groupoids, Lie algebroids and their role in differential geometry. Let M be an n-dimensional manifold and let $\tau \colon E \to M$ be a vector bundle with ℓ-dimensional fibers. A structure of *Lie algebroid on E* is given by a Lie algebra structure on the $C^\infty(M)$-module of sections of the bundle, $(\mathrm{Sec}(E),[\cdot\,,\cdot])$, together with a homomorphism $\rho\colon E\to TM$ of vector bundles (called the *anchor map*) satisfying the compatibility condition

$$[\sigma_1, F\sigma_2] = F[\sigma_1,\sigma_2] + \big(\rho(\sigma_1)F\big)\sigma_2\,.$$

Here F is a smooth function on M, σ_1, σ_2 are sections of E and we have denoted by $\rho(\sigma)$ the vector field on M given by $\rho(\sigma)(m)=\rho(\sigma(m))$. The homomorphism ρ is called the *anchor map*. From the compatibility condition and the Jacobi identity, it follows that the map $\sigma\mapsto\rho(\sigma)$ is a Lie algebra homomorphism from $\mathrm{Sec}(E)$ to $\mathfrak{X}(M)$.

It is convenient to think of a Lie algebroid $\tau\colon E\to M$ as a substitute of the tangent bundle of M. In this way, one regards an element a of E as a generalized velocity, and the actual velocity v is obtained when applying the anchor to a, i.e., $v=\rho(a)$.

The image of the anchor map, $\rho(E)$, defines an integrable smooth generalized distribution on M. Therefore, M is foliated by the integral leaves of $\rho(E)$, which are called the *leaves of the Lie algebroid*. A curve $a:[t_0,t_1]\to E$ is said to be *admissible* if $\dot m(t)=\rho(a(t))$, where $m(t)=\tau(a(t))$, $t\in[t_0,t_1]$. It follows that $a(t)$ is admissible if and only if the curve $m(t)$ lies on a leaf of the Lie algebroid, and that two points are in the same leaf if and only if they are connected by (the base curve of) an admissible curve.

A Lie algebroid is said to be *transitive* if it has only one leaf, which is obviously equal to M. It is easy to see that E is transitive if and only if ρ is surjective. If E is not transitive, then the restriction of the Lie algebroid to a leaf $L\subset M$, $E_{|L}\to L$ is transitive. In the latter case, one can show that $E_{|L}$, and hence $\ker\rho$, has constant dimension. We will say that a Lie algebroid is *locally transitive at a point* $m\in M$ if $\rho_m: E_m\to T_mM$ is surjective. In this way, m is contained in a leaf of maximal dimension.

Given a local basis $\{e_\alpha\}_{\alpha=1}^{\ell}$ of sections of E defined on an open set $V\subset M$, we can write $a=y^\alpha e_\alpha(\tau(a))$ for any $a\in E$ such that $\tau(a)\in V$. If (x^i), $i=1,\dots,n$ are local coordinates in the base M defined on V, we have local coordinates (x^i,y^α), $i=1,\dots,n$, $\alpha=1,\dots,\ell$ in E. The anchor map and the Lie bracket are then determined by the local functions ρ^i_α and $C^\alpha_{\beta\gamma}$ on M (called the *structure functions of the Lie algebroid*) defined by

$$\rho(e_\alpha)=\sum_{i=1}^{n}\rho^i_\alpha\frac{\partial}{\partial x^i}\qquad\text{and}\qquad [e_\alpha,e_\beta]=\sum_{\gamma=1}^{\ell}C^\gamma_{\alpha\beta}e_\gamma.$$

The structure functions satisfy the following relations

$$\rho^j_\alpha\frac{\partial\rho^i_\beta}{\partial x^j}-\rho^j_\beta\frac{\partial\rho^i_\alpha}{\partial x^j}=\rho^i_\gamma C^\gamma_{\alpha\beta}\,,\quad\text{and}\quad \sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)}\left[\rho^i_\alpha\frac{\partial C^\nu_{\beta\gamma}}{\partial x^i}+C^\mu_{\alpha\nu}C^\nu_{\beta\gamma}\right]=0\,,\tag{2.1}$$

where the summation over repeated indexes is understood. Equations (2.1) are usually called the *structure equations of the Lie algebroid*. Finally, the Lie bracket of two sections of E can be expressed in terms of the basis $\{e_\alpha\}_{\alpha=1}^{\ell}$ as

$$[\sigma,\eta]=\left(\sigma^\gamma\rho^k_\gamma\frac{\partial\eta^\alpha}{\partial x^k}-\eta^\gamma\rho^k_\gamma\frac{\partial\sigma^\alpha}{\partial x^k}+C^\alpha_{\beta\gamma}\sigma^\beta\eta^\gamma\right)e_\alpha\,.\tag{2.2}$$

If $\mathfrak{Y}$ is a family of sections of E, we will denote by $\overline{\mathrm{Lie}}(\mathfrak{Y})$ the distribution obtained by closing (the distribution defined by) $\mathfrak{Y}$ under the Lie bracket.

2.1 Admissible maps and morphisms of Lie algebroids

Let $\tau\colon E\to M$ and $\overline{\tau}\colon\overline{E}\to\overline{M}$ be two Lie algebroids with associated anchor maps $\rho\colon E\to TM$ and $\overline{\rho}\colon\overline{E}\to T\overline{M}$. A bundle map $\Psi\colon E\to\overline{E}$ is said to be *admissible* if $T\psi\circ\rho=\overline{\rho}\circ\Psi$. Equivalently Ψ is admissible if and only if it maps admissible curves into admissible curves. Indeed, if $a(t)$ is admissible on E and projects to $m(t)$, then $\overline{a}(t)=\Psi(a(t))$ projects to $\overline{m}(t)=\psi(m(t))$ and it is admissible, since

$$\overline{\rho}(\overline{a}(t))=\overline{\rho}(\Psi(a(t))=T\psi(\rho(a(t))=T\psi(\dot m(t))=\dot{\overline{m}}(t).$$

Denoting by $\psi\colon M \to \overline{M}$ the map on the base, one has the following commutative diagram

$$\begin{tikzcd}
 & TM \arrow[rr, "T\psi"] \arrow[ddl, "\tau_M"] & & T\overline{M} \arrow[ddl, "\tau_{\overline{M}}"] \\
E \arrow[ur, "\rho"] \arrow[rr, "\Psi"] \arrow[dd, "\tau"'] & & \overline{E} \arrow[ur, "\overline{\rho}"] \arrow[dd, "\overline{\tau}"'] & \\
 & & & \\
M \arrow[rr, "\psi"'] & & \overline{M} &
\end{tikzcd}$$

A map $\Psi\colon E \to \overline{E}$ is a *morphism of Lie algebroids* if it is admissible and preserves the Lie algebra structure of the algebroids [14], that is, for any σ and η sections of E such that there exist some sections $\{\zeta_l\}_{l=1}^{p}$ of $\overline{E}$ and some functions F_l, G_l, $l = 1, \ldots, p$ on M with

$$\Psi \circ \sigma = \sum_{l=1}^{p} F_l(\zeta_l \circ \psi)\,, \quad \Psi \circ \eta = \sum_{l=1}^{p} G_l(\zeta_l \circ \psi)\,,$$

then, the image of the Lie bracket of σ and η under Ψ is

$$\Psi \circ [\sigma, \eta] = \sum_{l=1}^{p} \left(\rho(\sigma)G_l - \rho(\eta)F_l\right)\,(\zeta_l \circ \psi) + \sum_{l_1,l_2=1}^{p} F_{l_1} G_{l_2}\,([\zeta_{l_1}, \zeta_{l_2}] \circ \psi).$$

Given local basis $\{e_\alpha\}_{\alpha=1}^{\ell}$ and $\{\overline{e}_\alpha\}_{\alpha=1}^{\overline{\ell}}$ of sections of E and $\overline{E}$, respectively, a bundle map Ψ can be written $\Psi(e_\alpha) = \Psi_\alpha^\beta \overline{e}_\beta$ for certain local functions Ψ_α on M. Then, one can check that Ψ is a morphism if and only if

$$\Psi_\gamma^\beta C_{\alpha\delta}^\gamma = \left(\rho_\alpha^i \frac{\partial \Psi_\delta^\beta}{\partial x^i} - \rho_\delta^i \frac{\partial \Psi_\alpha^\beta}{\partial x^i}\right) + \overline{C}_{\theta\sigma}^\beta \Psi_\alpha^\theta \Psi_\delta^\sigma\,. \tag{2.3}$$

Notice that if σ, η are Ψ-related to sections $\overline{\sigma}$, $\overline{\eta} \in \mathrm{Sec}(\overline{E})$, i.e., $\Psi \circ \sigma = \overline{\sigma} \circ \psi$ and $\Psi \circ \eta = \overline{\eta} \circ \psi$, then the Lie bracket $[\sigma, \eta]$ is Ψ-related to the Lie bracket $[\overline{\sigma}, \overline{\eta}]$, $\Psi \circ [\sigma, \eta] = [\overline{\sigma}, \overline{\eta}] \circ \psi$.

2.2 Linear connections

Here we briefly present the notion of E-connection on a vector bundle (cf. [11], see also [6, 12]), and discuss some related objects.

Definition 2.1. *Let $\tau : E \to M$ be a Lie algebroid. A linear E-connection on a vector bundle $\pi\colon P \to M$ is a $\mathbb{R}$-bilinear map $\nabla\colon \mathrm{Sec}(E) \times \mathrm{Sec}(P) \to \mathrm{Sec}(P)$ such that*

$$\nabla_{F\sigma}\alpha = F\nabla_\sigma\alpha \qquad \textit{and} \qquad \nabla_\sigma(F\alpha) = (\rho(\sigma)F)\alpha + F\nabla_\sigma\alpha$$

for any function $F \in C^\infty(M)$, section σ of E and section α of P.

Throughout the paper, we will restrict our attention to the case $P = E$, and by a *connection on E* we will understand a linear E-connection on $\tau : E \to M$. Given a local basis $\{e_\alpha\}_{\alpha=1}^{\ell}$ of sections of E, the local expression of the covariant derivative is

$$\nabla_\sigma \eta = \sigma^\alpha \Big(\rho_\alpha^i \frac{\partial \eta^\gamma}{\partial x^i} + \Gamma_{\alpha\beta}^\gamma \eta^\beta\Big) e_\gamma.$$

The terms $\Gamma^\gamma_{\alpha\beta}$ are called the *connection coefficients*. As in the study of tangent bundle geometry, the skew-symmetric part of the connection defines the so-called *torsion tensor*,

$$T(\sigma,\eta) = \nabla_\sigma\eta - \nabla_\eta\sigma - [\sigma,\eta]\,,$$

and the symmetric part of the connection determines what we call the *symmetric product*,

$$\langle\sigma : \eta\rangle = \nabla_\sigma\eta + \nabla_\eta\sigma\,.$$

The local expression of the symmetric product is (compare with the expression for the Lie bracket (2.2))

$$\langle\sigma : \eta\rangle = \left(\sigma^\gamma\rho^k_\gamma\frac{\partial\eta^\alpha}{\partial x^k} + \eta^\gamma\rho^k_\gamma\frac{\partial\sigma^\alpha}{\partial x^k} + S^\alpha_{\beta\gamma}\sigma^\beta\eta^\gamma\right)e_\alpha\,, \qquad \text{where} \quad S^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha}\,. \tag{2.4}$$

In particular, notice that $\langle e_\alpha : e_\beta\rangle = S^\gamma_{\alpha\beta}e_\gamma$. Similarly as with the involutive closure, if $\mathcal{Y}$ is a family of sections of E, we will denote by $\overline{\mathrm{Sym}}(\mathcal{Y})$ the distribution obtained by closing (the distribution defined by) $\mathcal{Y}$ under the symmetric product.

Since the covariant derivative is $C^\infty(M)$-linear in the first argument, it is possible to define the derivative of a section $\sigma \in \mathrm{Sec}(E)$ with respect to an element $a \in E_m$ by simply putting

$$\nabla_a\sigma = \nabla_\eta\sigma(m)\,,$$

where $\eta \in \mathrm{Sec}(E)$ is any section such that $\eta(m) = a$. Moreover, the covariant derivative allows us to take the derivative of sections along maps and, as a particular case, of sections along curves. If we have a morphism of Lie algebroids $\Phi\colon F \to E$ over the map $\varphi\colon N \to M$, then we can define the derivative of a section of E along φ as follows.

Definition 2.2. *Let $\sigma : N \to E$ be a section of E along φ, i.e., $\sigma(n) \in E_{\varphi(n)}$, $n \in N$. Then σ can be written in the form $\sigma = \sum_{l=1}^p F_l(\zeta_l \circ \varphi)$, for some sections $\{\zeta_1,\dots,\zeta_p\} \subset \mathrm{Sec}(E)$ and some functions $\{F_1,\dots,F_p\} \subset C^\infty(N)$. The* derivative of σ along φ *is defined by*

$$\nabla_b\sigma = \sum_{l=1}^p \left[(\rho(b)F_l)\zeta_l(\varphi(n)) + F_l(n)\nabla_{\Phi(b)}\,\zeta_l\right]\,, \quad b \in F\,.$$

Remark 2.3. Within this framework, one can consider time-dependent sections of E as follows: take the morphism $\Phi\colon T\mathbb{R}\times E \to E$, $\Phi(t,a) = a$ over the map $\varphi\colon \mathbb{R}\times M \to M$, $\varphi(t,m) = m$. The Lie algebroid structure on $T\mathbb{R}\times E$ is the direct product structure, that is, the anchor is $\rho_{T\mathbb{R}\times E}(\tau\frac{d}{dt},a) = \tau\frac{\partial}{\partial t} + \rho(a)$, and the bracket of projectable sections (on both factors) is the sum of the brackets on $T\mathbb{R}$ and E.

When studying mechanical control systems related by a morphism of Lie algebroids, we will resort to the following notion concerning the interplay between maps and linear connections.

Definition 2.4. *Let ∇ and $\overline{\nabla}$ be connections on E and $\overline{E}$, respectively, and let Ψ be a bundle map from E to $\overline{E}$. We say that Ψ* maps the connection ∇ to the connection $\overline{\nabla}$ *if*

$$\Psi\circ(\nabla_\sigma\eta) = \overline{\nabla}_\sigma(\Psi\circ\eta).$$

In coordinates this condition is equivalent to

$$\Psi^\beta_\gamma\Gamma^\gamma_{\alpha\delta} = \rho^i_\alpha\frac{\partial\Psi^\beta_\delta}{\partial x^i} + \overline{\Gamma}^\beta_{\theta\sigma}\Psi^\theta_\alpha\Psi^\sigma_\delta\,. \tag{2.5}$$

Geodesics

Consider the following situation: let $a : t \mapsto E$ be an admissible curve, and let $b : t \mapsto E$ be a curve in E, both of them projecting by τ onto the same base curve in M, $\tau(a(t)) = m(t) = \tau(b(t))$. Take the Lie algebroid structure $T\mathbb{R} \to \mathbb{R}$ and consider the morphism $\Phi : T\mathbb{R} \to E$, $\Phi(t, \dot{t}) = \dot{t}a(t)$ over $\varphi : \mathbb{R} \to M$, $\varphi(t) = m(t)$. Then one can define the *derivative of* $b(t)$ *along* $a(t)$ as $\nabla_{\frac{d}{dt}} b(t)$. In the literature, this derivative is usually denoted by $\nabla_{a(t)}b(t)$. In local coordinates, this reads

$$\nabla_{a(t)}b(t) = \left[\frac{db^\gamma}{dt} + \Gamma^\gamma_{\alpha\beta}a^\alpha b^\beta\right] e_\gamma(m(t)).$$

Definition 2.5. *Let $\tau : E \to M$ be a Lie algebroid and ∇ a connection on E. An admissible curve $a : t \mapsto E$ is said to be a* geodesic of ∇ *if $\nabla_{a(t)}a(t) = 0$.*

In local coordinates, the conditions for being a geodesic reads

$$\frac{da^\gamma}{dt} + \frac{1}{2}S^\gamma_{\alpha\beta}a^\alpha a^\beta = 0\,. \tag{2.6}$$

The Levi-Civita connection

Let $\mathcal{G}\colon E \times_M E \to \mathbb{R}$ be a bundle metric on a Lie algebroid $\tau : E \to M$. In a parallel way to the situation in tangent bundle geometry, one can see that there is a canonical connection on E associated with $\mathcal{G}$. The proof is analogous and will be omitted.

