THE CONSISTENCY PROBLEM IN OPTIMAL CONTROL: THE DEGENERATE CASE

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We examine the problem of the consistency of the second-order differential equations associated with optimal control problems. This problem can be treated in a presymplectic framework by means of a constraint algorithm. Two cases may arise: the regular one, already considered in the literature, and the degenerate one. The main contribution of this paper is the proposal of a discrete transition law for the optimal trajectories that reach a singular point. This discrete law respects both the geometry and the dynamical structure of the optimal control problem.

Keywords: optimal control, consistency problem, singular points, Transition Principle.

1. Introduction

As is well known, the application of tools from modern differential geometry in the fields of mechanics and control theory has meant a great advance in these research areas. In this spirit, we address here the consistency problem of the second-order differential equations describing the solution curves for optimal control problems. These equations can be obtained through either Pontryagin's Maximum Principle or constrained variational optimization, and constitute necessary conditions for optimality. Geometrically, the optimal control problem can be formulated as a vakonomic Hamiltonian system. This allows to give a precise mathematical interpretation of the consistency problem by means of a constraint algorithm, which eventually leads to a final submanifold where

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a well-defined optimal dynamics exists. We identify two overall situations: the regular case, in which the subsets obtained by this constraint algorithm have all a differentiable structure, and the degenerate case, in which this does not hold true. Note the difference in terminology with respect to classical optimal control theory [3], where the regular case precisely corresponds to the final constraint submanifold being the first one.

We focus on the study of the degenerate situation. Related work includes [9], where singular linear-quadratic optimal problems are considered, and [8, 21], where singular situations are treated for unconstrained and nonholonomically constrained Lagrangian systems. Here, we deal with (possibly nonlinear) general constraint functions and we do not make any assumption on the nature of the cost function. Building on previous work on ideas related to the Transition Principle [5, 21, 22], we propose a discrete law for the optimal trajectories which reach the singular set. In order to do so, we assume that the singularities are of fold type. We describe the structure of the singular set and identify a special class of curves, that we term *characteristics*, which are key in the description of the discrete law. We prove that the value of the momentum map of a Lie group action is preserved along the characteristics and define the notion of *decisive* points of a given singular point. Our results are illustrated in an optimal growth theory example [23]. We also motivate the need for further research by showing that the optimal control of nonholonomic systems with symmetry may exhibit singularities which are not fold.

The paper is organized as follows. Section 2. presents the geometric presymplectic framework suitable to deal with optimal control problems, along with examples coming from optimal growth theory and nonholonomic systems with symmetry. We examine the consistency of the optimal equations, and identify the regular and the degenerate cases. Section 3. presents the treatment of the latter one. Under suitable transversality conditions, we study the structure of the singular set and define the characteristic curves. We also introduce an appropriate dynamic relative vector field which enables us to define the notion of decisive points. Finally, we propose a version of the Transition Principle in this context. We apply the developments to the optimal growth problem, and show that the singularities of the class of nonholonomic systems with symmetry are not of fold type. Section 4. presents some conclusions and directions for future research.

2. A geometric formulation for optimal control problems

Roughly speaking, an optimal control problem in its simplest form consists of achieving a motion between two desired points q_0, q_1 in the configuration space Q, while extremizing certain functional determined by a cost function $C : TQ \to \mathbb{R}$ and satisfying certain differential and algebraic constraints. The great variety of examples range from the motion of robotic platforms to the behavior of economic growth models. The cost function can be associated, for instance, with energy expenditure or with net increment of certain capital good, whereas the constraints usually correspond to the differential equations governing the dynamics of the problem. The objective is then to extremize the functional $\mathcal{J} = \int_0^1 C dt$, among all twice differentiable curves $c(t)$ joining $c(0) = q_0$ and $c(1) = q_1$, and satisfying the constraints. These elements, the functional and the constraints, are precisely the ingredients on which *vakonomic dynamics* is based [2]. This dynamics, also termed *constrained variational optimization* [17], is obtained through the application of a constrained variational principle. In many situations, an optimal control problem can be recasted as a vakonomic problem. The standard way in which optimal control problems are treated is deriving necessary conditions in the form of differential equations that the possible solution curves must satisfy in order to be optimal. This is usually done by means of Pontryagin's Maximum Principle, although constrained variational optimization also yields the same result. In both cases, the problem of the *consistency* of the optimal equations may arise.

The intrinsic formulation for optimal control problems developed in [6, 17] (see also [12, 16]) will be the framework where our discussion will take place. Consider the Whitney sum $T^*Q \oplus TQ$, with projections $pr_1 : T^*Q \oplus TQ \to T^*Q$, $pr_2 : T^*Q \oplus TQ \to TQ$. Assume that the constraints are given by a submanifold M of TQ, locally defined by

$$\dot{q}^\alpha = \Psi^\alpha(q^A, \dot{q}^a)\,, \qquad 1 \leq \alpha \leq m\,, \quad m+1 \leq a \leq n\,, \quad 1 \leq A \leq n\,.$$

Consider the submanifold $W_0 = pr_2^{-1}(M) = T^*Q \times_Q M$, and denote $\pi_1 = pr_{1|W_0}$, $\pi_2 = pr_{2|W_0}$. Define on W_0 the presymplectic 2-form $\omega = \pi_1^*\omega_Q$, where ω_Q is the canonical symplectic form on T^*Q. Observe that $\operatorname{rank}\omega = 2n$. Define also the function

$$H_{W_0} = \langle \pi_1, \pi_2 \rangle - \pi_2^*\tilde{C}\,,$$

with $\tilde{C} : M \to \mathbb{R}$ the restriction of C to M. If (q^A) are local coordinates on a neighborhood U of Q, $(q^A, \dot{q}^a)$ coordinates on $TU \cap M$ and (q^A, λ_A) the induced coordinates on T^*U, then we have induced coordinates $(q^A, \lambda_A, \dot{q}^a)$ on $T^*U \times_Q (TU \cap M)$. Locally,

$$H_{W_0}(q^A, \lambda_A, \dot{q}^a) = \lambda_a \dot{q}^a + \lambda_\alpha \Psi^\alpha - \tilde{C}(q^A, \dot{q}^a)\,, \quad \omega = dq^A \wedge d\lambda_A\,.$$

