Motion control algorithms for simple mechanical systems with symmetry

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Abstract. We treat underactuated mechanical control systems with symmetry taking the viewpoint of the affine connection formalism. We first review the appropriate notions and tests of controllability associated with these systems, including that of fiber controllability. Secondly, we present a series expansion describing the evolution of the trajectories of general mechanical control systems starting from non-zero velocity. This series is then used to investigate the behavior of the system under small-amplitude periodic forcing. On this basis, motion control algorithms are designed for systems with symmetry to solve the tasks of point-to-point reconfiguration, static interpolation and stabilization problems. Several examples are given and the performance of the algorithms is illustrated in the blimp system.

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1. Introduction

Underactuated mechanical control systems constitute a research field of increasing interest in both theory and applications, see [4, 7, 8, 9, 11, 12, 15, 29, 30, 41, 42] and references therein. Examples include robots, airplanes, underwater and space vehicles, hovercrafts, and satellites. Accurate Lagrangian models exist for these systems when lift/drag effects are negligible.

From a theoretical viewpoint, underactuated mechanical control systems offer a control challenge. Mechanical systems are *dynamic* in nature, meaning that they are governed by second-order differential equations. This fundamental feature is lost when the system is transformed in first-order form, since the velocities are considered as part of the state and treated on the same footing as configurations. This is related to the limited success of the standard techniques in control

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theory [19, 37] when applied to the study of these systems in first-order form. Indeed, the resulting equations have non-zero drift, generically their linearization at zero velocity is not controllable, they are not stabilizable by continuous state feedback and they are not feedback linearizable as well. On the other hand, the combination of this approach (more precisely, the work in [18, 44, 45]) with a proper account of the geometry of mechanical systems has rendered new results in studying their controllability properties [22, 35, 39].

It is precisely the rich geometric structure of these systems which can be conveniently used to help focus the analysis. In doing so, we will employ the differential-geometric setting provided by the so-called affine connection formalism [30]. Within this framework, the special structure of mechanical control systems is properly taken into account and a set of new tools, such as the *symmetric product* [16, 30, 43], and notions, such as *configuration controllability*, are available to tackle problems in both analysis and design. Remarkably, the formalism is valid for both unconstrained and nonholonomically constrained systems [29].

Bullo, Leonard and Lewis [9] studied the nonlinear controllability problem and the constructive controllability problem (including both motion planning and stabilization) for mechanical systems evolving on a Lie group. As is shown in [10], the consideration of the symmetry properties of these systems leads to a simplified version of the controllability tests found in [30]. In [9], approximate local motion primitives and control algorithms were further developed to perform tasks such as stabilization (station keeping) and short range reconfigurations (parking, tracking), following a similar approach to Leonard and Krishnaprasad [26], see also [42, 47, 48].

Motivated by this previous work, here we focus our attention on mechanical control systems whose configuration space has a principal bundle structure. Roughly speaking, this means that the configuration manifold Q is the product of a Lie group G and a general smooth manifold M. Usually, G is called the *fiber* space (commonly corresponding to position and orientation of the system) and M the base or *shape* space (describing the actual shapes of the system). A large number of examples fall into this category, as for instance the bicycle [24], the motion of snakes and paramecia [22] or robotic locomotion systems [22, 40]. Moreover, mechanical systems evolving on a Lie group G could have input forces that are non-invariant by the left action of G due to a design choice, for example. In such a case, they are more naturally interpreted as systems evolving on a principal fiber bundle with a smaller Lie group $H \subset G$ as the symmetry group.

In [15], we studied the controllability problem for this class of mechanical control systems with symmetry. There, specific controllability tests were derived making use of the analysis described in [30, 44] and Lagrangian reduction techniques [6, 33, 38]. The notion of fiber controllability was also introduced, generalizing to mechanical systems the one in [22] specialized for driftless, kinematic systems.

In this paper, we build on these results to pay attention to the constructive controllability problem. The main contributions of this work are the development of a series expansion describing the evolution of general mechanical systems when starting from non-zero velocity (extending the work in [8]), and the construction of motion primitives and control algorithms for locally (fiber) controllable systems with symmetry. Both results are related, as we illustrate in the following.

On the basis of a suitable (fiber) controllability assumption, we generalize the motion primitives in [9] to perform the basic tasks of changing and maintaining velocity over one cycle. These primitives can then be used as the building blocks to design high-level motion algorithms. But, in order to do that, one needs a description of the evolution of the trajectories of the system when starting from non-zero velocity, since that will be the situation after the execution of one of those primitives. A key point in [9] was the use of perturbation techniques to investigate the response of the system under small-amplitude periodic forcing. However, this approach is not appropriate here as the systems considered evolve on general (not necessarily Lie groups) manifolds. The aforementioned series involving nested symmetric products of the inputs offers the appropriate description of the evolution of the trajectories of the system. Once this step is solved, one can design using discrete time feedback and multiple calls to the primitives, motion control algorithms that solve the point-to-point reconfiguration problem, the static interpolation problem and the local exponential stabilization problem for mechanical systems on principal fiber bundles. It is worth noting that the extension of these results turns out to be valid for fiber controllable systems, which is not a fact *a priori* guaranteed.

We organize the paper as follows. In Sections 2 and 3 we provide background on the affine connection formalism and Lagrangian reduction theory. In order to make a self-contained exposition, we also present some results from [15] on reduced representations of the symmetric product and on controllability of mechanical systems with symmetry. Several examples are discussed to illustrate the concepts. The following sections present the main contributions of this work. In Section 4 we develop a series that describes the evolution of the trajectories of general mechanical control systems starting from non-zero velocity. We use it to study the response of the system under small amplitude forcing. In Section 5 we focus on the construction of motion control algorithms for systems evolving on a principal fiber bundle. We generalize the procedures developed in [9] by means of an appropriate use of the infinite expansion. In Section 6 we illustrate the performance of the algorithms with the blimp example. Finally, we gather in an Appendix the proofs of the results of the paper.

2. Preliminaries on nonlinear control systems

Here we review the basic notions and results related to simple mechanical control systems and present a few examples that will be utilized to illustrate the controllability analysis and the motion algorithms in the following sections. We refer the reader to [8, 15, 29, 30] for more background on the subject.

2.1. Mechanical control systems

A *simple mechanical control system* is defined by a tuple $(Q, \mathcal{G}, V, \mathcal{F})$, where Q is a n-dimensional manifold defining the configuration space of the system, $\mathcal{G}$ is a Riemannian metric or kinetic energy on Q, V is a smooth function on Q or potential energy and $\mathcal{F} = \{F^1, \dots, F^m\}$ is a set of m linearly independent 1-forms on Q, which physically correspond to forces or torques. Here, we will simplify the treatment by assuming that $V = 0$. We remark that non-zero potential forces can be incorporated into the controllability analysis, as discussed in [30], whereas the extension of the series expansion and motion planning results presented below must still be addressed.

Let (q^a) be local coordinates on Q and denote by $\{\partial/\partial q^a\}_{1\leq a\leq n}$ the associated basis of vector fields spanning locally TQ, the tangent bundle of Q. In what follows, $C^\infty(Q)$ denotes the set of smooth functions on Q and $\mathfrak{X}(Q)$ the set of smooth vector fields on Q. Throughout the paper, the manifold Q and the mathematical objects defined on it will be assumed analytic.

Associated with the Riemannian metric $\mathcal{G}$ there is a natural affine connection, called the *Levi-Civita connection*. An *affine connection* [1, 23] is defined as an assignment

$$\begin{aligned} \nabla :\ \mathfrak{X}(Q) \times \mathfrak{X}(Q) &\longrightarrow \mathfrak{X}(Q) \\ (X, Y) &\longmapsto \nabla_X Y \end{aligned}$$

which is $\mathbb{R}$-bilinear and satisfies $\nabla_{fX} Y = f \nabla_X Y$ and $\nabla_X(fY) = f\nabla_X Y + X(f)Y$, for any X, $Y \in \mathfrak{X}(Q)$, $f \in C^\infty(Q)$. This implies that $\nabla_X Y(q)$ only depends on $X(q)$ and the value of Y along a curve which is tangent to X at q. Let $c : t \in [a, b] \to c(t) = (q^1(t), \dots, q^n(t)) \in Q$ be a curve on Q and W a vector field along c, i.e. a map $W : [a, b] \to TQ$

such that $\tau_Q(W(t)) = c(t)$ for all $t \in [a,b]$ (where $\tau_Q : TQ \to Q$ denotes the tangent bundle projection). Let V be a vector field that satisfies $V(c(t)) = W(t)$. The *covariant derivative of* W *along* c is defined by

$$\frac{DW(t)}{dt} = \nabla_{\dot{c}(t)} W(t) = \nabla_{\dot{c}(t)} V(q)\big|_{q=c(t)}.$$

This definition makes sense because of the defining properties of the affine connection. Now, we may take $W(t) = \dot{c}(t)$ and set up $\nabla_{\dot{c}(t)}\dot{c}(t) = 0$. This equation is called the *geodesic equation*, and its solutions are termed the *geodesics* of ∇. In local coordinates, this condition can be expressed as $\ddot{q}^a + \Gamma^a_{bc}\dot{q}^b\dot{q}^c = 0$, $1 \le a \le n$, where the $\Gamma^a_{bc}(q)$ are the *Christoffel symbols* of the affine connection, defined by

$$\nabla_{\frac{\partial}{\partial q^b}} \frac{\partial}{\partial q^c} = \Gamma^a_{bc}\frac{\partial}{\partial q^a}\,.$$

The vector field Z on TQ describing the geodesic equation is called the *geodesic spray* associated with ∇. In local coordinates,

$$Z = v^a\frac{\partial}{\partial q^a} - \Gamma^a_{bc}v^b v^c\frac{\partial}{\partial v^a}\,, \quad x = (q,v) \in TQ\,.$$

The Levi-Civita connection $\nabla^{\mathcal{G}}$ associated with the Riemannian metric is determined by the formula

$$\mathcal{G}(\nabla^{\mathcal{G}}_{X_1}X_2, X_3) = \frac{1}{2}(X_1(\mathcal{G}(X_2,X_3)) + X_2(\mathcal{G}(X_3,X_1)) - X_3(\mathcal{G}(X_1,X_2)) + \mathcal{G}(X_2,[X_3,X_1]) - \mathcal{G}(X_1,[X_2,X_3]) + \mathcal{G}(X_3,[X_1,X_2]))\,,$$

where $X_i \in \mathfrak{X}(Q)$. The Christoffel symbols of $\nabla^{\mathcal{G}}$ are

$$\Gamma^a_{bc} = \frac{1}{2}\mathcal{G}^{ad}\left(\frac{\partial \mathcal{G}_{db}}{\partial q^c} + \frac{\partial \mathcal{G}_{dc}}{\partial q^b} - \frac{\partial \mathcal{G}_{bc}}{\partial q^d}\right)\,,$$

where $(\mathcal{G}^{ad})$ denotes the inverse of the matrix $(\mathcal{G}_{da} = \mathcal{G}(\partial/\partial q^d, \partial/\partial q^a))$. The geodesics of $\nabla^{\mathcal{G}}$ are precisely the solutions of the classical Euler-Lagrange equations [1] for the kinetic energy Lagrangian, $L = \frac{1}{2}\mathcal{G}$.

Instead of the input forces $\{F^1,\ldots,F^m\}$, we shall make use of the input "accelerations" $\{Y_1,\ldots,Y_m\}$, defined as $\mathcal{G}(Y_i,\cdot) = F^i$, $1 \le i \le m$. If $Y_i = Y_i^a(q)\frac{\partial}{\partial q^a}$, the control equations read

$$\dot{q}^a = v^a\,, \quad \dot{v}^a = -\Gamma^a_{bc}\dot{q}^b\dot{q}^c + \sum_{i=1}^m u_i(t)Y_i^a(q)\,, \qquad 1 \le a \le n\,. \tag{1}$$

or, in vector field notation,

$$\dot{x}(t) = Z(x) + Y^{\mathrm{lift}}(q,t)\,, \quad x(0) = (q_0, v_0)\,, \tag{2}$$

where $Y^{\rm lift}(q,t) = \sum_{i=1}^m u_i(t) Y_i^a(q)\partial/\partial v^a = Y^a(q,t)\partial/\partial v^a$ denotes the vertical lift [30] of the control vector field, $Y(q,t) = \sum_i u_i(t)Y_i^a(q)\frac{\partial}{\partial q^a}$. This way of writing the control equations hides the second order nature of mechanical systems. The affine connection formalism, however, captures this special geometry while providing an intrinsic formulation of the problem. In fact, equation (1) can be written as

$$\nabla^{\mathcal{G}}_{\dot c(t)}\dot c(t) \;=\; \sum_{i=1}^m u_i(t) Y_i(c(t))\,. \tag{3}$$

Observe that we can use a general affine connection instead of the Levi-Civita one without changing the structure of the equation. This is particularly interesting, since nonholonomic mechanical control systems give also rise to equations of the form (3).

A *constrained mechanical control system* $(Q,\mathcal{G},V,\mathcal{F},\mathcal{D})$ is a simple mechanical control system $(Q,\mathcal{G},V,\mathcal{F})$ subject to the constraints given by the $(n-l)$-dimensional (nonholonomic) distribution $\mathcal{D}$ on Q. In a local description, $\mathcal{D}$ can be defined by the vanishing of l independent constraint functions $\omega_j(q)\dot q$, $1\le j\le l$. Under the assumption $V=0$, the application of Lagrange-d'Alembert's principle leads to

$$\ddot q^a + \Gamma^a_{bc}\dot q^b \dot q^c = \sum_{j=1}^l \lambda^j \omega_{je}\mathcal{G}^{ae} + \sum_{i=1}^m u_i(t) Y_i^a(q)\,, \quad 1\le a\le n\,, \tag{4}$$

which, together with the constraint equations $\omega_j(q)\dot q = 0$, describe the dynamics of the nonholonomic system. Here, the λ^j are the Lagrange multipliers. The term $\sum_{j=1}^l \lambda^j \omega_{je}$ represents the "reaction force" due to the constraints. This equation can alternatively be written as

$$\nabla^{\mathcal{G}}_{\dot c(t)}\dot c(t) = \lambda(t) + \sum_{i=1}^m u_i(t) Y_i(c(t))\,, \quad \dot c(t)\in\mathcal{D}_{c(t)}\,, \tag{5}$$

where now λ is seen as a section of $\mathcal{D}^\perp$, the $\mathcal{G}$-orthogonal complement to $\mathcal{D}$, along the curve c. Denoting by $\mathcal{P}: TQ\to\mathcal{D}$, $\mathcal{Q}: TQ\to\mathcal{D}^\perp$ the $\mathcal{G}$-orthogonal projectors, we can define an affine connection

$$\overline{\nabla}_X Y = \nabla^{\mathcal{G}}_X Y + (\nabla^{\mathcal{G}}_X\mathcal{Q})(Y) = \mathcal{P}(\nabla^{\mathcal{G}}_X Y) + \nabla^{\mathcal{G}}_X(\mathcal{Q}(Y))\,,$$

such that the nonholonomic control equations (5) can be rewritten as

$$\overline{\nabla}_{\dot c(t)}\dot c(t) \;=\; \sum_{i=1}^m u_i(t)\mathcal{P}(Y_i(c(t)))\,, \tag{6}$$

where we select the initial velocity in $\mathcal{D}$ (cf. [29] for details). Observe that the inputs Y_i act on the system only through their $\mathcal{D}$-components. The connection $\overline{\nabla}$ is called the *nonholonomic affine connection* [5, 29, 49]. Note that equations (3) and (6) have the same structure.

Remark 1. It can be easily deduced that $\overline{\nabla}$ restricts to $\mathcal{D}$, that is, $\overline{\nabla}_X Y = \mathcal{P}(\nabla^{\mathcal{G}}_X Y) \in \mathcal{D}$, for all $Y \in \mathcal{D}$, $X \in \mathfrak{X}(Q)$. This property implies (cf. [27]) that $\mathcal{D}$ is *geodesically invariant*, i.e. for every geodesic $c(t)$ of $\overline{\nabla}$ with initial velocity in $\mathcal{D}$, $\dot{c}(0) \in \mathcal{D}_{c(0)}$, we have $\dot{c}(t) \in \mathcal{D}_{c(t)}$.

A key tool in the description of mechanical control systems is the *symmetric product* $\langle \cdot : \cdot \rangle$ associated with a general affine connection ∇, see [16, 30, 43]. Given X, $Y \in \mathfrak{X}(Q)$, we define

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X \,.$$

As we will see in Section 3, the symmetric product plays a fundamental role in the controllability analysis of mechanical systems. Another interesting feature of the symmetric product is that characterizes geodesically invariant distributions: a distribution $\mathcal{U}$ on Q is geodesically invariant for ∇ if and only if $\langle X : Y \rangle \in \mathcal{U}$, $\forall X$, $Y \in \mathcal{U}$, see [27].

2.2. Systems with symmetry

A wide range of mechanical systems exhibit translational and rotational symmetries. Examining the configuration space Q, one observes that a splitting $Q = G \times M$ exists between variables describing the position and orientation of the system, or *pose* coordinates g in the Lie group G, and variables describing the internal shape of the system or *shape* coordinates $r \in M$. This exactly corresponds to the geometrical notion of a *trivial principal fiber bundle*, which we briefly describe next. For further details, we refer the reader to [1, 23].