Proposition 2.6. *Given a bundle metric $\mathcal{G}$ on E, there is a unique connection $\nabla^{\mathcal{G}}$ on E which is torsion-less and metric with respect to $\mathcal{G}$. The connection $\nabla^{\mathcal{G}}$ is determined by the formula*

$$2\mathcal{G}(\nabla_\sigma\eta, \zeta) = \rho(\sigma)\mathcal{G}(\eta, \zeta) + \rho(\eta)\mathcal{G}(\sigma, \zeta) - \rho(\zeta)\mathcal{G}(\eta, \sigma) + \mathcal{G}(\sigma, [\zeta, \eta]) + \mathcal{G}(\eta, [\zeta, \sigma]) - \mathcal{G}(\zeta, [\eta, \sigma])\,,$$

for $\sigma, \eta, \zeta \in \mathrm{Sec}(E)$.

Denoting by $\{e_\alpha\}_{\alpha=1}^{\ell}$ a local basis of sections of E, and by $\{e^\alpha\}_{\alpha=1}^{\ell}$ its dual basis, the bundle metric can be locally written as $\mathcal{G} = \mathcal{G}_{\alpha\beta}\, e^\alpha \otimes e^\beta$. The connection coefficients of $\nabla^{\mathcal{G}}$ are

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\mathcal{G}^{\alpha\nu}\Big([\nu, \beta; \gamma] + [\nu, \Gamma; \beta] + [\beta, \gamma; \nu]\Big)\,,$$

where $(\mathcal{G}^{\mu\nu})$ is the inverse matrix of $(\mathcal{G}_{\alpha\beta})$, and $[\alpha, \beta; \gamma]$ is a shorthand notation for

$$[\alpha, \beta; \gamma] = \frac{\partial \mathcal{G}_{\alpha\beta}}{\partial x^i}\rho^i_\gamma + C^\mu_{\alpha\beta}\mathcal{G}_{\mu\gamma}.$$

Associated with the bundle metric $\mathcal{G}$, one has the *musical isomorphisms*

$$\flat_{\mathcal{G}} : E \to E^*\,,\ \langle \flat_{\mathcal{G}}(a), b\rangle = \mathcal{G}(a, b)\,, \qquad \sharp_{\mathcal{G}} : E^* \to E\,,\ \sharp_{\mathcal{G}}(\theta) = \flat_{\mathcal{G}}^{-1}(\theta)\,,$$

where $E^* \to M$ denotes the dual bundle of E. Given a function V on M, the *gradient of* V, $\mathrm{grad}_{\mathcal{G}}\, V$, is the section of E defined by $\mathrm{grad}_{\mathcal{G}}\, V = \sharp_{\mathcal{G}}(\rho^* dV)$.

The constrained connection

Here, we introduce the notion of constrained connection on a Lie algebroid, which generalizes the concept of nonholonomic connection [18, 33]. This notion will be later useful to model control systems subject to nonholonomic constraints, following the developments in [1, 19]. Given an arbitrary connection and an arbitrary subbundle of a Lie algebroid, one can define a new connection which enjoys special properties with respect to the subbundle. This connection is determined by the choice of a projector. Let D be a subbundle of E and let P be a projector onto D, $P : E \to D$. We denote by Q the complementary projector of P, $Q = I - P$, and by D^c the complementary subbundle $D^c = \mathrm{Im}(Q)$. In this way, one has $D \oplus D^c = E$.

Definition 2.7. *Given a connection ∇ on E and a projector map $P : E \to D$, the* constrained connection *is the connection $\check{\nabla}$ on E defined by*

$$\check{\nabla}_\sigma\eta = P(\nabla_\sigma\eta) + \nabla_\sigma(Q\eta)\,, \quad \sigma, \eta \in \mathrm{Sec}(E)\,.$$

Some interesting properties of the constrained connection $\check{\nabla}$ are the following. Their proof is straightforward and will be omitted for brevity.

Proposition 2.8. *The following properties of the constrained connection $\check{\nabla}$ hold:*

(i) The connection $\check{\nabla}$ restricts to D, i.e., $\check{\nabla}_\sigma \eta \in D$ for any $\eta \in \mathrm{Sec}(D)$ and $\sigma \in \mathrm{Sec}(E)$.

(ii) The symmetric product $\langle \cdot : \cdot \rangle^{\!\vee}$ associated with $\check{\nabla}$ is given by

- $\langle \sigma : \eta \rangle^{\!\vee} = P(\langle \sigma : \eta \rangle)$ *for* $\sigma, \eta \in \mathrm{Sec}(D)$.
- $\langle \sigma : \eta \rangle^{\!\vee} = P(\langle \sigma : \eta \rangle) + \langle \sigma : \eta \rangle$ *for* $\sigma, \eta \in \mathrm{Sec}(D^c)$.
- $\langle \sigma : \eta \rangle^{\!\vee} = P(\langle \sigma : \eta \rangle) + \nabla_\sigma \eta$ *for* $\sigma \in \mathrm{Sec}(D)$ *and* $\eta \in \mathrm{Sec}(D^c)$.

(iii) The torsion tensor $\check{T}$ of $\check{\nabla}$ is given by

- $\check{T}(\sigma, \eta) = P(T(\sigma, \eta)) - Q([\sigma, \eta])$ *for* $\sigma, \eta \in \mathrm{Sec}(D)$.
- $\check{T}(\sigma, \eta) = P(T(\sigma, \eta)) + \nabla_\sigma \eta - \nabla_\eta \sigma - Q([\sigma, \eta])$ *for* $\sigma, \eta \in \mathrm{Sec}(D^c)$.
- $\check{T}(\sigma, \eta) = P(T(\sigma, \eta)) + \nabla_\sigma \eta - Q([\sigma, \eta])$ *for* $\sigma \in \mathrm{Sec}(D)$ *and* $\eta \in \mathrm{Sec}(D^c)$.

Proposition 2.9. *Let $\tau : E \to M$, $\tau : \overline{E} \to \overline{M}$ be two Lie algebroids, with projectors $P : E \to D$ and $P : \overline{E} \to \overline{D}$. Let ∇ and $\overline{\nabla}$ be connections on E and $\overline{E}$, respectively. Assume that a morphism of Lie algebroids $\Psi \colon E \to \overline{E}$ maps ∇ onto $\overline{\nabla}$. If $\Psi \circ P = \overline{P} \circ \Psi$ (equivalently $\Psi(D) \subset \overline{D}$ and $\Psi(D^c) \subset \overline{D}^c$), then Ψ maps $\check{\nabla}$ onto $\check{\overline{\nabla}}$.*

Proof. Since $Q = I - P$, it follows that $\Psi \circ Q = \overline{Q} \circ \Psi$. Therefore, for all $\eta \in \mathrm{Sec}(E)$ and $b \in E$

$$
\begin{aligned}
\Psi(\check{\nabla}_b \eta) &= \Psi(P(\nabla_b \eta)) + \Psi(\nabla_b(Q\eta)) = \overline{P}(\Psi(\nabla_b \eta)) + \overline{\nabla}_b(\Psi \circ (Q\eta)) \\
&= \overline{P}(\overline{\nabla}_b(\Psi \circ \eta)) + \nabla_b(\overline{Q}(\Psi \circ \eta)) = \check{\overline{\nabla}}_b(\Psi \circ \eta)\,.
\end{aligned}
$$

□

3 The prolongation of a Lie algebroid

Here we briefly review the notion of the prolongation of a Lie algebroid. For further details, see [25]. Given a Lie algebroid E, the underlying motivation behind the introduction of the prolongation of E is that of formulating second-order dynamical systems on E. Thinking of E as a substitute of the tangent bundle of M, the tangent bundle of E is not the appropriate space to describe second-order dynamics on E. This is clear if we note that the projection to M of a vector tangent to E is a vector tangent to M, and what one would like instead is an element of E, the *new* tangent bundle of M.

A space which takes into account this restriction is the E-tangent bundle of E, also called the *prolongation of E*, which we denote by $\mathcal{T}E$. This Lie algebroid is defined as the vector bundle $\tau_1 \colon \mathcal{T}E \to E$ whose fiber at a point $a \in E_m$ is the vector space

$$
\mathcal{T}_a E = \{\, (b, v) \in E_m \times T_a E \mid \rho(b) = T_a\tau(v) \,\}\ .
$$

Note that if the fibers of E are ℓ-dimensional, then the fibers of $\mathcal{T}E$ are 2ℓ-dimensional. We will use the redundant notation (a, b, v) to denote the element $(b, v) \in \mathcal{T}_a E$.

The anchor of $\mathcal{T}E$ is the map $\rho^1 \colon \mathcal{T}E \to TE$, defined by $\rho^1(a, b, v) = v$. We also consider the map $\mathcal{T}\tau \colon \mathcal{T}E \to E$ defined by $\mathcal{T}\tau(a, b, v) = b$. The Lie bracket associated with $\mathcal{T}E$ is defined as follows in terms of projectable sections. A section Z of $\mathcal{T}E$ is *projectable* if there exists a section σ of E such that $\mathcal{T}\tau \circ Z = \sigma \circ \tau$. Equivalently, a section Z is projectable if and only if it is of the form $Z(a) = (a, \sigma(\tau(a)), X(a))$, for some section σ of E and some vector field X on E. The Lie bracket of two projectable sections Z_1 and Z_2 is then given by

$$
[Z_1, Z_2](a) = (a, [\sigma_1, \sigma_2](m), [X_1, X_2](a))\,, \quad a \in E\,.
$$

It is easy to see that $[Z_1, Z_2](a)$ is an element of $\mathcal{T}E$ for every $a \in E$. Since any section of $\mathcal{T}E$ can be locally written as a linear combination of projectable sections, the definition of the Lie bracket for sections of $\mathcal{T}E$ follows.

Given local coordinates (x^i, y^α) associated with a basis $\{e_\alpha\}$ of sections of E, we can define a local basis $\{\mathcal{X}_\alpha, \mathcal{V}_\alpha\}_{\alpha=1}^{\ell}$ of sections of $\mathcal{T}E$ by

$$\mathcal{X}_\alpha(a) = \left(a, e_\alpha(\tau(a)), \rho^i_\alpha \frac{\partial}{\partial x^i}\Big|_a\right) \qquad \text{and} \qquad \mathcal{V}_\alpha(a) = \left(a, 0, \frac{\partial}{\partial y^\alpha}\Big|_a\right), \quad \alpha = 1, \ldots, \ell\,. \tag{3.1}$$

If (a, b, v) is an element of $\mathcal{T}E$, with $b = z^\alpha e_\alpha$ and $v = \rho^i_\alpha z^\alpha \frac{\partial}{\partial x^i} + v^\alpha \frac{\partial}{\partial y^\alpha}$, then we can write

$$(a, b, v) = z^\alpha \mathcal{X}_\alpha(a) + v^\alpha \mathcal{V}_\alpha(a).$$

The Lie brackets of the elements of the basis are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C^\gamma_{\alpha\beta}\, \mathcal{X}_\gamma\,, \qquad [\mathcal{X}_\alpha, \mathcal{V}_\beta] = 0\,, \qquad [\mathcal{V}_\alpha, \mathcal{V}_\beta] = 0\,.$$

Finally, notice that the anchor map ρ^1 applied to a section $Z = Z^\alpha \mathcal{X}_\alpha + V^\alpha \mathcal{V}_\alpha$ of $\mathcal{T}E$ defines a vector field on E whose coordinate expression is

$$\rho^1(Z) = \rho^i_\alpha Z^\alpha \frac{\partial}{\partial x^i} + V^\alpha \frac{\partial}{\partial y^\alpha}\,.$$

The following notion shows that morphisms of Lie algebroids can also be prolonged.

Definition 3.1. *Let $\Psi : E \to \overline{E}$ be a an admissible map between two Lie algebroids $\tau : E \to M$ and $\tau : \overline{E} \to \overline{M}$. The* prolongation of Ψ *is the mapping $\mathcal{T}\Psi \colon \mathcal{T}E \to \mathcal{T}\overline{E}$ defined by $\mathcal{T}\Psi(a, b, v) = (\Psi(a), \Psi(b), T_a\Psi(v))$.*

It is not difficult to see that if Ψ is a morphism of Lie algebroids, then its prolongation is also a morphism of Lie algebroids. Given local basis $\{e_\alpha\}_{\alpha=1}^{\ell}$ and $\{\overline{e}_\alpha\}_{\alpha=1}^{\overline{\ell}}$ of sections of E and $\overline{E}$, respectively, if $\Psi(e_\alpha) = \Psi^\beta_\alpha \overline{e}_\beta$, then the action of the prolongation of Ψ on $\mathcal{T}E$ is determined by

$$\begin{aligned} &\mathcal{T}\Psi(\mathcal{X}_\alpha(a)) = \Psi^\beta_\alpha(m)\overline{\mathcal{X}}_\beta(\Psi(a)) + \rho^i_\alpha(m)\frac{\partial \Psi^\gamma_\beta}{\partial x^i}(m) a^\beta \overline{\mathcal{V}}_\gamma(\Psi(a))\,,\ a \in E_m\,, \\ &\mathcal{T}\Psi(\mathcal{V}_\alpha(a)) = \Psi^\beta_\alpha(m)\,\overline{\mathcal{V}}_\beta(\Psi(a))\,,\ a \in E_m\,. \end{aligned} \tag{3.2}$$

3.1 Vertical and horizontal sections

An element $(a, b, v) \in \mathcal{T}E$ is said to be *vertical* if it is of the form $(a, 0, v)$, with v a vertical vector tangent to E at a. It follows that the vertical space of $\mathcal{T}E$ at the point $a \in E_m$, which we denote by $\mathrm{Ver}_a(\mathcal{T}E)$, can be identified with E_m by (a slight modification of) the usual vertical lifting map:

$$b \in E_m \longmapsto (a, 0, b^{\scriptscriptstyle V}_a) \in \mathrm{Ver}_a(\mathcal{T}E)\,,$$

where $b^{\scriptscriptstyle V}_a \in T_aE$ is the tangent vector to the curve $t \mapsto a + tb$ at $t = 0$. If σ is a section of E, then the section $\sigma^{\scriptscriptstyle V}$ of $\mathcal{T}E$ defined by $\sigma^{\scriptscriptstyle V}(a) = (a, 0, \sigma(m)^{\scriptscriptstyle V}_a)$ will be called the *vertical lift of* σ. Vertical elements are linear combinations of $\{\mathcal{V}_\alpha\}_{\alpha=1}^{\ell}$. Specifically, if the section σ of E has the local expression $\sigma = \sigma^\alpha e_\alpha$, then $\sigma^{\scriptscriptstyle V}$ is of the form

$$\sigma^{\scriptscriptstyle V} = \sigma^\alpha \mathcal{V}_\alpha.$$

Consider the *zero-section of* $\tau : E \to M$, that is

$$0_M : M \to E\,, \quad 0_M(m) = 0_m\,.$$

This section is a canonical embedding of M into E. Consequently, we can regard TM as a subspace of TE. Now, define the *horizontal space* along 0_M,

$$\mathrm{Hor}_m(\mathcal{T}E) = \{\, (0_m, b, v) \in \mathcal{T}_{0_m}E \ \mid v \in T_mM \subseteq T_{0_m}E\,\}\,, \qquad m \in M\,.$$

Note that $\tau_1(\mathrm{Hor}(\mathcal{T}E)) = \mathrm{Im}(0_M)$, i.e. $\mathrm{Hor}(\mathcal{T}E)$ is only defined along points of E that belong to the zero-section 0_M. Moreover, $\mathrm{Hor}_m(\mathcal{T}E)$ is a vector subspace of $\mathcal{T}_{0_m}E$. The following result shows that the horizontal space is complementary to the vertical space along the zero-section 0_M.