The dynamics of the vakonomic system is determined by studying the solutions of

$$i_X\omega = dH_{W_0}\,. \tag{1}$$

The presymplectic system (W_0, ω, H_{W_0}) is called *vakonomic Hamiltonian system.* Being the system presymplectic, we apply the Gotay-Nester constraint algorithm [10] to it. First, consider the points W_1 of W_0 where (1) has a solution,

$$W_1 = \{x \in T^*Q \times_Q M \mid dH_{W_0}(x)(V) = 0,\ \forall V \in \ker\omega(x)\}\,.$$

Locally, $\ker\omega = \langle \partial/\partial\dot{q}^a \rangle$, and the constraints defining W_1 are

$$\varphi_a = \lambda_a + \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial \dot{q}^a} - \frac{\partial \tilde{C}}{\partial \dot{q}^a} = 0\,, \quad m+1 \leq a \leq n\,. \tag{2}$$

Note that $\dim W_1 = 2n$. The equations of motion on W_1 are $\dot{q}^A = \partial H_{W_0}/\partial\lambda_A$, $\dot{\lambda}_A = -\partial H_{W_0}/\partial q^A$, which are equivalent to

$$\dot{q}^\alpha = \Psi^\alpha(q^A, \dot{q}^a)\,,\ \dot{\lambda}_\alpha = \frac{\partial \tilde{C}}{\partial q^\alpha} - \lambda_\beta \frac{\partial \Psi^\beta}{\partial q^\alpha}\,,\ \frac{d}{dt}\left(\frac{\partial \tilde{C}}{\partial \dot{q}^a} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial \dot{q}^a}\right) = \frac{\partial \tilde{C}}{\partial q^a} - \lambda_\beta \frac{\partial \Psi^\beta}{\partial q^a}\,. \tag{3}$$

These are the equations usually encountered in the literature for optimal control problems [6, 17]. Under a suitable transformation, it can be shown that they also correspond to the classical equations of vakonomic dynamics [2].

2.1. Examples

Here we present some examples to illustrate how the above exposed formalism can be employed to derive the equations of a constrained optimization problem.

Optimal growth theory

Consider the following example inspired upon closed von Neumann systems [23]. Let $F(K_1, K_2, K_3, \dot{K}_1, \dot{K}_2, \dot{K}_3)$ be a transformation function relating the capital goods K_1, K_2, K_3 and the net capital formations $\dot{K}_1, \dot{K}_2, \dot{K}_3$ of the form

$$F(K_1, K_2, K_3, \dot{K}_1, \dot{K}_2, \dot{K}_3) = \dot{K}_3 - \dot{K}_1^3(1 + K_2^2) - \dot{K}_2^3(1 + K_1^2)\,.$$

Consider the cost function $C(K_1, K_2, K_3, \dot{K}_1, \dot{K}_2, \dot{K}_3) = \dot{K}_2$. The problem consists of maximizing the functional

$$\int_0^T \dot{K}_2\,dt \quad \text{subject to} \quad F(K_1, K_2, K_3, \dot{K}_1, \dot{K}_2, \dot{K}_3) = 0\,,$$

with appropriate initial conditions. To derive the optimal equations, let $(K_1, K_2, K_3) \in Q = \mathbb{R}^3$ and take coordinates $(K_1, K_2, K_3, \dot{K}_1, \dot{K}_2, \lambda_1, \lambda_2, \lambda)$ on $W_0 \simeq \mathbb{R}^8$. The constraint $F = 0$ can be rewritten as

$$\dot{K}_3 = \Psi(K_1, K_2, \dot{K}_1, \dot{K}_2) = \dot{K}_1^3(1 + K_2^2) + \dot{K}_2^3(1 + K_1^2)\,.$$

The Hamiltonian function then reads $H_{W_0} = \lambda_1\dot{K}_1 + \lambda_2\dot{K}_2 + \lambda\Psi - \dot{K}_2$. The constraints defining the submanifold W_1 are

$$\lambda_1 = -\lambda\frac{\partial\Psi}{\partial\dot{K}_1}\,, \quad \lambda_2 = 1 - \lambda\frac{\partial\Psi}{\partial\dot{K}_2}\,.$$

Equations (3) take the form

$$\begin{aligned}
&\dot{K}_3 = \dot{K}_1^3(1 + K_2^2) + \dot{K}_2^3(1 + K_1^2)\,, \quad \dot{\lambda} = 0\,, \\
&3\dot{K}_1\ddot{K}_1(1 + K_2^2) + 3\dot{K}_1^2 K_2\dot{K}_2 = K_1\dot{K}_2^3\,, \quad 3\dot{K}_2\ddot{K}_2(1 + K_1^2) + 3\dot{K}_2^2 K_1\dot{K}_1 = K_2\dot{K}_1^3\,.
\end{aligned} \tag{4}$$

Note that $\ddot{K}_1$ and $\ddot{K}_2$ are not determined whenever $\dot{K}_1 = 0$ and $\dot{K}_2 = 0$, respectively.

Optimal control of nonholonomic systems with symmetry

Let Q be the configuration space of a mechanical system, and $L : TQ \to \mathbb{R}$ its Lagrangian function. The interaction of the system with its environment is modeled by a distribution $\mathcal{D}$ on Q, which establishes the allowed velocities. Assume that a Lie group G acts freely and properly on Q, leaving both the Lagrangian L and the constraints $\mathcal{D}$ invariant. This geometric picture is common to a wide variety of locomotion and robotic systems [13, 19]. The system is then endowed with a principal fiber bundle structure, $\pi : Q \to B$, with fiber G. We call $B = Q/G$ the *shape* space, a terminology inherited from locomotion systems, where $r \in B$ denotes the internal shape of the system and $g \in G$ its position and orientation. Locally, Q can be seen as the trivial bundle $B \times G$ (in locomotion, this trivialization is commonly global).

