Adaptive and distributed coordination algorithms for mobile sensing networks

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Summary. Consider n sites evolving within a convex polygon according to one of the following interaction laws: (i) each site moves away from the closest other site or polygon boundary, (ii) each site moves toward the furthest vertex of its own Voronoi polygon, or (iii) each site moves toward a geometric center (centroid, circumcenter, incenter, etc) of its own Voronoi polygon. These interaction laws give rise to strikingly simple dynamical systems whose behavior remains largely unknown. Which are their critical points? What is their asymptotic behavior? Are they optimizing any aggregate function? In what way do these local interactions give rise to distributed systems? Are they of any engineering use in robotic coordination problems and in the design of mobile sensor networks? This paper addresses these questions.

1 Research challenges in network coordination

Coordination problems are becoming increasingly important in numerous engineering disciplines. One fundamental capability of future networks of autonomous vehicles will be the ability to perform spatially-distributed sensing tasks including coverage, surveillance, exploration, target detection, and search. These future mobile and tunable sensor networks will be able to adapt to changing environments and dynamic situations, will provide guaranteed fault-tolerant quality of service, and will operate via limited-bandwidth ad-hoc communication links. To achieve these desirable capabilities, it is important to design multi-vehicle coordination algorithms that are adaptive, distributed, asynchronous, and verifiably correct. These constraints turn the coordination problem into a formidable scientific endeavor. In other words, the key scientific issues are encoded as requirements on the coordination algorithms.

From a broader perspective, research in multi-vehicle coordination requires concepts and methods from systems theory, distributed algorithms, geometry,

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and algorithmic robotics. Of particular importance is the algorithms' validation on multi-vehicle testbeds and on experimental architectures that include embedded software architectures, signal processing methods, and ad-hoc communication protocols.

Literature review

Cooperative control and sensing problems. Recent years have witnessed a large research effort focused on cooperative motion planning and formation control of multi-vehicle systems [17, 36, 41, 44]. Coverage algorithms (for systems with binary, limited-range sensors) are surveyed in [12]. The work in [34] proposes algorithms for an intelligent sensor array to climb gradients of spatially-distributed signals. It is only recently, however, that truly distributed control laws for dynamic networks are being proposed. Examples of control algorithms include [6, 15, 26, 43]; examples of communication and consensus protocols include [27, 37]. Heuristic approaches to the design of interaction rules and emerging behaviors have been throughly investigated in the behavior-based robotics literature [2, 4, 5, 22, 32, 38]. Along this line of research, algorithms have been designed for sophisticated cooperative tasks. However, no formal results are currently available on how to design reactive control laws, ensure their correctness, and guarantee their optimality with respect to an aggregate objective.

Geometric optimization, facility location and systems theory. Geometric optimization is a vast and exciting avenue of current research, see for example [1, 8, 33]. In particular, we shall focus on facility location problems, in which service sites are spatially allocated to fulfill a specified request [18, 35]. Relying on methods from computational geometry [16], certain resource allocation problems can be solved via the notion of Voronoi partition [19].

An approach to formalizing behavioral control has been pursued using tools from control theory and formal methods from computer science. Hybrid models of motion control systems are introduced in [11], motion description languages in [31], and hybrid automata are described in [20, 23]. An alternative set of useful tools comes from the "dynamical systems approach to algorithms" [10, 25]. Distributed dynamical systems are to be designed as gradient flows of appropriate aggregate functions. The approach taken in this paper is to combine these ideas with nonsmooth and convex optimization [9, 13].

Distributed algorithms. The study of distributed algorithms is concerned with providing mathematical models, devising precise specifications for their behavior, and formally proving their correctness and complexity. Distributed consensus, resource allocation, communication, and data consistency problems are treated via an automata-theoretic approach, see [30] and references therein. Numerical distributed asynchronous algorithms as networking algorithms, rate and flow control, and gradient descent flows are discussed in [7, 29, 45]. These references do not typically address algorithms over ad-hoc dynamically changing networks.

Distributed coordination algorithms for coverage control

We propose an innovative technical approach that relies on non-smooth distributed descent algorithms and on aggregate utility functions that encode optimal coverage and sensing policies. We characterize and optimize notions of quality-of-service provided by an adaptive sensor network in a dynamic environment. We consider that a multi-vehicle network with configuration $(p_1, \dots, p_n)$ provides optimal coverage of a domain of interest Q if (i) it minimizes the expected (according to certain density function ϕ) distance from any event in the domain to one of the vehicle locations, or; (ii) it minimizes the largest distance from any point in the domain to one of the vehicle locations, or; (iii) it maximizes the coverage of the domain in such a way that the various sensing radius do not overlap or leave the environment. Accordingly, we seek to extremize one of the *multi-center functions*

$$\text{(i)} \int_Q \min_{i\in\{1,\dots,n\}} \|q-p_i\|^2\phi(q)dq\,, \qquad \text{(ii)} \max_{q\in Q}\left[\min_{i\in\{1,\dots,n\}} \|q-p_i\|\right],$$
$$\text{(iii)} \min_{i\neq j\in\{1,\dots,n\}} \left\{\tfrac{1}{2}\|p_i-p_j\|, \mathrm{d}(p_i,\partial Q)\right\}\,.$$

We study the differentiable properties of these functions via nonsmooth analysis [13], and compute their (generalized) gradients. We show that their critical points are *center Voronoi configurations*. We study the corresponding nonsmooth gradient flows using the tools in [3, 21, 39]. We show that this flow is not amenable to a distributed implementation for two of the multi-center functions. Drawing connections with quantization theory [19, 24, 28], we then consider two distributed coordination algorithms: a novel strategy based on the generalized gradient and a strategy similar to the well-known Lloyd algorithm. We investigate their asymptotic behavior and show that both algorithms are guaranteed to continuously improve the network performance. A detailed analysis of all results presented in this paper can be found in [14, 15].

2 Preliminaries and problem setup

Let $\|\cdot\|$ denote the Euclidean distance function on $\mathbb{R}^N$ and let $v\cdot w$ denote the scalar product of $v, w \in \mathbb{R}^N$. Let versus(v) denote the unit vector in the direction of $0\neq v\in\mathbb{R}^N$, i.e., $\mathrm{versus}(v) = v/\|v\|$. Given $S\subset\mathbb{R}^N$, $\mathrm{co}(S)$ and $\mathrm{int}(S)$ denote its convex hull and interior set, respectively. If S is convex, let $\mathrm{proj}_S\colon\mathbb{R}^N\to S$ denote the orthogonal projection onto S and let $\mathrm{d}_S\colon\mathbb{R}^N\to\mathbb{R}$ denote the distance function to S. For $R>0$, $\overline{B}_N(p,R) = \{q\in\mathbb{R}^N \mid \|p-q\|\leq R\}$ and $B_N(p,R) = \mathrm{int}(\overline{B}_N(p,R))$. Let Q be a convex polygon in $\mathbb{R}^2$. We denote by $\mathrm{Ed}(Q) = \{e_1,\dots,e_M\}$ and $\mathrm{Ve}(Q) = \{v_1,\dots,v_L\}$ the set of edges and vertices of Q, respectively. Let $P = (p_1,\dots,p_n)\in Q^n\subset(\mathbb{R}^2)^n$ denote the location of n generators in Q. Finally, let $\pi_i : Q^n\to Q$ be the canonical projection onto the ith factor.