Lemma 3.2. *Along the zero-section of $\tau : E \to M$, we have the following direct sum decomposition*

$$\mathcal{T}_{0_M} E = \mathrm{Hor}(\mathcal{T}E) \oplus \mathrm{Ver}_{0_M}(\mathcal{T}E)\,.$$

Proof. Consider the map $\mathcal{H}\colon E \to \mathcal{T}_{0_M}E$ given by $\mathcal{H}(b) = (0_{\tau(b)}, b, T0_M(\rho(b)))$. The image of $\mathcal{H}$ is precisely the horizontal space, $\mathrm{Hor}_m(\mathcal{T}E) = \mathcal{H}(E_m)$, $m \in M$. Note that $\mathcal{H}$ is a splitting of the short exact sequence

$$0 \longrightarrow E \xrightarrow{V} \mathcal{T}_{0_M} E \xrightarrow{\mathcal{T}\tau} E \longrightarrow 0\,,$$

where $V(b) = (0_{\tau(b)}, 0, b^V_{0_{\tau(b)}})$. As a consequence, we have $\mathcal{T}_{0_m}E = \mathcal{H}(E_m) \oplus \mathrm{Ver}_{0_m}(\mathcal{T}E)$, $m \in M$. The restriction of $\mathcal{T}\tau$ to $\mathrm{Hor}(\mathcal{T}E)$ is an isomorphism $\mathcal{T}\tau\colon \mathrm{Hor}(\mathcal{T}E) \to E$, whose inverse map is $\mathcal{H}$. For $(0_m, b, v) \in \mathcal{T}_{0_m}E$, the decomposition is given by $(0_m, b, v) = (0_m, b, T(0_M\circ\tau)(v)) + (0_m, 0_m, v - T(0_M\circ\tau)(v)) \in \mathrm{Hor}(\mathcal{T}E) + \mathrm{Ver}_{0_M}(\mathcal{T}E)$. □

3.2 Homogeneity

A property that plays an important role in our later analysis is that of homogeneity. Consider the section Δ of $\mathcal{T}E$ defined by $\Delta(a) = (a, 0, a^V_a)$, $a \in E$. The section Δ is called the *Liouville section of $\mathcal{T}E$*. In coordinates, we have

$$\Delta = y^\alpha \mathcal{V}_\alpha \qquad \text{and} \qquad \rho^1(\Delta) = y^\alpha \frac{\partial}{\partial y^\alpha}.$$

A function $F \in C^\infty(E)$ is said to be *homogeneous of degree $s \in \mathbb{Z}$* if

$$\mathcal{L}_{\rho^1(\Delta)} F = sF\,.$$

where $\mathcal{L}$ stands for the Lie derivative operator[1]. In a local chart, a homogeneous function of degree $s \geq 0$ is a homogeneous polynomial in $\{y^\alpha\}_{\alpha=1}^{\ell}$ of degree s with arbitrary functions of $(x^i)_{i=1}^n$ as coefficients. Consequently, homogeneous functions of degree 0 are (pullbacks of) functions on the base M and there are not (smooth) non-trivial functions homogeneous of degree $s \leq -1$. A section Z of $\mathcal{T}E$ is said to be *homogeneous of degree $s \in \mathbb{Z}$* if

$$[\Delta, Z] = sZ.$$

We denote by $\mathcal{P}_s$ the set of homogeneous sections of $\mathcal{T}E$ of degree s. The following result describes the basic properties concerning homogeneous sections. The proof is omitted for brevity.

Lemma 3.3. *Let r, $s \in \mathbb{Z}$ and let Z be a section of $\mathcal{T}E$. Then*

(i) $[\mathcal{P}_s, \mathcal{P}_r] \subseteq \mathcal{P}_{s+r}$*, and* $\mathcal{P}_s = \{0\}$ *if* $s \leq 2$*,*

(ii) $Z \in \mathcal{P}_{-1}$ *if and only if there exists a section σ of E such that* $Z = \sigma^V$*,*

(iii) $Z \in \mathcal{P}_0$ *if and only if Z is a projectable section,*

(iv) $\mathcal{X}_\alpha \in \mathcal{P}_0$ *and* $\mathcal{V}_\alpha \in \mathcal{P}_{-1}$*, for* $\alpha = 1, \ldots, \ell$*. Moreover, if* $Z = Z^\alpha \mathcal{X}_\alpha + V^\alpha \mathcal{V}_\alpha$ *is the local expression of Z, then $Z \in \mathcal{P}_s$ if and only if the functions Z^α are homogeneous of degree s and the functions V^α are homogeneous of degree $s+1$.*

Note that for all $Z \in \mathcal{P}_s$, $s \geq 1$, we have $Z(0_m) = 0_{0_m}$, $m \in M$, that is, the homogeneous sections of $\mathcal{T}E$ of degree greater or equal than 1 vanish at the zero-section of E.

[1] Alternatively, one can see that homogeneous functions of degree s verify $d_\Delta F = sF$. (See [21] for the precise definition of the derivative operator d_Δ).

3.3 SODE sections

The dynamics of a mechanical system evolving on a certain configuration manifold is described by means of a vector field on the tangent bundle of the manifold which is a second order differential equation. Likewise, we will need the notion of SODE section of the prolongation to describe the behavior of mechanical systems evolving on Lie algebroids. In this section, we introduce this concept and discuss several geometric properties, resembling those of second order differential equations on manifolds.

Definition 3.4. *A section* Γ *of* $\mathcal{T}E$ *is a second-order differential equation (*SODE*) section on the Lie algebroid* E *if* $\mathcal{T}\tau \circ \Gamma = Id_E$.

A vector $v \in T_aE$ is called *admissible* if $T\rho(v) = \rho(a)$. Note that a curve in E is admissible if and only if its tangent vectors are admissible. We denote by $\mathrm{Adm}(E)$ the set of all admissible tangent vectors. Notice that v is admissible if and only if $(a, a, v) \in \mathcal{T}E$. Therefore we can identify $\mathrm{Adm}(E)$ with the subset of $\mathcal{T}E$ formed by all elements of that form,

$$\mathrm{Adm}(E) = \{z \in \mathcal{T}E \mid \tau_1(z) = \mathcal{T}\tau(z)\}\,.$$

Equivalently, a SODE section can be defined as a section of $\mathcal{T}E$ that takes values in $\mathrm{Adm}(E)$, i.e., $\Gamma(a) = (a, a, X(a))$, $a \in E$, where X is a vector field on E verifying $\rho(a) = T_a\tau(X(a))$. If Γ is a SODE section, then it has the local expression $\Gamma = y^\alpha \mathcal{X}_\alpha + F^\alpha(x, y)\mathcal{V}_\alpha$ and its associated vector field is of the form

$$\rho^1(\Gamma) = \rho^i_\alpha y^\alpha \frac{\partial}{\partial x^i} + F^\alpha(x, y)\frac{\partial}{\partial y^\alpha}.$$

The integral curves of this vector field satisfy the differential equations

$$\dot{x}^i = \rho^i_\alpha y^\alpha\,, \qquad \dot{y}^\alpha = F^\alpha(x, y)\,.$$

In particular, we are specially interested in homogeneous SODE sections with degree 1. We will refer to such SODE sections as *sprays*. Locally, a spray is such that the functions F^α are homogeneous with degree 2, $F^\alpha(x, y) = -\frac{1}{2}S^\alpha_{\beta\gamma}(x)y^\beta y^\gamma$ for some symmetric coefficients $S^\alpha_{\beta\gamma}$. The sprays are in one to one correspondence with torsion-less E-covariant derivatives (cf. Section 2.2), as we show in the following.

Let ∇ be a connection on E. The geodesic equations (2.6), together with the admissibility condition, correspond to the differential equations of the integral curves of a SODE section Γ_∇ of $\mathcal{T}E$, which is locally given by

$$\Gamma_\nabla = y^\alpha \mathcal{X}_\alpha - \frac{1}{2}\left(\Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\beta}\right) y^\beta y^\gamma \mathcal{V}_\alpha\,.$$

From the coordinate expression, we easily see that Γ_∇ is homogeneous with degree one, i.e., it is a spray. Moreover, notice that Γ_∇ is determined by the symmetric product associated with ∇ (cf. eq. (2.4)).

We now show how a spray Γ and a (2,1)-tensor field T determine a connection on E in a unique way. As an intermediate step, we first define the symmetric product associated with a spray. Given $\sigma, \eta \in \mathrm{Sec}(E)$, consider the section of $\mathcal{T}E$, $[\sigma^V, [\Gamma, \eta^V]]$. From the properties of the Lie bracket, we deduce that this section is (smooth and) homogeneous with degree -1. From Lemma 3.3(ii), it follows that $[\sigma^V, [\Gamma, \eta^V]]$ is the vertical lift of a section of E, which we denote by $\langle \sigma : \eta\rangle_\Gamma$. Therefore,

$$\langle \sigma : \eta\rangle^V = [\sigma^V, [\Gamma, \eta^V]].$$

From the Jacobi identity, one can deduce that the operation $\langle \sigma : \eta\rangle$ is symmetric. Moreover, if f is a function on M, we have $\langle \sigma : f\eta\rangle = \rho(\sigma)f\eta + f\langle \sigma : \eta\rangle$, since

$$\langle \sigma : f\eta\rangle^V = [[\sigma^V, \Gamma], (f\eta)^V] = \rho^1([\sigma^V, \Gamma])f\eta^V + f[\sigma^V, [\Gamma, \eta^V]] = (\rho(\sigma)f\eta + f\langle \sigma : \eta\rangle)^V\,.$$

Definition 3.5. *Given a spray* Γ *on* E, $\langle \cdot : \cdot \rangle_\Gamma$ *is the* symmetric product associated with Γ.

Taking a local basis of sections of E, one can compute

$$[\Gamma, \sigma^V] = [y^\alpha \mathcal{X}_\alpha - \frac{1}{2} S^\alpha_{\beta\gamma} y^\beta y^\gamma \mathcal{V}_\alpha, \sigma^\beta \mathcal{V}_\beta] = -\sigma^\alpha \mathcal{X}_\alpha + y^\alpha \left(\rho^i_\alpha \frac{\partial \sigma^\gamma}{\partial x^i} + S^\gamma_{\alpha\beta} \sigma^\beta \right) \mathcal{V}_\gamma, \tag{3.3}$$

$$[\eta^V, [\Gamma, \sigma^V]] = \left(\sigma^\gamma \rho^k_\gamma \frac{\partial \eta^\alpha}{\partial x^k} + \eta^\gamma \rho^k_\gamma \frac{\partial \sigma^\alpha}{\partial x^k} + S^\alpha_{\beta\gamma} \sigma^\beta \eta^\gamma \right) \mathcal{V}_\alpha \,. \tag{3.4}$$

From the local expression of the symmetric product, one can see that the symmetric product *determines* and *is determined* by the spray Γ.

Proposition 3.6. *Given a spray Γ and a skew-symmetric (2,1) tensor field T on E, there exists a unique connection $\nabla^{\Gamma,T} \colon \mathrm{Sec}(E) \times \mathrm{Sec}(E) \to \mathrm{Sec}(E)$ on E such that its associated spray is Γ and its torsion is T. This connection is given by*

$$\nabla^{\Gamma,T}_\sigma \eta = \frac{1}{2}\big([\sigma, \eta] + T(\sigma, \eta)\big) + \frac{1}{2} \langle \sigma : \eta \rangle_\Gamma \,. \tag{3.5}$$

Proof. From the properties of the symmetric product and the Lie bracket, it follows that $\nabla^{\Gamma,T}$ defined in (3.5) is a connection on E. Its torsion is given by

$$\nabla^{\Gamma,T}_\sigma \eta - \nabla^{\Gamma,T}_\eta \sigma - [\sigma, \eta] = \frac{1}{2}\big([\sigma, \eta] + T(\sigma, \eta)\big) - \frac{1}{2}\big([\eta, \sigma] + T(\eta, \sigma)\big) - [\sigma, \eta] = T(\sigma, \eta).$$

By definition, the spray associated with $\nabla^{\Gamma,T}$ is the SODE section Z whose projection $\rho^1(Z)$ is determined by the differential equation $\nabla^{\Gamma,T}_{a(t)} a(t) = 0$, for admissible curves $a : \mathbb{R} \to E$, $t \mapsto a(t)$. A simple coordinate calculation shows that these equations can be locally written as $\dot{y}^\alpha + \frac{1}{2} S^\alpha_{\beta\gamma} y^\beta y^\gamma = 0$, which concludes the proof. $\square$

The connection coefficients of $\nabla^{\Gamma,T}$ are given by

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\Big(S^\alpha_{\beta\gamma} + T^\alpha_{\beta\gamma} + C^\alpha_{\beta\gamma}\Big).$$

Note also that the symmetric product associated with Γ precisely corresponds to the symmetric product defined by $\nabla_{\Gamma,T}$.

$$\langle \sigma : \eta \rangle_\Gamma = \nabla^{\Gamma,T}_\sigma \eta + \nabla^{\Gamma,T}_\eta \sigma \,.$$

Proposition 3.7. *Let $\tau : E \to M$ and $\tau : \overline{E} \to \overline{M}$ be two Lie algebroids. Let ∇ and $\overline{\nabla}$ be connections on E and $\overline{E}$, respectively. A morphism Ψ maps ∇ onto $\overline{\nabla}$ if and only if Ψ maps the associated spray Γ into the associated spray $\overline{\Gamma}$ and maps the torsion tensor T into the torsion tensor $\overline{T}$.*

Proof. We prove in coordinates the above statement for torsion-less connections, from where the general result follows easily. If we take a SODE $\Gamma = y^\alpha \mathcal{X}_\alpha + F^\alpha \mathcal{V}_\alpha$, then

$$\mathcal{T}\Psi(\Gamma(a)) = \overline{a}^\alpha \overline{\mathcal{X}}_\alpha + \left[\rho^i_\alpha(m) a^\alpha \frac{\partial \Psi^\gamma_\beta}{\partial x^i}(m) a^\beta + F^\alpha(a) \Psi^\gamma_\alpha(m) \right] \overline{\mathcal{X}}_\gamma \,, a \in E_m \,.$$

This last expression is equal to $\overline{\Gamma}(\overline{a}) = \overline{a}^\alpha \overline{\mathcal{X}}_\alpha + \overline{F}^\alpha(\overline{a}) \overline{\mathcal{V}}_\alpha$ if and only if

$$\overline{F}^\alpha(\overline{a}) = \Psi^\gamma_\alpha F^\alpha(a) + a^\beta \rho^i_\alpha a^\alpha \frac{\partial \Psi^\gamma_\beta}{\partial x^i} \,.$$

In the case of two sprays, $F^\gamma = -\frac{1}{2} S^\gamma_{\alpha\beta} y^\alpha y^\beta$ and $\overline{F}^\gamma = -\frac{1}{2} \overline{S}^\gamma_{\alpha\beta} \overline{y}^\alpha \overline{y}^\beta$, so that the above equation reads

$$\Psi^\beta_\gamma S^\gamma_{\alpha\delta} = \left(\rho^i_\alpha \frac{\partial \Psi^\beta_\delta}{\partial x^i} + \rho^i_\delta \frac{\partial \Psi^\beta_\alpha}{\partial x^i} \right) + \overline{S}^\beta_{\theta\sigma} \Psi^\theta_\alpha \Psi^\sigma_\delta.$$

Since Ψ is a morphism, it verifies equation (2.3). Summing both expressions, we finally obtain equation (2.5), as claimed. $\square$

3.4 Geodesically invariant subbundles

Here we introduce the notion of geodesically invariant subbundles (which is a generalization of the concept of geodesically invariant distributions [18]), and establish its relation with the symmetric product associated with the connection.