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \mathbb{I}(r)^{-1}p\,, \tag{5}$$

$$\dot{p} = \frac{1}{2}\dot{r}^T\sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T\sigma_{pr}(r)\dot{r} + \frac{1}{2}p^T\sigma_{pp}(r)p\,, \tag{6}$$

$$M(r)\ddot{r} = -\mathsf{C}(r,\dot{r}) + N(r,\dot{r},p) + \tau\,. \tag{7}$$

Here, a nonholonomic momentum p is defined along the kinematic symmetry directions, with an associated governing equation (6) called the *momentum equation* [4]. $\mathbb{A}$ is the nonholonomic connection and τ represents the control forces applied to the system.

Assume that the shape space is fully controllable, that is, equation (7) can be rewritten as $\ddot{r} = u$. Given a positive definite quadratic function $C(\dot{r})$, we consider the following optimal control problem [14]: given q_0, $q_1 \in Q$, find the curves $r(t) \in B$ which steer the system from q_0 to q_1 while minimizing $\int_0^1 C(\dot{r})dt$, where $r = \pi(q)$, subject to the constraints (5) and the momentum equation (6). To deal with this problem, we treat p as a set of independent variables and the momentum equation as an additional set of constraints (cf. [14]). Then, the configuration space is $\tilde{Q} = Q \times \mathbb{R}^k$, with k the number of momentum directions. The function C can naturally be extended to $T\tilde{Q}$. The constraint submanifold $M \subset T\tilde{Q}$ is determined by (5) and (6). One has $\dim M = 2n + k - \dim G$. Note that the constraints are nonlinear in general, due to the presence of $\sigma_{\dot{r}\dot{r}}(r)$ in (6).

Now, we locally identify $T^*\tilde{Q}$ with $T^*(Q/G)\times T^*G\times T^*\mathbb{R}^k$. We further trivialize T^*G by left translations, $T^*G \cong G \times \mathfrak{g}^*$, $\lambda_g = (g, \lambda = L_g^*\lambda_g)$. Following the above exposed framework, the Hamiltonian reads

$$H = \lambda_a\dot{r}^a + \lambda_\beta(-\mathcal{A}_a^\beta\dot{r}^a + I^{\beta i}p_i) + \lambda_i\left(\frac{1}{2}\sigma_{iab}\dot{r}^a\dot{r}^b + \sigma_{ia}^j p_j\dot{r}^a + \frac{1}{2}\sigma_i^{jl}p_jp_l\right) - \frac{1}{2}C_{ab}\dot{r}^a\dot{r}^b\,,$$

where $1 \leq \beta \leq \dim G$, $1 \leq i,j,l \leq k$ and $1 \leq a,b \leq \dim Q/G$. The first constraint submanifold, W_1, is locally determined by the equations

$$\varphi_a = \lambda_a - \lambda_\beta\mathcal{A}_a^\beta + \lambda_i(\sigma_{iab}\dot{r}^b + \sigma_{ia}^j p_j) - C_{ab}\dot{r}^b = 0\,.$$

Equations (3) on W_1 take the form

$$\begin{aligned}
&(g^{-1}\dot{g})^\beta = -\mathcal{A}_a^\beta\dot{r}^a + I^{\beta i}p_i\,, \qquad \dot{p}_i = \frac{1}{2}\sigma_{iab}\dot{r}^a\dot{r}^b + \sigma_{ia}^j p_j\dot{r}^a + \frac{1}{2}\sigma_i^{jl}p_jp_l \\
&\dot{\lambda}_\beta = c_{\gamma\beta}^\delta\lambda_\delta(-A_a^\gamma\dot{r}^a + I^{\gamma j}p_j)\,, \qquad \dot{\lambda}_i = -\lambda_\beta I^{\beta i} - \lambda_j(\sigma_{ja}^i\dot{r}^a + \sigma_j^{il}p_l) \\
&\frac{d}{dt}\left(C_{ab}\dot{r}^b + \lambda_\beta\mathcal{A}_a^\beta - \lambda_i(\sigma_{iab}\dot{r}^b + \sigma_{ia}^j p_j)\right) \\
&\quad = \lambda_\beta\left(\frac{\partial\mathcal{A}_b^\beta}{\partial r^a}\dot{r}^b - \frac{\partial I^{\beta j}}{\partial r^a}p_j\right) - \lambda_i\left(\frac{1}{2}\frac{\partial\sigma_{ibc}}{\partial r^a}\dot{r}^b\dot{r}^c + \frac{\partial\sigma_{ib}^j}{\partial r^a}p_j\dot{r}^b + \frac{1}{2}\frac{\partial\sigma_i^{jl}}{\partial r^a}p_jp_l\right)\,,
\end{aligned} \tag{8}$$

where $c_{\gamma\beta}^\delta$ are the structure constants of the Lie algebra $\mathfrak{g}$. These equations are precisely the ones obtained in [14] through a completely different approach, namely reduced Lagrangian optimization.

2.2. Consistency of the optimal equations

An important feature of equations (3) is their implicit character, which in turn may impose some additional (and, in principle, "hidden") constraints on the optimal evolution. Overlooking these constraints would result in trajectories that do not have possible continuation after some time instant or numerical simulations that blow up. In order to guarantee that this does not happen, we must ensure that equations (3) determine a well-defined dynamics in some (probably strict) submanifold of W_1. This is the objective of the constraint algorithm.

In the discussion above, we have shown that on each point of W_1 there exists a vector X satisfying (1). If this does not define a dynamic vector field tangent to W_1, we have to restrict ourselves to the subset $W_2 \subset W_1$ where these solutions are tangent to W_1. Assume that W_2 is a submanifold. Proceeding further, we obtain a sequence

$$\cdots \hookrightarrow W_s \hookrightarrow \cdots \hookrightarrow W_2 \hookrightarrow W_1 \hookrightarrow W_0 = T^*Q \times_Q M \ ,$$

where we assume for the time being that all the subsets obtained are submanifolds. Algebraically, these constraint submanifolds may be described as

$$W_i = \{x \in W_{i-1} \mid dH_{W_0}(x)(v) = 0\,,\ \forall v \in T_x W_{i-1}^{\perp}\}\,,$$

where $T_x W_{i-1}^{\perp} = \{v \in T_x W_0 \mid \omega(x)(u,v) = 0\,,\ \forall u \in T_x W_{i-1}\}$, $i \geq 1$. If the algorithm stabilizes, i.e., if there exists $s \in \mathbb{N}$ with $W_{s+1} = W_s$ and $\dim W_s \neq 0$, then we obtain a final constraint submanifold $W_f = W_s$ on which a vector field X exists such that

$$(i_X \omega = dH_{W_0})_{|W_f} \ .$$

The problem of the consistency of the optimal equations is then solved on W_f, since on this submanifold a dynamic vector field exists which is tangent to W_f.

