Maximizing visibility in nonconvex polygons: nonsmooth analysis and gradient algorithm design

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***Abstract*— This paper presents a motion control algorithm for a planar mobile observer such as, e.g., a mobile robot equipped with an omni-directional camera. We propose a nonsmooth gradient algorithm for the problem of maximizing the area of the region visible to the observer in a simple nonconvex polygon. First, we show that the visible area is almost everywhere a locally Lipschitz function of the observer location. Second, we provide a novel version of LaSalle Invariance Principle for discontinuous vector fields and Lyapunov functions with a finite number of discontinuities. Finally, we establish the asymptotic convergence properties of the nonsmooth gradient algorithm and we illustrate numerically its performance.**

I. INTRODUCTION

Consider a single-point mobile robot in a planar nonconvex environment modeled as a simple polygon: how should the robot move in order to monotonically increase the area of its visible region (i.e., the region within its line of sight)? This problem is the subject of this paper, together with the following modeling assumptions. The dynamical model for the robot's motion is a first order system of the form $\dot{p} = u$, where p refers to the position of the robot in the environment and u is the driving input. The robot is equipped with an omni-directional camera and range sensor; the range of the sensor is larger than the diameter of the environment. The robot does not know the entire environment and its position in it, and its instantaneous motion depends only on what is within line of sight (this assumption restricts our attention to memoryless feedback laws).

In broad terms, this problem is related to numerous optimal sensor location and motion planning problems in the computational geometry, geometric optimization, and robotics literature. In computational geometry [1], the classical Art Gallery Problem amounts to finding the optimum number of guards in a nonconvex environment so that each point of the environment is visible by at least one guard. A heuristic for this problem is to use a greedy approach wherein the first robot (guard) is placed at the point where it sees the maximum area. The next robot is placed where it sees the maximum area not visible to the first and so on. In robotics, this approach is useful for 2D map building wherein a robot moves in such a way so that its next position is the best in terms of what it can see additionally. In this robotic context, these problems are referred to as Next Best View problems. The specific problem of interest in this paper is that of optimally locating a guard in a simple polygon. To the best of our knowledge, this problem is still open and is the subject of ongoing research; see [2], [3], [4], and the surveys on geometric optimization and art gallery problems [5], [6]. However, randomized algorithms for finding the optimal location up to a constant factor approximation exist; see [4]. These algorithms can be regarded as open-loop algorithms that require knowledge of the environment. Closed-loop heuristic algorithms for the Next Best View problem are proposed and simulated in [7] and in the early work [8].

A second set of relevant references are those on nonsmooth stability analysis. Indeed, our approach to maximizing visible area is to design a nonsmooth gradient flow. To define our proposed algorithm we rely on the notions of generalized gradient [9] and of Filippov solutions for differential inclusions [10]. To study our proposed algorithm we extend recent results on the stability and convergence properties of nonsmooth dynamical systems, as presented in [11], [12].

The contributions of this paper are threefold. First, we prove some basic properties of the area visible from a point observer in a nonconvex polygon Q, see Figure 1. Namely, we show that the area of the visibility polygon, as

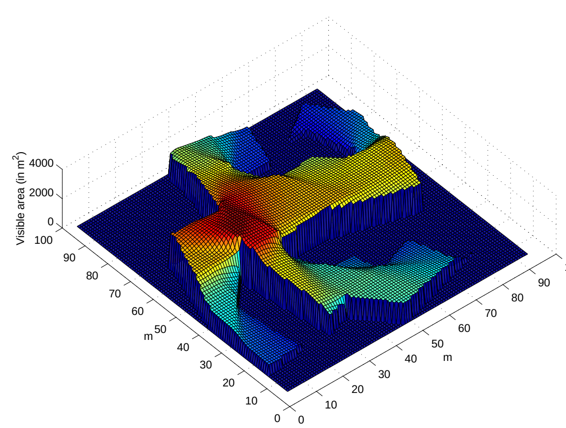

Fig. 1. The visible area function over a nonconvex polygon.

a function of the observer position, is a locally Lipschitz function almost everywhere, and that the finite set point of discontinuities are the reflex vertices of the polygon Q.

Additionally, we compute the generalized gradient of the function and show that it is, in general not regular. Second, we provide a generalized version of the certain stability theorems for discontinuous vector fields available in the literature [11], [12]. Specifically, we provide a generalized nonsmooth LaSalle Invariance Principle for discontinuous vector fields, Filippov solutions, and Lyapunov functions that are locally Lipschitz almost everywhere (except for a finite set of discontinuities). Third and last, we use these novel results to design a nonsmooth gradient algorithm that monotonically increases the area visible to a point observer. To the best of our knowledge, this is the first provably correct algorithm for this version of the Next Best View problem. We illustrate the performance of our algorithm via simulations for some interesting polygons.

The paper is organized as follows. Section II contains the analysis of the smoothness and of the generalized gradient of the function of interest. Section III contains the novel results on nonsmooth stability analysis. Section IV presents the nonsmooth gradient algorithm and the properties of the resulting closed-loop system. Finally, the simulations in Section V illustrate the convergence properties of the algorithm. The proofs for the results in Section II and III are included, whereas the proofs for the results in Section IV will be included in future submissions.

II. The area visible from an observer

In this section we study the area of the region visible to a point observer equipped with an omnidirectional camera. We show that the visible area, as a function of the location of the observer, is locally Lipschitz, except at a finite point set. We prove that, for general nonconvex polygons, the function is not regular. We also provide expressions for the generalized gradient of the visible area function wherever it is locally Lipschitz. We refer the reader to [9] for the notion of locally Lipschitz functions and related concepts.