Definition 3.8. *Let ∇ be a connection on E. A subbundle $D \subset E$ is* geodesically invariant for ∇ *if every geodesic $a\colon [t_0, t_1] \to E$ such that $a(t_0) \in D$ verifies that $a(t) \in D$ for all $t \in [t_0, t_1]$.*

Proposition 3.9. *Let ∇ be a connection on E. A subbundle $D \subset E$ is geodesically invariant for ∇ if and only if it is invariant under the symmetric product, i.e. $\langle \sigma : \eta \rangle \in \mathrm{Sec}(D)$ for every $\sigma, \eta \in \mathrm{Sec}(D)$.*

Proof. By definition, the subbundle D is geodesically invariant if and only if the spray of the connection Γ_∇ is tangent to D, i.e. $\rho^1(\Gamma)|_D \in TD$. The latter is equivalent to the condition $\big(\mathcal{L}_{\rho^1(\Gamma)}\phi\big)_{|D} = 0$ for every function ϕ on E such that $\phi_{|D} = 0$. Since D is linear, the constraint functions ϕ are of the form $\phi = \hat{\mu}$, with $\mu \in \mathrm{Sec}(D^\circ)$ and $\hat{\mu}(a) = \langle \mu_m, a \rangle$, $a \in E_m$. Therefore, D is geodesically invariant if and only if $(\mathcal{L}_{\rho^1(\Gamma)}\hat{\mu}) \circ \sigma = 0$ for all $\mu \in \mathrm{Sec}(D^\circ)$ and $\sigma \in \mathrm{Sec}(D)$. Let us see that $(\mathcal{L}_{\rho^1(\Gamma)}\hat{\mu}) \circ \sigma = -\frac{1}{2}\langle \mu, \nabla_\sigma \sigma \rangle$. In coordinates

$$\mathcal{L}_{\rho^1(\Gamma)}\hat{\mu} = \mathcal{L}_{\rho^1(\Gamma)}(\mu_\alpha y^\alpha) = \rho^i_\beta \frac{\partial \mu_\alpha}{\partial x^i} y^\alpha y^\beta - \frac{1}{2}\mu_\gamma S^\gamma_{\alpha\beta} y^\alpha y^\beta \,.$$

Taking the restriction to the image of σ, and writing explicitly the symmetric parts, we have

$$\mathcal{L}_{\rho^1(\Gamma)}\hat{\mu} \circ \sigma = \frac{1}{2}\left(\rho^i_\beta \frac{\partial \mu_\alpha}{\partial x^i} + \rho^i_\alpha \frac{\partial \mu_\beta}{\partial x^i} - \mu_\gamma S^\gamma_{\alpha\beta}\right)\sigma^\alpha \sigma^\beta \,.$$

On the other hand

$$\nabla_\sigma \mu = \sigma^\alpha \left(\rho^i_\alpha \frac{\partial \mu_\beta}{\partial x^i} - \mu_\gamma \Gamma^\gamma_{\beta\alpha}\right) e^\beta \,,$$

and thus

$$\langle \nabla_\sigma \mu, \sigma \rangle = \sigma^\alpha \sigma^\beta \left(\rho^i_\alpha \frac{\partial \mu_\beta}{\partial x^i} - \mu_\gamma \Gamma^\gamma_{\beta\alpha}\right) = \frac{1}{2}\sigma^\alpha \sigma_\beta \left(\rho^i_\alpha \frac{\partial \mu_\beta}{\partial x^i} + \rho^i_\beta \frac{\partial \mu_\alpha}{\partial x^i} - \mu_\gamma S^\gamma_{\beta\alpha}\right),$$

where we have used the fact that $\Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha} = S^\gamma_{\beta\alpha}$. Therefore $(\mathcal{L}_{\rho^1(\Gamma)}\hat{\mu}) \circ \sigma = \frac{1}{2}\langle \nabla_\sigma \mu, \sigma \rangle$, and the result follows by taking into account that $\langle \nabla_\sigma \mu, \sigma \rangle = \rho(\sigma)\langle \mu, \sigma \rangle - \langle \mu, \nabla_\sigma \sigma \rangle = -\langle \mu, \nabla_\sigma \sigma \rangle$, because $\langle \mu, \sigma \rangle = 0$. □

Given a connection ∇ on E and a projector map $P : E \to D$, consider the constrained connection $\check{\nabla}$ introduced in Section 2.2. Using the above result and Proposition 2.8(ii), one can deduce that D is geodesically invariant for $\check{\nabla}$.

4 General control systems on Lie algebroids

In this section we present the notion of a control system on a Lie algebroid. We introduce the concept of accessibility subbundle in the Lie algebroid and provide basic tests for controllability, building on the known results for control systems defined on manifolds. Finally, we study the controllability properties of control systems related by means of a morphism of Lie algebroids.

Consider a Lie algebroid $\tau : E \to M$, with anchor map $\rho : E \to TM$. Let $\sigma, \eta_1, \dots, \eta_k$ be sections of E. A *control problem on the Lie algebroid $E \to M$ with drift section σ and input sections $\eta_1, \dots, \eta_k$* is defined by the following equation on M,

$$\dot{m}(t) = \rho\Big(\sigma(m(t))) + \sum_{i=1}^{k} u_i(t)\eta_i(m(t))\Big)\,, \tag{4.1}$$

where $u = (u_1, \dots, u_k) \in U$, and U is an open set of $\mathbb{R}^k$ containing 0. The function $t \mapsto u(t) = (u_1(t), \dots, u_m(t))$ belongs to a certain class of functions of time, denoted by $\mathcal{U}$, called the *set of admissible* *controls*. For our purposes, we may restrict the admissible controls to be the piecewise constant functions with values in U. Notice that the trajectories of the control system are admissible curves of the Lie algebroid, and therefore they must lie on a leaf of E. It then follows that if E is not transitive, then there are points that cannot be connected by solutions of any control system defined on such a Lie algebroid. In particular, the system (4.1) cannot be locally accessible at points $m \in M$ where ρ is not surjective. Since the emphasis here is put on the controllability analysis, without loss of generality we will restrict our attention to locally transitive Lie algebroids.

Denoting by $f = \rho(\sigma)$ and $g_i = \rho(\eta_i)$, we can rewrite the system (4.1) as

$$\dot{m}(t) = f(m(t)) + \sum_{i=1}^{k} u_i(t) g_i(m(t)) \,, \tag{4.2}$$

which is a standard nonlinear control system on M affine in the inputs. Here we make use of the additional geometric structure provided by the Lie algebroid in order to carry over the analysis of the controllability properties of the control system (4.1). We refer to [28] for a comprehensive discussion of the notions of reachable sets, accessibility algebra and computable accessibility tests. A short list of definitions is provided in the appendix for reference.

Definition 4.1. *The* accessibility algebra $\mathcal{D}$ of the control system (4.1) in the Lie algebroid *is the smallest subalgebra of* $\operatorname{Sec}(E)$ *that contains the sections* $\sigma, \eta_1, \ldots, \eta_k$.

Using the Jacobi identity, one can deduce that any element of accessibility algebra $\mathcal{D}$ is a linear combination of repeated Lie brackets of sections of the form

$$[\zeta_l, [\zeta_{l-1}, [\ldots, [\zeta_2, \zeta_1] \ldots]]] \,,$$

where $\zeta_i \in \{\sigma, \eta_1, \ldots, \eta_k\}$, $1 \leq i \leq l$ and $l \in \mathbb{N}$.

Definition 4.2. *The* accessibility subbundle in the Lie algebroid*, denoted by* $\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})$*, is the vector subbundle of E generated by the accessibility algebra* $\mathcal{D}$*,*

$$\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\}) = \operatorname{span}\{\zeta(m) \mid \zeta \textit{ section of } E \textit{ in } \mathcal{D}\} \,, \quad m \in M \,.$$

If the dimension of $\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})$ is constant, then $\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})$ is the smallest Lie subalgebroid of E that has $\{\sigma, \eta_1, \ldots, \eta_k\}$ as sections. In the following result, we establish accessibility tests for control systems of the form (4.1).

Theorem 4.3. *Consider the system* (4.1)*. Let $m \in M$ and assume the Lie algebroid E is locally transitive at m. Then,* $\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})(m) + \ker \rho(m) = E_m$ *implies that the system is locally accessible from m.*

Proof. We first prove that every element X of the accessibility algebra $\mathcal{C}$ of (4.2) can be written as $X = \rho(\zeta)$, with $\zeta \in \mathcal{D}$ (see the appendix for the precise definition of the accessibility algebra $\mathcal{C}$). Take an element of $\mathcal{C}$ of the form $[X_l, [X_{l-1}, [\ldots, [X_2, X_1] \ldots]]]$, with $X_i \in \{f = \rho(\sigma), g_1 = \rho(\eta_1), \ldots, g_k = \rho(\eta_k)\}$. Denote $X_i = \rho(\zeta_i)$, with $\zeta_i \in \{\sigma, \eta_1, \ldots, \eta_k\}$. Since ρ is a Lie algebra homomorphism, we have that

$$[X_l, [X_{l-1}, [\ldots, [X_2, X_1] \ldots]]] = \rho\left([\zeta_l, [\zeta_{l-1}, [\ldots, [\zeta_2, \zeta_1] \ldots]]]\right) \,.$$

Since ρ is linear, we conclude the accessibility subbundle in the Lie algebroid is mapped by ρ onto the accessibility distribution C. Now, we show $C(m) = T_m M$ if and only if $\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})(m) + \ker \rho(m) = E_m$. Assume first that $C(m) = T_m M$. Let $e_m \in E_m$. Consider $\rho(e) \in T_m M = C(m) = \rho(\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})(m))$. There exists $a \in \overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})(m)$ such that $\rho(e) = \rho(a)$. Then, $e - a \in \ker \rho(m)$ and $e = a + (e - a)$. Therefore $E_m = \overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})(m) + \ker \rho(m)$. In addition, $\rho(E_m) = \rho(\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})(m)) = T_m M$. The other implication is trivial. Finally, the result follows from Chow's theorem [28]. □

Remark 4.4. If the dimension of $\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})$ is constant, then the above theorem expresses the following fact: if the Lie subalgebroid $\overline{\operatorname{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})$ is locally transitive at m, then the system (4.1) is locally accessible from m.

In practice, the most interesting property to establish is controllability. In what follows, we provide a controllability test for systems of the form (4.1) by adapting the notions of good and bad Lie brackets of vector fields [32] to the setting of Lie algebroids. Let $\{X_0, X_1, \ldots, X_k\}$ be a set of sections of the Lie algebroid E. The *degree* of an iterated Lie bracket B of elements in $\{X_0, X_1, \ldots, X_k\}$ is the number of occurrences of all its factors, and is therefore given by $\delta(B) = \delta_0(B) + \delta_1(B) + \cdots + \delta_k(B)$, where $\delta_i(B)$ is the number of times that X_i appears in B. A Lie bracket B is said to be *bad* if $\delta_0(B)$ is odd and $\delta_i(B)$ is even, $i \in \{1, \ldots, k\}$. Otherwise, B is said to be *good*. To make precise sense of these notions (degree, bad, good) one must resort to the concept of free Lie algebras, but it should be clear from the context what we mean here (see [32] for a detailed discussion).

Theorem 4.5. *Assume that the system* (4.1) *is locally accessible from $m \in M$. If every bad Lie bracket B in $\{\sigma, \eta_1, \ldots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good Lie brackets in $\{\sigma, \eta_1, \ldots, \eta_k\}$ of lower degree than B and elements in the kernel of the anchor map ρ at m,* $\ker \rho(m)$*, then the system is locally controllable from m.*

Proof. Under the hypothesis of the theorem, any bad Lie bracket of the vector fields $\{f = \rho(\sigma), g_1 = \rho(\eta_1), \ldots, g_k = \rho(\eta_k)\}$ evaluated at m can be written a $\mathbb{R}$-linear combination of good Lie brackets in $\{f, g_1, \ldots, g_k\}$ of lower degree. The application of Theorem 7.3 in [32] gives the result. □

4.1 Control systems related by a morphism of Lie algebroids

Let $\tau : E \to M$ and $\overline{\tau} : \overline{E} \to \overline{M}$ be two Lie algebroids. Consider a control system defined on E with drift section σ and independent input sections η_i, $i = 1, \ldots, k$, and a control system defined on $\overline{E}$ with drift section $\overline{\sigma}$ and independent input sections $\overline{\eta}_j$, $j = 1, \ldots, \overline{k}$. Let $\Psi : E \to \overline{E}$ be a morphism of Lie algebroids. Let $\mathrm{span}\{\overline{\eta}_1, \ldots \overline{\eta}_{\overline{k}}\}$ denote the linear subbundle of $\overline{E}$ generated by the sections $\overline{\eta}_j$, $j = 1, \ldots, \overline{k}$, and consider the affine subbundle $\xi + \mathrm{span}\{\overline{\eta}_1, \ldots \overline{\eta}_{\overline{k}}\}$.

Definition 4.6. *The control problems on E and $\overline{E}$ are* weakly Ψ-related *(or Ψ maps the system on E onto the system on $\overline{E}$) if $\Psi(\sigma(m)) \in \xi(\psi(m)) + \mathrm{span}\{\overline{\eta}_1, \ldots \overline{\eta}_{\overline{k}}\}(\psi(m))$ and $\Psi(\eta_i(m)) \in \mathrm{span}\{\overline{\eta}_1, \ldots \overline{\eta}_{\overline{k}}\}(\psi(m))$ for all $i = 1, \ldots, k$ and all $m \in M$.*

Equivalently, two systems are weakly Ψ-related if there exist functions C^i_j and b^j on M such that

$$\Psi \circ \sigma = \xi \circ \psi + \sum_{j=1}^{\overline{k}} b^j (\overline{\eta}_j \circ \psi) \quad \text{and} \quad \Psi \circ \eta_i = \sum_{j=1}^{\overline{k}} C^j_i (\overline{\eta}_j \circ \psi)\,, \quad i = 1, \ldots, k\,. \tag{4.3}$$

Definition 4.7. *The control problems on E and $\overline{E}$ are* Ψ-related *if σ is Ψ-related to a section of $\overline{E}$ with values in $\xi + \mathrm{span}\{\overline{\eta}_1, \ldots \overline{\eta}_{\overline{k}}\}$ and η_i is Ψ-related to a section of $\overline{E}$ with values in $\mathrm{span}\{\overline{\eta}_1, \ldots \overline{\eta}_{\overline{k}}\}$, for all $i = 1, \ldots, k$.*

Equivalently, two systems are Ψ-related if there exist functions $\overline{C}^i_j$ and $\overline{b}^j$ on $\overline{M}$ such that

$$\Psi \circ \sigma = \xi \circ \psi + \sum_{j=1}^{\overline{k}} (\overline{b}^j \overline{\eta}_j) \circ \psi \quad \text{and} \quad \Psi \circ \eta_i = \sum_{j=1}^{\overline{k}} (\overline{C}^j_i \overline{\eta}_j) \circ \psi\,, \quad i = 1, \ldots, k\,. \tag{4.4}$$

Clearly, Ψ-related systems are also weakly Ψ-related. The following result establishes the relation between the controllability properties of weakly Ψ-related systems.