It may happen that the final constraint submanifold coincides with the first one, $W_f = W_1$. This is the case, for instance, of sub-Riemannian geometry. In the following we examine this possibility. Denote $\omega_1 = j^*\omega$, with $j : W_1 \hookrightarrow W_0$. Consider the matrix

$$\bar{\mathcal{C}}_{ab} = \frac{\partial^2 \tilde{C}}{\partial \dot{q}^a \partial \dot{q}^b} - \lambda_\alpha \frac{\partial^2 \Psi^\alpha}{\partial \dot{q}^a \partial \dot{q}^b}\,, \tag{9}$$

Lemma 1. *Let $x \in W_1$. The orthogonal complement of $T_x W_1$ is given by*

$$T_x^{\perp} W_1 = \operatorname{span}\left\{ \frac{\partial}{\partial \dot{q}^a}, X_{jb}\left(\frac{\partial}{\partial q^b} + \frac{\partial \Psi^\alpha}{\partial \dot{q}^b}\frac{\partial}{\partial q^\alpha} + S_{Ab}\frac{\partial}{\partial \lambda_A}\right) \middle| \begin{array}{c} a = 1, \dots, n-m \\ j = 1, \dots, n-k \end{array} \right\}\,,$$

where X_{ja} are such that $\sum_a X_{ja}\bar{\mathcal{C}}_{ab} = 0$ and

$$S_{Ab} = \frac{\partial^2 \tilde{L}}{\partial q^A \partial \dot{q}^b} - \lambda_\gamma \frac{\partial^2 \Psi^\gamma}{\partial q^A \partial \dot{q}^b}\,. \tag{10}$$

Proof. Let us denote by $k \leq n-m$ the rank of $\bar{\mathcal{C}} = (\bar{\mathcal{C}}_{ab})$ at $x \in W_1$. Then, there exist $n-m-k$ linearly independent vectors $X_1, \ldots, X_{n-m-k}$ such that $X_j\bar{\mathcal{C}} = (\sum_a X_{ja}\bar{\mathcal{C}}_{ab}) = 0$, for each $j = 1, \ldots, n-m-k$. The result now follows by noting that the tangent bundle of W_1 is locally generated by the vector fields

$$\frac{\partial}{\partial q^A} + S_{Ab}\frac{\partial}{\partial \lambda_b}\,, \quad \frac{\partial}{\partial \dot{q}^a} + \bar{\mathcal{C}}_{ab}\frac{\partial}{\partial \lambda_b}\,, \quad \frac{\partial}{\partial \lambda_\alpha} - \frac{\partial \Psi^\alpha}{\partial \dot{q}^b}\frac{\partial}{\partial \lambda_b}\,,$$

where $A = 1, \ldots, n$, $a = 1, \ldots, n-m$ and $\alpha = 1, \ldots, m$. □

From Lemma 1, we can deduce that if $\bar{\mathcal{C}}$ is invertible for all $x \in W_1$, then $T^\perp W_1 = T^\perp W_0$, and therefore W_1 is the final constraint submanifold. Moreover, noting that

$$\frac{\partial}{\partial \dot{q}^a}(\varphi_b) = \bar{\mathcal{C}}_{ab}\,,$$

where the φ_a are given by (2), and $\ker \omega_1 = T_xW_1 \cap T_x^\perp W_1$, we conclude $\ker \omega_1 = \{0\}$. As a consequence we have the following result [6, 17],

Proposition 1. *If (W_1, ω_1) is a symplectic manifold, then $W_f = W_1$.*

The main limitation of the constraint algorithm is the assumption that the subsets W_i have a differentiable manifold structure. We call this situation the *regular* case. Some relevant examples (e.g. sub-Riemannian geometry, regular optimal control problems) fall into this case, but there are also important situations in which this does not hold true. We call this problem the *degenerate* case. For instance, it may happen that $\bar{\mathcal{C}}$ is nonsingular almost everywhere except at some points where its rank decreases. This will be the case that we will mainly treat here. Despite its apparent simplicity, the situation turns out to be rather involved. The following sections will discuss a way to overcome the consistency problem by means of an appropriate version of the Transition Principle [5].

To end this section, we present some results characterizing the various possibilities that may arise when studying the kernel of ω_1 if $\bar{\mathcal{C}}$ is singular.

Proposition 2. *Let $x \in W_1$ and $k = \operatorname{rank}(\bar{\mathcal{C}}) \leq n-m$. Then we have*

$$\begin{aligned} n-m-k \leq \dim\ker\omega_1 \leq 2(n-m-k)\,, &\quad \textit{if } n-m-k \textit{ is even}\,,\\ n-m-k+1 \leq \dim\ker\omega_1 \leq 2(n-m-k)\,, &\quad \textit{if } n-m-k \textit{ is odd}\,, \end{aligned}$$

and the dimension of the kernel can be any intermediate even value between the bounds.