Let us start by introducing the set of lines on the plane $\mathbb{R}^2$. For $(a,b,c) \in \mathbb{R}^3 \setminus \{(0,0,c) \in \mathbb{R}^3 \mid c \in \mathbb{R}\}$, define the equivalence class $[(a,b,c)]$ by

$$[(a,b,c)] \\ = \left\{(a',b',c') \in \mathbb{R}^3 \mid (a,b,c) = \lambda(a',b',c'),\ \lambda \in \mathbb{R}\right\}.$$

The *set of lines on* $\mathbb{R}^2$ is defined as

$$\mathbb{L} = \left\{[(a,b,c)] \subset \mathbb{R}^3 \mid (a,b,c) \in \mathbb{R}^3,\ a^2+b^2 \neq 0\right\}.$$

It is possible to show that $\mathbb{L}$ is a 2-dimensional manifold, sometimes referred to as the affine Grassmannian of lines in $\mathbb{R}^2$; see [13].

Next, two simple and useful functions are introduced. Let $f_{\text{pl}} : \mathbb{R}^2 \times \mathbb{R}^2 \setminus \left\{(p,p) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid p \in \mathbb{R}^2\right\} \to \mathbb{L}$ map two distinct points in $\mathbb{R}^2$ to the line passing through them. For $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, the function f_{pl} admits the expression

$$f_{\text{pl}}\left((x_1,y_1),(x_2,y_2)\right) = [(y_2-y_1, x_1-x_2, y_1x_2-x_1y_2)].$$

If $l_1 \parallel l_2$ denotes that the two lines $l_1, l_2 \in \mathbb{L}$ are parallel, let $f_{\text{lp}} : \mathbb{L}^2 \setminus \left\{(l_1,l_2) \in \mathbb{L}^2 \mid l_1 \parallel l_2\right\} \to \mathbb{R}^2$ map two lines that are not parallel to their unique intersection point. Given two lines $[(a_1,b_1,c_1)]$ and $[(a_2,b_2,c_2)]$ that are not parallel, the function f_{lp} admits the expression

$$f_{\text{lp}}\left([(a_1,b_1,c_1)],[(a_2,b_2,c_2)]\right) \\ = \left(\frac{b_2c_1-b_1c_2}{a_2b_1-a_1b_2}, \frac{a_1c_2-a_2c_1}{a_2b_1-a_1b_2}\right).$$

Note that the functions f_{pl} and f_{lp} are class C^ω, i.e., they are analytic over their domains.

Now, let us turn our attention to the polygonal environment. Let Q be a simple polygon, possibly nonconvex. Here as in [1], a simple polygon is a region enclosed by a single closed polygonal chain that does not intersect itself; thus, the region contains no holes. Let $\mathring{Q}$ and ∂Q denote the interior and the boundary of Q, respectively. Let $\text{Ve}(Q) = (v_1,\dots,v_n)$ be the list of vertices of Q ordered counterclockwise. The *interior angle of a vertex* v of Q is the angle formed inside Q by the two edges of the boundary of Q incident at v. The point $v \in \text{Ve}(Q)$ is a *reflex vertex* if its interior angle is strictly greater than π. Let $\text{Ve}_r(Q)$ be the list of reflex vertices of Q.

A point $q \in Q$ is *visible from* $p \in Q$ if the segment between q and p is contained in Q. The *visibility polygon* $S(p) \subset Q$ from a point $p \in Q$ is the set of points in Q visible from p. It is convenient to think of $p \mapsto S(p)$ as a map from Q to the set of polygons contained in Q. It must be noted that the visibility polygon is not necessarily a simple polygon.

Definition 2.1: Let v be a reflex vertex of Q, and let $w \in \text{Ve}(Q)$ be visible from v. The (v,w)*-generalized inflection segment* $I(v,w)$ is the set

$$I(v,w) = \{q \in S(v) \mid q = \lambda v + (1-\lambda)w, \lambda \geq 1\}.$$

A reflex vertex v of Q is an *anchor of* $p \in Q$ if it is visible from p and if $\{q \in S(v) \mid q = \lambda v + (1-\lambda)p, \lambda > 1\}$ is not empty.

In other words, a reflex vertex is an anchor of p if it occludes a portion of the environment from p. Given a point q and a line l, let $\text{dist}(q,l)$ denote the distance between them. Figure 2 illustrates the various quantities defined above.

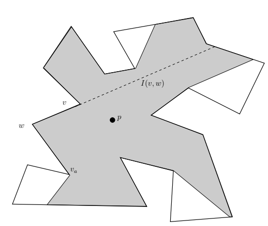

Fig. 2. A reflex vertex v, a generalized inflection segment $I(v,w)$, an anchor v_a of p and the visibility polygon (shaded region) from p.

Theorem 2.2: Let $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ be the set of generalized inflection segments of Q, and let P be a connected component of $Q \setminus \bigcup_{\alpha\in\mathcal{A}} I_\alpha$ For all $p \in P$, the visibility polygon $S(p)$ is simple and has a constant number of vertices, say $\mathrm{Ve}(S(p)) = \{u_1(p), \ldots, u_k(p)\}$. For all $i \in \{1, \ldots, k\}$, the map $P \ni p \mapsto u_i(p)$ is C^ω and either

$$\mathrm{d}u_i(p) = 0$$

if $u_i(p) \in \mathrm{Ve}(Q)$, or

$$\mathrm{d}u_i(p) = \frac{\mathrm{dist}(v_a, l)}{(\mathrm{dist}(p, l) - \mathrm{dist}(v_a, l))^2\sqrt{a^2+b^2}} \begin{bmatrix} -b \\ a \end{bmatrix} \begin{bmatrix} y - y_a \\ x_a - x \end{bmatrix}^T,$$

if $u_i(p) = f_{\mathrm{lp}}(f_{\mathrm{pl}}(v_a, p), l)$, where $v_a = (x_a, y_a)$ is an anchor of p and where $l = [(a, b, c)]$ is a line defined by an edge of Q.