Proposition 4.8. *Let $\Psi \colon E \to \overline{E}$ be a morphism of Lie algebroids such that the associated base map ψ is open. Consider two control systems on E and $\overline{E}$ that are weakly Ψ-related. If the system on E is locally accessible (respectively locally controllable) from $m \in M$, then the system on $\overline{E}$ is locally accessible (respectively locally controllable) from $\psi(m)$.*

Proof. First, note that if two control systems are Ψ-related, then the image by the morphism Ψ of a solution $a(t)$ of the system on E with control functions $t \mapsto u_i(t)$, $i = 1, \ldots, k$ is a solution of the system on $\overline{E}$ with control functions $t \mapsto \overline{u}_j(t) = b^j(\tau(a(t))) + \sum_{i=1}^{k} C_i^j(\tau(a(t)))u_i(t)$, $j = 1, \ldots, \overline{k}$.

Let $m \in M$ and take $\overline{V}$ an open neighborhood of $\psi(m)$ in $\overline{M}$. Consider the open neighborhood $V = \psi^{-1}(\overline{V})$ of m. If the system on E is locally accessible from m, then there is a non-empty open set $\mathcal{O}$ contained in $\mathcal{R}_M^V(m, \leq T)$ (where $\mathcal{R}_M^V(m, \leq T)$ denotes the reachable set in M starting from m in time less than or equal to T, see the appendix). Since ψ is an open map and $\psi(\mathcal{R}_M^V(m,t)) \subset \mathcal{R}_{\overline{M}}^{\psi(V)}(\psi(m),t)$, it follows that $\overline{\mathcal{O}} = \psi(\mathcal{O}) \subset \mathcal{R}_{\overline{M}}^{\overline{V}}(\psi(m), \leq T)$, and thus the system on $\overline{E}$ is locally accessible (respectively controllable) from $\psi(m)$. The argument for the controllable case is analogous. $\square$

A interesting particular case occurs when Ψ is an isomorphism between the fibers of the Lie algebroids. In such a case, if two systems are Ψ-related, one can see that the sufficient conditions for local accessibility (respectively controllability) are either simultaneously satisfied on E and $\overline{E}$ or simultaneously not satisfied.

Proposition 4.9. *Let $\Psi : E \to \overline{E}$ be a morphism of Lie algebroids which is an isomorphism on each fiber. Consider two control systems on E and $\overline{E}$, with $k \geq \overline{k}$, that are Ψ-related. Let $m \in M$. Then*

(a) $\overline{\mathrm{Lie}}(\{\sigma, \eta_1, \ldots, \eta_k\})(m) = E_m$ *if and only if* $\overline{\mathrm{Lie}}(\{\overline{\sigma}, \overline{\eta}_1, \ldots, \overline{\eta}_{\overline{k}}\})(\psi(m)) = \overline{E}_{\psi(m)}$,

(b) Every bad Lie bracket in $\{\sigma, \eta_1, \ldots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good Lie brackets of lower degree if and only if every bad Lie bracket in $\{\overline{\sigma}, \overline{\eta}_1, \ldots, \overline{\eta}_{\overline{k}}\}$ evaluated at $\psi(m) \in \overline{M}$ can be put as an $\mathbb{R}$-linear combination of good Lie brackets of lower degree.

Proof. Since the control systems are Ψ-related and Ψ is an isomorphism on each fiber, we have that $k = \overline{k}$, and one can assume without loss of generality that the drift sections σ and $\overline{\sigma}$ and the input sections η_i and $\overline{\eta}_i$ satisfy

$$\Psi \circ \sigma = \overline{\sigma} \circ \psi \qquad \text{and} \qquad \Psi \circ \eta_i = \overline{\eta}_i \circ \psi\,, \quad i = 1, \ldots, k\,. \tag{4.5}$$

From the properties of a morphism of Lie algebroids (see Section 2.1), we deduce that the Lie bracket of any subset of sections in $\{\sigma, \eta_1, \ldots, \eta_k\}$ is Ψ-related to the Lie bracket of the corresponding subset of sections in $\{\overline{\sigma}, \overline{\eta}_1, \ldots, \overline{\eta}_k\}$. Using now the fact that both Lie algebroids have fibers of the same dimension, we conclude the result. $\square$

As an example application, consider the case of a control system on a manifold M invariant under the action of a symmetry Lie group G on M. Assume that the action of G on M is free and proper, so that $\pi \colon M \to M/G = \overline{M}$ is a principal fiber bundle. Then the quotient map $\Psi \colon TM \to TM/G$ is a morphism of Lie algebroids between $E = TM \to M$ and $\overline{E} = TM/G \to T(M/G)$. Moreover, Ψ is an isomorphism in every fiber and its associated base map ψ is open. Being the control system on E invariant under the action of G, it induces a control system on $\overline{E}$. From the above results, we conclude that if the reduced system satisfies the sufficient conditions for local controllability (respectively accessibility), then the original system also satisfies such conditions. Moreover, since the map ψ is open, if the reduced system is not locally controllable (respectively accessible), then the original system cannot be locally controllable (respectively accessible).

5 Mechanical control systems

In this section, we consider control problems defined on the prolongation $\mathcal{T}E$ of E and we make use of the special geometry of this Lie algebroid to further investigate the controllability properties of the control systems defined on it. Given a Lagrangian function $L : E \to \mathbb{R}$ on the Lie algebroid, define the associated action functional $\mathcal{J} = \int_{t_0}^{t_1} L\, dt$. Consider the following constrained variational problem: find the extremals of $\mathcal{J}$ among the set of admissible curves with fixed endpoints m_0 and m_1 in the base M, i.e. curves $a : [t_0, t_1] \to E$, $m(t) = \tau(a(t))$, satisfying

$$\rho(a(t)) = \dot{m}(t)\,, \quad m(t_0) = m_0 \quad \text{and} \quad m(t_1) = m_1\,.$$

One can see that the infinitesimal variations $W : [t_0, t_1] \to \mathcal{T}E$ of a curve $a : [t_0, t_1] \to E$ corresponding to this constrained variational problem are of the form

$$W(t) = \rho^i_\alpha(m(t))\sigma^\alpha(t)\frac{\partial}{\partial x^i} + \left(\frac{d\sigma^\alpha}{dt} + C^\alpha_{\beta\gamma}a^\beta(t)\sigma^\gamma(t)\right)\frac{\partial}{\partial y^\alpha},$$

where $\sigma(t)$ is a curve on E over $m(t)$ which vanishes at t_0 and t_1. A simple calculation and the application of the Fundamental Theorem of Calculus show that the equations of motion describing the solutions of the constrained variational problem are

$$\frac{d}{dt}\frac{\partial L}{\partial y^\alpha} + C^\gamma_{\alpha\beta}y^\beta\frac{\partial L}{\partial y^\gamma} = \rho^i_\alpha\frac{\partial L}{\partial x^i}.$$

Euler-Lagrange operator. In alternative terms [7, 24, 35], the infinitesimal variations of a curve $a : [t_0, t_1] \to E$ can be written as the restriction to the curve of the complete lift of a general time-dependent section of E. In such a case, the above equations are precisely the components of the Euler-Lagrange operator $\delta L : \mathrm{Adm}(E) \to E^*$, which locally reads

$$\delta L = \left(\frac{d}{dt}\frac{\partial L}{\partial y^\alpha} + C^\gamma_{\alpha\beta}y^\beta\frac{\partial L}{\partial y^\gamma} - \rho^i_\alpha\frac{\partial L}{\partial x^i}\right)e^\alpha,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$. The equations of motion just read

$$\delta L = 0\,.$$

Equivalently, an admissible curve $a : [t_0, t_1] \to E$ is an extremal of the action functional $\mathcal{J}$ if the Euler-Lagrange operator δL vanishes at the points of the curve in $\mathrm{Adm}(E)$, $t \mapsto (a(t), a(t), \dot{a}(t))$.

Nonholonomic constraints. In the case of a system subject to (linear) nonholonomic constraints, in addition to the above data there is also a subbundle D of the Lie algebroid which prescribes the allowed velocities for the system. The equations of motion then are given by the application of the Lagrange-d'Alembert principle,

$$\delta L \in D^\circ\,, \qquad a \in D.$$

If there is a projector $P : E \to D$ onto D, denoting $Q = I - P$, the above equations can be rewritten as

$$P^*(\delta L) = 0\,, \qquad Q(a) = 0.$$

Here P^* stands for the dual linear map $\langle P^*(\theta), a\rangle = \langle\theta, P(a)\rangle$, $a \in E$, $\theta \in E^*$.

Control forces. In the presence of external forces, the equations of motion for both the unconstrained and the constrained situations have to be modified. Assume that some input forces $\{\theta_1, \ldots, \theta_m\} \subset \mathrm{Sec}(E^*)$ act on the Lagrangian system on E. Then, the equations of motion for the unconstrained control system read

$$\delta L = \sum_{l=1}^m u_l\,\theta_l\,, \tag{5.1}$$

and the equations of motion for the nonholonomically constrained control system are

$$P^*(\delta L) = \sum_{l=1}^m u_l P^*(\theta_l)\,, \quad Q(a) = 0\,. \tag{5.2}$$

Mechanical control systems. For the remainder of the paper, we will focus our attention on the class of Lagrangian control systems $(L, \{\theta_1, \dots, \theta_k\})$ whose Lagrangian function $L : E \to \mathbb{R}$ is of the form

$$L(a) = \frac{1}{2}\mathcal{G}(a,a) - V \circ \tau(a)\,, \quad a \in E,$$

with $\mathcal{G}\colon E \times_M E \to \mathbb{R}$ a bundle metric on E and V a function on M. We denote by $\{\eta_1, \dots, \eta_k\}$ the input sections of E determined by the control forces $\{\theta_1, \dots, \theta_k\}$ via the metric, i.e., $\eta_i = \sharp_{\mathcal{G}}(\theta_i)$. If $\Gamma_{\nabla^{\mathcal{G}}}$ denotes the spray associated with the Levi-Civita connection $\nabla^{\mathcal{G}}$, the controlled equations (5.1) can be written as

$$\dot{a}(t) = \rho^1\Big(\Gamma_{\nabla^{\mathcal{G}}}(a(t)) - (\operatorname{grad}_{\mathcal{G}} V)^{\scriptscriptstyle V}(a(t)) + \sum_{i=1}^{k} u_i(t)\eta_i^{\scriptscriptstyle V}(a(t))\Big). \tag{5.3}$$

Note that this system is a control problem on the Lie algebroid $\mathcal{T}E \to E$ as defined in Section 4. This is the reason why we will refer to the control problem with data $(\mathcal{G}, V, \{\theta_1, \dots, \theta_k\})$ as a *mechanical control system on a Lie algebroid*. Locally, the equations can be written as

$$\begin{aligned}
\dot{x}^i &= \rho^i_\alpha y^\alpha, \\
\dot{y}^\alpha &= -\frac{1}{2}\left(\Gamma^\alpha_{\beta\gamma}(x) + \Gamma^\alpha_{\gamma\beta}(x)\right) y^\beta y^\gamma - \mathcal{G}^{\alpha\beta}\rho^i_\beta \frac{\partial V}{\partial x^i} + \sum_{i=1}^{k} u_i(t)\eta_i^\alpha(x).
\end{aligned}$$

Alternatively, one can describe the dynamical behavior of the mechanical control system by means of an equation on E via the covariant derivative. An admissible curve $a : t \mapsto a(t)$ is a solution of the system (5.3) if and only if

$$\nabla^{\mathcal{G}}_{a(t)} a(t) + \operatorname{grad}_{\mathcal{G}} V(m(t)) = \sum_{i=1}^{k} u_i(t)\eta_i(m(t))\,. \tag{5.4}$$

Mechanical control systems with constraints. If the mechanical control system $(\mathcal{G}, V, \{\theta_1, \dots, \theta_k\})$ is subject to the constraints determined by a subbundle D of E, we can do the following. Consider the orthogonal decomposition $E = D \oplus D^\perp$, and the associated orthogonal projectors $P : E \to D$, $Q : E \to D^\perp$. Using the fact that $\mathcal{G}(P\cdot, \cdot) = \mathcal{G}(\cdot, P\cdot)$, one can write the controlled equations (5.2) as

$$P(\nabla^{\mathcal{G}}_{a(t)} a(t)) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = \sum_{i=1}^{k} u_i(t) P(\eta_i(m(t)))\,, \qquad Q(a) = 0\,.$$

In terms of the constrained connection $\check{\nabla}_\sigma \eta = P(\nabla^{\mathcal{G}}_\sigma \eta) + \nabla^{\mathcal{G}}_\sigma (Q\eta)$ (cf. Section 2.2), we can rewrite this equation as $\check{\nabla}_{a(t)} a(t) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = \sum_{i=1}^{k} u_i(t) P(\eta_i(m(t)))$, $Q(a) = 0$. Since the subbundle D is geodesically invariant for the connection $\check{\nabla}$, it follows that any integral curve of the spray $\Gamma_{\check{\nabla}}$ associated with $\check{\nabla}$ starting from $a_0 \in D$ is entirely contained in D (cf. Definition 3.8). Since the forcing terms in (5.5) coming from the potential and the inputs belong to D, the same property holds for the total controlled dynamics. As a consequence, the controlled equations can be simply stated as

$$\check{\nabla}_{a(t)} a(t) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = \sum_{i=1}^{k} u_i(t) P(\eta_i(m(t)))\,, \qquad a_0 \in D\,. \tag{5.5}$$

Note that one can write the controlled dynamics as a control system on the Lie algebroid $\mathcal{T}E \to E$,

$$\dot{a}(t) = \rho^1\Big(\Gamma_{\check{\nabla}}(a(t)) - P(\operatorname{grad}_{\mathcal{G}} V)^{\scriptscriptstyle V}(a(t)) + \sum_{i=1}^{k} u_i(t) P(\eta_i)^{\scriptscriptstyle V}(a(t))\Big). \tag{5.6}$$

The coordinate expression of these equations is greatly simplified if we take a basis $\{e_\alpha\} = \{e_a, e_A\}$ of E adapted to the orthogonal decomposition $E = D \oplus D^\perp$, i.e., $D = \operatorname{span}\{e_a\}$, $\mathcal{D}^\perp = \operatorname{span}\{e_A\}$. Denoting by

$(y^\alpha) = (y^a, y^A)$ the induced coordinates, the constraint equations $Q(a) = 0$ just read $y^A = 0$. The controlled equations (5.5) are then

$$\begin{aligned}
\dot{x}^i &= \rho^i_a y^a, \\
\dot{y}^a &= -\frac{1}{2} S^a_{bc} y^b y^c - \mathcal{G}^{a\beta} \rho^i_\beta \frac{\partial V}{\partial x^i} + \sum_{i=1}^k u_i(t) P(\eta_i)^a, \\
y^A &= 0.
\end{aligned}$$