Proof. Let $X_1, \ldots, X_{n-k}$ be linearly independent vectors such that $X_j\bar{\mathcal{C}} = (\sum_a X_{ja}\bar{\mathcal{C}}_{ab}) = 0$, for each $j = 1, \ldots, n-k$. To determine the kernel of ω_1, we have to identify those vectors in $T_x^\perp W_1$ which also belong to T_xW_1. Note that

$$\frac{\partial}{\partial \dot{q}^a}(\varphi_b) = \bar{\mathcal{C}}_{ab}\,, \quad X_{ja}\left(\frac{\partial}{\partial q^a} + \frac{\partial \Psi^\alpha}{\partial \dot{q}^a}\frac{\partial}{\partial q^\alpha} + S_{Aa}\frac{\partial}{\partial \lambda_A}\right)(\varphi_b) = X_{ja}T_{ab}\,,$$

where

$$T_{ab} = S_{ab} - S_{ba} + \frac{\partial \Psi^\alpha}{\partial \dot{q}^a} S_{\alpha b} - \frac{\partial \Psi^\alpha}{\partial \dot{q}^b} S_{\alpha a} \,. \tag{11}$$

The vectors $\left\{ X_{ja} \frac{\partial}{\partial \dot{q}^a} \right\}_{j=1}^{n-m-k}$ clearly belong to $\ker \omega_1$. The other possible elements in $\ker \omega_1$ should be a linear combination of elements in $T_x^\perp W_1$, i.e. of the form

$$Y_a \frac{\partial}{\partial \dot{q}^a} + Z_j X_{ja} \left(\frac{\partial}{\partial q^a} + \frac{\partial \Psi^\alpha}{\partial \dot{q}^a} \frac{\partial}{\partial q^\alpha} + S_{Aa} \frac{\partial}{\partial \lambda_A} \right) .$$

In order to be tangent to W_1, it should verify $Y\bar{\mathcal{C}} + \sum_j Z_j X_j T = 0$. Given that the map $Z \to Z\bar{\mathcal{C}}$ is degenerate, to find Y we must ensure that $\sum_j Z_j X_j T$ belongs to its image. Since $\bar{\mathcal{C}}$ is symmetric, this means $\langle \sum_j Z_j X_j T, \ker \bar{\mathcal{C}} \rangle = 0$, or equivalently,

$$\sum_{j=1}^{n-m-k} Z_j X_j T X_i = 0 \,, \quad \text{for each } i = 1, \dots, n-m-k \,. \tag{12}$$

Hence, $\dim \ker \omega_1$ relies on the number of solutions to this equation. Note that the skew-symmetry of T implies that the $(n-m-k) \times (n-m-k)$-matrix $\bar{T} = (X_j T X_i)$ is also skew-symmetric. This guarantees that if $n-m-k$ is odd, then there exists at least a non-trivial vector $Z = (Z_j)$ verifying (12) and therefore $\dim \ker \omega_1 \geq n-m-k+1$. □

Corollary 1. *Let $x \in W_1$ and assume* $\mathrm{rank}(\bar{\mathcal{C}}) = n-m-1$. *Then,*

$$\ker \omega_1(x) = \mathrm{span} \left\{ X_a \frac{\partial}{\partial \dot{q}^a}, Y_a \frac{\partial}{\partial \dot{q}^a} + X_a \left(\frac{\partial}{\partial q^a} + \frac{\partial \Psi^\alpha}{\partial \dot{q}^a} \frac{\partial}{\partial q^\alpha} + S_{Aa} \frac{\partial}{\partial \lambda_A} \right) \right\} ,$$

where X_a and Y_a are such that $X_a \bar{\mathcal{C}}_{ab} = 0$ *and* $Y_a \bar{\mathcal{C}}_{ab} + X_a T_{ab} = 0$.

Proof. In this case, $n-m-k = 1$ and the matrix $\bar{T}$ in the proof of Proposition 2 vanishes, so any $Z \in \mathbb{R}$ is a solution to equation (12). □

2.3. Symmetry and momentum maps

One of the advantages of dealing with optimal control problems by means of the above exposed formalism is the possibility of applying standard tools from Geometric Mechanics [1, 2] in their study. Here we briefly expose some facts related to symmetry which will be later used in the treatment of the degenerate case.

Assume that a Lie group G acts on Q and leaves the cost function $C : TQ \to \mathbb{R}$ and the constraints $M \subset TQ$ invariant. This action can be naturally lifted to an action Φ on W_0 leaving both the Hamiltonian $H : W_0 \to \mathbb{R}$ and the presymplectic 2-form ω invariant. Let $\mathfrak{g}$ denote the Lie algebra of G and $\mathfrak{g}^*$ its dual space. Consider the map

$$J : W_0 \longrightarrow \mathfrak{g}^* \,, \quad \langle J(\alpha_q, Z_q), \xi \rangle = \langle \alpha_q, \xi_Q(q) \rangle \,,$$

where $\xi \in \mathfrak{g}$ and ξ_Q corresponds to the fundamental vector field associated with the original action on Q. It is not difficult to see [7, 15] that J is a momentum map for the action Φ, that is, $i_{\xi_{W_0}}\omega = dJ_\xi$, with ξ_{W_0} the fundamental vector field associated with Φ and $J_\xi : W_0 \to \mathbb{R}$ given by $J_\xi(\alpha, X) = \langle J(\alpha, X), \xi\rangle$. The nice feature about the constraint algorithm is the fact that it respects both the action and the momentum map. This means that Φ restricts to a well-defined action on each submanifold W_i and that $J_{|W_i}$ is a momentum map for this restricted action. As a consequence, we have that $J_{W_f} : W_f \to \mathfrak{g}^*$ are conserved quantities for the optimal trajectories [7, 15].

3. The constraint algorithm: the degenerate case

In this section, we address the problem of determining the dynamics of the vakonomic Hamiltonian system when the assumptions of the constraint algorithm do not hold. We focus our attention on the case when the set W_2 is not a submanifold (the first type of singularity that may arise), which corresponds to the fact that the matrix $\bar{\mathcal{C}}$ does not have constant rank.