Assume that a Lie group G acts freely and properly on Q,

$$\begin{aligned} \Phi :\ G \times Q &\longrightarrow Q \\ (g, q) &\longmapsto \Phi(g, q) = \Phi_g(q) = gq \,, \end{aligned}$$

In this way, the quotient space $Q/G = M$ has a manifold structure such that the projection $\pi : Q \to M$ is a surjective submersion. We say then that $Q(M, G, \pi)$ is a principal bundle with bundle space Q, *base* space M, *fiber* space G and projection π. Note that $\ker \pi_*$ consists of the *vertical* tangent vectors, i.e., the vectors tangent to the orbits of G in Q. Locally, one can always trivialize Q and assume that $Q \equiv G \times M$. The associated bundle coordinates, $q = (g, r) \in Q$ satisfy $\pi(g, r) = r$. Usually G will be a subgroup of the matrix group $SE(3)$, such as $SE(2)$ for a snake or paramecium or $SO(3)$ for a satellite or a falling cat, and the action of G on Q reduces to matrix multiplication on the first factor.

A special coordinate system on G is given by the exponential mapping, $\exp : \mathfrak{g} \to G$ [50], where $\mathfrak{g}$ denotes the Lie algebra of G. In an open neighborhood U of $e \in G$, exp is invertible. We write $\exp^{-1} =$

$\log : U \subset G \to \mathfrak{g} \cong \mathbb{R}^k$, and for each $g \in V$, the exponential coordinates are defined as $x = \log(g)$. For instance, if $G = SE(2)$, and $g = (a, b, \theta)$ is such that $\theta \neq 0$, we have

$$\begin{aligned} x(1) &= \theta(a \sin\theta + b(1 - \cos\theta))/(2(1 - \cos\theta)) \,, \\ x(2) &= \theta(b \sin\theta - a(1 - \cos\theta))/(2(1 - \cos\theta)) \,, \\ x(3) &= \theta \,, \end{aligned}$$

and otherwise, $x = g$. See [36] for the corresponding expressions in other Lie groups.

Let the control system $(Q, \mathcal{G}, \mathcal{F})$ be *invariant* under the action of G, i.e. $(\Phi_g)^*\mathcal{G} = \mathcal{G}$ and $(\Phi_g)^* F^i = F^i$, for $1 \leq i \leq m$ and $g \in G$. Additionally, if the mechanical system is constrained, we must also have that $(\Phi_g)_*\mathcal{D} = \mathcal{D}$, for all $g \in G$. The invariance of the metric implies that the Lagrangian verifies $L(g, r, \dot{g}, \dot{r}) = L(e, r, g^{-1}\dot{g}, \dot{r}) = \ell(r, \dot{r}, \xi)$, where $\xi = g^{-1}\dot{g}$. The *reduced Lagrangian* $\ell : TQ/G \to \mathbb{R}$ is given by

$$\ell(r, \dot{r}, \xi) = \frac{1}{2}(\xi^T \, \dot{r}^T) \, \hat{\mathcal{G}} \begin{pmatrix} \xi \\ \dot{r} \end{pmatrix} \,,$$

where $\hat{\mathcal{G}}$ stands for the reduced metric [38]

$$\hat{\mathcal{G}} = \begin{pmatrix} I(r) & I(r)A(r) \\ A(r)^T I(r) & m(r) \end{pmatrix} \,. \tag{7}$$

Here, $I(r)$ and $A(r)$ denote the local form of the locked inertia tensor and the mechanical connection. $I(r)$ has the interpretation of the inertia of the system when frozen at shape r [1]. $A(r)$ plays a central role in understanding locomotion, since it determines how internal shape changes create net system motions [22, 40]. The mechanical connection is an example of a *principal connection* on a principal fiber bundle. Other related tensors and operations include the *curvature*, B, and the *derivative* along the principal connection, D, see [15] for further reference.

The reduced metric $\hat{\mathcal{G}}$ is block diagonalized when written in terms of the shape variables $(r, \dot{r})$ and the *locked body angular velocity*, $\Omega = \xi + A(r)\dot{r}$. Indeed, one can see that $\hat{\mathcal{G}}$ takes the form

$$\tilde{\mathcal{G}} = \begin{pmatrix} I(r) & 0 \\ 0 & m(r) - A^T(r)I(r)A(r) \end{pmatrix} = \begin{pmatrix} I(r) & 0 \\ 0 & \Delta(r) \end{pmatrix} \,.$$

The invariance of the problem induces a decomposition of the affine connections $\nabla^{\mathcal{G}}$ and $\overline{\nabla}$ (and, consequently, of the associated symmetric products) which reveals the special geometric structure of these systems. In addition, this also leads to important computational savings in the controllability analysis.

Proposition 1. ([15]) Given G-invariant $X = (g\xi(r), v(r))$ and $Y = (g\eta(r), w(r))$, the symmetric product associated with the Levi-Civita connection $\nabla^{\mathcal{G}}$ is

$$\langle X : Y\rangle_{\mathcal{G}} = g \begin{pmatrix} \langle \Omega : \Psi\rangle_I - I^{-1}\mathbb{L}^s \\ \langle v : w\rangle_\Delta - \Delta^{-1}\mathbb{S} \end{pmatrix} \tag{8}$$

where $\Omega = \xi + A(r)\dot{r}$, $\Psi = \eta + A(r)\dot{r}$ and

$$\begin{aligned} \mathbb{L}^s &= -D(I\Omega)(\cdot, w) - D(I\Psi)(\cdot, v) + IA\left(\langle v : w\rangle_\Delta - \Delta^{-1}\mathbb{S}\right) \in \mathfrak{g}^* \\ \mathbb{S} &= I(\Omega, B(w,\cdot)) + I(\Psi, B(v,\cdot)) + DI(\cdot)(\Omega, \Psi) \in T^*M\,, \end{aligned}$$

with $\langle \cdot : \cdot\rangle_I$, $\langle \cdot : \cdot\rangle_\Delta$ the symmetric products defined by the Levi-Civita connections ∇^I and ∇^Δ, respectively, and $\mathfrak{g}^*$ is the dual of the Lie algebra.

The principal connection which plays the role of the mechanical connection for nonholonomic systems is the *nonholonomic connection* [6, 38]. We denote its local form by $\mathcal{A}$, its curvature by $\mathbb{B}$ and the associated derivative by $\mathcal{D}$. Defining $\bar{\Omega} = \xi + \mathcal{A}(r)\dot{r}$, one can compute

$$\ell = \frac{1}{2}(\bar{\Omega}^T\, \dot{r}^T) \begin{pmatrix} I(r) & 0 \\ 0 & \tilde{\Delta}(r) \end{pmatrix} \begin{pmatrix} \bar{\Omega} \\ \dot{r} \end{pmatrix},$$

where $\tilde{\Delta}(r) = m(r) - A^T(r)I(r)A(r) + \tilde{A}^T(r)I(r)\tilde{A}(r)$ and $\tilde{A} = A - \mathcal{A}$. Finally, define $A^{\text{sym}} : TQ \to \mathcal{V}\cap\mathcal{D}$, as $A^{\text{sym}}(q)(\dot{q}) = (\tilde{I}^{-1}p)_Q$, where $\tilde{I}$ is the local form of the constrained inertia tensor and p is the constrained momentum, see [6, 38] for further details.

Proposition 2. ([15]) Given G-invariant $X = (g\xi(r), v(r)) \in \mathcal{D}$ and $Y = (g\eta(r), w(r)) \in \mathcal{D}$, the symmetric product associated with $\overline{\nabla}$ is

$$\langle X : Y\rangle = g \begin{pmatrix} A^{\text{sym}}(\langle \bar{\Omega} : \bar{\Psi}\rangle_I) - \tilde{I}^{-1}\tilde{\mathbb{L}}^s + \mathcal{A}\left(\langle v : w\rangle_{\tilde{\Delta}} - \tilde{\Delta}^{-1}\tilde{\mathbb{S}}^s\right) \\ \langle v : w\rangle_{\tilde{\Delta}} - \tilde{\Delta}^{-1}\tilde{\mathbb{S}}^s \end{pmatrix} \tag{9}$$

where $\bar{\Omega} = \xi + \mathcal{A}(r)\dot{r}$, $\bar{\Psi} = \eta + \mathcal{A}(r)\dot{r}$ and

$$\begin{aligned} \tilde{\mathbb{L}}^s &= -\mathcal{D}(I\bar{\Omega})(\cdot, w) \\ &\quad - \mathcal{D}(I\bar{\Psi})(\cdot, v) + I(\tilde{A}v, \gamma.w - [\cdot, \eta]) + I(\tilde{A}w, \gamma.v - [\cdot, \xi]) \in \mathfrak{g}^{\mathcal{D}^*} \\ \tilde{\mathbb{S}}^s &= I(\bar{\Psi}, B(v,\cdot)) + I(\bar{\Omega}, B(w,\cdot)) + I(\tilde{A}w, \mathbb{B}(v,\cdot)) \\ &\qquad + I(\tilde{A}v, \mathbb{B}(w,\cdot)) - D(I\bar{\Psi})(\tilde{A}\cdot, v) - D(I\bar{\Omega})(\tilde{A}\cdot, w) \\ &\qquad\quad + \mathcal{D}I(\cdot)(\bar{\Omega} + \tilde{A}v, \bar{\Psi} + \tilde{A}w) - \mathcal{D}I(\cdot)(\tilde{A}v, \tilde{A}w) \in T^*M\,, \end{aligned}$$

with $\langle \cdot : \cdot\rangle_{\tilde{\Delta}}$ the symmetric product defined by $\nabla^{\tilde{\Delta}}$.

2.3. Examples

Planar model for a blimp system. Consider a rigid body moving in $SE(2)$ with a thruster to adjust its pose (see Figure 1). The original motivation for this problem is the blimp system developed in [51] restricted to the horizontal plane. The control inputs are the thruster force F^1 and a torque F^2 that actuates its orientation with respect to the body axis. The acting point of the thruster is assumed to be located along the body's long axis, at a distance h from the center of mass.

The configuration of the blimp is given by $(x, y, \theta, \gamma) \in SE(2) \times \mathbb{S}^1$, where (x, y) is the position of the center of mass, θ is the orientation of the blimp with respect to the fixed basis $\{X^f, Y^f\}$ and γ is the orientation of the thrust with respect to the body basis $\{X^b, Y^b\}$.

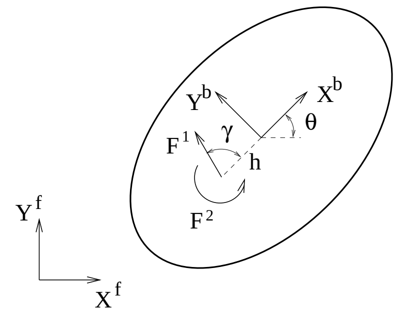

Figure 1. A planar blimp with rotating thruster

For simplicity, we assume that the thruster is massless. Then, the Riemannian metric of the system is given by

$$\mathcal{G} = (\mathcal{G}_{ab}) = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & J_1 + J_2 & J_2 \\ 0 & 0 & J_2 & J_2 \end{pmatrix},$$

where m denotes the mass of the blimp, J_1 is its moment of inertia and J_2 is the inertia of the thruster. The input forces are given by

$$F^1 = \cos(\theta + \gamma) dx + \sin(\theta + \gamma) dy - h \sin \gamma d\theta \,, \quad F^2 = d\gamma \,.$$

This simple mechanical control system is invariant under the left multiplication of the Lie group $G = SE(2)$ on Q, $\Phi((a, b, \alpha), (x, y, \theta, \gamma)) = (x \cos \alpha - y \sin \alpha + a, x \sin \alpha + y \cos \alpha + b, \theta + \alpha, \gamma)$. Hence, the blimp is a mechanical control system with symmetry. Given G-invariant vector fields $X = (g\xi, v)$, $Y = (g\eta, w)$, from Proposition 1 one can easily obtain,

$$\langle X : Y \rangle_{\mathcal{G}} = g \left\{ \begin{pmatrix} -\Omega^2\Psi^3 - \Omega^3\Psi^2 \\ \Omega^1\Psi^3 + \Omega^3\Psi^1 \\ 0 \\ \frac{\partial w}{\partial \gamma} v + \frac{\partial v}{\partial \gamma} w \end{pmatrix} - \begin{pmatrix} I^{-1}\mathbb{L}^s \\ 0 \end{pmatrix} \right\}, \qquad (10)$$

where

$$\mathbb{L}^s = - \begin{pmatrix} m\left(\frac{\partial \xi^1}{\partial \gamma} w + \frac{\partial \eta^1}{\partial \gamma} v\right) \\ m\left(\frac{\partial \xi^2}{\partial \gamma} w + \frac{\partial \eta^2}{\partial \gamma} v\right) \\ (J_1 + J_2)\left(\frac{\partial \xi^3}{\partial \gamma} w + \frac{\partial \eta^3}{\partial \gamma} v\right) \end{pmatrix} - \frac{mJ_2}{J_1 + J_2} \begin{pmatrix} \Omega^2 w + \Psi^2 v \\ -\Omega^1 w - \Psi^1 v \\ 0 \end{pmatrix}.$$

Planar robotic manipulator with a passive joint. Consider a three joint robotic manipulator operating on a horizontal plane (see Figure 2). We have taken this example from [11], where its kinematically controllability properties have been studied. The configuration of the manipulator is given by a triple $(\theta, r_2, r_3) \in \mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, where θ is the angle of the first link with the plane, and (r_2, r_3) are the joint angles between the links. The control inputs are the torques F^1 and F^2 which actuate the shape angles (r_2, r_3).

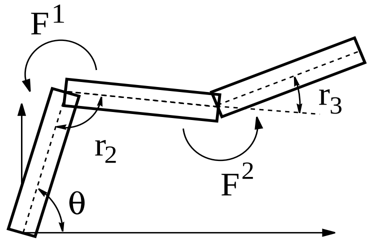

Figure 2. Three-link 3R planar robotic manipulator

The Riemannian metric of the system has the following structure, see [11] for the explicit expression of the coefficients:

$$\mathcal{G} = \begin{pmatrix} M_{11}(r_2, r_3) & M_{12}(r_2, r_3) & M_{13}(r_2, r_3) \\ M_{12}(r_2, r_3) & M_{22}(r_3) & M_{13}(r_3) \\ M_{13}(r_2, r_3) & M_{23}(r_3) & M_{33} \end{pmatrix}.$$

Ignoring joint friction, the dynamics of the system is determined by the kinetic energy Lagrangian associated with this metric. The input forces are given by $F_1 = dr_2$, $F_2 = dr_3$. The left multiplication of the Lie group $G = \mathbb{S}^1$ on Q, $\Phi : G \times Q \to Q$, $\Phi(a, (\theta, r_2, r_3)) = (a + \theta, r_2, r_3)$ leaves invariant this mechanical system.

The snakeboard. The snakeboard [31, 38] is a variant of the skateboard in which the passive wheel assemblies can pivot freely about a vertical axis. By coupling the twisting of the human torso with the appropriate turning of the wheels (where the turning is controlled by the rider's foot movement), the rider can generate a snake-like locomotion pattern without having to kick off the ground.

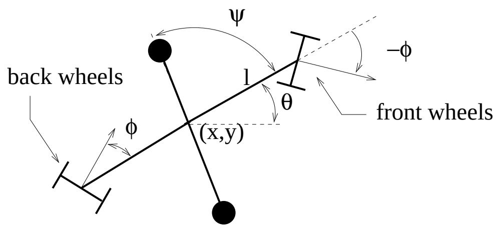

Figure 3. The snakeboard model

A model is shown in Figure 3. We make the simplifying assumption that the front and rear wheel axles move through equal and opposite rotations [6, 39], which eliminates terms in the derivations below and does not affect the essential features of the problem. A momentum wheel rotates about a vertical axis through the center of mass, simulating the motion of a human torso.

The position and orientation of the snakeboard is determined by the coordinates of the center of mass (x, y) and its orientation θ. The shape variables are (ψ, ϕ), so the configuration space is $Q = SE(2) \times \mathbb{S}^1 \times \mathbb{S}^1$. The physical parameters are the mass and the inertia of the board, m and J, respectively; the inertia of the rotor, J_r; the inertia of the wheels about the vertical axes, J_w; and the half-length of the board, l. To keep the rotor and body inertias on similar scales, we make the additional simplifying assumption [38, 39] that $J + J_r + 2J_w = ml^2$.

The Riemannian metric of this system is

$$\begin{aligned} \mathcal{G} \; = \; & m(dx \otimes dx + dy \otimes dy) + (J + J_r + 2J_w) d\theta \otimes d\theta \\ & + J_r(d\theta \otimes d\psi + d\psi \otimes d\theta) + J_r d\psi \otimes d\psi + 2J_w d\phi \otimes d\phi \,. \end{aligned}$$

The control torques are assumed to be applied to the rotation of the wheels and the rotor. Hence, we consider $F^1 = d\psi$, $F^2 = d\phi$.

The assumption that the wheels do not slip in the direction of the wheels axles yields the following two nonholonomic constraints

$$\begin{aligned} & -\sin(\theta + \phi)\dot{x} + \cos(\theta + \phi)\dot{y} - l\cos\phi\,\dot{\theta} = 0 \,, \\ & -\sin(\theta - \phi)\dot{x} + \cos(\theta - \phi)\dot{y} + l\cos\phi\,\dot{\theta} = 0 \,. \end{aligned}$$

A quick set of calculations shows that this constrained mechanical system is invariant under the left multiplication of $SE(2)$ (see [6, 38]).

3. Controllability properties of simple mechanical systems

In this section we present the notions of controllability for mechanical systems and briefly review the existing tests under the affine connection formalism [9, 15, 30]. We also illustrate these controllability tests with the examples introduced above.

3.1. Controllability notions and tests

In some mechanical systems, configurations may be controlled, but not configurations and velocities at the same time. The affine connection formalism helps focus the analysis on the set of configurations and states that are attainable from a given configuration starting from rest. Consider the control equation

$$\nabla_{\dot{c}(t)}\dot{c}(t) = \sum_{i=1}^{m} u_i(t) Y_i(c(t)) \,, \tag{11}$$

where the affine connection ∇ can be either the Levi-Civita affine connection associated with a kinetic energy metric or the nonholonomic affine connection for a constrained system.