2.1 Voronoi partitions

We refer the reader to [16, 35] for comprehensive treatments on Voronoi diagrams. The *Voronoi partition* $\mathcal{V}(P) = (V_1(P), \dots, V_n(P))$ of Q generated by the points $(p_1, \dots, p_n)$ is defined by

$$V_i(P) = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|, \forall j \neq i\}.$$

For simplicity, we refer to $V_i(P)$ as V_i. Since Q is convex, the boundary of each V_i is the union of a finite number of segments. If V_i and V_j share an edge, then p_i is a *(Voronoi) neighbor* of p_j (and vice-versa). All Voronoi neighboring relations are encoded in the map $\mathcal{N} : Q^n \times \{1, \dots, n\} \to 2^{\{1,\dots,n\}}$, where $\mathcal{N}(P, i)$ is the set of indexes of the Voronoi neighbors of p_i. We often write $\mathcal{N}(i)$ instead. A vertex $v \in \mathrm{Ve}(V_i(P))$ is *nondegenerate* if it is determined by exactly three elements (three generators, or to generators and an edge of Q, or one generator and two edges of Q). Otherwise it is *degenerate*. The configuration P is *nondegenerate* if all its vertices are nondegenerate, otherwise it is *degenerate*.

2.2 Multi-center functions as network performance measures

Let $\phi : Q \to \mathbb{R}_+$ be a distribution density function representing a probability that some event take place over the domain Q. Because of noise and loss of resolution, the sensing performance at point q taken from a sensor at the position p_i degrades with distance. Accordingly, $\|q - p_i\|$ gives a quantitative assessment of how poor the performance is. Consider the following notions of quality-of-service provided by a sensor network in a dynamic environment

$$\mathcal{H}_{\mathrm{C}}(P) = \int_Q \min_{i \in \{1,\dots,n\}} \|q - p_i\|^2 \phi(q) dq \,, \tag{1}$$

$$\mathcal{H}_{\mathrm{DC}}(P) = \max_{q \in Q} \left\{ \min_{i \in \{1,\dots,n\}} \|q - p_i\| \right\} , \tag{2}$$

$$\mathcal{H}_{\mathrm{SP}}(P) = \min_{\substack{i,j \in \{1,\dots,n\} \\ i \neq j}} \left\{ \tfrac{1}{2}\|p_i - p_j\|, \mathrm{d}(p_i, \partial Q) \right\} . \tag{3}$$

The performance measure $\mathcal{H}_{\mathrm{C}}$ corresponds to the expected distortion scenario, where the network tries to minimize the expected distance of any event in Q to one of the generators' locations given the information provided by ϕ , i.e.,

$$\min_{p_1,\dots,p_n} \left\{ \int_Q \min_{i \in \{1,\dots,n\}} \|q - p_i\|^2 \phi(q) dq \right\} .$$

In this way, the network optimizes its sensing performance. This problem is referred to as the p-median problem in [18]. Along the paper, we refer to it as the multi-centroid problem. The function $\mathcal{H}_{\mathrm{C}}$ can be rewritten as

$$\mathcal{H}_{\mathrm{C}}(P) = \sum_{i=1}^{n} \int_{V_i} \|q - p_i\|^2 \phi(q) dq \,.$$

Given a polytope W in $\mathbb{R}^N$, its centroid, $\mathrm{CM}(W)$, is the center of mass of W with respect to the density function ϕ, i.e.,

$$\mathrm{CM}(W) = \frac{1}{\mathrm{M}(W)} \int_W q\phi(q)dq\,, \quad \mathrm{M}(W) = \int_W \phi(q)dq\,.$$

Centroidal Voronoi configurations satisfy $p_i = \mathrm{CM}(V_i(P))$ for all $i \in \{1, \dots, n\}$.

The performance measure $\mathcal{H}_{\mathrm{DC}}$ corresponds to the worst case scenario, in which no information is available on the events taking place in Q. The network then tries to minimize the largest possible distance of any point in Q to one of the generators' locations, i.e.,

$$\min_{p_1,\dots,p_n} \left\{ \max_{q\in Q} \left\{ \min_{i\in\{1,\dots,n\}} \|q - p_i\| \right\} \right\} .$$

This problem is referred to as the p-center problem in [18, 42]. Along the paper, we refer to it as the multi-circumcenter problem. In terms of the Voronoi partition, the function $\mathcal{H}_{\mathrm{DC}}$ admits the following alternative expression

$$\mathcal{H}_{\mathrm{DC}}(P) = \max_{i\in\{1,\dots,n\}} \left\{ \max_{q\in V_i} \|q - p_i\| \right\} .$$

It is conjectured in [42] that the multi-circumcenter problem can be restated as a disk-covering problem: how to cover a region with disks of minimum radius. In Theorem 4 we provide a positive answer to this question. Given a polytope W in $\mathbb{R}^N$, its circumcenter, $\mathrm{CC}(W)$, is the center of the minimum-radius sphere that contains W. We say that P is a *circumcenter Voronoi configuration* if $p_i = \mathrm{CC}(V_i(P))$, for all $i \in \{1, \dots, n\}$.

The performance measure $\mathcal{H}_{\mathrm{SP}}$ corresponds to the situation where the network tries to maximize the coverage of Q so that the various sensing radius do not overlap, i.e.

$$\max_{p_1,\dots,p_n} \left\{ \min_{\substack{i,j\in\{1,\dots,n\}\\ i\neq j}} \left\{ \tfrac{1}{2}\|p_i - p_j\|, \mathrm{d}(p_i, \partial Q) \right\} \right\} .$$

We refer to this problem as the multi-incenter problem. In terms of the Voronoi partition, the function $\mathcal{H}_{\mathrm{SP}}$ admits the following alternative expression

$$\mathcal{H}_{\mathrm{SP}}(P) = \min_{i\in\{1,\dots,n\}} \left\{ \min_{q\notin \mathrm{int}(V_i)} \|q - p_i\| \right\} .$$

A similar conjecture to the one presented above is that the multi-incenter problem can be restated as a sphere-packing problem: how to maximize the coverage of a region with non-overlapping disks (contained in the region) of minimum radius. In Theorem 5 we provide a positive answer to this question. Given a polytope W in $\mathbb{R}^N$, its incenter set, $\mathrm{IC}(W)$, is the set of the centers of maximum-radius spheres contained in W. We say that $P \in Q^n$ is an *incenter Voronoi configuration* if $p_i \in \mathrm{IC}(V_i(P))$, for all $i \in \{1, \dots, n\}$. If P is an incenter Voronoi configuration, and each Voronoi region $V_i(P)$ has a unique incenter, $\mathrm{IC}(V_i(P)) = \{p_i\}$, then P is a *generic incenter Voronoi configuration.*

2.3 Nonsmooth analysis

Here we review some facts on nonsmooth analysis [13]. The *right* and *generalized directional derivative* of f at x in the direction of $v \in \mathbb{R}^N$ are, respectively,

$$f'(x,v) = \lim_{t\to 0^+} \frac{f(x+tv)-f(x)}{t}\,, \quad f^o(x;v) = \limsup_{\substack{y\to x\\ t\to 0^+}} \frac{f(y+tv)-f(y)}{t}\,.$$

The first limits does not always exist and this motivates the next definition.