Connection control systems. Given that the structure of the controlled equations is the same both in the absence (5.4) and in the presence of constraints (5.5), we will in general talk about *connection control systems on $\tau : E \to M$*. The dynamics of these systems is governed by an equation of the type

$$\nabla_{a(t)} a(t) + \eta(m(t))) = \sum_{i=1}^k u_i(t) \eta_i(m(t))) \,. \tag{5.7}$$

Here ∇ is a connection on E, and $\{\eta, \eta_1, \ldots, \eta_k\}$ are sections of E. We will often refer to η as the potential energy term in equations (5.7). Associated with this equation, there is always a control system on the Lie algebroid $\mathcal{T}E \to E$ given by

$$\dot{a}(t) = \rho^1 \Big((\Gamma_\nabla - \eta^{\scriptscriptstyle V})(a(t)) + \sum_{i=1}^k u_i(t) \eta_i^{\scriptscriptstyle V}(a(t)) \Big). \tag{5.8}$$

5.1 Accessibility and controllability notions

Here we introduce the notions of accessibility and controllability that are specialized to mechanical control systems on Lie algebroids. Let $m \in M$ and consider a neighborhood V of m in M. Define the set of reachable points in the base manifold M starting from m as

$$\begin{aligned}
\mathcal{R}^V_M(m, T) = \{ m' \in M \mid \exists u \in \mathcal{U} \text{ defined on } [0, T] \text{ such that the evolution of (5.8)} \\
\text{for } a(0) = 0_m \text{ satisfies } \tau(a(t)) \in V,\, t \in [0, T] \text{ and } \tau(a(T)) = m' \} \,.
\end{aligned}$$

Alternatively, one may write $\mathcal{R}^V_M(m, T) = \tau(\mathcal{R}^{\tau^{-1}(V)}_E(0_m, T))$. Denote

$$\mathcal{R}^V_M(m, \leq T) = \bigcup_{t \leq T} \mathcal{R}^V_M(m, t) \,.$$

Definition 5.1. *The system* (5.8) *is* locally base accessible from m *(respectively,* locally base controllable from m*) if $\mathcal{R}^V_M(m, \leq T)$ contains a non-empty open set of M (respectively, $\mathcal{R}^V_M(m, \leq T)$ contains a non-empty open set of M to which m belongs) for all neighborhoods V of m and all $T > 0$. If this holds for any $m \in M$, then the system is called* locally base accessible *(respectively,* locally base controllable*).*

In addition to the notions of base accessibility and base controllability, we shall also consider full-state accessibility and controllability starting from points of the form $0_m \in E$, $m \in M$ (note that full-state is meant here with regards to E, not to TM).

Definition 5.2. *The system* (5.8) *is* locally accessible from m at zero *(respectively,* locally controllable from m at zero*) if $\mathcal{R}^W_E(0_m, \leq T)$ contains a non-empty open set of E (respectively, $\mathcal{R}^W_E(0_m, \leq T)$ contains a non-empty open set of E to which 0_m belongs) for all neighborhoods W of 0_m in E and all $T > 0$. If this holds for any $m \in M$, then the system is called* locally accessible at zero *(respectively,* locally controllable at zero*).*

The relevance of the above definitions stems from the fact that, frequently, one needs to control a system by starting at rest. Nevertheless it is important to notice that not every equilibrium point at m corresponds to the point 0_m. Indeed, there might be other relative equilibrium points, explicitly all those points $a \in E$ such that $\rho^1(\Gamma(a)) = 0$, i.e., a is in the kernel of the anchor map ρ and $F^\alpha(a_m) = 0$, $\alpha = 1, \ldots, \ell$.

Finally, we also introduce the notion of accessibility and controllability with regards to a manifold.

Definition 5.3. *Let $\psi : M \to N$ be an open mapping. The system* (5.8) *is* locally base accessible from m with regards to N *(respectively,* locally base controllable from m *with regards to N) if $\psi(\mathcal{R}^V_M(m, \leq T))$ contains a non-empty open set of N (respectively, $\psi(\mathcal{R}^V_M(m, \leq T))$ contains a non-empty open set of N to which $\psi(m)$ belongs) for all neighborhoods V of m and all $T > 0$. If this holds for any $m \in M$, then the system is called* locally base accessible with regards to N *(respectively,* locally base controllable with regards to N*).*

Note that base accessibility and controllability with regards to M with $\mathrm{Id}_M : M \to M$ corresponds to the notions of base accessibility and controllability (cf. Definition 5.1). Moreover, if the system is base accessible, then it is base accessible with regards to N. The analogous implication for base controllability also holds true.

5.2 The structure of the control Lie algebra

The aim of this section is to show that the analysis of the structure of the control Lie algebra of affine connection control systems carried out in [20] can be further extended to control systems defined on a Lie algebroid. The enabling technical notion exploited here is that of homogeneity. As we will show later, this analysis will allow us to enlarge the class of systems to which the accessibility and controllability tests can be applied.

For the purpose of evaluating the brackets of the accessibility subbundle $\overline{\mathrm{Lie}}(\{\Gamma - \eta^V, \eta_1^V, \ldots, \eta_k^V\})$ at initial states of the form 0_m, $m \in M$, the discussion on the geometry of $\mathcal{T}E$ along the zero-section (cf. Section 3.1) will be most helpful. Since the Lie brackets in the accessibility subbundle of the mechanical control system are linear combinations of the brackets of the elements $\{\Gamma, \eta_1^V, \ldots, \eta_k^V, \eta_0^V\}$, as an intermediate step we will first analyze the structure of the subbundle $\overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \ldots, \eta_k^V, \eta^V\})$.

Let B be a Lie bracket formed with sections of the family $\mathcal{X} = \{\Gamma, \eta_1^V, \ldots, \eta_k^V, \eta^V\}$. For each l, consider the following sets

$$\mathrm{Br}^l(\mathcal{X}) = \{B \text{ bracket in } \mathcal{X} \mid \delta(B) = l\},$$
$$\mathrm{Br}_l(\mathcal{X}) = \{B \text{ bracket in } \mathcal{X} \mid B \in \mathcal{P}_l\}.$$

The notion of *primitive* bracket will also be useful. Given a bracket B in $\mathcal{X}$, it is clear that we can write $B = [B_1, B_2]$, with B_i brackets in $\mathcal{X}$. In turn, we can also write $B_\alpha = [B_{\alpha 1}, B_{\alpha 2}]$ for $\alpha = 1, 2$, and continue these decompositions until we end up with elements belonging to $\mathcal{X}$. The collection of brackets $B_1, B_2, B_{11}, B_{12}, \ldots$ are called the *components* of B. The components of B which do not admit further decompositions are called *irreducible*. A bracket B is called *primitive* if all of its components are brackets in $\mathrm{Br}_{-1}(\mathcal{X}) \cup \mathrm{Br}_0(\mathcal{X}) \cup \{\Gamma\}$.

We may now recall the following lemma taken from [17]. Although there it is stated for vector fields, the proof can be readily extended to sections of a Lie algebroid since it only relies on two facts: (i) the Jacobi identity, and (ii) the fact that $\mathcal{X}$ has the property that any bracket in $\mathrm{Br}_l(\mathcal{X})$, $l \leq 2$, is identically zero (which follows from Lemma 3.3(i)).

Lemma 5.4. *Any bracket in $\mathrm{Br}_0(\mathcal{X}) \cup \mathrm{Br}_{-1}(\mathcal{X})$ is a finite sum of primitive brackets.*

As a consequence of this lemma, and the fact that all the brackets in $\mathrm{Br}_l(\mathcal{X})$, with $l \geq 1$ vanish when evaluated at the zero-section of E, we conclude that the only brackets we need to consider are the primitive brackets in $\mathrm{Br}_{-1}(\mathcal{X}) \cup \mathrm{Br}_0(\mathcal{X})$. We do this next. First, observe the computation of the basic brackets (3.3) and (3.4). In particular, notice that $[\Gamma, \sigma^V]$ projects to $-\sigma$. Second, from Lemma 3.3(ii), we deduce that any bracket in $\mathrm{Br}_{-1}(\mathcal{X})$ is the vertical lift of a section of E. From Lemma 3.3(iii), we deduce that the brackets B belonging to $\mathrm{Br}_0(\mathcal{X})$ are projectable sections. We will denote by σ_B the section to which it projects. Thus, we have $B(0_m) = \sigma_B(m)$, $m \in M$. The following result completely unveils the structure of these brackets.

Lemma 5.5. *Let $B \in \mathrm{Br}_0(\mathcal{X})$ be a primitive bracket. Then either one of the following is true,*

(i) $B = [\Gamma, B_1]$ with $B_1 \in \mathrm{Br}_{-1}(\mathcal{X})$. If σ_1 is the section of E such that $B_1 = \sigma_1^V$, then $\sigma_B = -\sigma_1$. In addition, $[\sigma_2^V, B] = \langle \sigma_1 : \sigma_2 \rangle^V$ for all $\sigma_2 \in \mathrm{Sec}(E)$.

(ii) $B = [B_1, B_2]$ *with* $B_1, B_2 \in \mathrm{Br}_0(\mathcal{X})$*. Then,* $B(0_m) = [\sigma_{B_1}, \sigma_{B_2}](m)$ *for all* $m \in M$*.*

Proof. Let $B \in \mathrm{Br}_0(\mathcal{X})$. Then, either $B = [\Gamma, B_1]$ with $B_1 \in \mathrm{Br}_{-1}(\mathcal{X})$ primitive, or $B = [B_1, B_2]$ with $B_1, B_2 \in \mathrm{Br}_0(\mathcal{X})$ both primitive. In the first case, eq. (3.3) gives $B(0_m) = [\Gamma, B_1](0_m) = -\sigma_1(m)$, where $B_1 = \sigma_1^V$. From (3.4), we also deduce that $[\sigma_2^V, B] = [\sigma_2^V, [\Gamma, \sigma_1^V]] = \langle \sigma_1 : \sigma_2 \rangle^V$. In the second case, we have that B_1 and B_2 are projectable onto σ_{B_1} and σ_{B_2} respectively, and therefore $[B_1, B_2]$ projects to $[\sigma_{B_1}, \sigma_{B_2}]$. Consequently $B(0_m) = [B_1, B_2](0_m) = [\sigma_{B_1}, \sigma_{B_2}](m)$. □

Proposition 5.6. *Let* $m \in M$*. Then,*

$$\overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \dots, \eta_k^V, \eta^V\}) \cap \mathrm{Ver}_{0_m}(\mathcal{T}E) = \overline{\mathrm{Sym}}(\{\eta, \eta_1, \dots, \eta_k\})(m)^V ,$$
$$\overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \dots, \eta_k^V, \eta^V\}) \cap \mathrm{Hor}_{m}(\mathcal{T}E) = \overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta, \eta_1, \dots, \eta_k\}))(m) .$$

Proof. We prove the inclusion $\supseteq$ in the first equality by induction. Let us denote $\overline{\mathrm{Sym}}^{(1)}(\{\eta, \eta_1, \dots, \eta_k\}) = \mathrm{span}\{\eta, \eta_1, \dots, \eta_k\}$ and

$$\overline{\mathrm{Sym}}^{(l)}(\{\eta, \eta_1, \dots, \eta_k\}) = \mathrm{span}\{\langle \sigma_1 : \sigma_2 \rangle \mid \sigma_i \in \overline{\mathrm{Sym}}^{(l_i)}(\{\eta, \eta_1, \dots, \eta_k\}) ,\ l_1 + l_2 = l\} .$$

The result is trivially true for $l = 1$, $\overline{\mathrm{Sym}}^{(1)}(\{\eta, \eta_1, \dots, \eta_k\})^V \subseteq \overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \dots, \eta_k^V, \eta^V\})$. Assume it is true for l and let us prove it for $l+1$. Take $\sigma_i \in \overline{\mathrm{Sym}}^{(l_i)}(\{\eta, \eta_1, \dots, \eta_k\})$, $i = 1, 2$, with $l_1 + l_2 = l + 1$ and observe that, using (3.4),

$$\langle \sigma_1 : \sigma_2 \rangle^V = [[\sigma_1^V, \Gamma], \sigma_2^V] .$$

By induction hypothesis, $\sigma_i^V \in \overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \dots, \eta_k^V, \eta^V\})$, and therefore, $\langle \sigma_1 : \sigma_2 \rangle^V \in \overline{\mathrm{Lie}}(\{\Gamma, \eta_1^V, \dots, \eta_k^V, \eta^V\})$. From the previous discussion, we know that for the opposite inclusion it is sufficient to look at the primitive brackets in $\mathrm{Br}_{-1}(\mathcal{X})$. Let $B = [B_1, B_2] \in \mathrm{Br}_{-1}(\mathcal{X})$ primitive, with $B_1 \in \mathrm{Br}_{-1}(\mathcal{X})$, $B_2 \in \mathrm{Br}_0(\mathcal{X})$ primitive. From Lemma 5.5, we have either $B_2 = [\Gamma, B_2']$ or $B_2 = [B_2', B_2'']$. In the first case, we have

$$B = [B_1, [\Gamma, B_2']] .$$

Using again (3.4), we conclude that $B \in \overline{\mathrm{Sym}}(\{\eta, \eta_1, \dots, \eta_k\})^V$. As for the second case, using the Jacobi identity, we have

$$B = [B_1, [B_2', B_2'']] = -[B_2'', [B_1, B_2']] + [B_2', [B_1, B_2'']] .$$

Applying repeatedly the above argument to $[B_1, B_2']$ and $[B_1, B_2'']$ until they are expressed in terms of symmetric products, we see that B can be expressed as a linear combination of elements in $\overline{\mathrm{Sym}}(\{\eta, \eta_1, \dots, \eta_k\})^V$, and hence we conclude the result. The second equality is a direct consequence of the first one and Lemma 5.5. □

Now, consider the set $\mathcal{X}' = \{\Gamma - \eta_0^V, \eta_1^V, \dots, \eta_k^V\}$. As noted before, the elements in $\overline{\mathrm{Lie}}(\mathcal{X}')$ are linear combinations of the elements in $\overline{\mathrm{Lie}}(\mathcal{X})$. In fact, for each bracket B' of elements in $\mathcal{X}'$, let us define the subset $S(B') \subset \mathrm{Br}(\mathcal{X})$ formed by all possible brackets $B \in \mathrm{Br}(\mathcal{X})$ obtained by replacing each occurrence of $\Gamma - \eta_0^V$ in B' by either Γ or η_0^V. Then, one can prove by induction (cf. [17]) that

$$B' = \sum_{B \in S(B')} (-1)^{\delta_{k+1}(B)} B , \tag{5.9}$$

where recall that $\delta_{k+1}(B)$ stands for the number of occurrences of η^V in B. Reciprocally, given an element $B \in \mathrm{Br}(\mathcal{X})$, one can determine the bracket B' of elements in $\mathcal{X}'$ such that $B \in S(B')$ simply by substituting each occurrence of Γ or η_0^V in B by $\Gamma - \eta_0^V$. We denote this operation by $\mathrm{pseudoinv}(B) = B'$.