Take $q_0 \in Q$, $(q_0, 0_{q_0}) \in T_{q_0}Q$ and let $U \subset Q$, $\bar{U} \subset TQ$ be neighborhoods of q_0 and $(q_0, 0_{q_0})$, respectively, with $U = \tau_Q(\overline{U})$. Define

$$\mathcal{R}^{\overline{U}}_{TQ}(q_0, T) = \left\{ (q, v) \in TQ \;\middle|\; \begin{array}{l} \text{there exists a solution } (c, u) \text{ of (11)} \\ \text{such that } \dot{c}(0) = 0_{q_0}, (c(t), \dot{c}(t)) \in \overline{U} \\ \text{for } t \in [0, T] \text{ and } \dot{c}(T) = v \in T_qQ \end{array} \right\} ,$$

and $\mathcal{R}^U_Q(q_0, T) = \tau_Q(\mathcal{R}^{\overline{U}}_{TQ}(q_0, T))$. Let $\mathcal{R}^U_Q(q_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}^U_Q(q_0, t)$, $\mathcal{R}^{\overline{U}}_{TQ}(q_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}^{\overline{U}}_{TQ}(q_0, t)$. We will focus our attention on the following notions of accessibility and controllability [30].

Definition 1. The system (11) is *locally configuration accessible (LCA) at $q_0 \in Q$* (resp. *locally accessible (LA) at $q_0 \in Q$ and zero velocity*) if there exists $T > 0$ such that $\mathcal{R}^U_Q(q_0, \leq t)$ (resp. $\mathcal{R}^{\overline{U}}_{TQ}(q_0, \leq t)$) contains a non-empty open set of Q (resp. of TQ), for all neighborhoods U of q_0 (resp. $\overline{U}$ of $(q_0, 0_{q_0})$) and all $0 \leq t \leq T$. If this holds for any $q_0 \in Q$ then the system is called *locally configuration accessible* (resp. *locally accessible at zero velocity*).

Definition 2. The system (11) is *small-time locally configuration controllable (STLCC) at $q_0 \in Q$* (resp. *small-time locally controllable (STLC) at $q_0 \in Q$ and zero velocity*) if there exists $T > 0$ such that $\mathcal{R}^U_Q(q_0, \leq t)$ (resp. $\mathcal{R}^{\overline{U}}_{TQ}(q_0, \leq t)$) contains a non-empty open set of Q (resp. of TQ)

to which q_0 (resp. $(q_0, 0_{q_0})$) belongs, for all neighborhoods U of q_0 (resp. $\overline{U}$ of $(q_0, 0_{q_0})$) and all $0 \leq t \leq T$. If this holds for any $q_0 \in Q$ then the system is called *small-time locally configuration controllable* (resp. *small-time locally controllable at zero velocity*).

Given the set $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$, denote by $\overline{\mathrm{Sym}}(\mathcal{Y})$ the distribution obtained by closing the set $\mathcal{Y}$ under the symmetric product and by $\overline{\mathrm{Lie}}(\mathcal{Y})$ the involutive closure of $\mathcal{Y}$. If P is a symmetric product of vector fields in $\mathcal{Y}$, we let $\gamma_i(P)$ denote the number of occurrences of Y_i in P. The *degree* of P will be $\sum_{j=1}^{m} \gamma_j(P)$. P is *bad* if $\gamma_i(P)$ is even for each $1 \leq i \leq m$. Otherwise, P is *good*. We refer the reader to [30] for the precise statement of these notions (degree, bad, good) in terms of free symmetric algebras.

The (configuration) accessibility and controllability tests for mechanical systems developed in [30] have been further refined for systems evolving on Lie groups [9, 10] and on principal fiber bundles [15]. Assume that the control system (11) is invariant under the action of a Lie group G. Let us denote by $\mathcal{B} = \{B_1, \ldots, B_m\}$ the representatives of the input vector fields $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ at $\mathfrak{g} \times TM$, that is,

$$Y_i(r, g) = gB_i(r, e) = g \begin{pmatrix} \xi_i(r) \\ v_i(r) \end{pmatrix}, \quad 1 \leq i \leq m\,.$$

Due to the invariance of the system, $\langle Y_i : Y_j \rangle = \langle gB_i : gB_j \rangle = g \langle B_i : B_j \rangle$ for all $1 \leq i, j \leq m$. The explicit expression in bundle coordinates for this symmetric product is given by Proposition 1 for unconstrained systems and by Proposition 2 for constrained ones. Note also that the Lie brackets $[Y_i, Y_j]$ can be written as

$$[Y_i, Y_j] = g\,[B_i, B_j] = g \begin{pmatrix} [B_i, B_j]_{\mathfrak{g}} \\ [B_i, B_j]_{TM} \end{pmatrix} = g \begin{pmatrix} [\xi_i, \xi_j]_{\mathfrak{g}} + \frac{\partial \xi_j}{\partial r} v_i - \frac{\partial \xi_i}{\partial r} v_j \\ [v_i, v_j]_{TM} \end{pmatrix}$$

Theorem 1. ([15]) Let the system (11) be invariant under G.

1. The system is LCA at $q = (g, r)$ (resp. LA at q and zero velocity) if $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{B}))_{(e,r)} = \mathfrak{g} \times T_r M$ (resp. $\overline{\mathrm{Sym}}(\mathcal{B})_{(e,r)} = \mathfrak{g} \times T_r M$).

2. Suppose that the system is LCA at (e, r) (resp. LA at (e, r) and zero velocity) and that every bad symmetric product P at (e, r) in $\mathcal{B}$ can be written as a linear combination of good symmetric products at (e, r) of lower degree than P. Then (11) is STLCC at $q = (g, r)$, $g \in G$ (resp. STLC at q and zero velocity).

Note that these tests remove completely the dependence on the Lie group elements $g \in G$ from the computations. An additional computational simplification occurs for many dynamic locomotion systems, where some of the input forces are of the form $F^j = dr^j$ for certain j. The associated input vector fields have vanishing locked body angular velocities [15], which eliminates many terms in the computation of the symmetric products (cf. Propositions 1 and 2).

The concept of *weak controllability* for kinematic systems [22] can also be defined this kind of control systems [15]. This notion essentially means controllability in the fiber, without regards to the intermediate or final positions of the shape variables. This concept is meaningful for locomotion systems, where the group elements correspond to position and orientation. Let $Q = G \times M$, and V^τ (resp. $\overline{V}^\tau$) denote any subset of Q (resp. TQ) such that $\tau(V^\tau)$ (resp. $\tau_*(\overline{V}^\tau)$) is an open subset of G (resp. TG), where $\tau : Q \to G$ denotes the natural projection. Let $q_0 = (r_0, g_0)$ and $U \subset Q$, $\overline{U} \subset TQ$ as before. Then,

Definition 3. The system (11) is *locally fiber configuration accessible (LFCA) at $q_0 \in Q$* (resp. *locally fiber accessible (LFA) at $q_0 \in Q$ and zero velocity*) if there exists $T > 0$ such that $\mathcal{R}_Q^U(q_0, \leq t)$ (resp. $\mathcal{R}_{TQ}^{\overline{U}}(q_0, \leq t)$) contains a subset $\emptyset \neq V^\tau$ of Q (resp. $\overline{V}^\tau$ of TQ), for all neighborhoods U of q_0 (resp. $\overline{U}$ of $(q_0, 0_{q_0})$) and all $0 \leq t \leq T$. If this holds for any $q_0 \in Q$ then the system is called *locally fiber configuration accessible* (resp. *locally fiber accessible at zero velocity*).

Definition 4. The system (11) is *small-time locally fiber configuration controllable (STLFCC) at $q_0 \in Q$* (resp. *small-time locally fiber controllable (STLFC) at $q_0 \in Q$ and zero velocity*) if there exists $T > 0$ such that $\mathcal{R}_Q^U(q_0, \leq t)$ (resp. $\mathcal{R}_{TQ}^{\overline{U}}(q_0, \leq t)$) contains a subset $\emptyset \neq V^\tau$ of Q (resp. $\overline{V}^\tau$ of TQ) such that $g_0 \in \tau(V^\tau)$ (resp. $(g_0, 0_{g_0}) \in \tau_*(\overline{V}^\tau)$), for all neighborhoods U of q_0 (resp. $\overline{U}$ of $(q_0, 0_{q_0})$) and all $0 \leq t \leq T$. If this holds for any $q_0 \in Q$ then the system is called *small-time locally fiber configuration controllable* (resp. *small-time locally fiber controllable at zero velocity*).

Theorem 2. ([15]) Let the system (11) be invariant under G.

1. The system is LFCA at $q = (g, r)$ (resp. LFA at q and zero velocity) if $\tau_* \overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\boldsymbol{\mathcal{B}}))_{(e,r)} = \mathfrak{g}$ (resp. $\tau_*(\overline{\mathrm{Sym}}(\boldsymbol{\mathcal{B}})_{(e,r)}) = \mathfrak{g} \times T_r M$).
2. Suppose that the system is LFCA at (e, r) (resp. LFA at (e, r) and zero velocity) and that the projection through τ of every bad symmetric product P at (e, r) in $\boldsymbol{\mathcal{B}}$, $\tau_* P$, can be written as a linear combination of projections through τ of good symmetric products at (e, r) of lower degree than P. Then (11) is STLFCC at $q = (r, g)$, for all $g \in G$ (resp. STLFC at q and zero velocity).

3.2. Application to the examples

Planar model of the blimp system. The reduced representation of the input vector fields at $\mathfrak{g} \times TM$ is given by

$$B_1 = \frac{1}{m}\cos\gamma\frac{\partial}{\partial x} + \frac{1}{m}\sin\gamma\frac{\partial}{\partial y} - \frac{h}{J_1}\sin\gamma\frac{\partial}{\partial\theta} + \frac{h}{J_1}\sin\gamma\frac{\partial}{\partial\gamma}\,,$$
$$B_2 = -\frac{1}{J_1}\frac{\partial}{\partial\theta} + \frac{J_1+J_2}{J_1J_2}\frac{\partial}{\partial\gamma}\,,$$

and some of their symmetric products are

$$\langle B_1 : B_1\rangle_{\mathcal{G}} = \frac{h^2}{J_1^2}\sin(2\gamma)\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}, \quad \langle B_1 : B_2\rangle_{\mathcal{G}} = \begin{pmatrix}-\frac{1}{mJ_2}\sin\gamma\\ \frac{1}{mJ_2}\cos\gamma\\ -\frac{h(J_1+J_2)}{J_1^2J_2}\cos\gamma\\ \frac{h(J_1+J_2)}{J_1^2J_2}\cos\gamma\end{pmatrix},$$

$$\langle B_2 : B_2\rangle_{\mathcal{G}} = 0\,, \quad \Big\langle B_2 : \langle B_1 : B_1\rangle_{\mathcal{G}}\Big\rangle_{\mathcal{G}} = 2\frac{h^2}{J_1^2}\frac{J_1+J_2}{J_1J_2}\cos(2\gamma)\begin{pmatrix}0\\0\\-1\\1\end{pmatrix}.$$

Now, $\{B_1, B_2, \langle B_1 : B_2\rangle_{\mathcal{G}}\,, \langle B_1 : B_1\rangle_{\mathcal{G}}\,, \Big\langle B_2 : \langle B_1 : B_1\rangle_{\mathcal{G}}\Big\rangle_{\mathcal{G}}\}$ spans $\mathfrak{g}\times TM$ at every (e,r) and hence the system is LA at zero velocity. However, the bad product $\langle B_1 : B_1\rangle_{\mathcal{G}}$ is not a linear combination of the lower order good products B_1 and B_2. Therefore, we can not conclude that the system is STLC at zero velocity. The case $\gamma = 0$ is an exception since $\langle B_1 : B_1\rangle_{\mathcal{G}}(e,0) = 0$ (thus the system is STLC at $(g,0)$ and zero velocity, for all $g \in G$). On the other hand, if we restrict our attention to fiber controllability, we observe that $\tau_*\langle B_1 : B_1\rangle_{\mathcal{G}} \in \mathrm{span}\{\tau_* B_2\}$ and therefore the blimp is STLFC at zero velocity.

The planar robotic manipulator. The expression for the corresponding reduced input vector fields, B_1, B_2 is too lengthy to report it here. It is enough to note that the computational simplification mentioned above is in place: B_1 and B_2 have vanishing locked body angular velocities $\Omega_1 = 0 = \Omega_2$. As a consequence, the locked body angular velocity corresponding to the symmetric product of any input vector fields also vanishes. Denoting by $\mathcal{H} = \mathrm{span}\{B_1, B_2\}$, we conclude

$$X, Y \in \mathcal{H} \Rightarrow \langle X : Y\rangle_{\mathcal{G}} \in \mathcal{H}\,.$$

Otherwise said, the input distribution is geodesically invariant. Another way of reaching the same conclusion is observing that the input distribution is the horizontal space of the mechanical connection [1].

This implies that the system is indeed kinematic as defined in [28] (a fact already observed in [11]).

Computations with Mathematica show that span $\{B_1, B_2, [B_1, B_2]\} = \mathfrak{g} \times TM$ and therefore the system is LCA. As every bad symmetric product can be obviously written as a combination of good ones by the previous discussion, we have that the system is indeed STLCC.

The snakeboard. The projections to $\mathcal{D}$ of the reduced representatives of the inputs under the decomposition $TQ = \mathcal{D} \oplus \mathcal{D}^\perp$ are

$$\begin{aligned} \mathsf{B}_1 &= \mathcal{P}(B_1) = \frac{ml^2}{J_r(ml^2 - J_r \sin^2 \phi)} \left(\frac{\partial}{\partial \psi} + \frac{J_r}{ml^2} \sin \phi e_1 \right) , \\ \mathsf{B}_2 &= \mathcal{P}(B_2) = \frac{1}{2J_w} \frac{\partial}{\partial \phi} . \end{aligned}$$

Taking into account that $\bar{\Omega}_i = 0$ for B_i, $i = 1, 2$, the amount of calculations for the controllability tests following Proposition 2 is quite light. Indeed, it is easy to see that the controllability analysis yields the following results at the point $\mathbf{0} = (0,0,0,0,0)$

$$\begin{aligned} \langle \mathsf{B}_1 : \mathsf{B}_1 \rangle (\mathbf{0}) &= 0 , \qquad \langle \mathsf{B}_1 : \mathsf{B}_2 \rangle (\mathbf{0}) = \frac{1}{2J_w ml} e_x , \\ \langle \mathsf{B}_2 : \mathsf{B}_2 \rangle (\mathbf{0}) &= 0 , \qquad [\mathsf{B}_1, \mathsf{B}_2] (\mathbf{0}) = \frac{1}{2J_w ml} e_x , \\ [\mathsf{B}_2, [\mathsf{B}_1, \mathsf{B}_2]] (\mathbf{0}) &= -\frac{1}{2J_w^2 ml^2} e_\theta , \\ [\mathsf{B}_2, [\mathsf{B}_1, [\mathsf{B}_2, [\mathsf{B}_1, \mathsf{B}_2]]]] (\mathbf{0}) &= -\frac{1}{4J_w^3 m^2 l^3} e_y - \frac{1}{2J_w^3 m^2 l^4} e_\theta . \end{aligned}$$

Since $\{\mathsf{B}_1, \mathsf{B}_2, \langle \mathsf{B}_1 : \mathsf{B}_2 \rangle , [\mathsf{B}_2, [\mathsf{B}_1, \mathsf{B}_2]] , [\mathsf{B}_2, [\mathsf{B}_1, [\mathsf{B}_2, [\mathsf{B}_1, \mathsf{B}_2]]]]\}$ spans $\mathfrak{g} \times T_{(0,0)}M$, the system is LCA at $(g, 0, 0)$, for all $g \in G$. Moreover, the bad symmetric products $\langle \mathsf{B}_1 : \mathsf{B}_1 \rangle$ and $\langle \mathsf{B}_2 : \mathsf{B}_2 \rangle$ vanish at $\mathbf{0}$ and the remaining ones are either 0 or in span$\{\mathsf{B}_2(\mathbf{0}), \langle \mathsf{B}_1 : \mathsf{B}_2 \rangle (\mathbf{0})\}$, so we have STLCC at $(g, 0, 0)$, for all $g \in G$.

4. Series expansion starting from non-zero velocity

Within the realm of geometric control theory, series expansions play a key role in the study of nonlinear controllability [3, 20, 44, 45], trajectory generation and motion planning problems [9, 25, 26, 42], etc. Magnus [32] describes the evolution of systems on a Lie group. In [13, 17, 21, 46] a general framework is developed to describe the evolution of a nonlinear system via the so-called Chen-Fliess series and its factorization.

In the context of mechanical control systems, the work by Bullo in [8] describes the evolution of the trajectories with zero initial velocity. In this section, we give a partial generalization of this result for trajectories with non-zero initial velocity, which will be useful later in establishing the motion control algorithms.

In the sequel, for any time-dependent vector field $X(q,t)$, we will use the notation $\overline{X}(q,t)$ to refer to the operation

$$\overline{X}(q,t) = \int_0^t X(q,s)ds\,, \tag{12}$$

where the base point $q \in Q$ is held fixed in the time integral.