Definition 1. *The function* $f : \mathbb{R}^N \to \mathbb{R}$ *is* regular at $x \in \mathbb{R}^N$ *if for all* $v \in \mathbb{R}^N$, $f'(x;v)$ *exists and* $f^o(x;v) = f'(x;v)$.

From Rademacher's Theorem [13], locally Lipschitz functions are differentiable a.e. If Ω_f denotes the set of points in $\mathbb{R}^N$ where f fails to be differentiable and S is any set of measure zero, the *generalized gradient* of f is

$$\partial f(x) = \operatorname{co}\{\lim_{i\to+\infty} df(x_i) \mid \ x_i \to x\,,\ x_i \notin S \cup \Omega_f\}\,.$$

A point $x \in \mathbb{R}^N$ with $0 \in \partial f(x)$ is a *critical point of* f.

Proposition 1. *Let* $\{f_k : \mathbb{R}^N \to \mathbb{R} \mid k \in \{1,\dots,m\}\}$ *be locally Lipschitz functions at* $x \in \mathbb{R}^N$. *Then,* $f : x' \mapsto \min\{f_k(x') \mid k \in \{1,\dots,m\}\}$ *is locally Lipschitz at* x, *and if* $I(x')$ *is the set of indexes* k *such that* $f_k(x') = f(x')$,

$$\partial f(x) \subset \operatorname{co}\{\partial f_i(x) \mid \ i \in I(x)\}\,, \tag{4}$$

and if f_i *is regular at* x *for* $i \in I(x)$, *then equality holds and* f *is regular at* x.

Proposition 2. *Let* f *be a locally Lipschitz function at* $x \in \mathbb{R}^N$. *If* f *attains a local minimum or maximum at* x, *then* $0 \in \partial f(x)$, *i.e.,* x *is a critical point.*

Let $\operatorname{Ln} : 2^{\mathbb{R}^N} \to \mathbb{R}$ be the map that associates to each convex set $S \subset \mathbb{R}^N$ its least-norm element, $\operatorname{Ln}(S) = \operatorname{proj}_S(0)$. For a locally Lipschitz function f, we consider the *generalized gradient vector field* $\operatorname{Ln}(\partial f) : \mathbb{R}^N \to \mathbb{R}^N$ given by $x \mapsto \operatorname{Ln}(\partial f)(x) = \operatorname{Ln}(\partial f(x))$.

Theorem 1. *Let* f *be a locally Lipschitz function at* x. *Assume* $0 \notin \partial f(x)$. *Then, there exists* $T > 0$ *such that for all* $0 < t < T$

$$f(x - t\,\operatorname{Ln}(\partial f)(x)) \le f(x) - \frac{t}{2}\|\operatorname{Ln}(\partial f)(x)\|^2\,.$$

2.4 Stability analysis via nonsmooth Lyapunov functions

For differential equations with discontinuous right-hand sides, solutions are defined in terms of differential inclusions [21]. Let $F : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ be a set-valued map. A solution to the differential inclusion $\dot{x} \in F(x)$ on an interval

$[t_0, t_1] \subset \mathbb{R}$ is defined as an absolutely continuous function $x : [t_0, t_1] \to \mathbb{R}^N$ such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in [t_0, t_1]$. Now, consider the equation

$$\dot{x}(t) = X(x(t)), \tag{5}$$

where $X : \mathbb{R}^N \to \mathbb{R}^N$ is measurable and essentially locally bounded. Let

$$K[X](x) = \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \operatorname{co}\{X(B_N(x, \delta) \setminus S)\}, \quad x \in \mathbb{R}^N.$$

A Filippov solution of (5) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion $\dot{x} \in K[X](x)$. A set M is *weakly invariant* (resp. *strongly invariant*) for (5) if for each $x_0 \in M$, contains a maximal solution (resp. all maximal solutions) of (5). Given a locally Lipschitz function f : $\mathbb{R}^N \to \mathbb{R}$, define the *set-valued Lie derivative of f with respect to X* at x as

$$\widetilde{\mathcal{L}}_X f(x) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ such that } \zeta \cdot v = a, \ \forall \zeta \in \partial f(x)\}.$$

For each $x \in \mathbb{R}^N$, $\widetilde{\mathcal{L}}_X f(x)$ is a closed and bounded interval in $\mathbb{R}$, possibly empty. The following result generalizes LaSalle Invariance Principle for differential equations of the form (5) with nonsmooth Lyapunov functions.

Theorem 2 (LaSalle Invariance Principle [3, 39]). *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz and regular function. Let $x_0 \in \mathbb{R}^N$ and let $f^{-1}(\leq f(x_0), x_0)$ be the connected component of $\{x \in \mathbb{R}^N \mid f(x) \leq f(x_0)\}$ containing x_0. Assume either $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ or $\widetilde{\mathcal{L}}_X f(x) = \emptyset$ for all $x \in f^{-1}(\leq f(x_0), x_0)$, and that this set is bounded. Then $f^{-1}(\leq f(x_0), x_0)$ is strongly invariant for (5). Let*

$$Z_{X,f} = \{x \in \mathbb{R}^N \mid 0 \in \widetilde{\mathcal{L}}_X f(x)\}.$$

Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^N$ of (5) starting from x_0 converges to the largest weakly invariant set M contained in $\overline{Z}_{X,f} \cap f^{-1}(\leq f(x_0), x_0)$.

2.5 Nonsmooth gradient flows

Given a locally Lipschitz and regular function f, consider

$$\dot{x}(t) = -\operatorname{Ln}(\partial f)(x(t)). \tag{6}$$

Theorem 1 guarantees that, unless the flow is at a critical point, $-\operatorname{Ln}(\partial f)(x)$ is a direction of descent at x. In general, the vector field $\operatorname{Ln}(\partial f)$ is discontinuous, and therefore the solution of (6) must be understood in the Filippov sense. Since f is locally Lipschitz, $\operatorname{Ln}(\partial f) = df$ a.e. The following result guarantees the convergence to the set of critical points of f.