Proof: The first part of the proof is by contradiction. Let $|\mathrm{Ve}(S(p'))| > |\mathrm{Ve}(S(p))|$ for some point $p' \in P$. This means that at least one additional vertex is visible from p' that was occluded by an anchor of p. Two cases may arise. First, when the additional vertex belongs to $\mathrm{Ve}(Q)$, then by our definition, p and p' must lie on opposite sides of a generalized inflection segment. This is a contradiction. Secondly, if the additional vertex does not belong to $\mathrm{Ve}(Q)$, it must be the projection of a reflex vertex (acting as an anchor). Here again two cases may arise: (1) the reflex vertex is visible from p, and (2) it is not. The first case is possible only if the reflex vertex is visible but does not act as an anchor. So, positive lengths of both sides adjoining the reflex vertex must also be visible from p and at least one of the sides is completely not visible from p' since there is a projection. This means that p and p' lie on opposite sides of a generalized inflection segment generated by the reflex vertex and one of its adjacent vertices. This is a contradiction. The second case is possible if the reflex vertex in question is occluded by another reflex vertex. But this means that p and p' lie on opposite sides of the generalized inflection segment from the reflex vertex to the anchor occluding the reflex vertex; again this is a contradiction. If, on the other hand, $|\mathrm{Ve}(S(p'))| < |\mathrm{Ve}(S(p))|$, then the above arguments hold by interchanging p and p'. Hence, p and p' lie on opposite sides of a generalized inflection segment which is a contradiction. This completes the proof that $|\mathrm{Ve}(S(p'))|$ is constant for all $p' \in P$.

Let $p \in P$. Since the visibility polygon $S(p)$ is star-shaped and since any ray emanating from p can intersect the environment at most at two distinct points, then $S(p)$ is simple. (Indeed, if the ray emanating from p intersect the environment at three points, then p must belong to a generalized inflection segment.)

Regarding the second statement, it is clear that if $u_i(p)$ is a vertex of Q then it is independent of p. Instead, if $u_i(p) \notin \mathrm{Ve}(Q)$, then

$$u_i(p) = f_{\mathrm{lp}}(f_{\mathrm{pl}}((x, y), (x_a, y_a)), \ell)$$

where $p = (x, y)$, $v_a = (x_a, y_a)$ is an anchor of p, and ℓ is the line, determined by an edge of Q, that identifies u_i. Now, $p \in P$ implies $p \neq v_a$. It follows that $f_{\mathrm{pl}}(p, v_a)$ is C^ω for all $p \in P$. Also, from the definition of $u_i(p)$, it is clear that $f_{\mathrm{pl}}(p, v_a) \nparallel \ell$. Therefore, for all $p \in P$, $f_{\mathrm{lp}}(f_{\mathrm{pl}}(p, v_a), \ell)$ is C^ω; this implies that $p \mapsto u_i(p)$ is also C^ω. The formula for the derivative can be verified directly. ■

Next, the area of a visibility polygon as a function of the observer location is studied, see Figure 1. Recall that the area of a simple polygon Q with counterclockwise-ordered vertices $\mathrm{Ve}(Q) = ((x_1, y_1), \ldots, (x_n, y_n))$ is given by

$$A(Q) = \sum_{i=1}^{n} x_i(y_{i-1} - y_{i+1}),$$

where $(x_0, y_0) = (x_n, y_n)$ and $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. As in the previous theorem, let $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ be the set of generalized inflection segments of Q and let P be a connected component of $Q \setminus \bigcup_{\alpha\in\mathcal{A}} I_\alpha$. Next, if $p \in P$, the visibility polygon from p has a constant number of vertices, say $k = |\mathrm{Ve}(S(p))|$, is simple, and satisfies $A \circ S(p) = \sum_{i=1}^{k} x_i(y_{i-1} - y_{i+1})$ where $\mathrm{Ve}(S(p)) = (u_1, \ldots, u_k)$ are ordered counterclockwise, $u_i(p) = (x_i, y_i)$, $u_0 = u_k$, and $u_{k+1} = u_1$. Therefore, $P \ni p \mapsto A \circ S(p)$ is also C^ω and

$$\mathrm{d}(A \circ S)(p) = \sum_{i=1}^{k} \frac{\partial A(u_1, \ldots, u_k)}{\partial u_i} \mathrm{d}u_i(p). \tag{1}$$

To illustrate this equality, it is convenient to introduce the *versor* operator defined by $\mathrm{vers}(X) = X/\|X\|$ if $X \in \mathbb{R}^2 \setminus \{0\}$ and by $\mathrm{vers}(0) = 0$. We depict the normalized gradient $\mathrm{vers}(\mathrm{d}(A \circ S))$ of the visible area function in Figure 3.

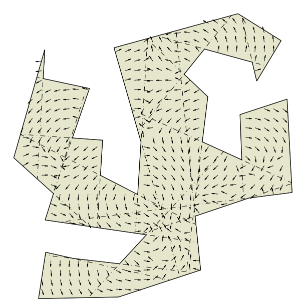

Fig. 3. Normalized gradient of the visible area function over the nonconvex polygon depicted in Figure 1. The dashed lines represent some of the generalized inflection segments.

Theorem 2.3: The map $A \circ S$ restricted to $Q \setminus \mathrm{Ve}_r(Q)$ is locally Lipschitz.