Let us consider the system (2) where the initial velocity v_0 is not necessarily vanishing. In a coordinate notation the system is written as

$$\begin{aligned} &\dot{q}^a(t) = v^a,\;\; \dot{v}^a(t) = -\Gamma^a_{bc}(q)v^b v^c + Y^a(q,t), \\ &q(0) = q_0,\;\; v(0) = v_0. \end{aligned} \tag{13}$$

Take now the change of coordinates given by $w = v - v_0$. Then, equation (13) becomes

$$\begin{aligned} &\dot{q}^a(t) = w^a(t) + v_0^a\,, \\ &\dot{w}(t) = -\Gamma^a_{bc}(q)v_0^b v_0^c - (\Gamma^a_{bc}(q) + \Gamma^a_{cb}(q))v_0^b w^c \\ &\qquad\qquad - \Gamma^a_{bc}(q)w^b w^c + Y^a(q,t)\,, \\ &q(0) = q_0\,, \quad w(0) = 0\,, \end{aligned} \tag{14}$$

where v_0^a are the coordinates of v_0 on the specific chart. Now, define the local vector fields J_0, N on Q, and S on TQ,

$$\begin{aligned} J_0(q) = v_0^a \frac{\partial}{\partial q^a}\,, \qquad N(q,t) = Y(q,t) - \frac{1}{2}\langle J_0 : J_0\rangle\,, \\ S(q,w) = w^a \frac{\partial}{\partial q^a} - \Gamma^a_{bc}(q)w^b w^c \frac{\partial}{\partial w^a}\,, \end{aligned} \tag{15}$$

We have that

$$\left[J_0^{\text{lift}}, S\right](q,w) = v_0^a \frac{\partial}{\partial q^a} - (\Gamma^a_{bc}(q) + \Gamma^a_{cb}(q))v_0^b w^c \frac{\partial}{\partial w^a}$$

and then equations (14) are rewritten as

$$\begin{aligned} y(t) &= S(y) + \left[J_0^{\text{lift}}, S\right](y) + N^{\text{lift}}(q,t)\,, \\ y(0) &= \begin{pmatrix} q_0 \\ 0 \end{pmatrix}, \end{aligned} \tag{16}$$

where we have denoted $y(t) = (q(t), w(t))^T$. Now, by means of the Chronological Calculus [2], we can derive an infinite expansion describing $w(t)$, and thus $v(t)$.

Theorem 3. Let $(q(t), v(t))$ be the solution of equation (2) with initial conditions $q(0) = q_0$, $v(0) = v_0$. Let the Christoffel symbols $\Gamma^a_{bc}(q)$ and the vector input vector field $Y(q,t)$ be uniformly integrable and bounded analytic in a neighborhood of q_0. Let J_0, N, S be as in (15) and define recursively the time-varying vector fields $Y_k(q,t)$ by

$$\begin{aligned} Y_1(q,t) &= Y(q,t)\,, \\ Y_{k+1}(q,t) &= -\left\langle \overline{Y}_k(q,t) : \frac{1}{2}\overline{Y}_k(q,t) + \sum_{m=1}^{k-1} \overline{Y}_m(q,t) \right\rangle, \quad k \geq 1\,, \end{aligned} \tag{17}$$

and $S_k(q,t)$ by

$$\begin{aligned} S_1 &= -\frac{1}{2}\langle J_0 : J_0 \rangle\,, \\ S_{k+1} &= -\left\langle \overline{S}_k + \overline{Y}_k : J_0 + \sum_{m=1}^{k-1} \overline{S}_m \right\rangle - \\ &\qquad \left\langle \overline{S}_k : \frac{1}{2}\overline{S}_k + \overline{Y}_k + \sum_{m=1}^{k-1} (\overline{S}_m + \overline{Y}_m) \right\rangle, \quad k \geq 1\,, \end{aligned} \tag{18}$$

where we are using the operation defined on (12). Then, there exists a sufficiently small T_c such that the series $Y_\infty(q,t) = \sum_{k=1}^{\infty} Y_k(q,t)$ and $S_\infty(q,t) = \sum_{k=1}^{\infty} S_k(q,t)$ converge absolutely and uniformly in t and q, for all $t \in [0, T_c]$ and for all q in a neighborhood of q_0. Denote by $\overline{Y}_\infty(q,t)$ and $\overline{S}_\infty(q,t)$ the vector fields that result from taking the time integral of $Y_\infty(q,t)$ and $S_\infty(q,t)$ respectively, with q held fixed, following (12). Then, over the same temporal interval, the solution $(q(t), v(t))$ satisfies

$$v(t) = J_0(q(t)) + \overline{S}_\infty(q(t),t) + \overline{Y}_\infty(q(t),t)\,. \tag{19}$$

Theorem 3 is a generalization of Theorem 3.3 in [8], which was derived for mechanical systems starting from rest, i.e. $v_0 = 0$. The treatment in [8] does not resort to the local definitions in equation (15), although both the statement and the proof of the convergence result for the infinite expansion use local charts. Here, expansion (19) consists of three types of terms: the ones corresponding to the initial velocity, those of $\overline{Y}_\infty(q,t)$ and the "mixture" terms of $\overline{S}_\infty(q,t)$ arising from the interaction between the initial velocity and the input functions.

We refer the reader to the Appendix for the formal proof of the theorem. The absolute convergence of the infinite series in (19) is a direct consequence of [8], since it is assumed that $J_0(q) + Y(q,t)$, or equivalently $Y(q,t)$, is uniformly integrable and bounded analytic on a neighborhood of q_0. We also remark that the homogeneous nature of the vector fields J_0, N, S and Y becomes critical in deriving the expressions and bounds for the convergence of $\overline{S}_\infty$ and $\overline{Y}_\infty$, a property that would be lost with the inclusion of potential energy functions.

Controllability of the blimp. II. Using Theorem 3 with $v_0 = 0$, we can easily prove that the example of the blimp is not STLCC at (g, γ), with $\gamma \neq k\pi/2$, $k \in \mathbb{Z}$. In fact, consider the basis of TQ, $\{Y_1 = gB_1, Y_2 = gB_2, \langle Y_1 : Y_2\rangle_{\mathcal{G}} = g\,\langle B_1 : B_2\rangle_{\mathcal{G}}\,,\ \langle Y_1 : Y_1\rangle_{\mathcal{G}} = g\,\langle B_1 : B_1\rangle_{\mathcal{G}}\}$ on a neighborhood of (g, γ), with $\gamma \neq k\pi/2$. Let $\phi : Q \to \mathbb{R}$ be a function satisfying the properties: (i) $\phi(g, \gamma) = 0$, (ii) $Y_1(\phi) = Y_2(\phi) = \langle Y_1 : Y_2\rangle_{\mathcal{G}}(\phi) = 0$ and $\langle Y_1 : Y_1\rangle_{\mathcal{G}}(\phi)(g, \gamma) = -1$ on a neighborhood V of (g, γ), (iii) within any neighborhood of (g, γ) there exists points q where $\phi(q) < 0$ and $\phi(q) > 0$. The proof of the existence of such functions is simple and can be found in [44].

Now, developing the first terms of $\overline{Y}_\infty(q, t)$, we can see that

$$\begin{aligned}\dot{c}(t)(\phi) = &\Big(\bar{u}_1 Y_1 + \bar{u}_2 Y_2 - \frac{1}{2}\overline{(\bar{u}_1^2\,\langle Y_1 : Y_1\rangle + 2\bar{u}_1\bar{u}_2\,\langle Y_1 : Y_2\rangle} \\ &+ \overline{\bar{u}_2^2\,\langle Y_2 : Y_2\rangle} + O(\|Z\|^3 t^5)\Big)\,(\phi) = \frac{1}{2}\overline{\bar{u}_1^2} + O(\|Z\|^3 t^5)\,,\end{aligned}$$

where $Z = u_1 Y_1 + u_2 Y_2$ and we have used property (ii) above. Since $\dot{c}(t)(\phi) = d(\phi(c(t)))/dt$, then $d(\phi(c(t)))/dt$ will not change its sign for sufficiently small t, no matter what function u_1 we use. Therefore,

$$\phi(c(t)) = \phi(q_0) + \int_0^t \frac{d}{ds}\,(\phi(c(s)))ds = \int_0^t \frac{d}{ds}\,(\phi(c(s)))ds > 0$$

will have positive sign for t small enough and therefore all the points in a neighborhood of q_0 where ϕ has the negative sign are unreachable in small time. This shows that the blimp is not STLCC at (g, γ), for $\gamma \neq k\pi/2$, $k \in \mathbb{Z}$. The series expansions in [8] has also been used in [14] to characterize the configuration controllability properties of mechanical systems underactuated by one control.

Now, we make use of Theorem 3 to derive an expansion for the solutions of mechanical control systems under small amplitude forcing. Let ϵ be a small parameter $0 < \epsilon \ll 1$ and assume that the input vector field is of the form

$$Y(q, t; \epsilon) = \sum_{i=1}^m u_i(t; \epsilon) Y_i(q) = \sum_{i=1}^m (\epsilon u_i^1(t) + \epsilon^2 u_i^2(t)) Y_i(q)\,,$$

where $u_i^j(t)$ are $O(1)$, for $t \in [0, T]$, $j = 1\,, 2$. We will denote by $X^j(q, t)$, $j = 1\,, 2$ the inputs to first and second order; respectively, i.e. $X^j(q, t) = \sum_{i=1}^m u_i^j(t) Y_i(q)$, $j = 1,\ 2$, so that

$$Y(q, t; \epsilon) = \epsilon X^1(q, t) + \epsilon^2 X^2(q, t)\,. \tag{20}$$

Proposition 3. (Approximate evolution) Consider a mechanical system (2) with input function given by (20), where $0 < \epsilon \ll 1$ is a small parameter. Let $(q(t), v(t))$ be a solution with initial velocity of small magnitude, $v_0 = \epsilon v_1 + \epsilon^2 v_2$, v_1, $v_2 = O(1)$. Accordingly, denote $J_0 = \epsilon J_1 + \epsilon^2 J_2$. Then, for $t \in [0, T]$ we have $v(t;\epsilon) = \epsilon v_1(t) + \epsilon^2 v_2(t) + \epsilon^3 v_3(t) + O(\epsilon^4)$, with

$$\begin{aligned}
v_1(t) &= J_1 + \overline{X}^1(q(t),t)\,, \\
v_2(t) &= J_2 - \frac{1}{2}\langle J_1 : J_1\rangle\, t \\
&\quad - \left\langle J_1 : \overline{\overline{X}}^1 \right\rangle (q(t),t) + \left(\overline{X}^2 - \frac{1}{2}\overline{\left\langle \overline{X}^1 : \overline{X}^1 \right\rangle}\right)(q(t),t)\,, \\
v_3(t) &= -\langle J_1 : J_2\rangle\, t + \langle J_1 : \langle J_1 : J_1\rangle\rangle \frac{t^2}{4} - \left\langle J_2 : \overline{\overline{X}}^1 \right\rangle (q(t),t) \\
&\quad + \left\langle J_1 : \left\langle J_1 : \overline{\overline{\overline{X}}}^1 \right\rangle\right\rangle (q(t),t) + \frac{1}{2}\overline{\left\langle \langle J_1 : J_1\rangle\, t : \overline{X}^1 \right\rangle}(q(t),t) \\
&\quad - \left\langle J_1 : \overline{\overline{X}}^2 \right\rangle - \frac{1}{2}\overline{\overline{\left\langle \overline{X}^1 : \overline{X}^1 \right\rangle}}(q(t),t) + \overline{\left\langle \left\langle J_1 : \overline{\overline{X}}^1 \right\rangle : \overline{X}^1 \right\rangle}(q(t),t) \\
&\quad - \overline{\left\langle \overline{X}^1 : \overline{X}^2 - \frac{1}{2}\overline{\left\langle \overline{X}^1 : \overline{X}^1 \right\rangle} \right\rangle}(q(t),t)\,.
\end{aligned} \tag{21}$$

Proof. The first terms in the expansion of $\overline{Y}_\infty(q,t)$ and $\overline{S}_\infty(q,t)$ are

$$\begin{aligned}
\overline{Y}_\infty &= \epsilon \overline{X}^1 + \epsilon^2 \left(\overline{X}^2 - \frac{1}{2}\overline{\left\langle \overline{X}^1 : \overline{X}^1 \right\rangle}\right) \\
&\qquad - \epsilon^3 \overline{\left\langle \overline{X}^1 : \overline{X}^2 - \frac{1}{2}\overline{\left\langle \overline{X}^1 : \overline{X}^1 \right\rangle} \right\rangle} + O(\epsilon^4)\,, \\
\overline{S}_\infty &= -\epsilon^2 \left(\frac{1}{2}\langle J_1 : J_1\rangle\, t + \left\langle J_1 : \overline{\overline{X}}^1 \right\rangle\right) + \epsilon^3 \Big(-\langle J_1 : J_2\rangle\, t \\
&\qquad + \langle J_1 : \langle J_1 : J_1\rangle\rangle \frac{t^2}{4} - \left\langle J_2 : \overline{\overline{X}}^1 \right\rangle + \left\langle J_1 : \left\langle J_1 : \overline{\overline{\overline{X}}}^1 \right\rangle\right\rangle \\
&\qquad + \overline{\left\langle \left\langle J_1 : \overline{\overline{X}}^1 \right\rangle : \overline{X}^1 \right\rangle} + \frac{1}{2}\overline{\left\langle \langle J_1 : J_1\rangle\, t : \overline{X}^1 \right\rangle} \\
&\qquad - \left\langle J_1 : \overline{\overline{X}}^2 - \frac{1}{2}\overline{\overline{\left\langle \overline{X}^1 : \overline{X}^1 \right\rangle}} \right\rangle\Big) + O(\epsilon^4)\,.
\end{aligned}$$

Substituting these into (19) we get the result. □

This expansion is convergent under two complementary sets of hypotheses: either t is small enough as in Theorem 3, or t is bounded and $\epsilon \ll 1$. In this way, Proposition 3 can be used to determine how the system is affected after time T by small magnitude inputs. To simplify computations, we will take periodic inputs of period $T = 2\pi$.

Remark 2. Proposition 3 extends the result in [9] to approximate the evolution under small amplitude forcing of the body velocities of mechanical control systems defined on a Lie group. Observe the difference in the methodologies employed with respect to this paper. While in [9], a regular perturbative approach applied directly to the control equations leads to an analogous expansion to that of Proposition 3, here such technique would be quite useless due to the explicit dependence of the inputs on the configuration variables. For instance, one would obtain expressions like $\int_0^t X^1(q(t),t)dt$, where now q is seen as a function of time. Such kind of expressions are not amenable to the construction of motion algorithms. This observation is yet another indication of the value of the results obtained in [8] and Theorem 3.

5. Coming back to mechanical systems with symmetry

In this section we develop motion control algorithms for underactuated mechanical systems evolving on a principal fiber bundle. In doing so, we build on the previous work in [9] for systems evolving on Lie groups. As is evident from Proposition 3 and the examples above, the behavior of the velocities $v(t)$ is influenced by the evolution of the configurations $q(t)$. This is a drawback that becomes apparent when designing motion algorithms for systems evolving on a general manifold Q. However, for systems invariant under a Lie group action, one can use the symmetry to "get rid of" some of the configuration variables. Indeed, let $q = (g,r) \in Q = G \times M$ denote the coordinates of the system. We have

$$Y(q,t) = \epsilon X^1(q,t) + \epsilon^2 X^2(q,t) = \epsilon g B^1(r,t) + \epsilon^2 g B^2(r,t) \tag{22}$$

where $B^i(r,t)$ take values in $\mathfrak{g} \times T_r M$, $B^i(r,t) = (b^i(r,t), R^i(r,t))$. Then, the solutions $(g(t), r(t)) \in G \times M$ of the system are also G-invariant, and we can restrict our attention to $(g^{-1}g(t), r(t), \xi(t), \dot{r}(t)) \in \mathfrak{g} \times TM$. Splitting in (21) the evolution of the position and shape velocities, $v(t) = (g\xi(t), \dot{r}(t))$, and reducing by the group action, we obtain $\xi(t) \equiv \xi(t;\epsilon) = \epsilon\xi_1(t) + \epsilon^2\xi_2(t) + O(\epsilon^3)$, where

$$\begin{aligned}
\xi_1(t) &= \xi_1 + \overline{b}^1(r(t),t)\,,\\
\xi_2(t) &= \xi_2 - \frac{1}{2}\left\langle(\xi_1,\dot{r}_1) : (\xi_1,\dot{r}_1)\right\rangle_{\mathfrak{g}} t \\
&\qquad - \left\langle(\xi_1,\dot{r}_1) : \overline{\overline{B}}^1\right\rangle_{\mathfrak{g}}(r(t),t) + \left(\overline{b}^2 - \frac{1}{2}\overline{\left\langle\overline{B}^1 : \overline{B}^1\right\rangle_{\mathfrak{g}}}\right)(r(t),t)\,,
\end{aligned}$$

and an analogous expression for $\dot{r}(t;\epsilon)$. Observe that if we had $Q = G$, we would get to remove completely the configuration variables of (21) and $\xi(t;\epsilon)$ would depend on time only.

Yet in this situation, in spite of the advantage that symmetry of the system provides, we would need to know the final shape $r(T)$ in order to obtain an approximation of evolution of the Lie group and shape variables after one cycle. But $r(T)$ itself depends on the controls we want to design. This difficulty can be overcome by finding an adequate approximation of $r(t)$. There are several ways to do this. We have chosen the following that makes use of the expansion for $\dot{r}(t)$.

Following Proposition 3, $\dot{r}(t) = \epsilon(\dot{r}_1 + \overline{R}^1(r(t),t)) + O(\epsilon^2)$. Integrating this equation, we obtain

$$r(t) = r_0 + \epsilon\left(\dot{r}_1 t + \int_0^t \overline{R}^1(r(s),s)ds\right) + O(\epsilon^2)\,.$$

Observe that here we are assuming that after a small input forcing, the final shape $r(2\pi)$ remains in the coordinate chart at r_0 that we first selected when deriving the series expansion. Taylor expanding $\overline{R}^1(r(t),t)$ about r_0, we have

$$\int_0^t \overline{R}^1(r(s),s)ds = \int_0^t \overline{R}^1(r_0,s)ds + O(\epsilon) = \overline{\overline{R}}^1(r_0,t) + O(\epsilon)\,.$$

Therefore, we find that

$$r(t) = r_0 + \epsilon\left(\dot{r}_1 t + \overline{\overline{R}}^1(r_0,t)\right) + \epsilon^2 s_2(t)\,, \tag{23}$$

where $s_2(t)$ denotes the terms that appear to second order.