Proposition 3. *Let $x_0 \in \mathbb{R}^N$ and $f^{-1}(\leq f(x_0), x_0)$ is bounded. Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^N$ of eq. (6) starting from x_0 converges asymptotically to the set of critical points of f contained in $f^{-1}(\leq f(x_0), x_0)$.*

3 The 1-center problems

Here we consider the multi-center problems with a single generator in order to gain insight into the general case. When $n = 1$, the function $\mathcal{H}_{\mathrm{C}}$ at p corresponds to the polar moment of inertia of the polygon Q about the point p, $\mathcal{H}_{\mathrm{C}}(p) = J_{Q,p}$. From the parallel axis theorem, one deduces that

$$\mathcal{H}_{\mathrm{C}}(p) = \mathcal{H}_{\mathrm{C}}(\mathrm{CM}(Q)) + \mathrm{M}(Q)\,\|p - \mathrm{CM}(Q)\|^2\,.$$

As a consequence, the minimization of $\mathcal{H}_{\mathrm{C}}$ consists of finding the centroid of Q. In this case, the gradient of $\mathcal{H}_{\mathrm{C}}$ is simply

$$\frac{\partial \mathcal{H}_{\mathrm{C}}}{\partial p}(p) = 2\,\mathrm{M}(Q)(p - \mathrm{CM}(Q))\,,$$

and therefore, $\partial \mathcal{H}_{\mathrm{C}}/\partial p\ (p) = 0$ if and only if $p = \mathrm{CM}(Q))$.

The minimization of $\mathcal{H}_{\mathrm{DC}}$ consists of finding the center of the minimum-radius sphere enclosing the polygon Q. On the other hand, the maximization of $\mathcal{H}_{\mathrm{SP}}$ corresponds to determining the center of the maximum-radius sphere contained in Q. Let us therefore define the functions

$$\begin{aligned} \mathrm{lg}_Q(p) &= \max\{\|q - p\| \mid\ q \in Q\} = \max\{\|v - p\| \mid\ v \in \mathrm{Ve}(Q)\}\,, \\ \mathrm{sm}_Q(p) &= \min\{\|q - p\| \mid\ q \notin \mathrm{int}(Q)\} = \min\{\mathrm{d}_e(p) \mid\ e \in \mathrm{Ed}(Q)\}\,. \end{aligned} \tag{7}$$

When $n = 1$, $\mathcal{H}_{\mathrm{DC}} = \mathrm{lg}_Q : Q \to \mathbb{R}$ and $\mathcal{H}_{\mathrm{SP}} = \mathrm{sm}_Q : Q \to \mathbb{R}$. Since the function lg_Q is the maximum of a (finite) set of convex functions in p, it is also a convex function [9]. Therefore, any local minimum of lg_Q is also global. Furthermore, one can show that the function lg_Q has a unique global minimum, which is the circumcenter of the polygon Q. The function sm_Q is the minimum of a (finite) set of affine (hence, concave) functions defined on the half-planes determined by the edges of Q, and hence it is also a concave function [9] on the intersection of their domains, which is precisely Q. Therefore, any local maximum of sm_Q is also global. However, this maximum is not unique in general. One can prove that the incenter set of the polygon Q corresponds to the set of maxima of the function sm_Q.

Proposition 4. *The functions* lg_Q, sm_Q *are locally Lipschitz and regular, and their generalized gradients are given by*

$$\partial\,\mathrm{lg}_Q(p) = \mathrm{co}\{\mathrm{versus}(p - v) \mid\ v \in \mathrm{Ve}(Q)\,,\ \mathrm{lg}_Q(p) = \|p - v\|\}\,, \tag{8}$$

$$\partial\,\mathrm{sm}_Q(p) = \mathrm{co}\{n_e \mid\ e \in \mathrm{Ed}(Q)\,,\ \mathrm{sm}_Q(p) = \mathrm{d}_e(p)\}\,. \tag{9}$$

Moreover, $0 \in \partial\,\mathrm{lg}_Q(p) \iff p = \mathrm{CC}(Q)$, $0 \in \partial\,\mathrm{sm}_Q(p) \iff p \in \mathrm{IC}(Q)$, *and, if* $0 \in \mathrm{int}(\partial\,\mathrm{sm}_Q(p))$, *then* $\mathrm{IC}(Q) = \{p\}$.

Next, let us study the generalized gradient flow arising from the 1-center functions. Clearly, the gradient descent of the function $\mathcal{H}_{\mathrm{C}}$ converges asymptotically to the centroid $\mathrm{CM}(Q)$. On the other hand, as a consequence of Propositions 3 and 4, we have the following result.

Proposition 5. *The gradient flows of the functions* lg_Q *and* sm_Q

$$\dot{x}(t) = -\operatorname{Ln}(\partial\,\mathrm{lg}_Q)(x(t))\,, \tag{10}$$

$$\dot{x}(t) = \operatorname{Ln}(\partial\,\mathrm{sm}_Q)(x(t))\,, \tag{11}$$

converge asymptotically to the circumcenter $\mathrm{CC}(Q)$ *and the incenter set* $\mathrm{IC}(Q)$*, respectively. Moreover, if* $0 \in \mathrm{int}(\partial\,\mathrm{lg}_Q(\mathrm{CC}(Q)))$*, then the flow* (10) *reaches* $\mathrm{CC}(Q)$ *in finite time. The flow* (11) *always reaches the set* $\mathrm{IC}(Q)$ *in finite time.*

Note that if $0 \in \partial\,\mathrm{lg}_Q(\mathrm{CC}(Q)) \setminus \mathrm{int}(\partial\,\mathrm{lg}_Q(\mathrm{CC}(Q)))$, then generically convergence of (10) is achieved over an infinite time horizon. Fig. 1 shows an example of the implementation of the gradient descent (10) and (11). Note that if the circumcenter $\mathrm{CC}(Q)$ (respectively the incenter set $\mathrm{IC}(Q)$) is first computed offline, then the strategy of directly going toward it would converge in a less "erratic" way.

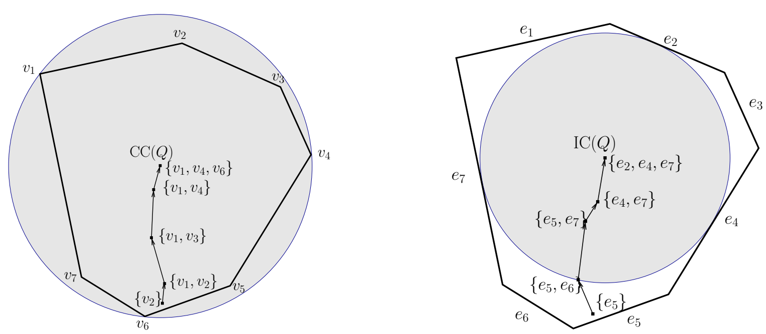

Fig. 1. Illustration of the gradient descent of lg_Q and sm_Q. The points where the curve $t \mapsto p(t)$ fails to be differentiable correspond to points where there is a new vertex v of Q such that $\|p(t) - v\| = \mathrm{lg}_Q(p(t))$ (respectively a new edge e of Q such that $\mathrm{d}_e(p(t)) = \mathrm{sm}_Q(p(t))$). The circumcenter and the incenter are attained in finite time according to Proposition 5.

4 Analysis of the multi-center functions

Here we analyze the locational optimization functions $\mathcal{H}_{\mathrm{C}}$, $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$. We characterize their smoothness properties, generalized gradients, and critical points for an arbitrary numbers of generators.