Proof: By Theorem 2.2, it suffices to consider points lying on generalized inflection segments. Let p belongs to multiple, say m, generalized inflection segments $\{I_\alpha\}_{\alpha\in\{1,\ldots,m\}}$. Let $B(p, \epsilon)$ be the open ball of radius ϵ centered at p; let ϵ be small enough such that no generalized inflection segments intersect $B(p, \epsilon)$ other than $\{I_\alpha\}_{\alpha\in\{1,\ldots,m\}}$. For $\alpha \in \{1, \ldots, m\}$, let v_{k_α} be the anchor determining the generalized inflection segment I_α. Without loss of generality, it can be assumed that no anchor is visible from p other than $v_{k_1}, \ldots, v_{k_m}$. For $\alpha \in \{1, \ldots, m\}$, lines $l_\alpha \perp f_{\mathrm{pl}}(p, v_{k_\alpha})$ can be constructed with the property that $l_\alpha \cap Q = \emptyset$ and the vector $v_{k_\alpha} - p$ points toward l_α. Let, h_α

be the line parallel to l_α, tangent to $B(\epsilon, p)$, and intersecting the segment from p to v_{k_α}. Let p' and p'' belong to $B(p,\epsilon) \cap (Q \setminus \mathrm{Ve}_r(Q))$. Next, let $q'_\alpha = f_{\text{lp}}(f_{\text{pl}}(p', v_{k_\alpha}), l_\alpha)$ and $q''_\alpha = f_{\text{lp}}(f_{\text{pl}}(p'', v_{k_\alpha}), l_\alpha)$; see Figure 4. Let v'_α and v''_α

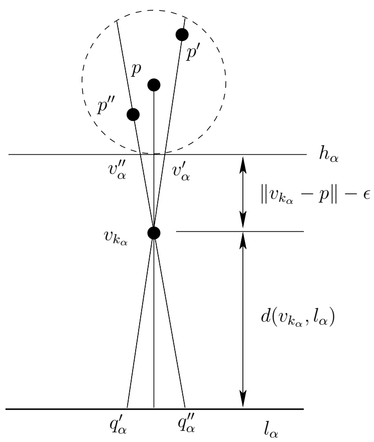

Fig. 4. Definition of the lines l_α, h_α, and the points $q'_\alpha, q''_\alpha, v'_\alpha, v''_\alpha$.

be the intersections between h_α and the lines $f_{\text{pl}}(p', v_{k_\alpha})$ and $f_{\text{pl}}(p'', v_{k_\alpha})$, respectively.

Now, $|A(v_{k_\alpha}, q'_\alpha, q''_\alpha)| = \frac{1}{2}\|q'_\alpha - q''_\alpha\| \operatorname{dist}(v_{k_\alpha}, l_\alpha)$. But from Figure 4, it is easy to see that $\|q'_\alpha - q''_\alpha\| = \frac{\operatorname{dist}(v_{k_\alpha}, l_\alpha)}{\|v_{k_\alpha} - p\| - \epsilon}\|v'_\alpha - v''_\alpha\|$ and that $\|v'_\alpha - v''_\alpha\| < \|p' - p''\|$. For $K_\alpha(p) = \frac{1}{2}\frac{\operatorname{dist}(v_{k_\alpha}, l_\alpha)}{\|v_{k_\alpha} - p\| - \epsilon}\operatorname{dist}(v_{k_\alpha}, l_\alpha)$, the following is true:

$$
\begin{aligned}
|A(S(p')) - A(S(p''))| &\leq \sum_{\alpha=1}^{m} |A(v_{k_\alpha}, q'_\alpha, q''_\alpha)| \\
&\leq \sum_{\alpha=1}^{m} K_\alpha(p)\|p' - p''\|.
\end{aligned}
$$

This fact is illustrated by Figure 5. This completes the proof

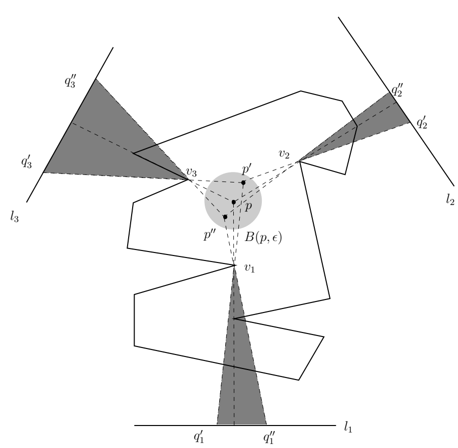

Fig. 5. Upper bounds on the change in area. Here $m = 3$.

that $Q \setminus \mathrm{Ve}_r(Q) \ni p \mapsto A \circ S(p)$ is locally Lipschitz. ■

To obtain the expression for the generalized gradient of $A \circ S$, the polygon Q is partitioned as follows.

Lemma 2.4: Let $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of generalized inflection segments of Q. There exists a unique partition $\{\overline{P_\beta}\}_{\beta \in \mathcal{B}}$ of Q where P_β is a connected component of $Q \setminus \bigcup_{\alpha \in A} I_\alpha$ and $\overline{P_\beta}$ denotes its closure.