Proposition 4. Consider a G-invariant mechanical system with inputs given by (22). Let $(g(t), r(t), g(t)\xi(t), \dot{r}(t))$ be a solution of the system with initial conditions $g(0) = g_0$, $r(0) = r_0$, $\xi(0) = \epsilon\xi_1 + \epsilon^2\xi_2$ and $\dot{r}(0) = \epsilon\dot{r}_1 + \epsilon^2\dot{r}_2$, where $\dot{r}_i$, $\xi_i = O(1)$ for $i = 1, 2$. Then, $\xi(t) \equiv \xi(t;\epsilon) = \epsilon\xi_1(t) + \epsilon^2\xi_2(t) + \epsilon^3\xi_3(t) + O(\epsilon^4)$ and $\dot{r}(t) \equiv \dot{r}(t;\epsilon) = \epsilon\dot{r}_1(t) + \epsilon^2\dot{r}_2(t) + \epsilon^3\dot{r}_3(t) + O(\epsilon^4)$, where

$$\begin{aligned}
\xi_1(t) &= \xi_1 + \overline{b}^1(r_0,t)\,,\\
\xi_2(t) &= \xi_2 - \frac{1}{2}\langle(\xi_1,\dot{r}_1):(\xi_1,\dot{r}_1)\rangle_{\mathfrak{g}}\, t - \left\langle(\xi_1,\dot{r}_1):\overline{\overline{B}}^1\right\rangle_{\mathfrak{g}}(r_0,t)\\
&\quad + \left(\overline{b}^2 - \frac{1}{2}\overline{\left\langle\overline{B}^1:\overline{B}^1\right\rangle_{\mathfrak{g}}}\right)(r_0,t) + \frac{\partial\overline{b}^1}{\partial r}_{|r_0}\left(\dot{r}_1 t + \overline{\overline{R}}^1(r_0,t)\right)\,,
\end{aligned}$$

$$\dot r_1(t) = \dot r_1 + \overline{R}^1(r_0,t)\,,$$
$$\dot r_2(t) = \dot r_2 - \frac{1}{2}\left\langle (\xi_1,\dot r_1) : (\xi_1,\dot r_1)\right\rangle_{TM} t - \left\langle (\xi_1,\dot r_1) : \overline{\overline{B}}^1 \right\rangle_{TM} (r_0,t)$$
$$+ \left(\overline{R}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle_{TM}}\right)(r_0,t) + \frac{\partial \overline{R}^1}{\partial r}_{|r_0} \left(\dot r_1 t + \overline{\overline{R}}^1(r_0,t)\right)\,,$$

and, to third order of ϵ,

$$\xi_3(t) = -\left\langle (\xi_1,\dot r_1) : (\xi_2,\dot r_2)\right\rangle_{\mathfrak g} t - \left\langle (\xi_1,\dot r_1) : \overline{\overline{B}}^2 - \frac{1}{2}\overline{\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}}\right\rangle_{\mathfrak g}(r_0,t)$$
$$- \left\langle (\xi_2,\dot r_2) : \overline{\overline{B}}^1 \right\rangle_{\mathfrak g} + \left\langle (\xi_1,\dot r_1) : \left\langle (\xi_1,\dot r_1) : (\xi_1,\dot r_1)\right\rangle\right\rangle_{\mathfrak g} \frac{t^2}{4}$$
$$+ \left\langle \left\langle (\xi_1,\dot r_1) : \overline{\overline{\overline{B}}}^1 \right\rangle : (\xi_1,\dot r_1) \right\rangle_{\mathfrak g} (r_0,t)$$
$$+ \frac{1}{2}\overline{\left\langle \left\langle (\xi_1,\dot r_1) : (\xi_1,\dot r_1)\right\rangle t : \overline{B}^1 \right\rangle_{\mathfrak g}}(r_0,t) + \overline{\left\langle \left\langle (\xi_1,\dot r_1) : \overline{\overline{B}}^1 \right\rangle : \overline{B}^1 \right\rangle_{\mathfrak g}}(r_0,t)$$
$$- \overline{\left\langle \overline{B}^1 : \overline{B}^2 - \frac{1}{2}\overline{\langle \overline{B}^1 : \overline{B}^1\rangle} \right\rangle_{\mathfrak g}}(r_0,t) + \frac{\partial \overline{b}^1}{\partial r}_{|r_0} s_2(t)$$
$$+ \frac{\partial}{\partial r}_{|r_0}\left(\overline{b}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle_{\mathfrak g}} - \left\langle (\xi_1,\dot r_1) : \overline{\overline{B}}^1 \right\rangle_{\mathfrak g}\right)\left(\dot r_1 t + \overline{\overline{R}}^1(r_0,t)\right)$$
$$+ \frac{1}{2}\left(\dot r_1 t + \overline{\overline{R}}^1(r_0,t)\right)^T \frac{\partial^2 \overline{b}^1}{\partial r^2}_{|r_0}\left(\dot r_1 t + \overline{\overline{R}}^1(r_0,t)\right)\,,$$

$$\dot r_3(t) = -\left\langle (\xi_1,\dot r_1) : (\xi_2,\dot r_2)\right\rangle_{TM} t$$
$$- \left\langle (\xi_1,\dot r_1) : \overline{\overline{B}}^2 - \frac{1}{2}\overline{\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}}\right\rangle_{TM}(r_0,t) - \left\langle (\xi_2,\dot r_2) : \overline{\overline{B}}^1 \right\rangle_{TM}$$
$$+ \left\langle (\xi_1,\dot r_1) : \left\langle (\xi_1,\dot r_1) : (\xi_1,\dot r_1)\right\rangle\right\rangle_{TM} \frac{t^2}{4} + \left\langle \left\langle (\xi_1,\dot r_1) : \overline{\overline{\overline{B}}}^1 \right\rangle : (\xi_1,\dot r_1) \right\rangle_{TM}(r_0,t)$$
$$+ \frac{1}{2}\overline{\left\langle \left\langle (\xi_1,\dot r_1) : (\xi_1,\dot r_1)\right\rangle t : \overline{B}^1 \right\rangle_{TM}}(r_0,t) + \overline{\left\langle \left\langle (\xi_1,\dot r_1) : \overline{\overline{B}}^1 \right\rangle : \overline{B}^1 \right\rangle_{TM}}(r_0,t)$$
$$- \overline{\left\langle \overline{B}^1 : \overline{B}^2 - \frac{1}{2}\overline{\langle \overline{B}^1 : \overline{B}^1\rangle} \right\rangle_{TM}}(r_0,t) + \frac{\partial \overline{R}^1}{\partial r}(r_0,t) s_2(t)$$
$$+ \frac{\partial}{\partial r}_{|r_0}\left(\overline{R}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle_{TM}} - \left\langle (\xi_1,\dot r_1) : \overline{B}^1 \right\rangle_{TM}\right)\left(\dot r_1 t + \overline{\overline{R}}^1(r_0,t)\right)$$
$$+ \frac{1}{2}\left(\dot r_1 t + \overline{\overline{R}}^1(r_0,t)\right)^T \frac{\partial^2 \overline{R}^1}{\partial r^2}_{|r_0}\left(\dot r_1 t + \overline{\overline{R}}^1(r_0,t)\right)\,.$$

Let $x(t)$ be the exponential coordinates of $g(t)$ about the initial condition $g(0) = g_0$. Then, $x(t;\epsilon) = \epsilon x_1(t) + \epsilon^2 x_2(t) + O(\epsilon^3)$ where

$$
\begin{aligned}
x_1(t) &= \xi_1 t + \overline{\overline{b}}^1(r_0,t)\,,\\
x_2(t) &= \xi_2 t - \langle(\xi_1,\dot r_1) : (\xi_1,\dot r_1)\rangle_{\mathfrak{g}} \frac{t^2}{4}\\
&\quad - \left\langle (\xi_1,\dot r_1) : \overline{\overline{\overline{B}}}^1 \right\rangle_{\mathfrak{g}} (r_0,t) + \left(\overline{\overline{b}}^2 - \frac{1}{2}\overline{\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle_{\mathfrak{g}}}}\right)(r_0,t)\\
&\quad + \overline{\frac{\partial \overline{b}^1}{\partial r}_{|r_0}\left(\dot r_1 t + \overline{\overline{R}}^1(r_0,t)\right)} - \frac{1}{2}\overline{\left[\xi_1 + \overline{b}^1, \xi_1 t + \overline{\overline{b}}^1\right]_{\mathfrak{g}}}(r_0,t)\,.
\end{aligned}
$$

Integrating the expression for $\dot r(t;\epsilon)$, we get a similar formula for $r(t)$.

Proof. Use Proposition 3 and approximation (23) to expand all the functions depending on $r(t)$ around r_0. For $x(t)$, use the expansion obtained for $\xi(t)$ and Magnus formula [32], which gives the expression of the exponential coordinates of the solution of $g^{-1}(t)\dot g(t) = \xi(t)$ as $x(t;\epsilon) = \overline{\xi}(t) - \frac{1}{2}\overline{\left[\xi,\overline{\xi}\right]}(t) + O(\epsilon^3)$, valid if $\xi(t) = O(\epsilon)$. □

Remark 3. An analogous proposition can be proven for systems defined on any manifold Q. In this respect, observe that with the implementation of the symmetry we gain accuracy in the expansions, since we do not have to approximate the variables of the Lie group. It is in this sense that we say that we got rid of the position variables.

5.1. Inversion Algorithm

From now on we will assume that the control system under consideration satisfies the property of being STLC at zero velocity with symmetric products of second order. That is,

> **A.** The subspace $\mathrm{span}\{B_i(r), \langle B_j : B_k\rangle(r) \,|\, 1 \le i \le m\,,\ 1 \le j < k \le m\}$ has maximal rank for every $r \in M$, and every bad symmetric product of second order $\langle B_i : B_i\rangle(r)$ can be put as a linear combination of the input vector fields $\{B_i(r)\}_{i=1}^m$.

The following result is a generalization of the Inversion Algorithm developed in [9] for systems evolving on Lie groups.

Lemma 1. (Inversion Algorithm) Assume that the mechanical control system satisfies property A and let $(\eta, v) \in \mathfrak{g}\times T_{r_0}M$ be an arbitrary velocity, where r_0 is any shape in M. Design the input accelerations $B^1(r,t)$, $B^2(r,t)$ according to the following steps:

1. Let P be the set of $N = \frac{m(m-1)}{2}$ ordered pairs $\{(j,k) \mid 1 \leq j < k \leq m\}$ indexing the good symmetric products of second order. Assign to each element in P an integer $a(j,k)$ in the set $\{1,\ldots,N\}$, i.e. the map $a : P \to \{1,\ldots,N\}$ is an enumeration of P. For each $\alpha = 1,\ldots,N$, define the function

$$\psi_\alpha(t) = \frac{1}{\sqrt{2\pi}} \left(\alpha \sin(\alpha t) - \sin((\alpha+N)t\right) ,$$

2. Under the assumption A, the matrix with columns $B_i(r_0)$, $\forall 1 \leq i \leq m$, and $\langle B_j : B_k \rangle (r_0)$ $\forall 1 \leq j < k \leq m$, is full rank. Then, one can compute the numbers $z_i(r_0)$ and $z_{jk}(r_0)$ such that

$$\begin{pmatrix} \eta \\ v \end{pmatrix} = \sum_{1 \leq i \leq m} z_i(r_0) B_i(r_0) + \sum_{1 \leq j < k \leq m} z_{jk}(r_0) \langle B_j : B_k \rangle (r_0) .$$

3. Finally, set

$$B^1(r,t) = \sum_{1 \leq j<k \leq m} \sqrt{|z_{jk}(r_0)|} (B_j(r) - \mathrm{sg}(z_{jk}(r_0)) B_k(r)) \psi_{a(j,k)}(t) \quad (24)$$

$$B^2(r,t) \equiv B^2(r) = \frac{1}{2\pi} \sum_{1 \leq i \leq m} z_i(r_0) B_i(r) \quad (25)$$
$$+ \frac{1}{4\pi} \sum_{1 \leq j<k \leq m} |z_{jk}(r_0)| \left(\langle B_j : B_j \rangle (r) + \langle B_k : B_k \rangle (r) \right) .$$

where $\mathrm{sg}(z)$ stands for the sign of z. The functions in (24), (25) verify

$$\left(\overline{B}^2 - \frac{1}{2} \overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle} \right) (r_0, 2\pi) = (\eta, v) . \quad (26)$$

See the proof of the lemma in the Appendix. In what follows, we will refer to this procedure by means of the function

$$(B^1(r,t), B^2(r,t)) = \mathbf{Inverse}(r_0, (\eta, v)) .$$

Observe that the lemma can be adapted for systems which are STLFC with second order symmetric products. Since $\mathfrak{g} = \mathrm{span}\{\tau_*(B_i(r_0)), \tau_*(\langle B_j : B_k \rangle (r_0)) \mid 1 \leq i \leq m, 1 \leq j < k \leq m\}$, $\forall r_0 \in M$, we choose $z_i(r_0)$ and $z_{jk}(r_0)$ such that for a given $\eta \in \mathfrak{g}$, $B^1(r,t)$, $B^2(r,t)$ satisfy

$$\eta = \tau_* \left(\left(\overline{B}^2 - \frac{1}{2} \overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle} \right) (r_0, 2\pi) \right) .$$

Accordingly, we use the notation $(B^1(r,t), B^2(r,t)) = \mathbf{Inverse}_{\mathfrak{g}}(r_0, \eta)$.

5.2. Primitives of Motion

With the aid of Lemma 1 and Proposition 4, we can define two basic motion primitives that will enable us to maintain and change the system's velocity. These manoeuvres will be the grounds on which to build more complex algorithms. Let the initial velocity of the system be $O(\sigma)$, where $0 < \sigma \ll 1$ is a small constant such that the series expansion (19) converges for $T = 2\pi$ (this is guaranteed when σ is within the order of L/T^2, where L is a constant depending on the affine connection and the local coordinate chart selected at q_0, cf. Section 8.1 and [8]). After a period T, we will have maintained or changed the system's velocity to $O(\sigma)$ by applying an input force of $O(\sigma)$ or $O(\sqrt{\sigma})$, respectively.

Maintain-Velocity$(\sigma, (\xi_{\mathrm{rf}}, v_{\mathrm{rf}}))$:
Keeps velocity $(\xi(t), v(t))$ close to a reference value $\sigma(\xi_{\mathrm{rf}}, v_{\mathrm{rf}})$.

Initial state	$g(0) = g_0$, $r(0) = r_0$, $\xi(0) = \sigma\xi_{\mathrm{rf}} + \sigma^2\xi_{\mathrm{err}}$, $v(0) = \sigma v_{\mathrm{rf}} + \sigma^2 v_{\mathrm{err}}$,
Input	$\epsilon = \sigma$, $(B^1, B^2) = \textbf{Inverse}(r_0, \pi\, \langle(\xi_{\mathrm{rf}}, v_{\mathrm{rf}}) : (\xi_{\mathrm{rf}}, v_{\mathrm{rf}})\rangle - (\xi_{\mathrm{err}}, v_{\mathrm{err}}))$,
Final state	$\log(g_0^{-1} g(2\pi)) = 2\pi\sigma\xi_{\mathrm{rf}} + \pi\sigma^2\xi_{\mathrm{err}} + O(\sigma^3)$, $r(2\pi) = r_0 + 2\pi\sigma v_{\mathrm{rf}} + \pi\sigma^2 v_{\mathrm{err}} + O(\sigma^3)$, $\xi(2\pi) = \sigma\xi_{\mathrm{rf}} + O(\sigma^3)$, $v(2\pi) = \sigma v_{\mathrm{rf}} + O(\sigma^3)$.

Change-Velocity$(\sigma, (\xi_{\mathrm{f}}, v_{\mathrm{f}}))$:
Steers velocity $(\xi(t), v(t))$ to a final value $\sigma(\xi_{\mathrm{f}}, v_{\mathrm{f}})$.

Initial state	$g(0) = g_0$, $r(0) = r_0$, $\xi(0) = \sigma\xi_0$, $v(0) = \sigma v_0$,
Input	$\epsilon = \sqrt{\sigma}$, $(B^1, B^2) = \textbf{Inverse}(r_0, (\xi_{\mathrm{f}}, v_{\mathrm{f}}) - (\xi_0, v_0))$,
Final state	$\log(g_0^{-1} g(2\pi)) = \pi\sigma(\xi_{\mathrm{f}} + \xi_0) + O(\sigma^{3/2})$, $r(2\pi) = r_0 + \pi\sigma(v_{\mathrm{f}} + v_0) + O(\sigma^{3/2})$, $\xi(2\pi) = \sigma\xi_{\mathrm{f}} + O(\sigma^2)$, $v(2\pi) = \sigma v_{\mathrm{f}} + O(\sigma^2)$.

For fiber controllable systems, we can influence the group variables similarly and keep track of the shape evolution as follows.

Maintain-Velocity$_{\mathfrak{g}}(\sigma, \xi_{\rm rf})$:
Keeps velocity $\xi(t)$ close to a reference value $\sigma\xi_{\rm rf}$.