Consider the restriction of the projection map $\pi_2 : W_0 \to T^*Q$ to the submanifold W_1, which we denote as $\pi : W_1 \to T^*Q$. Note that $\dim W_1 = \dim T^*Q = 2n$. Locally,

$$\pi(q^A, \dot{q}^a, \lambda_\alpha) = (q^A, \frac{\partial \tilde{L}}{\partial \dot{q}^a} - \lambda_\alpha \frac{\partial \Psi^\alpha}{\partial \dot{q}^a}, \lambda_\alpha)\,, \quad \ker \pi_* = \operatorname{span}\{X_a \frac{\partial}{\partial \dot{q}^a} \mid \sum_a X_a \bar{\mathcal{C}}_{ab} = 0\}\,.$$

Consequently, by the inverse function theorem, the matrix $\bar{\mathcal{C}}$ is singular at $x \in W_1$ if and only if π is not a local diffeomorphism at x. Let S be the singular subset of π, i.e.

$$S = \{x \in W_1 \mid \mathcal{H}(x) = \det(\bar{\mathcal{C}}(x)) = 0\}\,.$$

In what follows, we assume that this subset is a submanifold of codimension 1 in W_1. We also assume that it is regular (i.e. $d\mathcal{H}(x) \neq 0$, for all $x \in S$) and that the following transversality condition holds

$$\ker \pi_x \cap T_x S = \{0\}\,, \quad x \in S\,. \tag{13}$$

It follows from (13) that the rank of the matrix $\bar{\mathcal{C}}$ at each $x \in S$ is $n - m - 1$, namely, $\dim \ker \pi_x = 1$, and that $\pi_{|S}$ is a local diffeomorphism between S and $\pi(S)$.

From a geometrical point of view, we are dealing with the following situation: the 2-form ω_1 is non-degenerate at W_1/S and has rank $2n-2$ at S. From a dynamical viewpoint, the optimal trajectories are well-defined on W_1/S, but we do not know what happens if they reach the singular set, since X is undetermined there. In the following, we investigate the local structure of S by means of the theory of stable mappings. This enables us to identify a set of special curves, called *characteristics*, which are key in the description of the behavior of the optimal trajectories when they reach S. Finally, the introduction of an appropriate relative dynamics vector field will complete the set of necessary tools to formulate the Transition Principle in this context.

3.1. Local structure of the singular set

Next, we briefly review some facts from the theory of stable mappings [11] which will be most helpful to unveil the local structure of the mapping π around the points in S. The exposition here follows [21]. Let N_1 and N_2 be two smooth manifolds of dimensions $n_1 \geq n_2$, respectively. Denote by $J^1(N_1, N_2)$ the space of 1-jets of maps from N_1 to N_2. A pair of local charts, $(x_1, \ldots, x_{n_1})$ on N_1 and $(y_1, \ldots, y_{n_2})$ on N_2, induces a local chart (x, y, p) on $J^1(N_1, N_2)$. Consider the 1-jet graph of a smooth map $F : N_1 \to N_2$,

$$j^1F : N_1 \longrightarrow J^1(N_1, N_2)\,, \quad j^1F(z) = j^1F_*(z)\,.$$

In local coordinates, this map reads $j^1F(x) = (x, F(x), \frac{\partial F}{\partial x}(x))$.

Let $S_1 \subset J^1(N_1, N_2)$ be the submanifold of 1-jets of co-rank 1, $S_1 = \{(x, y, p) \in J^1(N_1, N_2) \mid \mathrm{rank}(p) = n_2 - 1\}$ and denote $S_1(F) = (j^1F)^{-1}(S_1)$. Recall that a point z in $S_1(F)$ is called a *fold point* if $T_zS_1(F) + \ker F_*(z) = T_zN_1$.

Definition 1. A smooth map $F : N_1 \to N_2$ is called a *submersion with folds* if its singularities are all fold points and it satisfies the transversality condition

$$\mathrm{Im}(j^1F)_*(z) + T_{F(z)}S_1 = T_{F(z)}J^1(N_1, N_2)\,, \quad \forall z \in S_1(F)\,.$$

Theorem 1. *Let $F : N_1 \to N_2$ be a submersion with folds and let $z \in S_1(F)$. Then there exist a system of local coordinates $(x_1, \ldots, x_{n_1})$ in a neighborhood of z and a system of local coordinates $(y_1, \ldots, y_{n_2})$ in a neighborhood of $F(z)$ such that*

(i) $z = (0, \ldots, 0)$, $F(z) = (0, \ldots, 0)$,

(ii) the coordinate expression of F is $y_1 = x_1, \ldots, y_{n_2-1} = x_{n_2-1}$, $y_{n_2} = x_{n_2}^2 \pm \cdots \pm x_{n_1}^2$.

From the previous discussion, we can conclude that the mapping π is a submersion with folds and, according to Theorem 1, it can be locally represented in the normal form

$$y_1 = x_1\,, \ \ldots \quad y_{2n-1} = x_{2n-1}\,, \quad y_{2n} = x_{2n}^2\,,$$

with respect to appropriate charts $(x_1, \ldots, x_{2n})$ and $(y_1, \ldots, y_{2n})$ in W_1 and T^*Q, respectively. As a consequence, for each point $x \in S$, there exists a neighborhood U of x in W_1 and a neighborhood V of $\pi(x)$ in $\pi(W_1)$ such that U/S splits into two connected components U_1, U_2 such that $\pi_{|U_i}$ is a diffeomorphism and $\pi(U_1) = \pi(U_2) = V/\pi(U \cap S)$.

3.2. Characteristics of the singular set

From the proof of Proposition 2, we can deduce that $\ker \pi_*(x) \subset \ker \omega_1(x)$. In the case under consideration, we have $\mathrm{rank}(\bar{\mathcal{C}}) = n - m - 1$ at each $x \in S$ and hence, by Corollary 1, $\dim \ker \omega_1(x) = 2$. As a consequence, taking into account assumption (13),

$$\dim(\ker \omega_1(x) \cap T_xS) = 1\,,$$

that is, there exists a one-dimensional distribution on S, $\ell : x \mapsto \ell_x$, which we call the *characteristic distribution*, a terminology inherited from [5].

DEFINITION 2. The integral curves of the characteristic distribution ℓ are the *characteristic curves* of the singular hypersurface S.

The characteristic curves will play a key role in the formulation of the Transition Principle for the vakonomic Hamiltonian system. Note that these curves are special in the sense that they are *dynamically unnoticed* by the problem since they belong to the kernel of ω_1. Another important property of these curves is following one.