Figure 6 illustrates this partition for the given nonconvex polygon. For $\beta \in \mathcal{B}$, define $A_\beta : \overline{P_\beta} \to \mathbb{R}_+$ by

$$
A_\beta(p) = A \circ S(p), \quad \text{for } p \in P_\beta,
$$

and by continuity on the boundary of P_β. It turns out that the maps A_β, $\beta \in \mathcal{B}$, are continuously differentiable[1] on $\overline{P}_\beta$. Equation (1) gives the value of the gradient for $p \in P_\beta$. However, in general, for $p \in \overline{P_{\beta_1}} \bigcap \dots \bigcap \overline{P_{\beta_m}} \setminus \mathrm{Ve}_r(Q)$, based on Theorem 2.3 and Lemma 2.4, it can be written that

$$
\partial(A \circ S)(p) = \operatorname{co}\left\{\mathrm{d}A_{\beta_1}(p), \dots, \mathrm{d}A_{\beta_m}(p)\right\}. \qquad (2)
$$

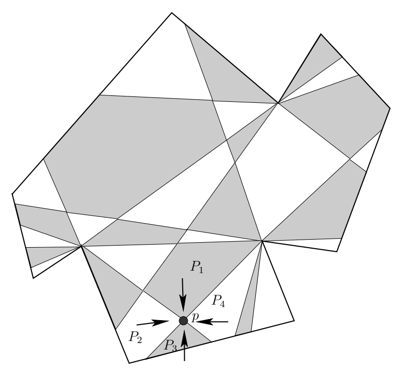

Fig. 6. Partition of Q. The generalized gradient of the area function at p is the convex hull of the gradient of four functions $A_1, \dots, A_4$ at p.

This completes our study of the generalized gradient of the locally Lipschitz function $A \circ S$. Next, it is shown how this function is not regular in many interesting situations.

Lemma 2.5: There exists a nonconvex polygon Q such that the maps $A \circ S$ and $-A \circ S$ restricted to $Q \setminus \mathrm{Ve}_r(Q)$ are not regular.

Proof: We present two examples to justify the above statement. In Figure 7, $\partial(A \circ S)(p') = \operatorname{co}\{\mathrm{d}A_1, \mathrm{d}A_2\}$ where $\|\mathrm{d}A_1\| \gg \|\mathrm{d}A_2\|$. Take a vector η' perpendicular to the generalized inflection segment to which p' belongs (see Figure 7). It is clear that $(A \circ S)'(p; \eta') \approx \|\mathrm{d}A_2\|$. However, $(A \circ S)^0(p'; \eta') = \max\{\langle\zeta, \eta'\rangle | \zeta \in \partial(A \circ S)(p')\} \approx \|\mathrm{d}A_1\| > \|\mathrm{d}A_2\|$. Again, in Figure 7, $\partial(-A \circ S)(p'') = \operatorname{co}\{-\mathrm{d}A_3, -\mathrm{d}A_4\}$, where $\|-\mathrm{d}A_4\| \gg \|-\mathrm{d}A_3\|$. Take a vector η'' perpendicular to the generalized inflection segment to which p'' belongs (see Figure 7). It is clear that $-(A \circ S)'(p''; \eta'') \approx -\|\mathrm{d}A_4\|$. However, $(A \circ S)^0(p''; \eta'') = \max\{\langle\zeta, \eta''\rangle | \zeta \in \partial(A \circ S)(p'')\} \approx -\|\mathrm{d}A_3\| > -\|\mathrm{d}A_4\|$. ■

[1] A function is continuously differentiable on a closed set if (1) it is continuously differentiable on the interior, and (2) the limit of the derivative at a point in the boundary does not depend on the direction from which the point is approached.

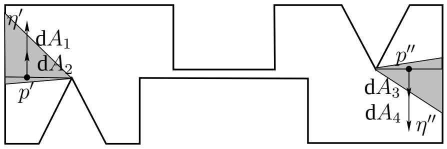

Fig. 7. Example polygon for which $A \circ S$ and $-A \circ S$ restricted to $Q \setminus \mathrm{Ve}_r(Q)$ are not regular. Note here that $\mathrm{d}A_1$ and $\mathrm{d}A_2$ are not perfectly aligned with η'. Also, $\mathrm{d}A_3$ and $\mathrm{d}A_4$ are not perfectly aligned with η''.

III. An invariance principle in nonsmooth stability analysis

This section presents results on stability analysis for discontinuous vector fields via nonsmooth Lyapunov functions. The results extend the work in [12] and will be useful in the next control design section. We refer the reader to [10] for some useful nonsmooth analysis concepts.

In what follows we shall study differential equations of the form

$$\dot{x}(t) = X(x(t)),$$

where X is a discontinuous vector field on $\mathbb{R}^N$.

Lemma 3.1: Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be measurable and essentially locally bounded and let $f : \mathbb{R}^N \to \mathbb{R}$ be locally Lipschitz. Let $\gamma : [t_0, t_1] \to \mathbb{R}^N$ be a Filippov solution of X such that $f(\gamma(t))$ is regular for almost all $t \in [t_0, t_1]$. Then

(i) $\frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)))$ exists for almost all $t \in [t_0, t_1]$, and

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t))) \in \widetilde{\mathcal{L}}_X f(\gamma(t))$ for almost all $t \in [t_0, t_1]$.

Proof: The result is an immediate consequence of Lemma 1 in [12]. ■

The following result is a generalization of the classic LaSalle Invariance Principle for smooth vector fields and smooth Lyapunov functions to the setting of discontinuous vector fields and nonsmooth Lyapunov functions.

Theorem 3.2 (LaSalle Invariance Principle): Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be measurable and essentially locally bounded and let $S \subset \mathbb{R}^N$ be compact and strongly invariant for X. Let $C \subset S$ consist of a finite number of points and let $f : S \to \mathbb{R}$ be locally Lipschitz on $S \setminus C$ and bounded from below on S. Assume the following properties hold:

(A1) if $x \in S \setminus C$, then either $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ or $\widetilde{\mathcal{L}}_X f(x) = \emptyset$,

(A2) if $x \in C$ and if γ is a Filippov solution of X with $\gamma(0) = x$, then $\lim_{t \to 0^-} f(\gamma(t)) \geq \lim_{t \to 0^+} f(\gamma(t))$, and

(A3) if $\gamma : \overline{\mathbb{R}}_+ \to S$ is a Filippov solution of X, then $f \circ \gamma$ is regular almost everywhere.