4.1 Smoothness and generalized gradients

We start by studying the function $\mathcal{H}_{\mathrm{C}}$. Because the map $(q, P) \mapsto \min_{i\in\{1,\dots,n\}} \|q - p_i\|$ is locally Lipschitz in P with Lipschitz constant equal to 1, one can show that $\mathcal{H}_{\mathrm{C}}$ is locally Lipschitz on Q^n with Lipschitz constant $\mathrm{M}(Q)$. Now, let $S = \{P \in Q^n \mid \text{ there exists } i, j \in \{1, \dots, n\} \text{ s.t. } p_i = p_j\}$. Over the set $Q^n \setminus S$, one can show [19, 35] that for all $i \in \{1, \dots, n\}$,

$$\frac{\partial \mathcal{H}_{\mathrm{C}}}{\partial p_i}(P) = \int_{V_i} \frac{\partial}{\partial p_i} \|q - p_i\|^2 \phi(q) dq = 2\,\mathrm{M}(V_i)(p_i - \mathrm{CM}(V_i))\,, \qquad (12)$$

where $P \mapsto \mathrm{M}(V_i(P))$ and $P \mapsto \mathrm{CM}(V_i(P))$ are continuous functions of P. Therefore, over $Q^n \setminus S$, $\mathcal{H}_{\mathrm{C}}$ is continuously differentiable.

To study the properties of the functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$, let us consider the following alternative expressions and useful quantities. Write

$$\mathcal{H}_{\mathrm{DC}}(P) = \max_{i \in \{1,\dots,n\}} G_i(P)\,, \quad \mathcal{H}_{\mathrm{SP}}(P) = \min_{i \in \{1,\dots,n\}} F_i(P)\,,$$

where we define

$$G_i(P) = \max_{q \in V_i(P)} \|q - p_i\|\,, \quad F_i(P) = \min_{q \notin \mathrm{int}(V_i(P))} \|q - p_i\|\,.$$

Note that $G_i(P) = \mathrm{lg}_{V_i(P)}(p_i)$ and $F_i(P) = \mathrm{sm}_{V_i(P)}(p_i)$. Proposition 4 provides an explicit expression for the generalized gradients of lg_{V_i} and sm_{V_i} when the Voronoi cell V_i is held fixed. Despite the slight abuse of notation, it is convenient to let $\partial\,\mathrm{lg}_{V_i(P)}(p_i)$ denote $\partial\,\mathrm{lg}_V(p_i)_{|V=V_i(P)}$, and let $\partial\,\mathrm{sm}_{V_i(P)}(p_i)$ denote $\partial\,\mathrm{sm}_V(p_i)_{|V=V_i(P)}$.

In contrast to this analysis at fixed Voronoi partition, the properties of the functions G_i and F_i are strongly affected by the dependence on the Voronoi partition $\mathcal{V}(P)$. We characterize these properties in the following result.

Proposition 6. *The functions $G_i, F_i : Q^n \to \mathbb{R}$ are locally Lipschitz and regular. As a consequence, the locational optimization functions $\mathcal{H}_{DC}, \mathcal{H}_{SP} : Q^n \to \mathbb{R}$ are locally Lipschitz and regular.*

The generalized gradients of the functions of G_i and F_i can be described in a very precise way by means of a careful analysis of the vertexes and edges where their values are attained, and of the degenerate/nondegenerate character of the Voronoi partition. We refer the interested reader to [14] for a detailed discussion along these lines. Here, we will only highlight the fact that the knowledge of the generalized gradients of G_i and F_i is key to describe the generalized gradients of the functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$. This is a consequence of Propositions 1 and 6, which imply

$$\partial \mathcal{H}_{\mathrm{DC}}(P) = \mathrm{co}\{\partial G_i(P) \mid i \in I(P)\}\,, \quad \partial \mathcal{H}_{\mathrm{SP}}(P) = \mathrm{co}\{\partial F_i(P) \mid i \in I(P)\}\,.$$

4.2 Critical points

Having characterized the (generalized) gradients of $\mathcal{H}_{\mathrm{C}}$, $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$, we now turn to studying their critical points.

Theorem 3 (Minima of $\mathcal{H}_{\mathbf{C}}$). *Let $P \in Q^n$ be a local minimum of $\mathcal{H}_C$. Then P is a centroidal Voronoi configuration.*

Theorem 4 (Minima of $\mathcal{H}_{\mathbf{DC}}$). *Let $P \in Q^n$ be nondegenerate and $0 \in \operatorname{int}(\partial\mathcal{H}_{DC}(P))$. Then, P is a strict local minimum of $\mathcal{H}_{DC}$, all generators are active and P is a circumcenter Voronoi configuration.*

Theorem 5 (Maxima of $\mathcal{H}_{\mathbf{SP}}$). *Let $P \in Q^n$ and $0 \in \operatorname{int}(\partial\mathcal{H}_{SP}(P))$. Then, P is a strict local maximum of $\mathcal{H}_{SP}$, all generators are active and P is a generic incenter Voronoi configuration.*

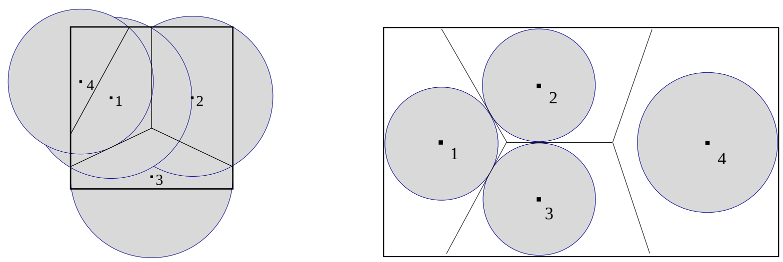

Fig. 2. Local extrema of $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$ in a convex polygonal environment. The configuration on the left corresponds to a local minimum of $\mathcal{H}_{\mathrm{DC}}$ with $0 \in \partial\mathcal{H}_{\mathrm{DC}}(P)$ and $\operatorname{int}(\partial\mathcal{H}_{\mathrm{DC}}(P)) = \emptyset$. The configuration on the right corresponds to a local maximum of $\mathcal{H}_{\mathrm{SP}}$ with $0 \in \partial\mathcal{H}_{\mathrm{SP}}(P)$ and $\operatorname{int}(\partial\mathcal{H}_{\mathrm{SP}}(P)) = \emptyset$. In both cases, the 4th generator is inactive and non-centered.

Remark 1. Theorems 4 and 5 provide the interpretation of the multi-center problems in Section 2.2: since all generators are active, they share the same radius. Dropping the hypothesis that 0 belongs to the interior of the generalized gradient gives rise to simple examples where P is a local minimum of $\mathcal{H}_{\mathrm{DC}}$ (respectively a local maximum of $\mathcal{H}_{\mathrm{SP}}$), and there are generators which are inactive and non-centered, see Fig. 2.