Initial state	$g(0) = g_0$, $r(0) = r_0$, $\xi(0) = \sigma\xi_{\rm rf} + \sigma^2\xi_{\rm err}$, $v(0) = \sigma v_0$,
Input	$\epsilon = \sigma$, $(B^1, B^2) = \mathbf{Inverse}_{\mathfrak{g}}(r_0, \pi \left\langle (\xi_{\rm rf}, v_0) : (\xi_{\rm rf}, v_0)\right\rangle_{\mathfrak{g}} - \xi_{\rm err})$,
Final state	$\log(g_0^{-1}g(2\pi)) = 2\pi\sigma\xi_{\rm rf} + \pi\sigma^2\xi_{\rm err} + O(\sigma^3)$, $r(2\pi) = r_0 + 2\pi\sigma v_0 + \sigma^2\pi\{ - \pi \left\langle (\xi_{\rm rf}, v_0) : (\xi_{\rm rf}, v_0)\right\rangle_{TM}$ $\quad + \left(\overline{R}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}_{TM}\right)(r_0, 2\pi)\Big\} + O(\sigma^3)$, $\xi(2\pi) = \sigma\xi_{\rm rf} + O(\sigma^3)$, $v(2\pi) = \sigma v_0 + \sigma^2\{ - \pi^2 \left\langle (\xi_{\rm rf}, v_0) : (\xi_{\rm rf}, v_0)\right\rangle_{TM}$ $\quad + \left(\overline{R}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}_{TM}\right)(r_0, 2\pi)\Big\} + O(\sigma^3)$.

Change-Velocity$_{\mathfrak{g}}(\sigma, \xi_{\rm f})$: Steers $\xi(t)$ to a final value $\sigma\xi_{\rm f}$.

Initial state	$g(0) = g_0$, $r(0) = r_0$, $\xi(0) = \sigma\xi_0$, $v(0) = \sigma v_0$,
Input	$\epsilon = \sqrt{\sigma}$, $((B^1, B^2) = \mathbf{Inverse}_{\mathfrak{g}}(r_0, \xi_{\rm f} - \xi_0)$,
Final state	$\log(g_0^{-1}g(2\pi)) = \pi\sigma(\xi_{\rm f} + \xi_0) + O(\sigma^{3/2})$, $r(2\pi) = r_0 + \sigma\pi\Big\{2v_0 + \left(\overline{R}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}_{TM}\right)(r_0, 2\pi)\Big\}$ $\quad +O(\sigma^{3/2})$, $\xi(2\pi) = \sigma\xi_{\rm f} + O(\sigma^2)$, $v(2\pi) = \sigma\Big\{v_0 + \left(\overline{R}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}_{TM}\right)(r_0, 2\pi)\Big\} + O(\sigma^2)$.

5.3. Control Algorithms

Using a proper combination of the motion primitives presented above, we can now design basic control algorithms to solve various motion planning problems for systems evolving on a general manifold Q generalizing the ones of [9]. In what follows, we present these plans formulated for systems given on principal fibered bundles $G \times M$. To make sure that the approximations remain valid over periods of time of

Table I. Constant velocity algorithm.

Goal:	drive the system from $(g_0, r_0, 0_{\mathfrak{g}}, 0_{TM})$ to $(g_1, r_1, 0_{\mathfrak{g}}, 0_{TM})$
Arguments:	$(g_0, r_0, g_1, r_1, \sigma)$
Require:	$\log(g_0^{-1} g_1)$ well defined and $r_1 \in \mathrm{Dom}\, \phi_{r_0}$.

$N = \max\{\mathrm{floor}(\|\log(g_0^{-1}g_1)\|/(2\pi\sigma)), \mathrm{floor}(\|r_1 - r_0\|/(2\pi\sigma))\}$
$\xi_{\mathrm{nom}} = \log(g_0^{-1}g_1)/(2\pi\sigma N)$
$v_{\mathrm{nom}} = (r_1 - r_0)/(2\pi\sigma N)$
Change-Velocity$(\sigma, (\xi_{\mathrm{nom}}, v_{\mathrm{nom}}))$
for $k = 1$ to $(N-1)$ **do**
  Maintain-Velocity$(\sigma, (\xi_{\mathrm{nom}}, v_{\mathrm{nom}}))$
end for
Change-Velocity$(\sigma, (0_{\mathfrak{g}}, 0_{TM}))$

order $1/\sigma$, a discrete feedback is employed after the application of each primitive. These algorithms can also be adapted for STLFC systems. We refer the reader to [34] for the statement and the convergence proofs of the corresponding versions for fiber controllable systems.

Point-to-point reconfiguration problem. This motion plan allows us to reconfigure the system starting and ending at zero velocity: given an initial position $(g_0, r_0, 0_{\mathfrak{g}}, 0_{TM})$, we are able to take the system to a final desired state $(g_1, r_1, 0_{\mathfrak{g}}, 0_{TM})$. It is assumed that $\log(g_0^{-1}g_1)$ is well defined and $r_1 \in \mathrm{Dom}\, \phi_{r_0}$, for some coordinate chart ϕ_{r_0} at r_0. If this is not true, one can always steer the system through intermediate states using the same procedure repeatedly.

The algorithm consists of three steps. Over the first 2π units of time, we communicate the system an adequate velocity to achieve the final configuration. This nominal velocity is then maintained during several cycles until reaching a convenient configuration, when we turn again the velocity to zero. Table I specifies the details and the following lemma proves the convergence of the algorithm.

Lemma 2. (Constant Velocity Algorithm) Let σ be a small positive constant. Let $g(0) = g_0$, $r(0) = r_0$, $\xi(0)$, $v(0) = O(\sigma^2)$ be the initial state of the system and let (g_1, r_1) be a final configuration in Q such that $\log(g_0^{-1}g_1)$ is well defined and $r_1 \in \mathrm{Dom}\, \phi_{r_0}$, where ϕ_{r_0} is a coordinate chart about r_0. Let N be a positive integer and define the inputs $(B^1(r,t), B^2(r,t))$ for $t \in [0, 2\pi(N+1)]$ according to Table I. Then,

$$\log(g^{-1}(2\pi(N+1))g_1) = O(\sigma^{\frac{3}{2}}), \qquad r(2\pi(N+1)) = r_0 + O(\sigma^{\frac{3}{2}}),$$
$$\xi(2\pi(N+1)) = O(\sigma^2), \qquad v(2\pi(N+1)) = O(\sigma^2).$$

Table II. Local Exponential Stabilization Algorithm.

Goal:	drive the system to $(g_{\mathrm{f}}, r_{\mathrm{f}}, 0_{\mathfrak{g}}, 0_{TM})$ exponentially as $t \to +\infty$.
Arguments:	$(g_{\mathrm{f}}, r_{\mathrm{f}}, \sigma)$.
Require:	$\|(\log(g_{\mathrm{f}}^{-1} g(0)), r_{\mathrm{f}} - r(0), \xi(0), v(0))\| \leq \sigma$.

$N = \max\{\mathrm{floor}(\|\log(g_0^{-1} g_1)\|/(2\pi\sigma)), \mathrm{floor}(\|r_1 - r_0\|/(2\pi\sigma))\}$
for $k = 1$ to $+\infty$ **do**
  $t_k = 4\pi k$ $\{t_k$ is the current time$\}$
  $\sigma_k = \|(\log(g_{\mathrm{f}}^{-1} g(t_k)), r(t_k) - r_{\mathrm{f}}, \xi(t_k), v(t_k))\|$
  Change-Velocity$(\sigma_k, -(\log(g_{\mathrm{f}}^{-1} g(t_k)) + \pi\xi(t_k), r(t_k) - r_{\mathrm{f}} + \pi v_{\mathrm{nom}})./(2\pi\sigma_k))$
  Change-Velocity$(\sigma_k, (0_{\mathfrak{g}}, 0_{TM}))$
end for

The proof of this lemma is given in the Appendix. Observe that the final velocity is not exactly zero. To solve this problem we have the stabilization algorithm described next.

Point stabilization problem. By means of this algorithm, the initial configuration (g_0, r_0) is stabilized to a final value $(g_{\mathrm{f}}, r_{\mathrm{f}})$ whenever the condition

$$\|(\log(g_{\mathrm{f}}^{-1} g_0), r_0 - r_{\mathrm{f}}, \xi_0, v_0)\| \;\leq\; \sigma\,, \tag{27}$$

is satisfied in order to assure convergence. Again, if this equation does not hold, one can use the first algorithm to steer the system to an intermediate configuration.

In this case, an iteration procedure is applied until a state close to the desired one is reached. The criterium to stop the iteration is that the magnitude

$$\|(\log(g_{\mathrm{f}}^{-1} g_*), r_* - r_{\mathrm{f}}, \xi_*, v_*)\| \leq \mathrm{tol}\,,$$

be less than certain tolerance "tol". Each iteration consists of two **Change-Velocity** primitives: first, the system is given some velocity aimed to approach the final state and then we change this velocity to zero with the second primitive. The iteration takes 4π units of time. Table II indicates how this is done. As for the convergence of the algorithm, we have the following lemma.

Lemma 3. (Local Exponential Stabilization Algorithm) Let $\sigma > 0$ be small enough and assume that the initial state of the system satisfies (27), for $(g_{\mathrm{f}}, r_{\mathrm{f}})$ a final desired configuration. Let the inputs $(B^1(r,t), B^2(r,t))$ be determined as in Table II and let $t_k = 4\pi k$. Then, there exists a $\lambda > 0$

Table III. Static interpolation algorithm.

Goal:	drive the system through $\{(g_0, r_0), (g_1, r_1), \ldots, (g_p, r_p)\}$.
Arguments:	$(g_0, r_0, g_1, r_1, \ldots, g_p, r_p, \sigma)$
Require:	$(g(0), r(0), \xi(0), v(0)) = (g_0, r_0, 0_{\mathfrak{g}}, 0_{TM})$, $\log(g_i^{-1} g_{i+1})$ well defined and $r_{i+1} \in \mathrm{Dom}(\phi_{r_i})$, $1 \le i \le p$.

```
for j = 1 to p do
    g_tmp,j = g(t) exp(πξ(t))
    r_tmp,j = r(t) + πv(t)
    N_j = max{floor(‖log(g_tmp,j^{-1} g_j)‖/(2πσ)), floor(‖r_j − r_tmp,j‖/(2πσ))}
    ξ_nom,j = log(g_tmp,j^{-1} g_j)/(2πσN_j)
    v_nom,j = (r_j − r_tmp,j)/(2πσN_j)
    Change-Velocity(σ, (ξ_nom,j, v_nom,j))
    for k = 1 to (N_j − 1) do
        Maintain-Velocity(σ, (ξ_nom,j, v_nom,j))
    end for
end for
Change-Velocity(σ, (0_g, 0_TM))
```

such that

$$\begin{aligned}&\|(\log(g_{\mathrm{f}}^{-1} g(t_k)), r(t_k) - r_{\mathrm{f}}, \xi(t_k), v(t_k)))\| \\ &\le \|(\log(g_{\mathrm{f}}^{-1} g(0)), r(0) - r_{\mathrm{f}}, \xi(0), v(0)))\| \, e^{-\lambda t_k}, \quad \forall k \in \mathbb{N}.\end{aligned}$$

Additionally, for $t \in [4k\pi, 4(k+1)\pi]$ it holds that $\|(\log(g_{\mathrm{f}}^{-1} g(t)), r(t) - r_{\mathrm{f}}, \xi(t), v(t)))\| = O(e^{-\lambda k/2})$.

The proof of this lemma is also given in the Appendix.

Static interpolation problem. This motion task is an alternative to the point-to-point reconfiguration problem and it steers the system through several ordered configurations $\{(g_i, r_i) \in G \times M \,|\, 0 \le i \le p\}$. It is required that $\log(g_i^{-1} g_i)$ be well defined and that $r_{i+1} \in \mathrm{Dom}\, \phi_i$, with ϕ_i a coordinate chart about r_i, $\forall 1 \le i \le p$.

The algorithm consists of a repetition of the point-to-point reconfiguration problem that takes the system between (g_i, r_i) and (g_{i+1}, r_{i+1}), but instead of stopping at each configuration, the nominal velocity for the next step is directly acquired, see Table III. It can be proven that the system passes through the determined configurations with an error of $O(\sigma)$. The proof of the convergence of the algorithm is analogous to that of the constant velocity algorithm in Lemma 2.

6. Numerical simulations

For the sake of completeness, we illustrate in this section the performance of the motion algorithms. We have implemented the version for a fiber controllable system: the blimp example introduced above.

The different parameters of the problem take the values $m = 2$, $J_1 = 2$, $J_2 = 1$, $h = 0.5$ in normalized units. The input magnitude σ was set to $\sigma = 0.07$. In all the plots, the front part of the blimp is drawn with a symbol 'o' to track the orientation changes. The relative angle of the thruster is also shown. The figure of the blimp is superimposed on the trajectory after the execution of each of the primitives.

Constant Velocity Algorithm. The goal is to drive the system from the initial fiber configuration $(0, 0, 0)$, with $\gamma(0) = 0$, and zero initial velocity, to the final fiber value $(2, 0, \pi)$. Recall that the blimp is STFLCC at zero velocity but not STLCC and hence we cannot reach our goal point with zero velocity in *all* the configuration variables.

The task is carried out with an oscillatory motion induced by the form of the inputs of the Inversion Algorithm. Observe in Figure 4 that the fiber velocities, and even the shape velocity, oscillate around a constant value (the nominal velocity) along the execution.

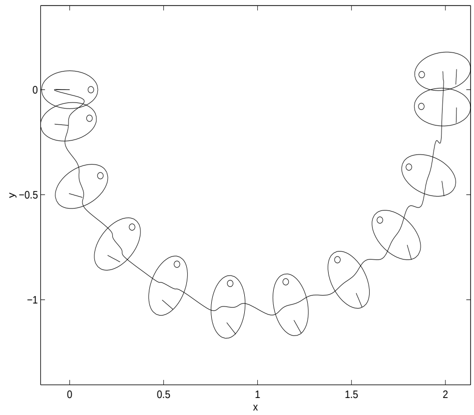

Figure 4. Illustration of the Constant Velocity Algorithm for the blimp system. The starting configuration is $(0, 0, 0, 0) \in SE(2) \times \mathbb{S}^1$ and the goal is the position $(2, 0, \pi) \in SE(2)$

Static Interpolation Algorithm. This task is executed between the fiber configurations $(0, 0, 0)$, $(1, 1, \pi/2)$ and $(2, 0, \pi)$ with zero initial shape and velocity (see Figure 6). Notice that the committed error around each configuration is of order σ, in agreement with the theoretical analysis.

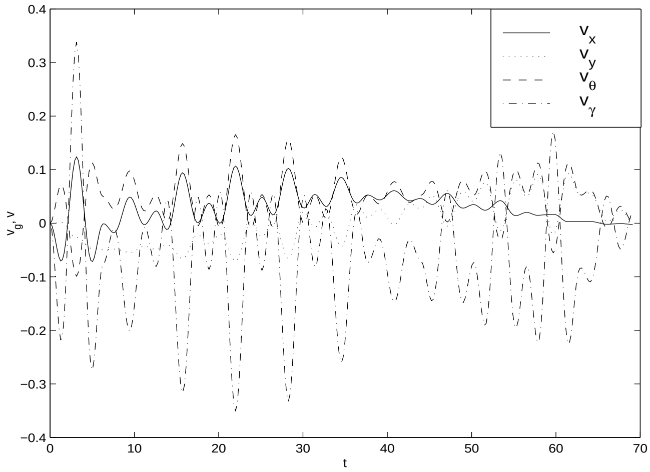

Figure 5. Evolution of the velocity variables during the execution of the Constant Velocity Algorithm in Figure 4

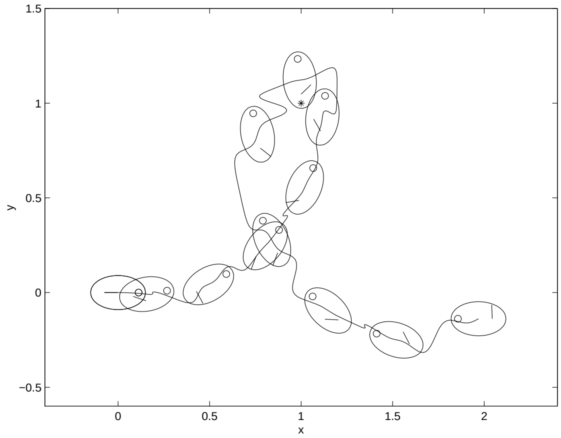

Figure 6. Illustration of the Static Interpolation Algorithm for the blimp system. The starting configuration is $(0,0,0,0) \in SE(2) \times \mathbb{S}^1$ and the goal is the position $(2,0,\pi) \in SE(2)$, passing through $(1,1,\pi/2)$ signaled in the figure with a star

Local Exponential Stabilization. We apply this algorithm after the results obtained with the previous execution of the Static Interpolation Algorithm.

Figure 6 shows that the nature of the convergence is indeed exponential. Recall that exponential stabilization cannot be achieved by smooth time-varying feedback, and indeed the motion primitives are continuous, but not smooth, functions of the state. The stabilization relies on discrete time continuous feedback and multiple calls to the Change-Velocity primitive.