For each $k \in \mathbb{N}$, define the following families of sections in E,

$$\mathcal{C}_{\mathrm{ver}}^{(k)}(\eta; \eta_1, \dots, \eta_k) = \{\sigma \in \mathrm{Sec}(E) \mid \sigma^V = B'', B'' = \sum_{\substack{\tilde{B} \in S(\mathrm{pseudoinv}(B)) \\ \cap\, \mathrm{Br}_{-1}(\mathcal{X}) \cap \mathrm{Br}_0(\mathcal{X})}} (-1)^{\delta_{k+1}(\tilde{B})} \tilde{B} ,\ B \in \mathrm{Br}^{2k-1}(\mathcal{X}) \text{ primitive}\} ,$$

$$\mathcal{C}_{\mathrm{hor}}^{(k)}(\eta; \eta_1, \dots, \eta_k) = \{\sigma \in \mathrm{Sec}(E) \mid \sigma = \sigma_{B''}, B'' = \sum_{\substack{\tilde{B} \in S(\mathrm{pseudoinv}(B)) \\ \cap\, \mathrm{Br}_{-1}(\mathcal{X}) \cap \mathrm{Br}_0(\mathcal{X})}} (-1)^{\delta_{k+1}(\tilde{B})} \tilde{B} ,\ B \in \mathrm{Br}^{2k}(\mathcal{X}) \text{ primitive}\} .$$

Let $\mathcal{C}_{\rm ver}(\eta;\eta_1,\ldots,\eta_k)=\cup_{k\in\mathbb{N}}\mathcal{C}^{(k)}_{\rm ver}(\eta;\eta_1,\ldots,\eta_k)$, $\mathcal{C}_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)=\cup_{k\in\mathbb{N}}\mathcal{C}^{(k)}_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)$, and denote by $C_{\rm ver}(\eta;\eta_1,\ldots,\eta_k)$ and $C_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)$, respectively, the subbundles of the Lie algebroid E generated by the latter families.

Taking into account the previous discussion, we are now ready to compute $\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\})$ for a mechanical control system defined on a Lie algebroid.

Proposition 5.7. *Let $m\in M$. Then,*

$$\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\})\cap {\rm Ver}_{0_m}(\mathcal{T}E)=C_{\rm ver}(\eta;\eta_1,\ldots,\eta_k)(m)^V\,,$$
$$\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\})\cap {\rm Hor}_{m}(\mathcal{T}E)=C_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)(m)\,.$$

Proof. From the definition of the families $\mathcal{C}_{\rm ver}(\eta;\eta_1,\ldots,\eta_k)$ and $\mathcal{C}_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)$, one sees that for each $B\in{\rm Br}(\mathcal{X})$ primitive, we have computed the $\mathbb{R}$-linear combinations from ${\rm Br}(\mathcal{X})$ that appear along with B in the decomposition of pseudoinv(B) according to (5.9). Since it is only these primitive brackets which appear when making Lie brackets of $\{\Gamma-\eta_0^V,\eta_1^V,\ldots,\eta_k^V\}$, this will generate $\overline{\rm Lie}(\{\Gamma-\eta_0^V,\eta_1^V,\ldots,\eta_k^V\})$ along the zero-section of $\mathcal{T}E$. □

Remark 5.8. In the absence of potential terms, i.e., $\eta=0$, one has that

$$C_{\rm ver}(0;\eta_1,\ldots,\eta_k)=\overline{\rm Sym}(\{\eta_1,\ldots,\eta_k\})\,,\quad C_{\rm hor}(0;\eta_1,\ldots,\eta_k)=\overline{\rm Lie}(\overline{\rm Sym}(\{\eta_1,\ldots,\eta_k\}))\,.$$

It is worth noticing that, in this case, $C_{\rm ver}(0;\eta_1,\ldots,\eta_k)\subseteq C_{\rm hor}(0;\eta_1,\ldots,\eta_k)$. This is not true in general.

5.3 Accessibility and controllability tests

In this section we merge the notions introduced in Section 5.1 with the results obtained in Section 5.2 to give tests for accessibility and controllability.

Proposition 5.9. *Let $m\in M$ and assume the Lie algebroid E is locally transitive at m. Then the mechanical control system* (5.8) *is*

- *locally base accessible from m if $C_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)(m)+\ker\rho=E_m$,*
- *locally accessible from m at zero if $C_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)(m)+\ker\rho=E_m$ and $C_{\rm ver}(\eta;\eta_1,\ldots,\eta_k)(m)=E_m$.*

Proof. Consider the accessibility subbundle $\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\})$ in the Lie algebroid $\mathcal{T}E$. Since E is locally transitive at m and $C_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)(m)+\ker\rho_m=E_m$ by hypothesis, and $C_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)(m)\subseteq\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\})(0_m)$ (cf. Proposition 5.7), then $\rho^1(\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\}))(0_m)=T_m(0_M(M))$. As a consequence, there exists an open connected submanifold N of M containing m such that $0_M(N)$ is an integral manifold of $\rho^1(\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\}))$. Let λ be the maximal integral manifold of E which contains $0_M(N)$. Since $\rho^1(\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\}))$ is the accessibility distribution, λ is invariant under (5.8) and the system is locally accessible when restricted to λ. Thus the set $\mathcal{R}^U_E(0_m,\leq T)$ is open for all $U\subseteq\lambda$ neighborhood of 0_m and sufficiently small T. Let V be a neighborhood of m in M and define $U=\tau^{-1}(V)\cap\lambda$. The result now follows from the fact that τ is an open mapping and hence the set $\tau(\mathcal{R}^U_E(0_m,\leq T))\subset\mathcal{R}^V_M(m,\leq T)$ is open in M for T sufficiently small.

As for the second statement, note that the fact that E is locally transitive at m implies that $\mathcal{T}E$ is locally transitive at 0_m. In addition, since $C_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)(m)+\ker\rho=E_m$ and $C_{\rm ver}(\eta;\eta_1,\ldots,\eta_k)(m)=E_m$ by hypothesis, and $C_{\rm hor}(\eta;\eta_1,\ldots,\eta_k)(m)$, $C_{\rm ver}(\eta;\eta_1,\ldots,\eta_k)(m)^V\subseteq\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\})(0_m)$ by Proposition 5.7, we conclude $\overline{\rm Lie}(\{\Gamma-\eta^V,\eta_1^V,\ldots,\eta_k^V\})(0_m)+\ker\rho^1=\mathcal{T}_{0_m}E$. The result now follows from Theorem 4.3. □

In the absence of potential energy terms, the accessibility tests presented above simply read as follows.

Proposition 5.10. *Let $m\in M$ and assume the Lie algebroid E is locally transitive at m. A mechanical control system* (5.8) *with no potential terms is*

- *locally base accessible from m if* $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta_1,\dots,\eta_k\}))(m)+\ker\rho=E_m$.
- *locally accessible from m at zero if* $\overline{\mathrm{Sym}}(\{\eta_1,\dots,\eta_k\})(m)=E_m$.

Remark 5.11. Alternatively, if the sufficient condition for locally base accessibility in Proposition 5.10 is not met, i.e., $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta_1,\dots,\eta_k\}))(m)+\ker\rho\neq E_m$, the corresponding proof also yields the following result. Let N denote the maximal integral manifold of $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\{\eta_1,\dots,\eta_k\}))(m)$ passing through m. Then, for each neighborhood V of m in M and each T sufficiently small, $\mathcal{R}^V_M(m,\leq T)\subset N$, and $\mathcal{R}^V_M(m,\leq T)$ contains a non-empty open subset of N.

The notions of good and bad symmetric products can be stated in a similar way as for Lie brackets. We say that a symmetric product P in the sections $\{\eta,\eta_1,\dots,\eta_k\}$ is *bad* if the number of occurrences of each η_i in P is even. Otherwise, P is *good*. Accordingly, $\langle\eta_i:\eta_i\rangle$ is bad and $\langle\langle\eta:\eta_j\rangle:\langle\eta_i:\eta_i\rangle\rangle$ is good. The following theorem gives sufficient conditions for local controllability.

Proposition 5.12. *Let $m\in M$. The mechanical control system* (5.8) *is*

- *locally base controllable from m if it is locally base accessible from m and every bad symmetric product in $\{\eta,\eta_1,\dots,\eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree and elements of* $\ker\rho$,
- *locally controllable from m at zero if it is locally accessible from m at zero and every bad symmetric product in $\{\eta,\eta_1,\dots,\eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree.*

Proof. The proof follows from the following considerations. First, note that every bad Lie bracket in $\{\Gamma-\eta^V\eta_1^V,\dots,\eta_k^V\}$ gives rise to the vertical lift of a bad symmetric product in $\{\eta,\eta_1,\dots,\eta_k\}$. Second, observe that every good symmetric product in $\{\eta,\eta_1,\dots,\eta_k\}$ can be alternatively written as a good Lie bracket in $\{\Gamma-\eta^V,\eta_1^V,\dots,\eta_k^V\}$ evaluated at the zero section 0_M (modulo a minus sign). The result is now an application of Proposition 4.5 to the setting of mechanical control systems on Lie algebroids. □

The corresponding tests for base accessibility and controllability with regards to a manifold can be proved in a similar way.

Proposition 5.13. *Let $\psi:M\to N$ be an open map. Let $m\in M$ and assume $\psi_*(\rho(E_m))=T_{\psi(m)}N$. Then the mechanical control system* (5.8) *is*

- *locally base accessible from m with regards to N if* $C_{\mathrm{hor}}(\eta;\eta_1,\dots,\eta_k)(m)+\rho^{-1}(\ker\psi_*)=E_m$,
- *locally base controllable from m with regards to N if the system is locally base accessible from m with regards to N and every bad symmetric product in $\{\eta,\eta_1,\dots,\eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree and elements of* $\rho^{-1}(\ker\psi_*)$.

5.4 Mechanical control systems related by a morphism of Lie algebroids

In this section we study the relation between the controllability properties of two mechanical control systems related by a morphism of Lie algebroids. Consider a mechanical control system on $\tau:E\to M$ with $\Gamma-\eta^V$ as drift section (where Γ is a spray) and inputs $\{\eta_1^V,\dots,\eta_k^V\}$, and a mechanical control system on $\tau:\overline{E}\to\overline{M}$ with $\overline{\Gamma}-\overline{\eta}^V$ as drift section (where $\overline{\Gamma}$ is a spray) and inputs $\{\overline{\eta}_1^V,\dots,\overline{\eta}_{\overline{k}}^V\}$.

Let $\Psi:E\to\overline{E}$ be a morphism of Lie algebroids. Assume the two mechanical control systems are weakly $\mathcal{T}\Psi$-related. Because of homogeneity, one can deduce that $\mathcal{T}\Psi\circ\Gamma=\overline{\Gamma}\circ\Psi$, so that Ψ maps the corresponding associated connection ∇ onto the associated connection $\overline{\nabla}$. Moreover, using the definition of morphism of Lie algebroids, one can conclude that

$$\Psi\circ\eta=\overline{\eta}\circ\Psi+\sum_{j=1}^{\overline{k}}b^j(\overline{\eta}_j\circ\Psi)\,,\quad \Psi\circ\eta_i=\sum_{j=1}^{\overline{k}}C_i^j(\overline{\eta}_j\circ\Psi)\,,\quad i=1,\dots,k\,,$$

for some functions C_i^j on M, i.e., the relation by $\mathcal{T}\Psi$ between the vertical lifts of the potential terms and the input sections of the mechanical control systems translates into a relation by Ψ of the potential terms and the input sections themselves. Following the steps of the proof of Proposition 4.8, one can also infer the relationship between the base accessibility and controllability properties of two Ψ-weakly related mechanical systems. As above, the control functions of the second system are related to the control functions of the first one by means of $\overline{u}_j(t) = b^j(m(t)) + \sum_{i=1}^k C_i^j(m(t))u_i(t)$.

Proposition 5.14. *Let $\Psi\colon E \to \overline{E}$ be a morphism of Lie algebroids such that the base map ψ is an open map. Consider two mechanical control systems which are weakly Ψ-related. If the system on E is locally base accessible (respectively locally base controllable) from m then the system on $\overline{E}$ is locally base accessible (respectively locally base controllable) from $\psi(m)$.*

In the particular case when Ψ is an isomorphism between the fibers of the Lie algebroids, if the two systems are Ψ-related, then the sufficient conditions for local (base) accessibility (respectively controllability) are either simultaneously satisfied on E and $\overline{E}$ or simultaneously not satisfied, as stated in the following result.

Proposition 5.15. *Let $\Psi : E \to \overline{E}$ be a morphism of Lie algebroids which is an isomorphism on each fiber. Consider two mechanical control systems on E and $\overline{E}$, with $k \geq \overline{k}$, that are Ψ-related. Let $m \in M$. Then*

(i) $C_{\mathrm{ver}}(\eta; \eta_1, \ldots, \eta_k)(m) = E_m$ *if and only if* $C_{\mathrm{ver}}(\overline{\eta}; \overline{\eta}_1, \ldots, \overline{\eta}_{\overline{k}})(\psi(m)) = \overline{E}_{\psi(m)}$,

(ii) $C_{\mathrm{hor}}(\eta; \eta_1, \ldots, \eta_k)(m) + \ker\rho = E_m$ *if and only if* $C_{\mathrm{hor}}(\overline{\eta}; \overline{\eta}_1, \ldots, \overline{\eta}_{\overline{k}})(\psi(m)) + \ker\overline{\rho} = \overline{E}_{\psi(m)}$,

(iii) Every bad symmetric product in $\{\eta, \eta_1, \ldots, \eta_k\}$ evaluated at m can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree if and only if every bad symmetric product in $\{\overline{\eta}, \overline{\eta}_1, \ldots, \overline{\eta}_{\overline{k}}\}$ evaluated at $\psi(m)$ can be put as an $\mathbb{R}$-linear combination of good symmetric products of lower degree.

Proof. Note that the assumptions of the proposition imply $k = \overline{k}$. Therefore, the sections η_i and $\overline{\eta}_i$ can be chosen to be Ψ-related

$$\Psi \circ \eta_i = \overline{\eta}_i \circ \psi\,,\ 1 \leq i \leq k\,. \tag{5.10}$$

Taking into account (5.10), one can verify that

$$\Psi \circ C_{\mathrm{ver}}(\eta; \eta_1, \ldots, \eta_k) = C_{\mathrm{ver}}(\overline{\eta}; \overline{\eta}_1, \ldots, \overline{\eta}_{\overline{k}}) \circ \psi\,, \tag{5.11a}$$

$$\Psi \circ C_{\mathrm{hor}}(\eta; \eta_1, \ldots, \eta_k) = C_{\mathrm{hor}}(\overline{\eta}; \overline{\eta}_1, \ldots, \overline{\eta}_{\overline{k}}) \circ \psi\,. \tag{5.11b}$$

The proof now follows from (5.11) and the hypothesis that Ψ is an isomorphism on every fiber. □

6 Applications to simple mechanical control systems and semidirect products

In this section we show how the formalism of mechanical control systems on Lie algebroids unifies the treatment of several situations which have been previously considered in the literature. We recover known accessibility and controllability results for simple mechanical control systems and develop some new ones.

Simple mechanical control systems

Let Q be a n-dimensional manifold. A *simple mechanical control system* is defined by a tuple $(Q, \mathcal{G}, V, \mathcal{F})$, where Q is the manifold of configurations of the system, $\mathcal{G}$ is a Riemannian metric on Q (the kinetic energy metric of the system), $V \in C^\infty(Q)$ is the potential function and $\mathcal{F} = \{F^1, \ldots, F^k\}$ is a set of k linearly independent 1-forms on Q, which physically correspond to forces or torques. The dynamics of simple mechanical control systems is classically described by the forced Euler-Lagrange's equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \sum_{i=1}^{k} u_i(t) F^i\,, \tag{6.1}$$

where $L : TQ \to \mathbb{R}$, $L(q, \dot q) = \frac{1}{2}\mathcal{G}(\dot q, \dot q) - V(q)$ is the Lagrangian function of the system.