5 Dynamical systems for the multi-center problems

Here, we describe three algorithms that (locally) extremize the multi-center functions. We present continuous-time versions of the algorithms and discuss their convergence properties. The generators' location obeys a first order dynamical behavior described by

$$\dot{p}_i = u_i(p_1, \dots, p_n)\,, \quad i \in \{1, \dots, n\}\,. \tag{13}$$

The dynamical system (13) is said to be *centralized* if there exists at least an $i \in \{1, \dots, n\}$ such that $u_i(p_1, \dots, p_n)$ cannot be written as a function of the form $u_i(p_i, p_{i_1}, \dots, p_{i_m})$, with $m < n-1$. The dynamical system (13) is said to be *Voronoi-distributed* if each $u_i(p_1, \dots, p_n)$ can be written as a function of the form $u_i(p_i, p_{i_1}, \dots, p_{i_m})$, with $i_k \in \mathcal{N}(P, i)$, $k \in \{1, \dots, m\}$. We refer to [15] for more details on the distributed character of Voronoi neighborhood relationships. Finally, the dynamical system (13) is said to be *nearest-neighbor-distributed* if each $u_i(p_1, \dots, p_n)$ can be written as a function of the form $u_i(p_i, p_{i_1}, \dots, p_{i_m})$, with $\|p_i - p_{i_k}\| \leq \|p_i - p_j\|$ for all $j \in \{1, \dots, n\}$, and $k \in \{1, \dots, m\}$. A nearest-neighbor-distributed dynamical system is also Voronoi-distributed.

5.1 Gradient dynamical systems

Consider the (signed) generalized gradient descent flow (6) for the locational optimization functions $\mathcal{H}_{\mathrm{C}}$, $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$,

$$\dot{P} = -\operatorname{Ln}(\partial \mathcal{H}_{\mathrm{C}})(P)\,, \quad \dot{P} = -\operatorname{Ln}(\partial \mathcal{H}_{\mathrm{DC}})(P)\,, \quad \dot{P} = \operatorname{Ln}(\partial \mathcal{H}_{\mathrm{SP}})(P)\,.$$

Alternatively, we may write for each $i \in \{1, \dots, n\}$,

$$\dot{p}_i = -\frac{\partial \mathcal{H}_{\mathrm{C}}}{\partial P}(p_1, \dots, p_n) = 2\,\mathrm{M}(V_i)\,(\mathrm{CM}(V_i) - p_i)\,, \tag{14}$$

$$\dot{p}_i = -\pi_i(\operatorname{Ln}(\partial \mathcal{H}_{\mathrm{DC}})(p_1, \dots, p_n))\,, \tag{15}$$

$$\dot{p}_i = \pi_i(\operatorname{Ln}(\partial \mathcal{H}_{\mathrm{SP}})(p_1, \dots, p_n))\,. \tag{16}$$

One can show that the set $Q^n \setminus S$ is positively invariant for the flow (14). Therefore, this dynamical system corresponds to a standard gradient descent flow. As noted in Section 2.4, the vector fields (15) and (16) are discontinuous, and therefore their solution must be understood in the Filippov sense. One needs to first compute the generalized gradients at P, $\partial \mathcal{H}_{\mathrm{DC}}(P)$ and $\partial \mathcal{H}_{\mathrm{SP}}(P)$, then compute the least-norm element, and finally project to each of the n components. Note that the least-norm element of convex sets can be computed efficiently, see [9], however closed-form expressions are not available in general. One can also see that the compact set Q^n is strongly invariant for both vector fields $-\operatorname{Ln}(\partial \mathcal{H}_{\mathrm{DC}})$ and $\operatorname{Ln}(\partial \mathcal{H}_{\mathrm{SP}})$ (cf. [14]).

Proposition 7. *For the dynamical system* (14) *(respectively* (15), (16)*), the generators' location $P = (p_1, \dots, p_n)$ converges asymptotically to the set of critical points of $\mathcal{H}_C$ (respectively, of $\mathcal{H}_{DC}$, $\mathcal{H}_{SP}$).*

Remark 2. The gradient dynamical system (14) is Voronoi-distributed since the partial derivative of $\mathcal{H}_{\mathrm{C}}$ with respect to the ith sensor location only depends on its own position and the position of its Voronoi neighbors. On the other hand, the gradient dynamical systems (15) and (16) enjoy convergence guarantees, but their implementation is centralized because of two reasons. First, all functions $G_i(P)$ (respectively $F_i(P)$) need to be compared in order to determine which generator is active. Second, the least-norm element of the generalized gradients depends on the relative position of the active generators with respect to each other and to the environment.

5.2 Nonsmooth dynamical systems based on distributed gradients

In this section, we propose a distributed implementation of the previous gradient dynamical systems and explore their relation with behavior-based rules. Consider the following variations of the gradient dynamical systems (15)-(16),

$$\dot{p}_i = -\operatorname{Ln}(\partial \operatorname{lg}_{V_i(P)})(P)\,, \tag{17}$$

$$\dot{p}_i = \operatorname{Ln}(\partial \operatorname{sm}_{V_i(P)})(P)\,, \tag{18}$$

for $i \in \{1, \dots, n\}$. Note that the system (17) is Voronoi-distributed, since $\operatorname{Ln}(\partial \operatorname{lg}_{V_i(P)})(P)$ is determined only by the position of p_i and of its Voronoi neighbors $\mathcal{N}(P, i)$. On the other hand, the system (18) is nearest-neighbor-distributed, since $\operatorname{Ln}(\partial \operatorname{sm}_{V_i(P)})(P)$ is determined only by the position of p_i and its nearest neighbors. For future reference, let $\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})(P)$ denote $(\operatorname{Ln}(\partial \operatorname{lg}_{V_1(P)})(P), \dots, \operatorname{Ln}(\partial \operatorname{lg}_{V_n(P)})(P))$, and let $\operatorname{Ln}(\partial \operatorname{sm}_{\mathcal{V}})(P)$ denote $(\operatorname{Ln}(\partial \operatorname{sm}_{V_1(P)})(P), \dots, \operatorname{Ln}(\partial \operatorname{sm}_{V_n(P)})(P))$, and write

$$\dot{P} = -\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})(P)\,, \quad \dot{P} = \operatorname{Ln}(\partial \operatorname{sm}_{\mathcal{V}})(P)\,.$$

As for the previous dynamical systems, note that these vector fields are discontinuous, and therefore their solutions must be understood in the Filippov sense. One can see that the compact set Q^n is strongly invariant for both vector fields. This fact is a consequence of the expressions for the generalized gradients of lg and sm in Proposition 4. Note that in the 1-center case, (15) (respectively (16)) coincides with (17) (respectively with (18)).