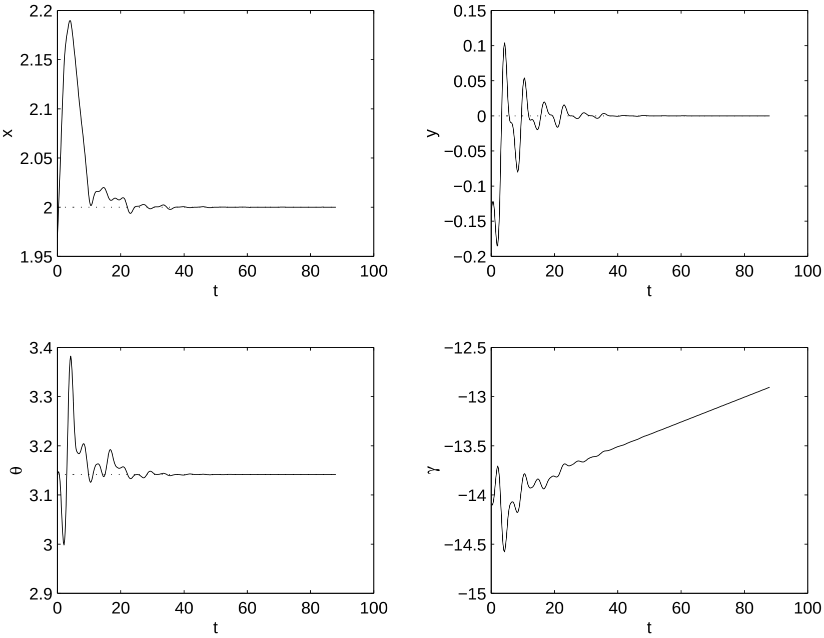

Figure 7. Illustration of the Local Exponential Stabilization Algorithm for the blimp system. The starting state is given by the end of the execution of the Static Interpolation Algorithm of Figure 6

7. Conclusions

On the basis of previous work [8, 9, 30], and as a continuation of the results found in [15], we have studied the motion planning problem for underactuated mechanical control systems evolving on a principal fiber bundle. In doing so, we have extended the series expansion of [8], obtaining a local description of the evolution of the trajectories of a general mechanical control system starting from non-zero velocity. This series makes use of the theoretical framework of affine connection control systems, and in particular of the notion of symmetric product, and provides a powerful tool for further analyzing the controllability aspects of mechanical systems.

As a consequence of these developments, we have been able to describe the response of the system under small amplitude input forcing. This has been key to define two motion primitives we can operate the system with at a local or low level to maintain and change velocity. The primitives have been used as the building blocks to define three motion control algorithms solving at a higher level the tasks of point-to-point reconfiguration, static interpolation and stabilization.

Several examples are presented to illustrate the theoretical results. The performance of the three algorithms is shown in the blimp system.

There is quite a number of issues that remain to be investigated. Related with the motion algorithms, we mention the relaxation of the controllability assumptions on which they are based, how to improve convergence and complexity aspects and the most ambitious one of overcoming the assumption of small amplitude forcing. Future research should also be devoted to the extension of the framework to include more complex and accurate models (e.g. dissipation, viscous forces), hybrid systems or systems with switching regimes.

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8. Appendix

8.1. Proof of Theorem 3

Consider the differential equations in $\mathbb{R}^{2n}$

$$\dot{y}_k = (S + [J_k^{\text{lift}}, S] + N_k^{\text{lift}})(y_k, t)\,, \quad y_k(0) = \begin{pmatrix} q_0 \\ 0 \end{pmatrix}. \tag{28}$$

For $k = 1$ we recover (16) by taking $J_1(q) = J_0(q)$, $N_1(q,t) = N(q,t)$ and $y_1 = y$. In equation (28), we can regard the vector field $f(y) = S(y) + [J_k^{\text{lift}}, S](y)$ as a perturbation to the vector field $g(y) = N_k(y)$. This enables us to describe the flow of the equation in terms of a nominal and a perturbed flow, using tools from Chronological Calculus [2]. Indeed, we have the formula

$$\Phi_{0,t}^{f+g} = \Phi_{0,t}^{g} \circ \Phi_{0,t}^{(\Phi_{0,t}^{g})^* f}\,, \tag{29}$$

where $\Phi_{0,t}^{g}$ denotes the flow of the vector field g and $(\Phi_{0,t}^{g})^* f$ is the pullback of f by $\Phi_{0,t}^{g}$, i.e. $(\Phi_{0,t}^{g})^* f(y) = (T_y \Phi_{0,t}^{g})^{-1} \circ f \circ \Phi_{0,t}^{g}(y)$. Then, we set $y_k(t) = \Phi_{0,t}^{N_k^{\text{lift}}}(y_{k+1}(t))$ and

$$\dot{y}_{k+1} = \left(\left(\Phi_{0,t}^{N_k^{\text{lift}}} \right)^* \left(S + [J_k^{\text{lift}}, S] \right) \right)(y_{k+1})\,, \quad y_{k+1}(0) = \begin{pmatrix} q_0 \\ 0 \end{pmatrix}. \tag{30}$$

To obtain an explicit expression of $(\Phi_{0,t}^{g})^* f$, we can resort to the formal expansion of the pullback along a flow [2]. It turns out that

$$(\Phi_{0,t}^{g})^* f = f + \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{s_{n-1}} (ad_{g(s_n)} \dots ad_{g(s_1)} f) ds_n \dots ds_1\,, \tag{31}$$

where we have dropped the argument y inside the integral for simplicity. The convergence of the series in (31) is in general a delicate issue. Nevertheless, if the Lie brackets $ad_{g(s_n)} \dots ad_{g(s_1)} f$ vanish for all n greater than a given N, then the series becomes a finite sum and converges.

This is precisely our case, because of the properties of homogeneous vector fields with respect to the Lie bracket [8]. We have that

$$\mathrm{ad}^m_{N_k^{\mathrm{lift}}} S = 0\,, \quad m \geq 3\,, \quad \mathrm{ad}^m_{N_k^{\mathrm{lift}}}[J_k^{\mathrm{lift}}, S] = 0\,, \quad m \geq 2\,,$$

and our series reduces to a finite sum. In fact,

$$\begin{aligned}(\Phi_{0,t}^{N_k^{\mathrm{lift}}})^*(S + [J_k^{\mathrm{lift}}, S]) &= S + [J_k^{\mathrm{lift}} + \overline{N}_k^{\mathrm{lift}}, S] + [\overline{N}_k^{\mathrm{lift}}, [J_k^{\mathrm{lift}}, S]] \\ &+ \int_0^t \int_0^{s_1} [N_k^{\mathrm{lift}}(s_2), [N_k^{\mathrm{lift}}(s_1), S]] ds_2 ds_1 = \\ & S + [J_k^{\mathrm{lift}} + \overline{N}_k^{\mathrm{lift}}, S] - \left\langle \overline{N}_k^{\mathrm{lift}} : J_k^{\mathrm{lift}} \right\rangle - \frac{1}{2} \left\langle \overline{N}_k^{\mathrm{lift}} : \overline{N}_k^{\mathrm{lift}} \right\rangle .\end{aligned}$$

Thus, the differential equation for $y_{k+1}(t)$ is of the same form as (28) with the vector fields

$$J_{k+1} = J_k + \overline{N}_k\,, \quad N_{k+1} = -\left\langle \overline{N}_k : J_k + \frac{1}{2}\overline{N}_k \right\rangle\,,$$

which is equivalent to

$$J_{k+1} = \sum_{m=1}^{k-1} \overline{N}_m + J_0\,, \quad N_{k+1} = -\left\langle \overline{N}_k : \frac{1}{2}\overline{N}_k + J_0 + \sum_{m=1}^{k-1} \overline{N}_m \right\rangle .$$

Thus, by an iteration procedure, we have that the solution of the original differential equation $y(t) = y_1(t)$ is given by

$$y(t) = \left(\Phi_{0,t}^{N_1^{\mathrm{lift}}} \circ \Phi_{0,t}^{N_2^{\mathrm{lift}}} \circ \cdots \circ \Phi_{0,t}^{N_{k-1}^{\mathrm{lift}}} \right) (y_k(t))\,, \quad \text{for all } k\,,$$

where $y_k(t)$ is the solution of (28). Since the flows of N_k^{lift} and N_l^{lift} commute, i.e. $[N_k^{\mathrm{lift}}(q,s_1), N_l^{\mathrm{lift}}(q,s_2)] = 0$, for all k, l, we get

$$y(t) = \Phi_{0,t}^{\sum_{m=1}^{k-1} N_m^{\mathrm{lift}}} (y_k(t))\,, \quad \text{for all } k\,. \tag{32}$$

On the other hand, since $J_0(q) + Y(q,t)$ is analytic and bounded, then $N_\infty(q,t) = \sum_{m=1}^\infty N_m(q,t)$ converges absolutely and uniformly in q and t, for q in a neighborhood of q_0 and $t \in [0, T_{c_1}]$, for some $T_{c_1} > 0$, by similar arguments to those used in [8] to prove the convergence of $Y_\infty(q,t)$. Therefore,

$$\lim_{k\to\infty} N_k(q,t) = 0\,, \quad \lim_{k\to\infty} J_k(q,t) = J_0(q) + \overline{N}_\infty(q,t)\,,$$

uniformly in q and for a small time t, and $y_\infty(t) = \lim_{k\to\infty} y_k(t)$ satisfies

$$\dot{y}_\infty(t) = (S + [(J_0 + \overline{N}_\infty)^{\mathrm{lift}}, S])(y_\infty, t)\,, \quad y_\infty(0) = \begin{pmatrix} q_0 \\ 0 \end{pmatrix} .$$

As the initial velocity is zero, we can integrate $y_\infty(t)$ as

$$y_\infty(t) \;=\; \begin{pmatrix} q_\infty(t) \\ w_\infty(t) \end{pmatrix} = \begin{pmatrix} \Phi_{0,t}^{(J_0+\overline{N}_\infty)}(q_0) \\ 0 \end{pmatrix} . \qquad (33)$$

On the other hand for the initial condition $(q_0, 0)^T$, we have

$$\Phi_{0,t}^{N_\infty^{\mathrm{lift}}} \begin{pmatrix} q_0 \\ 0 \end{pmatrix} = \begin{pmatrix} q_0 \\ \overline{N}_\infty(q_0,t) \end{pmatrix} . \qquad (34)$$

Now, using (32), (33) and (34), we conclude that $q(t) = q_\infty(t)$ and $w(t) = v(t) - J_0 = \overline{N}_\infty(q,t)$. Therefore, $v(t) = J_0 + \overline{N}_\infty(q(t),t)$. Observe that the series $N_\infty(q,t)$ can be split into $N_\infty(q,t) = S_\infty(q,t) + Y_\infty(q,t)$, with $S_\infty(q,t) = \sum_{k=1}^\infty S_k(q,t)$ and $Y_\infty(q,t) = \sum_{k=1}^\infty Y_k(q,t)$ given by

$$S_1 = -\frac{1}{2}\langle J_0 : J_0 \rangle \, , \quad S_{k+1} = -\left\langle \overline{S}_k + \overline{Y}_k : J_0 + \sum_{m=1}^{k-1} \overline{S}_m \right\rangle \qquad (35)$$
$$- \left\langle \overline{S}_k : \frac{1}{2}\overline{S}_k + \overline{Y}_k + \sum_{m=1}^{k-1} (\overline{S}_m + \overline{Y}_m) \right\rangle ,$$

where we are using (12), and $Y_k(q,t)$ is defined recursively by (17). Clearly, the series $S_\infty(q,t) = \sum_{k=1}^\infty S_k(q,t)$, is absolutely and uniformly convergent where both $Y_\infty(q,t)$ and $N_\infty(q,t)$ are.

8.2. Proof of the Inversion Algorithm

It is straightforward to check that the functions $\psi_a(t)$ of the Inversion Algorithm [9] satisfy the following properties,

(P.1) $\overline{\psi}_a(2\pi) = \overline{\overline{\psi}}_a(2\pi) = \overline{\overline{\overline{\psi}}}_a(2\pi) = 0$

(P.2) $\overline{\overline{\psi}_a \overline{\psi}_b}(t) = \frac{\delta_{ab} t}{2\pi} + r_{ab}(t)$, with $r_{ab}(2\pi) = \overline{r}_{ab}(2\pi) = 0$

(P.3) $\overline{\overline{\psi}_a t}(2\pi) = \overline{\overline{\overline{\psi}}_a \overline{\psi}_b}(2\pi) = \overline{\psi_a r_{bc}}(2\pi) = 0$.

With these choice of inputs, the proof of the Inversion Algorithm remains the same as in [9]. Indeed, as a consequence of (P.2), we have

$$\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}_{|2\pi} = \Bigg\langle \sum_{1 \le j < k \le m} \sqrt{|z_{jk}(r_0)|}(B_j(r) - sg(z_{jk}(r_0))B_k(r)) :$$
$$\sum_{1 \le p < q \le m} \sqrt{|z_{pq}(r_0)|}(B_p(r) - sg(z_{pq}(r_0))B_q(r)) \Bigg\rangle \overline{\overline{\psi_{a(j,k)}}\;\overline{\psi_{a(p,q)}}}(2\pi) =$$
$$\sum_{1 \le j < k \le m} |z_{jk}(r_0)| \left(\langle B_j : B_j \rangle + \langle B_k : B_k \rangle \right) - 2 \sum_{1 \le j < k \le m} z_{jk}(r_0) \langle B_j : B_k \rangle \, ,$$

that along with

$$\overline{B}^2(r_0, 2\pi) = \sum_{1\leq i\leq m} z_i(r_0) B_i(r_0) + \frac{t}{4\pi} \sum_{1\leq j<k\leq m} |z_{jk}(r_0)| \left(\langle B_j : B_j \rangle (r_0) + \langle B_k : B_k \rangle (r_0) \right)$$

makes $\overline{B}^2(r_0, 2\pi) - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}(r_0, 2\pi) = (\eta, v)$.

8.3. Primitives of motion

The primitives of motion are a direct consequence of the next result.

Proposition 5. Let $r_0 \in M$, $(B^1(r,t), B^2(r,t)) = \mathbf{Inverse}(r_0, (\eta, v))$. Assume $\xi(0) = \epsilon\xi_1 + \epsilon^2\xi_2$, $\dot{r}(0) = \epsilon\dot{r}_1 + \epsilon^2\dot{r}_2$ and that the statements in Proposition 4 hold. Then, $\xi(2\pi) = \epsilon\xi_1(2\pi) + \epsilon^2\xi_2(2\pi) + \epsilon^3\xi_3(2\pi) + O(\epsilon^4)$, $\dot{r}(2\pi) = \dot{r}_1(2\pi) + \dot{r}_2(2\pi) + \dot{r}_3(2\pi) + O(\epsilon^4)$ with

$$\xi_1(2\pi) = \xi_1\,, \qquad \xi_2(2\pi) = \xi_2 - \pi \left\langle (\xi_1, \dot{r}_1) : (\xi_1, \dot{r}_1) \right\rangle_{\mathfrak{g}} + \eta$$

$$\begin{aligned} \xi_3(2\pi) = &-2\pi \left\langle (\xi_1, \dot{r}_1) : (\xi_2, \dot{r}_2) \right\rangle_{\mathfrak{g}} + \pi^2 \left\langle (\xi_1, \dot{r}_1) : \left\langle (\xi_1, \dot{r}_1) : (\xi_1, \dot{r}_1) \right\rangle \right\rangle_{\mathfrak{g}} \\ &- \left\langle (\xi_1, \dot{r}_1) : \overline{\overline{B}}^2 - \frac{1}{2}\overline{\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}} \right\rangle_{\mathfrak{g}} (r_0, , 2\pi) \\ &+ 2\pi\dot{r}_1 \frac{\partial}{\partial r} \left(\overline{b}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle_{\mathfrak{g}}} \right) (r_0, 2\pi)\,, \end{aligned}$$

$$\dot{r}_1(2\pi) = \dot{r}_1\,, \qquad \dot{r}_2(2\pi) = \dot{r}_2 - \pi \left\langle (\xi_1, \dot{r}_1) : (\xi_1, \dot{r}_1) \right\rangle_{TM} + v$$

$$\begin{aligned} \dot{r}_3(2\pi) = &- \left\langle (\xi_1, \dot{r}_1) : (\xi_2, \dot{r}_2) \right\rangle_{TM} + \pi \left\langle (\xi_1, \dot{r}_1) : \left\langle (\xi_1, \dot{r}_1) : (\xi_1, \dot{r}_1) \right\rangle \right\rangle_{TM} \\ &- \left\langle (\xi_1, \dot{r}_1) : \overline{\overline{B}}^2 - \frac{1}{2}\overline{\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle}} \right\rangle_{TM} (r_0, , 2\pi) \\ &+ 2\pi\dot{r}_1 \frac{\partial}{\partial r} \left(\overline{R}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle_{TM}} \right) (r_0, 2\pi)\,, \end{aligned}$$

$$x(2\pi) = 2\pi\xi_1\epsilon + \pi\epsilon^2 \left(2\xi_2 - \pi \left\langle (\xi_1, \dot{r}_1) : (\xi_1, \dot{r}_1) \right\rangle_{\mathfrak{g}} + \eta \right)\,,$$

$$r(2\pi) = r_0 + 2\pi\dot{r}_1\epsilon + \pi\epsilon^2 \left(2\dot{r}_2 - \pi \left\langle (\xi_1, \dot{r}_1) : (\xi_1, \dot{r}_1) \right\rangle_{TM} + v \right)\,.$$