Define $Z_{X,f} = \left\{ x \in S \setminus C \mid 0 \in \widetilde{\mathcal{L}}_X f(x) \right\}$ and let M be the largest weakly invariant set contained in $(\overline{Z}_{X,f} \cup C)$. Then the following statements hold:

(i) if $\gamma : \overline{\mathbb{R}}_+ \to S$ is a Filippov solution of X, then $f \circ \gamma$ is monotonically nonincreasing;

(ii) each Filippov solution of X with initial condition in S approaches M as $t \to +\infty$;

(iii) if M consists of a finite number of points, then each Filippov solution of X with initial condition in S converges to a point of M as $t \to +\infty$.

Proof: Fact (i) is a consequence of Assumptions (A1), (A2) and (A3), and of Lemma 3.1.

In what follows we shall require the following notion. Given a curve $\gamma : \mathbb{R}_+ \to \mathbb{R}^N$, the positive limit set of γ, denoted by $\Omega(\gamma)$, is the set of $y \in \mathbb{R}^N$ for which there exists a sequence $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}$ such that $t_k < t_{k+1}$, for $k \in \mathbb{N}$, $\lim_{k \to +\infty} t_k = +\infty$, and $\lim_{k \to +\infty} \gamma(t_k) = y$. For $x \in S$, let γ_1 be a Filippov solution of X with $\gamma_1(0) = x$ and let $\Omega(\gamma_1)$ be the limit set of γ_1. Under this setting, $\Omega(\gamma_1)$ is nonempty, bounded, connected and weakly invariant, see [10]. Furthermore, $\Omega(\gamma_1) \subset S$ because S is strongly invariant.

To prove fact (ii), it suffices to show that $\Omega(\gamma_1) \subset \overline{Z}_{X,f} \cup C$. Trivially, $\Omega(\gamma_1) \cap C \subset C$. Let $y \in \Omega(\gamma_1) \setminus C$ so that f is locally Lipschitz at y. There exists a sequence $\{t_k\}_{k \in \mathbb{N}}$ such that $\lim_{k \to +\infty} \gamma_1(t_k) = y$. Because $f \circ \gamma_1$ is monotonically nonincreasing and f is bounded from below, $\lim_{t \to +\infty} f(\gamma_1(t))$ exists and is equal to, say, $a \in \mathbb{R}$. Now, by continuity of f, $a = \lim_{k \to +\infty} f \circ \gamma_1(t_k) = f(y)$. This proves that $f(y) = a$ for all $y \in \Omega(\gamma_1) \setminus C$. At this point we distinguish two cases. First, assume that y is an isolated point in $\Omega(\gamma_1)$. Then clearly, there exists a Filippov solution of X, say γ_2, such that $\gamma_2(t) = y$ for all $t \geq 0$. Hence $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_2(t)) = 0$, and, by Lemma 3.1, $0 \in \widetilde{\mathcal{L}}_X f(\gamma_2(t))$ or in other words $y \in Z_{X,f}$. Second, assume that y is not isolated in $\Omega(\gamma_1)$, and let γ_2 be a Filippov solution of X with $\gamma_2(0) = y$. Since f is continuous at y and $\Omega(\gamma_1)$ contains a finite number of points of discontinuity of f, there exists $\delta > 0$ such that $f(y') = a$ for all $y' \in B(y, \delta) \cap \Omega(\gamma_1)$. Therefore, there exists $t' > 0$ such that $f(\gamma_2(t)) = a$ for all $t \in [0, t']$. Hence, we have $\frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_2(t)) = 0$ for all $t \in [0, t']$. It follows from Lemma 3.1 that for all $t \in [0, t']$, we have $0 \in \widetilde{\mathcal{L}}_X f(\gamma_2(t))$ or in other words $\gamma_2(t) \in Z_{X,f}$. By continuity of γ_2 at $t = 0$, we have that $\gamma_2(0) = y \in \overline{Z}_{X,f}$. Since $\Omega(\gamma_1)$ is weakly invariant, we have $\Omega(\gamma_1) \subset M$ and hence γ_2 approaches M.

We now prove fact (iii). If M consists of a finite number of points, and since $\Omega(\gamma_1) \subset M$ is connected, $\Omega(\gamma_1)$ is a point. Hence, by the argument in the preceding paragraph, each Filippov solution of X approaches a point of M. In other words, it converges to a point of M. ■

IV. Maximizing the area visible from a mobile observer

In this section we build on the analysis results obtained thus far to design an algorithm that maximizes the area visible to a mobile observer. We aim to reach local maxima of the visible area $A \circ S$ by designing some appropriate form of a gradient flow for the discontinuous function $A \circ S$. We now present an *introductory and incomplete* version of the algorithm: the objective is to steer the mobile observer along a path for which the visible area is guaranteed to be nondecreasing.

Name: Increase visible area for Q
Goal: Maximize the area visible to a mobile observer
Assumption: Generalized inflection segments of Q do not intersect.
Initial position does not belong to a generalized inflection segment.

Let $p(t)$ denote the observer position at time t inside the nonconvex polygon Q. The observer performs the following tasks at each time instant:

compute visibility polygon $S(p(t)) \subset Q$,
if $p(t)$ does not belong to any generalized inflection segment or to the boundary of Q **then**
  move along the versor of the gradient of $A \circ S$
else if $p(t)$ belongs to a generalized inflection segment but not to the boundary of Q **then**
  depending on the generalized gradient of $A \circ S$, either slide along the segment or leave the segment in an appropriate direction
else if $p(t)$ belongs to the boundary of Q but not to a reflex vertex, **then**
  depending on the projection of the generalized gradient along the boundary, either slide along the boundary or move in an appropriate direction toward the interior of Q
else
  either follow a direction of ascent of $A \circ S$ or stop
end if

The remainder of this section is dedicated to formalizing these loose ideas.