There are several ways of intrinsically writing these equations. Here, we present a formulation following the Lie algebroid formalism explained above. Consider $E = TQ$, $M = Q$ and the mappings $\tau = \tau_Q : TQ \to \mathbb{R}$, $\rho = \mathrm{Id}_{TQ} : TQ \to TQ$. Then, it is easy to see that TQ is a Lie algebroid with anchor map Id_{TQ}. In this setting, the forces in $\mathcal{F}$ correspond to sections of the dual bundle $E^* = T^*Q$. By means of the musical isomorphisms associated with the kinetic energy $\mathcal{G}$, we can consider them as sections $\mathcal{Y} = \{Y_1, \dots, Y_k\}$ of $E = TQ$ (i.e. vector fields). Then, equations (6.1) are equivalently given by

$$\dot a(t) = \rho^1\Big(\Gamma(a(t)) - (\mathrm{grad}_{\mathcal{G}} V)^V(a(t)) + \sum_{i=1}^k u_i(t) Y_i^V(a(t))\Big)\,, \tag{6.2}$$

where Γ is the second order equation associated with $\nabla^{\mathcal{G}}$. Here, the mapping ρ^1 is just the identity in TTM.

The notion of base accessibility and controllability (resp. accessibility and controllability at zero) in $M = Q$ precisely corresponds to the concept of configuration accessibility and controllability (resp. accessibility and controllability at zero velocity) in Q as introduced in [20]. Also, since ρ is an isomorphism, we conclude that the application of Propositions 5.9 and 5.12 to this case just renders the known tests for accessibility and controllability [20].

Simple mechanical control systems with symmetry I

Assume that a simple mechanical control system $(Q, \mathcal{G}, V, \mathcal{F})$ is *invariant under the action of a Lie group G on Q*,

$$\begin{array}{rccl} \Phi : & G \times Q & \longrightarrow & Q \\ & (g, q) & \longmapsto & \Phi(g, q) = \Phi_g(q) = gq\,. \end{array}$$

Invariance for the control system means that $\Phi_g^*\mathcal{G} = \mathcal{G}$, $\Phi_g^* V = V$ and $\Phi_g^* F^i = F^i$, for $1 \le i \le k$ and all $g \in G$. The orbit through a point q is $\mathrm{Orb}_G(q) = \{gq \mid g \in G\}$. We denote by $\mathfrak{g}$ the Lie algebra of G. For any element $\xi \in \mathfrak{g}$, let ξ_Q denote the corresponding infinitesimal generator of the group action on Q. Then,

$$T_q(\mathrm{Orb}_G(q)) = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}\,.$$

If the action Φ is free and proper, we can endow the quotient space Q/G with a manifold structure such that the canonical projection $\pi : Q \longrightarrow Q/G$ is a surjective submersion. Then, we have that $Q(Q/G, G, \pi)$ is a principal bundle with bundle space Q, base space Q/G, structure group G and projection π. Note that the kernel of $T\pi$ consists of the vertical tangent vectors, i.e., the vectors tangent to the orbits of G in Q. We denote the bundle of vertical vectors by $\mathcal{V}$, with $\mathcal{V}_q = T_q(\mathrm{Orb}_G(q))$, $q \in Q$.

The action Φ induces the *lifted action of G on TQ*, $\hat\Phi : G \times TQ \to TQ$, defined by $\hat\Phi_g = T\Phi_g$. Assuming Φ is free and proper, we have that $\hat\Phi$ is also free and proper, and hence $p : TQ \to TQ/G$, $p(v_q) = [v_q]$, is a surjective submersion. Consider then the Lie algebroid defined by $E = TQ/G$, $M = Q/G$ and

$$\tau([v_q]) = [q]\,, \quad \rho([v_q]) = T\pi(v_q)\,.$$

It is not difficult to verify that both τ and ρ are well-defined. The Lagrangian function L being G-invariant, it induces a reduced Lagrangian function $\ell : TQ/G \to \mathbb{R}$. The invariance of the forces in $\mathcal{F}$, or equivalently the fact that the sections in $\mathcal{Y}$ are invariant, implies that there exist well-defined sections $\mathcal{B} = \{B_i : Q/G \to TQ/G\}$, $1 \le i \le k$, such that $p \circ Y_i = B_i \circ \pi$. Finally, the invariance of the potential function and the Riemannian metric implies that there exists $\overline{\mathrm{grad}_{\mathcal{G}} V}$ such that $p \circ \mathrm{grad}_{\mathcal{G}} V = \overline{\mathrm{grad}_{\mathcal{G}} V} \circ \pi$. In this setting, the notion of base accessibility and controllability precisely corresponds to configuration accessibility and controllability in Q/G. However, note that the notions of accessibility and controllability at zero in E are stronger than the notions of accessibility and controllability at zero velocity in Q/G, since the former ones imply that the reachable sets contain open sets in $E = TQ/G$, whereas the latter only involve open sets in $T(Q/G)$.

If the mechanical control system is defined on a trivial principal fiber bundle $Q = G \times Q/G$, one may consider the canonical projection $\tau : G \times Q/G \to G$. This mapping is open, and the notions of base accessibility and base controllability with regards to G (cf. Section 5.1) precisely correspond to the concepts of *fiber configuration accessibility* and *fiber configuration controllability* as introduced in [9]. Proposition 5.13 renders appropriate tests to check these properties.

A different (and natural) question, however, concerns the search for tests of accessibility or controllability in Q which make use of the symmetry properties of the mechanical control system. This is what we analyze next.

Simple mechanical control systems with symmetry II

Now, we will apply the results of Section 5.4 to simple mechanical control systems with symmetry. Denote by $E_1 \to M_1$ the Lie algebroid $TQ \to Q$, and by $E_2 \to M_2$ the Lie algebroid $TQ/G \to Q/G$. Note that both algebroids have fibers of the same dimension $n = \dim Q$. Let $\Psi : TQ \to TQ/G$ be the projection mapping associated with the lifted action $\hat{\Phi}$, $\Psi = p$. It can be easily verified that Ψ defines a morphism of Lie algebroids. The base mapping $\psi : Q \to Q/G$ corresponds precisely to the projection π. Finally, observe that Ψ is surjective.

As a consequence of Proposition 5.15(ii), we have that the criterion to test base accessibility in $M_1 = Q$ (cf. Proposition 5.9),

$$C_{\text{hor}}(\operatorname{grad}_{\mathcal{G}} V; \mathcal{Y}) = E_1 (= TQ)\,,$$

is equivalent to

$$C_{\text{hor}}(\overline{\operatorname{grad}_{\mathcal{G}} V}; \mathcal{B}) = E_2 (= TQ/G)\,.$$

Hence, in this way we simplify the computational cost to test the controllability properties of the system, since one deals with the reduced representation (and therefore, one works in a space of smaller dimension). These are precisely the results obtained in [9, 26] (although here the analysis is more general since we consider nontrivial potential terms). The same simplification occurs for accessibility at zero velocity (cf. Proposition 5.15(i)) ,

$$C_{\text{ver}}(\operatorname{grad}_{\mathcal{G}} V; \mathcal{Y}) = TQ \quad \text{if and only if} \quad C_{\text{ver}}(\overline{\operatorname{grad}_{\mathcal{G}} V}; \mathcal{B}) = TQ/G\,.$$

As for controllability in Q, Proposition 5.15(iii) ensures that it is enough to check that the bad symmetric products in $\{\overline{\operatorname{grad}_{\mathcal{G}} V}, B_1, \dots, B_k\}$ are $\mathbb{R}$-linear combinations of good ones in TQ/G. Finally, Proposition 5.14 ensures that if the reduced system is not base accessible (resp. controllable), then the original system is not base accessible (resp. controllable).

Semidirect products

Let $\mathfrak{g}$ be a real Lie algebra acting transitively on a manifold M, that is, let $\mathfrak{g} \to \mathfrak{X}(M)$ be a surjective Lie algebra homomorphism mapping each element ξ of $\mathfrak{g}$ to a vector field ξ_M on M. Define the following Lie algebroid structure. The bundle space is $E = M \times \mathfrak{g}$ and the mapping $\tau : E \to M$ is just the projection onto the first factor. The anchor map is given by $\rho(m, \xi) = \xi_M(m)$. The Lie bracket is defined via the anchor map ρ and the bracket of constant sections. The latter is defined as the constant section corresponding to the bracket on $\mathfrak{g}$, that is, if $\sigma_1(m) = (m, \xi_1)$ and $\sigma_2(m) = (m, \xi_2)$ are two constant sections, then $[\sigma_1, \sigma_2](m) = (m, [\xi_1, \xi_2]_{\mathfrak{g}})$. Note that the case of a linear action on a vector space $\mathfrak{V}$ can be treated as a semidirect product by taking an orbit of the action as the base manifold M.

If we identify $TE \equiv TM \times T\mathfrak{g} \equiv TM \times \mathfrak{g} \times \mathfrak{g}$ using the left multiplication, an element of $\mathcal{T}E$ is of the form $(a, b, v) = \big((m, \xi), (m, \eta), (v_m, \xi, \zeta)\big)$. The condition $T\tau(v) = \rho(b)$ simply implies that $v_m = \eta_M(m)$. Therefore, we can identify $\mathcal{T}E$ with $M \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$, and the corresponding maps are

$$\tau_1(m, \xi, \eta, \zeta) = (m, \xi), \quad \mathcal{T}\tau(m, \xi, \eta, \zeta) = (m, \eta), \quad \rho^1(m, \xi, \eta, \zeta) = (\eta_M(m), \xi, \zeta)\,.$$

Let $(\mathcal{G}, V, \{\theta_1, \dots, \theta_k\})$ be a mechanical control system on the Lie algebroid E. Assume the bundle metric $\mathcal{G}$ comes from an inner product on $\mathfrak{g}$ (and therefore does not depend on the base point), $\mathcal{G}((m, \xi_1), (m, \xi_2)) =$

$\mathcal{G}(\xi_1, \xi_2)$. For each $\xi \in \mathfrak{g}$, define $\mathrm{ad}_\xi^\dagger : \mathfrak{g} \to \mathfrak{g}$ by $\mathcal{G}(\mathrm{ad}_\xi^\dagger \eta_1, \eta_2) = \mathcal{G}(\eta_1, [\xi, \eta_2]_\mathfrak{g})$. The spray associated with $\nabla^\mathcal{G}$ then reads

$$\Gamma_{\nabla^\mathcal{G}}(m, \xi) = (m, \xi, \xi, \mathrm{ad}_\xi^\dagger \xi),$$

and the controlled equations of motion are explicitly given by

$$\dot{a} - \mathrm{ad}_a^\dagger a = -\operatorname{grad}_\mathcal{G} V(m) + \sum_{i=1}^k u_i \eta_i(m)\,.$$

If one takes constant sections $\sigma_i(m) = (m, \xi_i)$, $i = 1, 2$, the expression of the Levi-Civita connection is

$$\nabla^\mathcal{G}_{\sigma_1} \sigma_2(m) = \Big(m, \frac{1}{2}[\xi_1, \xi_2]_\mathfrak{g} - \frac{1}{2}(\mathrm{ad}_{\xi_1}^\dagger \xi_2 + \mathrm{ad}_{\xi_2}^\dagger \xi_1)\Big)\,,$$

and the symmetric product is

$$\langle \sigma_1 : \sigma_2 \rangle(m) = \Big(m, -(\mathrm{ad}_{\xi_1}^\dagger \xi_2 + \mathrm{ad}_{\xi_2}^\dagger \xi_1)\Big)\,.$$

The above-developed tests can be easily applied to this kind of problems in order to determine whether or not the system is base accessible (resp. controllable).

Systems of the above type appears frequently as mechanical systems defined on homogeneous spaces for a given group action. If a group G acting transitively on a manifold M, one can consider the Lie algebroid $TG \times M \to G \times M$, with anchor map $\rho(v_g, m) = (v_g, 0_m)$. The bracket of the Lie algebroid is just the Lie bracket on the manifold G, where the coordinates of M are considered as *parameters*. Typically, one is thinking of mechanical systems defined on TG that depend on certain parameters which are modeled by the coordinates of M. These systems are not invariant under the right action on the group G on itself, but they are invariant under the action on G on itself and M *at the same time*. Therefore, one can consider the map $\Psi\colon TG \times M \to \mathfrak{g} \times M$, $\Psi(v_g, m) = (v_g g^{-1}, gm)$, which is a morphism of Lie algebroids with base map $\psi\colon G \times M \to M$, $\phi(g, m) = gm$. One can verify that the map Ψ is an isomorphism in every fiber and that its associated base map ψ is open. Therefore, one can apply the results of Section 5.4 to determine whether a mechanical system on TG depending on certain parameters modeled by M and which is not G-invariant, is locally controllable by analyzing the corresponding system on the Lie algebroid $\mathfrak{g} \times M$.

An explicit example of the above type of system is the case of a rigid body $G = SO(3)$ subject to control forces with fixed direction (in space), which are the elements in M. Then the above procedure precisely consists of studying the control problem in body coordinates. The variables in M evolve dynamically due to the non-inertial rotating frame.

7 Conclusions

We have investigated the controllability properties of control systems defined on Lie algebroids. We have established some general controllability results for nonlinear affine control systems. We have also introduced the concept of mechanical control system evolving on a Lie algebroid. After defining appropriate accessibility and controllability notions, we have investigated sufficient tests guaranteeing them. We have paid special attention to the situation where two control systems are related by means of a morphism of Lie algebroids. Finally, we have illustrated the results with the class of simple mechanical control systems and the class of systems evolving on semidirect products. Future directions of research will include the investigation of controllability tests along relative equilibria of mechanical control systems on Lie algebroids and the treatment of models that include gyroscopic forces and dissipation.

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8 Appendix

Here we gather some basic definitions concerning control systems defined on manifolds [28]. Let $\mathcal{R}_M^V(m,T)$ be the reachable set from a point $m \in M$ at time $T > 0$, following trajectories which remain in the neighborhood V of m in M for $t \leq T$. Denote

$$\mathcal{R}_M^V(m, \leq T) = \bigcup_{t \leq T} \mathcal{R}_M^V(m,t)\,.$$

Definition 8.1. *The system* (4.1) *is* locally accessible from $m \in M$ *if* $\mathcal{R}_M^V(m, \leq T)$ *contains a non-empty open set of* M *for all neighborhoods* V *of* m *and all* $T > 0$*. If this holds for any* $m \in M$*, then the system is called* locally accessible.

Definition 8.2. *The system* (4.1) *is* locally controllable from $m \in M$ *if* $\mathcal{R}_M^V(m, \leq T)$ *contains a non-empty open set of* M *to which* m *belongs for all neighborhoods* V *of* m *and all* $T > 0$*. If this holds for any* $m \in M$*, then the system is called* locally controllable.

The *accessibility algebra* $\mathcal{C}$ *of the control system* (4.2) is defined as the smallest subalgebra of $\mathfrak{X}(M)$ containing $f, g_1, \ldots, g_k$. It is not difficult to show that every element of $\mathcal{C}$ is a linear combination of repeated Lie brackets of the form

$$[X_l, [X_{l-1}, [\ldots, [X_2, X_1] \ldots]]] ,$$

where $X_i \in \{f, g_1, \ldots, g_k\}$, $1 \leq i \leq l$ and $l \in \mathbb{N}$. The *accessibility distribution* C is defined as the distribution on M generated by the accessibility algebra $\mathcal{C}$,

$$C(m) = \operatorname{span}\{X(m) \mid X \text{ vector field in } \mathcal{C}\} , \quad m \in M .$$