Proposition 8. *Let $P \in Q^n$. Then the solutions of the dynamical systems* (17) *and* (18) *starting at P are unique.*

Remark 3 (Relation with behavior-based robotics: move toward the furthest-away vertex). The distributed gradient control law in the disk-covering setting (17) has an interesting "behavioral" interpretation. For the ith generator, if the maximum of $\operatorname{lg}_{V_i(P)}$ is attained at a single vertex v of its Voronoi cell V_i, then $\operatorname{lg}_{V_i(P)}$ is differentiable at that configuration, and its derivative corresponds to $\operatorname{versus}(p_i - v)$. Therefore, the control law (17) corresponds to the behavior "move toward the furthest vertex in own Voronoi cell." If there are two or more vertexes of V_i where the value $\operatorname{lg}_{V_i(P)}(p_i)$ is attained, then (17) provides an average behavior by computing the least-norm element in the convex hull of all $\operatorname{versus}(p_i - v)$ such that $\|p_i - v\| = \operatorname{lg}_{V_i(P)}(p_i)$.

Remark 4 (Relation with behavior-based robotics: move away from the nearest neighbor). The distributed gradient control law in the sphere-packing setting (18) has also an interesting interpretation. For the ith generator, if the minimum of $\operatorname{sm}_{V_i(P)}$ is attained at a single edge e, then $\operatorname{sm}_{V_i(P)}$ is differentiable at that configuration, and its derivative is n_e. The control law (18) corresponds to the behavior "move away from the nearest neighbor" (where a neighbor can also be the boundary of the environment). If there are two or more edges where the value $\mathrm{sm}_{V_i(P)}(p_i)$ is attained, then (18) provides an average behavior in an analogous manner as before.

Proposition 9. *For the dynamical system* (17) *(resp. the dynamical system* (18)*), the generators' location $P = (p_1, \dots, p_n)$ converges asymptotically to the largest weakly invariant set contained in the closure of $A_{\mathrm{DC}}(Q) = \{P \in Q^n \mid i \in I(P) \implies p_i = \mathrm{CC}(V_i)\}$ (resp. the largest weakly invariant set contained in the closure of $A_{\mathrm{SP}}(Q) = \{P \in Q^n \mid i \in I(P) \implies p_i \in \mathrm{IC}(V_i)\}$).*

5.3 Distributed dynamical systems based on geometric centering

Here, we propose alternative distributed dynamical systems for the multicenter functions. Our design is directly inspired by the results in Theorems 4 and 5 on the critical points of the functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$. For $i \in \{1, \dots, n\}$, consider the dynamical systems

$$\dot{p}_i = \mathrm{CC}(V_i) - p_i \,, \tag{19}$$
$$\dot{p}_i \in \mathrm{IC}(V_i) - p_i \,. \tag{20}$$

Alternatively, we write $\dot{P} = \mathrm{CC}(\mathcal{V}(P)) - P$ and $\dot{P} \in \mathrm{IC}(\mathcal{V}(P)) - P$. Note that both systems are Voronoi-distributed. The vector field (19) is continuous, since the circumcenter of a polygon depends continuously on the location of its vertexes, and the location of the vertexes of the Voronoi partition depends continuously on the location of the generators; see [35]. However, eq. (20) is a differential inclusion, since the incenter sets may not be singletons. Following [21], the existence of solutions is guaranteed by the following result.

Proposition 10. *Consider the set-valued map* $\mathrm{IC}(\mathcal{V}) - \mathrm{Id} : Q^n \to 2^{(\mathbb{R}^2)^n}$ *given by* $P \mapsto \mathrm{IC}(\mathcal{V}(P)) - P$*. Then* $\mathrm{IC}(\mathcal{V}) - \mathrm{Id}$ *is upper semicontinuous with nonempty, compact and convex values.*

One can also see that the compact set Q^n is strongly invariant for the vector field $\mathrm{CC}(\mathcal{V}) - \mathrm{Id}$ and for the differential inclusion $\mathrm{IC}(\mathcal{V}) - \mathrm{Id}$. Next, we characterize the asymptotic convergence of these dynamical systems.

Proposition 11. *For the dynamical system* (19) *(respectively* (20)*), the generators' location $P = (p_1, \dots, p_n)$ converges asymptotically to the largest weakly invariant set contained in the closure of $A_{\mathrm{DC}}(Q)$ (respectively in the closure of $A_{\mathrm{SP}}(Q)$).*

5.4 Simulations

To illustrate the performance of the distributed coordination algorithms, we include some simulation results. The algorithms are implemented in `Mathematica` as a single centralized program. We compute the bounded

Voronoi diagram of a collection of points using the `ComputationalGeometry` package. We compute the mass, centroid, and polar moment of inertia of polygons via the numerical integration routine `NIntegrate`. We compute the circumcenter of a polygon via the algorithm in [40] and the incenter set via the `LinearProgramming` solver in `Mathematica`. Measuring displacements in meters, we consider the domain determined by the vertexes

$$\{(0,0),(2.5,0),(3.45,1.5),(3.5,1.6),(3.45,1.7),(2.7,2.1),(1.,2.4),(.2,1.2)\}.$$

In Fig. 3 we illustrate the performance of the dynamical system (14), in Figs. 4 and 5 we illustrate the performance of the dynamical systems (17) and (19), and in Figs. 6 and 7 we illustrate the performance of the dynamical systems (18) and (20). Observing the final configurations in the four figures, one can verify, visually and numerically, that the active sensors are asymptotically centered as forecast by our analysis.

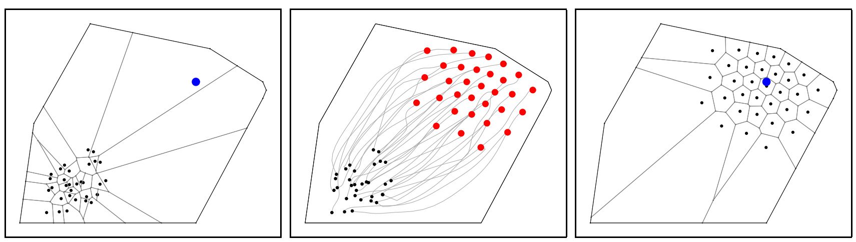

Fig. 3. "Move-toward-the-centroid" algorithm for 32 sensors in a convex polygonal domain with Gaussian density function $\phi = \exp(-x^2 - y^2)$ centered at the larger ball. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution.

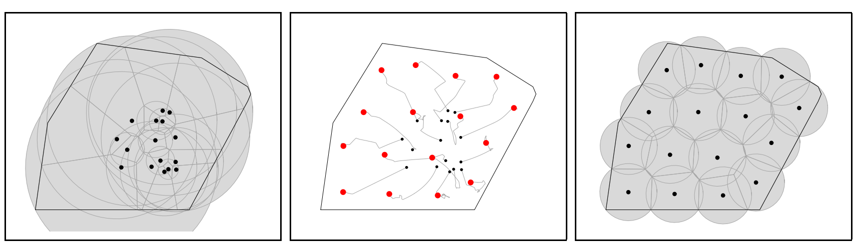

Fig. 4. "Toward the furthest" algorithm for 16 sensors in a convex polygonal domain. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After 2 sec., the multi-circumcenter function is approximately .39504 m.