A. A modified gradient vector field

Before describing the algorithm to maximize the area visible to the mobile observer, we introduce the following useful notions. Given a simple polygon Q with $\mathrm{Ve}(Q) = (v_1, \dots, v_n)$ and $\epsilon > 0$, define the following quantities:

(i) let the *ϵ-expansion of* Q be $Q^\epsilon = \{p \mid ||p - q|| \le \epsilon \text{ for some } q \in Q\}$,
(ii) for $i \in \{1, \dots, n\}$, let P_i^ϵ be the open set delimited by the edge $\overline{v_i v_{i+1}}$, the bisectors of the external angles at v_i and v_{i+1} and the boundary of Q^ϵ,
(iii) for ϵ small enough and for any point p in Q^ϵ, let $\mathrm{prj}_Q(p)$ be uniquely equal to $\arg\min\{||p' - p|| \mid p' \in \partial Q\}$, and
(iv) let the *outward normal* $n(\mathrm{prj}_Q(p))$ be the unit vector directed from $\mathrm{prj}_Q(p)$ to p.

We illustrate these notions in Figure 8. Note that $\mathrm{prj}_Q(p)$ can never be a reflex vertex. We can now define a vector field on Q^ϵ as follows:

$$X_Q(p) = \begin{cases} \mathrm{vers}(\mathrm{d}(A \circ S)(p)), & \text{if } p \in \mathring{Q} \setminus \{I_\alpha\}_{\alpha \in \mathcal{A}}, \\ -n(\mathrm{prj}_Q(p)), & \text{if } p \in P_i^\epsilon, \\ 0, & \text{otherwise}. \end{cases}$$

(Recall that the versor operator is defined by $\mathrm{vers}(Y) = Y/\|Y\|$ if $Y \in \mathbb{R}^2 \setminus \{0\}$ and by $\mathrm{vers}(0) = 0$.) Note that

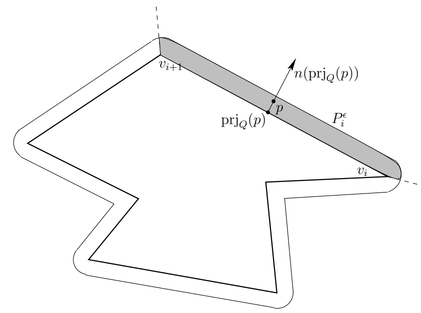

Fig. 8. The ϵ-expansion Q^ϵ of the simple polygon Q, an open set P_i^ϵ and the corresponding outward normal $n(\mathrm{prj}_Q(p))$.

X_Q is well-defined because at $p \in \mathring{Q} \setminus \{I_\alpha\}_{\alpha \in \mathcal{A}}$ the function $A \circ S$ is analytic. Clearly, X_Q is not continuous on Q^ϵ. However, the set of points where it is discontinuous is of measure zero. Almost everywhere in the interior of Q, the vector field X_Q is equal to the normalized gradient of $A \circ S$ as depicted in Figure 3. We now present the differential equation describing the motion of the observer:

$$\dot{p}(t) = X_Q(p(t)). \tag{3}$$

A Filippov solution of (3) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion

$$\dot{p}(t) \in K[X_Q](p(t)), \tag{4}$$

where the operator $K[X_Q]$ is the usual Filippov differential inclusion associated with X_Q, see [10]. Since X_Q is measurable and bounded, the existence of a Filippov solution is guaranteed. We study uniqueness and completeness as follows.

Lemma 4.1: The following statements hold true:

(i) there exists a simple polygon Q for which the corresponding vector field X_Q admits multiple Filippov solutions;
(ii) any simple polygon Q is a strongly invariant set for the corresponding vector field X_Q, so that any Filippov solution is defined over $\overline{\mathbb{R}}_+$.

Proof: We present an example to justify the statement (i). In Figure 9, at the point p_0 on the generalized inflection segment, both directions η_1 and η_2 belong to $\partial(A \circ S)(p_0)$. Three distinct Filippov solutions of equation (3) exist. Two of the solutions start from p_0 along the two directions η_1 and η_2 while the third solution is $p(t) = p_0$ for all $t \ge 0$. Statement (ii) is a consequence of the definition of X_Q on

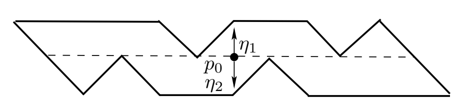

Fig. 9. Three Filippov solutions exist starting from the point p_0.

P_i^ϵ for $i \in \{1, \dots, n\}$. ∎

Remark 4.2: An important observation in this setting is that at all points p where $A \circ S$ is locally Lipschitz, we have

$K[\mathrm{d}(A \circ S)](p) = \partial(A \circ S)(p)$. In such a case it is also true that for all $\eta \in \partial(A \circ S)(p)$, there exists at least one $\delta > 0$ such that $\delta\eta \in K[X_Q](p)$ and vice versa.

We now claim that any solution of the differential inclusion (4) has the property that the visible area increases monotonically. To prove these desirable properties, we first present the following results in nonsmooth analysis.