PROPOSITION 3. *The momentum map $J : W_1 \longrightarrow \mathbb{R}$ is conserved along the characteristic curves of the singular hypersurface S.*

Proof. Consider $Y \in \mathfrak{X}(S)$ spanning the characteristic distribution ℓ and let $\xi \in \mathfrak{g}$,

$$Y(J_\xi) = dJ_\xi(Y) = (i_{\xi_{W_1}}\omega_1)(Y) = -(i_Y\omega_1)(\xi_{W_1}) = 0\,,$$

where in the last equality we have used $\ell = \operatorname{span}\{Y\} \subset \ker\omega_1$. □

3.3. Vector field along the map $\pi : W_1 \to T^*Q$

The notion of a vector field along a map is a generalization of the concept of vector field [18, 20, 21]. Given a smooth map $F : M \to N$, a vector field X along F is a smooth map $X : M \to TN$ such that $\tau_N \circ X = F$. Vector fields along the identity map Id_M are standard vector fields. Consider now the map $X_R : W_1 \longrightarrow T(T^*Q)$ defined by

$$X_R(q^A, \dot{q}^a, \lambda_\alpha) = \frac{\partial H_{W_0}}{\partial \lambda_A}\frac{\partial}{\partial q^A} - \frac{\partial H_{W_0}}{\partial q^A}\frac{\partial}{\partial \lambda_A} = \dot{q}^a\frac{\partial}{\partial q^a} + \Psi^\alpha\frac{\partial}{\partial q^\alpha} + \left(\frac{\partial \tilde{L}}{\partial q^A} - \lambda_\alpha\frac{\partial \Psi^\alpha}{\partial q^A}\right)\frac{\partial}{\partial \lambda_A}.$$

It can be seen that this definition does not depend on the choice of local coordinates. Moreover, X_R is a vector field along $\pi : W_1 \to T^*Q$. In case π is a diffeomorphism, a well-defined dynamics X of the vakonomic Hamiltonian system exists on W_1, and one can verify that $X_R = X \circ \pi$. The map X_R allows to define the notion of in and out-points.

DEFINITION 3. Let $x \in S$ and consider local coordinates $(x_1, \ldots, x_{2n})$ in W_1, and $(y_1, \ldots, y_{2n})$ in T^*Q such that the map $\pi : W_1 \to T^*Q$ locally reads $y_1 = x_1$, …, $y_{2n-1} = x_{2n-1}$, $y_{2n} = x_{2n}^2$. Then, $x \in S$ is an *in-point of the vakonomic Hamiltonian system* if $X_R(x)$ is directed towards the neighborhood $V = \{(y_1, \ldots, y_{2n}) \mid y_{2n} \geq 0\}$ of $\pi(x)$ in $\pi(W_1)$, and *out-point* otherwise.

The last ingredient that we need to formulate the Transition Principle is the notion of decisive points associated with a given point $x \in S$.

DEFINITION 4. Let $x \in S$ and denote by γ_x the characteristic curve in S passing through x. A point $y \in \gamma_x$ is called *decisive for x* if it is an in-point and belongs to the same level set of the Hamiltonian H, i.e. $H(y) = H(x)$.

3.4. The Transition Principle

In this section, the developments of the preceeding sections are put together to give an appropriate formulation of the Transition Principle for the vakonomic Hamiltonian problem. In doing so, we build on previous formulations of this principle for discontinuous Hamiltonian systems [5], singular Lagrangians [21] and nonholonomic systems [8].

First, note that as a consequence of our assumptions, there is a well-defined dynamics X along W_1/S. On the other hand, the behavior of the system cannot be determined on the singular surface S by means of the constraint algorithm. Accelerations in the coordinates q^a are undetermined on S and there may be discontinuities in the motion. The Transition Principle is a natural way to determine these discontinuities.

Transition Principle. *When an optimal solution of the vakonomic Hamiltonian system reaches the hypersurface S at a point x, it can then continue its motion along all trajectories of the dynamics vector field X coming out from any decisive point for x.*

Remark 1. Given the formulation of this principle, it is clear that the Hamiltonian H is preserved by the discrete transition to the decisive points. Moreover, Proposition 3 implies that the momentum map is also preserved. From this point of view, we see that the discrete transition in the singular set S has the same dynamical behavior as the vector field X on W_1/S.

Optimal growth problem revisited

Consider the problem stated in Section 2.1. In this case, $\pi : W_1 \to T^*Q$ reads

$$(K_1, K_2, K_3, \dot{K}_1, \dot{K}_2, \lambda) \longmapsto (K_1, K_2, K_3, -3\lambda \dot{K}_1^2(1+K_2^2), 1-3\lambda\dot{K}_2^2(1+K_1^2), \lambda)\,.$$

The matrix $\bar{\mathcal{C}}$ defined in (9) is given by

$$\bar{\mathcal{C}} = -6\lambda \begin{pmatrix} \dot{K}_1(1+K_2^2) & 0 \\ 0 & \dot{K}_2(1+K_1^2) \end{pmatrix}$$

The singular set of π is then $S = \{(K_1, K_2, K_3, \dot{K}_1, \dot{K}_2, \lambda) \mid \dot{K}_1\dot{K}_2\lambda = 0\}$. Note that S is the union of three hyperplanes, $S_1 = \{\dot{K}_1 = 0\}$, $S_2 = \{\dot{K}_2 = 0\}$ and $S_3 = \{\lambda = 0\}$. The treatment of both S_1 and S_2 is analogous, whereas S_3 does not fall into our hypothesis (since the transversality condition (13) is violated on S_3). We restrict our attention to $U = S_1/\{\dot{K}_2 \neq 0, \lambda \neq 0\}$. Following Corollary 1, at $z \in U$, we have that

$$\ker \omega_1(z) = \operatorname{span}\{\frac{\partial}{\partial \dot{K}_1}, -\frac{1}{\lambda}\frac{K_1\dot{K}_2}{1+K_1^2}\frac{\partial}{\partial \dot{K}_2} + \frac{\partial}{\partial K_1}\}\,.$$