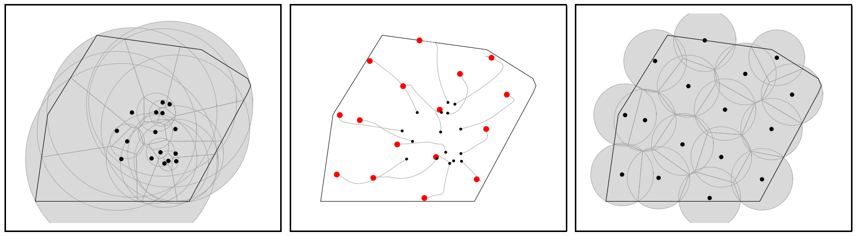

Fig. 5. "Move-toward-the-circumcenter" algorithm for 16 sensors in a convex polygonal domain. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After 20 sec., the multi-circumcenter function is approximately 0.43273 m.

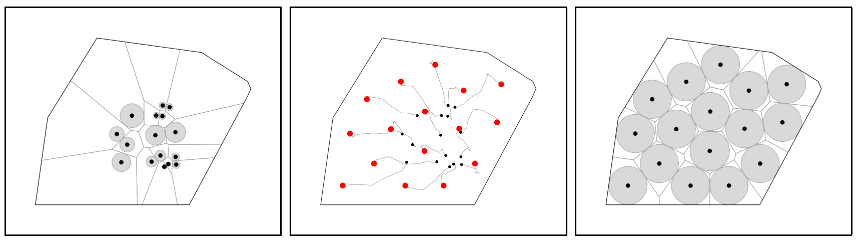

Fig. 6. "Away-from-closest" algorithm for 16 sensors in a convex polygonal domain. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After 2 sec., the multi-incenter function is approximately .26347 m.

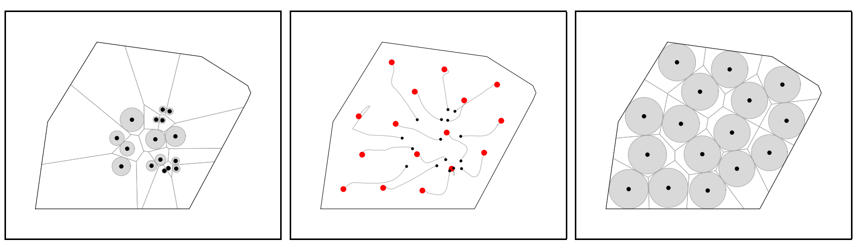

Fig. 7. "Move-toward-the-incenter" algorithm for 16 sensors in a convex polygonal domain. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After 20 sec., the multi-incenter function is approximately .2498 m.

6 Conclusions and future work

We have introduced multi-center functions that provide quality-of-service measures for mobile networks. We have established that $\mathcal{H}_{\mathrm{C}}$ is locally Lipschitz on Q^n and continuously differentiable on $Q^n \setminus S$, and that both $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$ are locally Lipschitz and regular, having computed their generalized gradients. Furthermore, under certain technical conditions, we have characterized their critical points as center (centroidal, circumcenter and incenter, respectively) Voronoi configurations, and have shown their correspondence with the solutions of expected distortion, disk-covering and sphere-packing problems (see Table 1). We have also considered various algorithms that extremize the multi-center functions. First, we considered the (nonsmooth) gradient flows induced by their respective (generalized) gradients. Second, for the nonsmooth multi-center functions, we devised a novel strategy based on the generalized gradients of the 1-center functions of each generator. Third, we introduced and characterized a geometric centering strategy. We have unveiled the remarkable geometric interpretations of these algorithms, discussed their distributed character and analyzed their asymptotic behavior using nonsmooth stability analysis (see Tables 2 and 3).

Future directions of research include: (i) sharpening the asymptotic convergence results for the proposed dynamical systems, (ii) considering the setting of convex polytopes in $\mathbb{R}^N$, for $N > 2$, (iii) investigating the effect of measurement errors on the proposed algorithms and quantifying their closed-loop robustness, and (iv) analyzing other meaningful geometric optimization problems and their relations with cooperative behaviors.

Table 1. Nonsmooth analysis of the multi-center cost functions.

	$\mathcal{H}_{\mathrm{C}}$	$\mathcal{H}_{\mathrm{DC}}$	$\mathcal{H}_{\mathrm{SP}}$
SMOOTHNESS	continuously differentiable*	regular, globally Lipschitz	regular, globally Lipschitz
CRITICAL POINTS	Centroidal Voronoi configurations	Circumcenter Voronoi configurations**	Incenter Voronoi configurations**
DESCRIPTION	expected distortion	disk covering	sphere packing

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* $\mathcal{H}_{\mathrm{C}}$ is continuously differentiable over the positively invariant set $Q^n \setminus S$.
** Provided 0 is in the interior of the corresponding generalized gradient.

Table 2. Properties of the dynamical systems based on nonsmooth gradient information that extremize the multi-center problems.

	$\dot{p}_i = -\operatorname{Ln}[\partial \lg_{V_i(P)}](P)$	$\dot{p}_i = \operatorname{Ln}[\partial \operatorname{sm}_{V_i(P)}](P)$	$\dot{p}_i = \pi_i\left(\operatorname{Ln}[\partial\mathcal{H}](P)\right)$
SMOOTHNESS	discontinuous	discontinuous	discontinuous
DISTRIBUTIVITY	Voronoi neighbors	closest neighbors	centralized
CRITICAL POINTS	Circumcenter Voronoi configurations	Incenter Voronoi configurations	n/a
COST CRITERIA	$\mathcal{H}_{\mathrm{DC}}$	$\mathcal{H}_{\mathrm{SP}}$	$\mathcal{H}$
HEURISTIC DESCRIPTION	"move toward furthest vertex of own cell"	"move away from closest neighbor"	"nonsmooth gradient descent"
ASYMPTOTIC BEHAVIOR	Active go to circumcenter of own Voronoi cell	Active go to incenter of own Voronoi cell	Set of critical points

Table 3. Properties of the dynamical systems based on geometric centering that extremize the multi-center problems.

	$\dot{p}_i = \mathrm{CM}(V_i(P)) - p_i$	$\dot{p}_i = \mathrm{CC}(V_i(P)) - p_i$	$\dot{p}_i \in \mathrm{IC}(V_i(P)) - p_i$
SMOOTHNESS	continuous	continuous	upper-semicontinuous
DISTRIBUTIVITY	Voronoi neighbors	Voronoi neighbors	Voronoi neighbors
CRITICAL POINTS	Centroidal Voronoi configurations	Circumcenter Voronoi configurations	Incenter Voronoi configurations
COST CRITERIA	$\mathcal{H}_{\mathrm{C}}$	$\mathcal{H}_{\mathrm{DC}}$	$\mathcal{H}_{\mathrm{SP}}$
HEURISTIC DESCRIPTION	"move toward centroid of own cell"	"move toward circumcenter of own cell"	"move toward incenter set of own cell"
ASYMPTOTIC BEHAVIOR	All go to centroid of own Voronoi cell	Active go to circumcenter of own Voronoi cell	Active go to incenter of own Voronoi cell

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