Proof. By (P.1), we deduce $\overline{B}^1(r_0, 2\pi) = \overline{\overline{B}}^1(r_0, 2\pi) = \overline{\overline{\overline{B}}}^1(r_0, 2\pi) = 0$. By applying the Taylor expansion given in Proposition 4, we obtain

$$\xi_1(2\pi) = \xi_1\,, \qquad \dot{r}_1(2\pi) = \dot{r}_1\,,$$

$$\xi_2(2\pi) = \xi_2 - \pi \left\langle (\xi_1, \dot{r}_1) : (\xi_1, \dot{r}_1) \right\rangle_{\mathfrak{g}} + \left(\overline{b}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle_{\mathfrak{g}}} \right) (r_0, 2\pi),$$

$$\dot{r}_2(2\pi) = \dot{r}_2 - \pi \left\langle (\xi_1, \dot{r}_1) : (\xi_1, \dot{r}_1) \right\rangle_{TM} + \left(\overline{R}^2 - \frac{1}{2}\overline{\left\langle \overline{B}^1 : \overline{B}^1 \right\rangle_{TM}} \right) (r_0, 2\pi),$$

and, using (P.3),

$$
\begin{aligned}
\xi_3(2\pi) = &-2\pi \left\langle (\xi_1, \dot r_1) : (\xi_2, \dot r_2) \right\rangle_{\mathfrak{g}} + \pi^2 \left\langle (\xi_1, \dot r_1) : \langle (\xi_1, \dot r_1) : (\xi_1, \dot r_1) \rangle \right\rangle_{\mathfrak{g}} \\
&+ \Big[\overline{\Big\langle \langle (\xi_1, \dot r_1) : (\xi_1, \dot r_1) \rangle\, t : \overline{B}^1 \Big\rangle_{\mathfrak{g}}} - \Big\langle (\xi_1, \dot r_1) : \overline{\overline{B}}^2 - \frac{1}{2} \overline{\big\langle \overline{B}^1 : \overline{B}^1 \big\rangle} \Big\rangle_{\mathfrak{g}} \\
&+ \overline{\Big\langle \Big\langle (\xi_1, \dot r_1) : \overline{\overline{B}}^1 \Big\rangle : \overline{B}^1 \Big\rangle_{\mathfrak{g}}} - \overline{\Big\langle \overline{B}^1 : \overline{B}^2 - \frac{1}{2} \overline{\langle \overline{B}^1 : \overline{B}^1 \rangle} \Big\rangle_{\mathfrak{g}}} \\
&+ 2\pi \dot r_1 \frac{\partial}{\partial r} \left(\overline{b}^2 - \frac{1}{2} \overline{\langle \overline{B}^1 : \overline{B}^1 \rangle_{\mathfrak{g}}} \right) \Big] (r_0, 2\pi) \,,
\end{aligned}
$$

$$
\begin{aligned}
\dot r_3(2\pi) = &-2\pi \left\langle (\xi_1, \dot r_1) : (\xi_2, \dot r_2) \right\rangle_{TM} + \pi^2 \left\langle (\xi_1, \dot r_1) : \langle (\xi_1, \dot r_1) : (\xi_1, \dot r_1) \rangle \right\rangle_{TM} \\
&+ \Big[\overline{\Big\langle \langle (\xi_1, \dot r_1) : (\xi_1, \dot r_1) \rangle\, t : \overline{B}^1 \Big\rangle_{TM}} - \Big\langle (\xi_1, \dot r_1) : \overline{\overline{B}}^2 - \frac{1}{2} \overline{\big\langle \overline{B}^1 : \overline{B}^1 \big\rangle} \Big\rangle_{TM} \\
&+ \overline{\Big\langle \Big\langle (\xi_1, \dot r_1) : \overline{\overline{B}}^1 \Big\rangle : \overline{B}^1 \Big\rangle_{TM}} (r_0, 2\pi) - \overline{\Big\langle \overline{B}^1 : \overline{B}^2 - \frac{1}{2} \overline{\langle \overline{B}^1 : \overline{B}^1 \rangle} \Big\rangle_{TM}} \\
&+ 2\pi \dot r_1 \frac{\partial}{\partial r} \left(\overline{R}^2 - \frac{1}{2} \overline{\langle \overline{B}^1 : \overline{B}^1 \rangle_{TM}} \right) \Big] (r_0, 2\pi) \,.
\end{aligned}
$$

On the other hand, for the exponential coordinates $x(t)$ of $g(t)$ we get

$$
\begin{aligned}
x_1(2\pi) = 2\pi \xi_1 \,, \qquad x_2(2\pi) = &\; 2\pi \xi_2 - \pi^2 \langle (\xi_1, \dot r_1) : (\xi_1, \dot r_1) \rangle_{\mathfrak{g}} + \pi \eta \\
&- \frac{1}{2} \overline{\left[\xi_1 + \overline{b}^1, \xi_1 t + \overline{\overline{b}}^1 \right]_{\mathfrak{g}}} (r_0, 2\pi) \,,
\end{aligned}
$$

and, similarly, for the shape variables,

$$
r_1(2\pi) = 2\pi \dot r_1 \,, \qquad r_2(2\pi) = 2\pi \dot r_2 - \pi^2 \langle (\xi_1, \dot r_1) : (\xi_1, \dot r_1) \rangle_{TM} + \pi v \,.
$$

Finally, because of (P.3), we have that

$$
\begin{aligned}
\overline{\Big\langle \langle (\xi_1, \dot r_1) : (\xi_1, \dot r_1) \rangle\, t : \overline{B}^1 \Big\rangle} (r_0, 2\pi) &= \overline{\Big\langle \Big\langle (\xi_1, \dot r_1) : \overline{\overline{B}}^1 \Big\rangle : \overline{B}^1 \Big\rangle} (r_0, 2\pi) \\
&= \overline{\Big\langle \overline{B}^1 : \overline{B}^2 - \frac{1}{2} \overline{\langle \overline{B}^1 : \overline{B}^1 \rangle} \Big\rangle} (r_0, 2\pi) = 0 \,,
\end{aligned}
$$

and also

$$
\begin{aligned}
\overline{\left[\xi_1 + \overline{b}^1, \xi_1 t + \overline{\overline{b}}^1 \right]_{\mathfrak{g}}} (r_0, 2\pi) &= 2\pi^2 \left[\xi_1, \xi_1 \right]_{\mathfrak{g}} + \overline{\left[\xi_1, \overline{\overline{b}}^1 \right]_{\mathfrak{g}}} (r_0, 2\pi) \\
&+ \overline{\left[\overline{b}^1 t, \xi_1 \right]_{\mathfrak{g}}} (r_0, 2\pi) + \overline{\left[\overline{b}^1, \overline{\overline{b}}^1 \right]_{\mathfrak{g}}} (r_0, 2\pi) = 0 \,.
\end{aligned}
$$

8.4. Constant Velocity Algorithm

To compute the estimates of the position variables for this and the following algorithms, we will use the next result, corollary of the Campbell-Baker-Hausdorff formula.

Lemma 4. Let $0 < \sigma \ll 1$ be a parameter and g_0, g_1 two group variables with $y_0 = \log(g_0)$ and $y_1 = \log(g_1)$ their exponential coordinates in $\mathfrak{g}$. Let $[y_0, y_1] = O(\sigma^p)$ and $y_0 + y_1 = O(\sigma^s)$. If $p > s$, then

$$\log(g_0 g_1) = y_0 + y_1 + O(\sigma^p)\,.$$

After following each step in Table I, using the primitives **Change-Velocity** and **Maintain-Velocity**, we have that
After $t = 2\pi$,

$$\log(g_0^{-1} g(2\pi)) = \pi\sigma\xi_{\text{nom}} + O(\sigma^{\frac{3}{2}}), \quad \xi(2\pi) = \sigma\xi_{\text{nom}} + O(\sigma^2),$$
$$r(2\pi) - r_0 = \pi\sigma v_{\text{nom}} + O(\sigma^{\frac{3}{2}}), \qquad v(2\pi) = \sigma v_{\text{nom}} + O(\sigma^2),$$

after $t = 4\pi$,

$$\log(g^{-1}(2\pi) g(4\pi)) = 2\pi\sigma\xi_{\text{nom}} + O(\sigma^2), \quad \xi(2\pi) = \sigma\xi_{\text{nom}} + O(\sigma^3),$$
$$r(4\pi) - r(2\pi) = 2\pi\sigma v_{\text{nom}} + O(\sigma^2), \qquad v(2\pi) = \sigma v_{\text{nom}} + O(\sigma^3),$$

at time $t = 2\pi(k+1)$,

$$\log(g^{-1}(2\pi k) g(2\pi(k+1))) = 2\pi\sigma\xi_{\text{nom}} + O(\sigma^3),$$
$$\xi(2\pi(k+1)) = \sigma\xi_{\text{nom}} + O(\sigma^3),$$
$$r(2\pi(k+1)) - r(2\pi k) = 2\pi\sigma v_{\text{nom}} + O(\sigma^3),$$
$$v(2\pi(k+1)) = \sigma v_{\text{nom}} + O(\sigma^3),$$

and, after $t = 2\pi(N+1)$,

$$\log(g^{-1}(2\pi N) g(2\pi(N+1))) = \pi\sigma\xi_{\text{nom}} + O(\sigma^{\frac{3}{2}}),$$
$$\xi(2\pi(N+1)) = O(\sigma^2),$$
$$r(2\pi(N+1)) - r(2\pi N) = \pi\sigma v_{\text{nom}} + O(\sigma^{\frac{3}{2}}),$$
$$v(2\pi(N+1)) = O(\sigma^2),$$

so the final velocity is as claimed in Lemma 2. In the shape variables,

$$r(2\pi(N+1)) - r_0 = \sum_{k=0}^{N} (r(2\pi(k+1)) - r(2k\pi)) = \pi\sigma v_{\text{nom}}$$
$$+ 2\pi\sigma(N-1) v_{\text{nom}} + \pi\sigma v_{\text{nom}} + O(\sigma^{\frac{3}{2}}) = 2\pi\sigma N \frac{r_{\text{f}} - r_0}{2\pi\sigma N} + O(\sigma^{\frac{3}{2}})\,.$$

To check the position variables, we use Lemma 4. Put $\log(g_0^{-1} g(2\pi)) = \pi\sigma\xi_{\text{nom}} + \sigma^{\frac{3}{2}}\eta_0$ and $\log(g^{-1}(2\pi) g(4\pi)) = 2\pi\sigma\xi_{\text{nom}} + \sigma^2\eta_1$, for some η_0,

$\eta_1 \in \mathfrak{g}$. Since

$$\left[\pi\sigma\xi_{\mathrm{nom}} + \sigma^{\frac{3}{2}}\eta_0, 2\pi\sigma\xi_{\mathrm{nom}} + \sigma^2\eta_1\right]_{\mathfrak{g}} = 2\pi^2\sigma^2\left[\xi_{\mathrm{nom}}, \xi_{\mathrm{nom}}\right]_{\mathfrak{g}} + 2\pi\sigma^{\frac{5}{2}}\left[\eta_0, \xi_{\mathrm{nom}}\right]_{\mathfrak{g}} + \pi\sigma^3\left[\xi\mathrm{nom}, \eta_1\right]_{\mathfrak{g}} + \sigma^{\frac{9}{2}}\left[\eta_0, \eta_1\right]_{\mathfrak{g}} = O(\sigma^{\frac{5}{2}}),$$

and $\pi\sigma\xi_{\mathrm{nom}} + 2\pi\sigma\xi_{\mathrm{nom}} = O(\sigma)$, we have $\log(g_0^{-1}g(4\pi)) = 3\pi\sigma\xi_{\mathrm{nom}} + O(\sigma^{\frac{5}{2}})$. This procedure can be iterated applying Lemma 4 to obtain that $\log(g_0^{-1}g(2\pi(N+1))) = 2N\pi\sigma\xi_{\mathrm{nom}} + O(\sigma^{\frac{3}{2}})$. Finally,

$$\log(g^{-1}(2\pi(N+1))g_1) = \log((g_0^{-1}g(2\pi(N+1)))^{-1}(g_0^{-1}g_1)) = O(\sigma^{\frac{3}{2}}).$$

8.5. Local Exponential Stabilization Algorithm

First we prove that if $\|(\log(g_{\mathrm{f}}^{-1}g(t_k)), r(t_k) - r_{\mathrm{f}}, \xi(t_k), v(t_k))\| = O(\sigma_k) \ll 1$ at time t_k, then

$$\|(\log(g_{\mathrm{f}}^{-1}g(t_k + 4\pi)), r(t_k + 4\pi) - r_{\mathrm{f}}, \xi(t_k + 4\pi), v(t_k + 4\pi))\| = O(\sigma_k^{\frac{3}{2}}) \ll 1,$$

By hypothesis, there exist x_{err}, r_{err}, ξ_{err} and v_{err} of $O(1)$, such that

$$\log(g_{\mathrm{f}}^{-1}g(t_k)) = \sigma_k x_{\mathrm{err}}, \qquad r(t_k) - r_{\mathrm{f}} = \sigma_k r_{\mathrm{err}},$$
$$\xi(t_k) = \sigma_k \xi_{\mathrm{err}}, \qquad v(t_k) = \sigma_k v_{\mathrm{err}}.$$

Then,

$$\frac{\log(g_{\mathrm{f}}^{-1}g(t_k)) + \pi\xi(t_k)}{2\pi\sigma_k} = \frac{x_{\mathrm{err}} + \pi\xi_{\mathrm{err}}}{2\pi}$$
$$\frac{r(t_k) - r_{\mathrm{f}} + \pi v(t_k)}{2\pi\sigma_k} = \frac{r_{\mathrm{err}} + \pi v_{\mathrm{err}}}{2\pi}.$$

After the first **Change-Velocity** primitive, the final configurations will have changed according to

$$\log(g^{-1}(t_k)g(t_k + 2\pi)) = \frac{\sigma_k}{2}(\pi\xi_{\mathrm{err}} - x_{\mathrm{err}}) + O(\sigma_k^{\frac{3}{2}}),$$
$$r(t_k + 2\pi) - r(t_k) = \frac{\sigma_k}{2}(\pi v_{\mathrm{err}} - r_{\mathrm{err}}) + O(\sigma_k^{\frac{3}{2}}).$$

Thus,

$$r(t_k + 2\pi) - r_{\mathrm{f}} = r(t_k + 2\pi) - r(t_k) + r(t_k) - r_{\mathrm{f}} = \frac{\sigma_k}{2}(r_{\mathrm{err}} + \pi v_{\mathrm{err}}) + O(\sigma_k^{\frac{3}{2}}),$$

and, using Lemma 4,

$$\log(g_{\rm f}^{-1}g(t_k+2\pi)) = \log((g_{\rm f}^{-1}g(t_k))(g^{-1}(t_k)g(t_k+2\pi)) \\ = \frac{\sigma_k}{2}(x_{\rm err}+\pi\xi_{\rm err}) + O(\sigma_k^{\frac{3}{2}})\,.$$

After applying the second primitive, the configurations become

$$\log(g^{-1}(t_k+2\pi)g(t_k+4\pi)) = -\frac{\sigma_k}{2}(x_{\rm err}+\pi\xi_{\rm err}) + O(\sigma_k^{\frac{3}{2}})\,,$$
$$r(t_k+4\pi) - r(t_k+2\pi) = -\frac{\sigma_k}{2}(r_{\rm err}+\pi v_{\rm err}) + O(\sigma_k^{\frac{3}{2}})\,,$$

so, using Lemma 4, one can deduce that at time $t_{k+1} = t_k + 4\pi$,

$$\log(g_{\rm f}^{-1}g(t_{k+1})) = O(\sigma_k^{\frac{3}{2}})\,, r(t_{k+1}) - r_{\rm f} = O(\sigma_k^{\frac{3}{2}})\,.$$

This is equivalent to say that

$$\|(\log(g_{\rm f}^{-1}g(t_k+4\pi)), r(t_k+4\pi) - r_{\rm f}, \\ \xi(t_k+4\pi), v(t_k+4\pi))\| = M_k\sigma_k^{\frac{3}{2}} \ll 1\,,$$

where M_k depends continuously on σ_k, the initial state and the coefficients of $B^1(r,t)$, $B^2(r,t)$. Being $M_k(g(t_k), r(t_k), \xi(t_k), v(t_k))$ continuous in a neighborhood of $(g_{\rm f}, r_{\rm f}, 0_{\mathfrak{g}}, 0_{TM})$, it is also bounded. Thus, there exist positive constants C_1, C_2 such that

$$\|(\log(g_{\rm f}^{-1}g(t_k)), r(t_k) - r_{\rm f}, \xi(t_k), v(t_k))\| < C_1 \implies \\ M_k(g(t_k), r(t_k), \xi(t_k), v(t_k)) < C_2\,.$$

Let $\alpha < 1$ be some positive constant verifying $\sigma = \alpha \min\{C_1\,, 1/C_2^2\}$ and set $\beta = \alpha^{1/2}$. Now we prove by induction that $\sigma_k < \sigma$ and $M_k\sigma_k^{1/2} \leq \beta$, $\forall k$. For $k=0$, we have that

$$\sigma_0 = \|(\log(g_{\rm f}^{-1}g(0)), r(0) - r_{\rm f}, \xi(0), v(0))\| \leq \sigma < C_1\,,$$

so $M_0 < C_2$ and

$$M_0\,\sigma_0^{1/2} < C_2\sigma^{1/2} \leq \beta < 1$$

and our claim is true. Assume that it holds for k, then we have to verify it for $k+1$. In first place, observe that

$$\sigma_{k+1} = \|(\log(g_{\rm f}^{-1}g(t_{k+1})), r(t_{k+1}) - r_{\rm f}, \xi(t_{k+1}), v(t_{k+1}))\| \\ = (M_k\sigma_k^{1/2})\sigma_k \leq \beta\sigma < \sigma < C_1\,,$$

then, $\sigma_{k+1}^{1/2} M_{k+1} \leq \sigma^{1/2} C_2 \leq \beta$. This proves $\sigma_k^{1/2} M_k \leq \beta$, $\forall k$ and the sequence $\{\sigma_k \,|\, k \geq 0\}$ is such that $\sigma_k \leq \beta^k \sigma_0$. Thus, for $\lambda = -\ln \beta > 0$,

$$\begin{aligned} &\|(\log(g_{\mathrm{f}}^{-1} g(t_k)), r(t_k) - r_{\mathrm{f}}, \xi(t_k), v(t_k)\| \\ &\qquad \leq \|(\log(g_{\mathrm{f}}^{-1} g(0)), r(0) - r_{\mathrm{f}}, \xi(0), v(0))\| e^{-\lambda k} \,. \end{aligned}$$

The last claim in Lemma 3 is consequence of the following. Observe that in each step of the loop, the system is affected by two **Change-Velocity** primitives, evolving initially from a state of order $O(\sigma_k) = O(e^{-\lambda k})$. The magnitude of the input in the two primitives is of order $\sqrt{\sigma_k} = e^{-\lambda k/2}$. Therefore the expansions in Proposition 4 show that during $t \in [t_k, t_{k+1}]$ the state is of order $\sqrt{\sigma_k} = e^{-\lambda k/2}$.