Only the second vector field is tangent to S_1, so it spans the characteristic distribution ℓ. The characteristic curve passing through $z = (K_1(0), K_2(0), K_3(0), 0, \dot{K}_2(0), \lambda(0))$ is

$$K_1(s) = K_1(0) + b\,s\,, \quad \dot{K}_2(s) = \dot{K}_2(0)\left(\frac{1+K_1(0)^2}{1+K_1(s)^2}\right)^{1/2\lambda(0)}, \qquad b \in \mathbb{R}/\{0\}\,.$$

Consider the following changes of coordinates on $W_1/\{\dot{K}_2 > 0, \lambda \neq 0\}$ and $\pi(W_1/\{\dot{K}_2 > 0, \lambda \neq 0\}) \subset T^*Q$, respectively,

$$
\begin{aligned}
&x_1 = K_1\,, \quad x_4 = \dot{K}_1\sqrt{3(1+K_2^2)}\,, && y_1 = K_1\,, \quad y_4 = -p_1/p_3\,,\\
&x_2 = K_2\,, \quad x_5 = 3\lambda\dot{K}_2^2(1+K_1^2)\,, && y_2 = K_2\,, \quad y_5 = 1-p_2\,,\\
&x_3 = K_3\,, \quad x_6 = \lambda\,, && y_3 = K_3\,, \quad y_6 = p_3\,.
\end{aligned}
$$

In these new coordinates, π reads $(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (y_1, y_2, y_3, y_4, y_5, y_6)$, with

$$
y_1 = x_1\,, \quad y_2 = x_2\,, \quad y_3 = x_3\,, \quad y_4 = x_4^2\,, \quad y_5 = x_5\,, \quad y_6 = x_6\,,
$$

and the relative vector field X_R is then given by

$$
\begin{aligned}
X_R = {}& \frac{x_4}{\sqrt{3(1+x_2^2)}}\frac{\partial}{\partial y_1} + \left(\frac{x_5}{3x_6(1+x_1^2)}\right)^{1/2}\frac{\partial}{\partial y_2} + \left(\frac{x_4^3}{(27(1+x_2^2))^{1/2}}\right. \\
& \left. + \left(\frac{x_5^3}{27\lambda^3(1+x_1^2)}\right)^{1/2}\right)\frac{\partial}{\partial y_3} + 2x_1\left(\frac{x_5}{3x_6(1+x_1^2)}\right)^{3/2}\frac{\partial}{\partial y_4} + 2x_2x_6\left(\frac{x_4}{\sqrt{3(1+x_2^2)}}\right)^3\frac{\partial}{\partial y_5}\,.
\end{aligned} \tag{14}
$$

Assume a trajectory reaches S at a point belonging to S_1, $z^* = (K_1^*, K_2^*, K_3^*, 0, \dot{K}_2^*, \lambda^*) = (x_1^*, x_2^*, x_3^*, 0, x_5^*, x_6^*)$, with $\dot{K}_2^* > 0$ and $\lambda^* \neq 0$. Since z is an out-point, from (14) we deduce that $x_1^* < 0$. Given the expression of the characteristic curves, the unique decisive point associated with z^* is $z = (-K_1^*, K_2^*, K_3^*, 0, \dot{K}_2^*, \lambda^*)$. Following the Transition Principle, the motion can be prolonged along any trajectory of the dynamics vector field X coming from z. Figure 1 shows an example of the application of this principle.

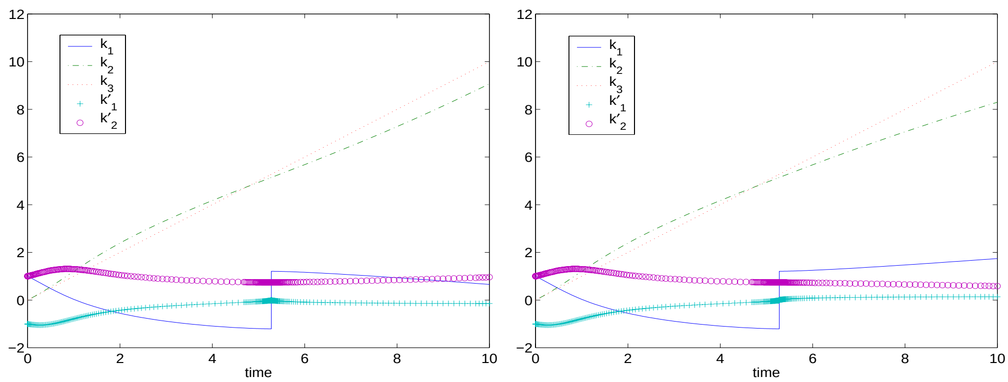

Fig. 1: Two prolongations of an optimal trajectory which reaches the singular set at S_1.

In this case, the economical interpretation of the Transition Principle would be that in order to keep the economy maximizing the capital K_2 while behaving according to $\dot{K}_3 = \Psi$, we have to 'inject' at some point in time a specific amount of capital K_1. From then on, there are two optimal choices. Examining Figure 1, one observes that the first choice will eventually lead us to another future 'injection' of capital in K_1, so the second possibility remains the most stable one.

Optimal control of nonholonomic systems revisited

Consider the optimal control problem for nonholonomic systems with symmetry as presented in Section 2.1. In this case, the matrix given by (9) has entries

$$\bar{\mathcal{C}}_{ab} = C_{ab}(r) - \lambda_i \sigma_{iab}(r)\,.$$

Therefore, the equations defining the singular set S of π are of the form $\psi(r,\lambda) = 0$. As a consequence, we have that the transversality condition (13) is violated, since the vectors in $\ker \pi$ will necessarily belong to TS. Therefore, the singular points are not of fold type. Further research is needed in order to deal with this important class of systems.

4. Conclusions

We have investigated the consistency of the equations associated with optimal control problems. This has been done making use of a constraint algorithm in a presymplectic framework, which has enabled us to identify two overall situations: the regular case and the degenerate case. Special attention has been put on the study of the latter one. Under some transversality conditions, we have described the structure of the singular set, and identified a particular class of curves on it called characteristics. Building on these developments, we have proposed a discrete transition law for the trajectories which reach the singular set, in such a way that both the geometry and the dynamics of the problem are respected. Further research directions would include the consideration of more general types of singularities to overcome the transversality conditions and the application of the results to the optimal control of nonholonomic systems with symmetry.

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