B. Properties of solutions

To prove the convergence properties of the solution of (4) using the results presented in Section III, we must first define a suitable Lyapunov function. Intuitively since our objective is to maximize the visible area, our Lyapunov function should be closely related to it. For $\epsilon > 0$, we now define the *extended area function* A_Q^ϵ at all points $p \in Q \bigcup \{\cup_i P_i^\epsilon\}$. The extended function coincides with the original function on the interior and on the boundary of Q and is defined appropriately outside:

$$A_Q^\epsilon(p) = \begin{cases} A \circ S(p), & p \in Q, \\ A \circ S(\mathrm{prj}_Q(p)) - ||p - \mathrm{prj}_Q(p)||, & p \in \cup_i P_i^\epsilon. \end{cases}$$

For all $p \in \partial Q \setminus \mathrm{Ve}\, Q$, A_Q^ϵ satisfies (see Figure 10):

$${A_Q^\epsilon}'(p; n(\mathrm{prj}_Q(p))) = -1.$$

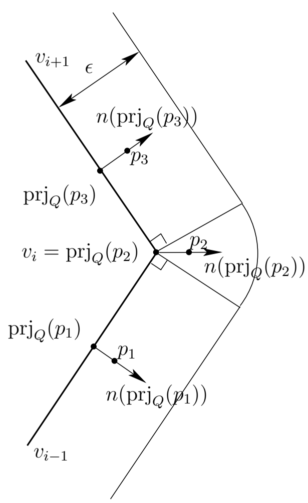

Fig. 10. Extending the function $A \circ S$ to A_Q^ϵ. Note the direction of $n(\mathrm{prj}_Q(p_i))$ at all points p_i.

Remark 4.3: The extended area function A_Q^ϵ is locally Lipschitz on $(Q \setminus \mathrm{Ve}_r(Q)) \bigcup \{\cup_i P_i^\epsilon\}$ and analytic almost everywhere on $Q \bigcup \{\cup_i P_i^\epsilon\}$.

The following theorem is important to prove that such a function leads to a monotonically nondecreasing value of the area of the visibility polygon.

Theorem 4.4: Let $G(Q)$ be the set of points where both maps $p \mapsto -A_Q^\epsilon(p)$ and $p \mapsto A_Q^\epsilon(p)$ are not regular. Then any Filippov solution $\gamma : \overline{\mathbb{R}}_+ \to Q$ of X_Q has the property that $\gamma(t) \notin G(Q)$ for almost all $t \in \overline{\mathbb{R}}_+$ unless γ reaches a critical point of $K[X_Q]$.

In the following theorem, the functions A_Q^ϵ and $-A_Q^\epsilon$ are used as candidate Lyapunov functions to show the convergence properties of Filippov solutions of X_Q.

Theorem 4.5: Any Filippov solution $\gamma : \overline{\mathbb{R}}_+ \to Q$ of X_Q has the following properties:

(i) $t \mapsto A \circ S(\gamma(t))$ is continuous and monotonically nondecreasing,

(ii) γ approaches the set of critical points of $K[X_Q]$.

Theorem 4.5 implies that the single observer converges to a critical point of $A \circ S$ or to a reflex vertex of Q. However, as shown in Figure 11, the presence of noise or computational inaccuracies actually work to drive the observer away from a reflex vertex that is not a local maximum. This will be true for other critical points too that are not local maxima.

V. Simulation results

Figures 11 and 14 illustrate the performance of the gradient algorithm in equation (4). The algorithm is implemented in `Matlab`®. The vertices of the environment that are visible to a point observer is first obtained by means of a simple $O(n^2)$ algorithm, where n is the number of vertices of the polygonal environment. This is then used to compute the visibility polygon of the observer. The calculation of the generalized gradient of the visible area function is then a natural outcome of (1) and (2). Computational inaccuracies in the implementation of the algorithm to calculate the visibility polygon have been noticed in some configurations; see Figure 12 and the plot of the variation of visible area with time in Figure 11. See Figure 13 for the phase portrait of the vector field X_Q for the polygon in Figure 12. Our experiments suggest that the observer reaches a local maximum of the visible area in finite time, however this can be shown not to be true in general.

VI. Conclusions

This paper introduces a gradient-based algorithm to optimally locate a mobile observer in a nonconvex environment. We presented nonsmooth analysis and control design results. The simulation results illustrate that, in the presence of noise, the observer reaches a local maximum of the visible area. In an "highly nonconvex" environment, a single observer may not be able to see a large fraction of the environment. In such a case, a team of observers can be deployed to achieve the same task. We therefore plan to investigate this same visibility objective for teams of observers. Other directions of future research include practical robotic implementation issues as well as other combined mobility and visibility problems.

Acknowledgments

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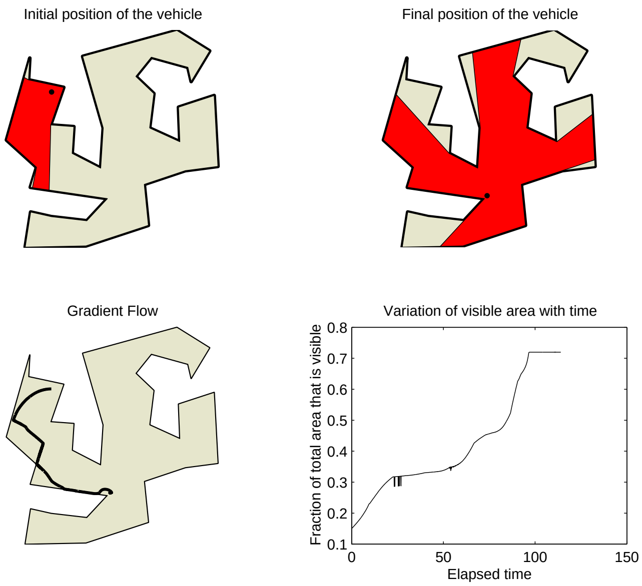

Fig. 11. Simulation results of the gradient algorithm for the nonconvex polygon depicted in Figure 1. The observer arrives, in finite time, at a local maximum. Note here that the observer visits a reflex vertex at some point in its trajectory but comes out of it due to computational inaccuracies because it is not a local maximum.

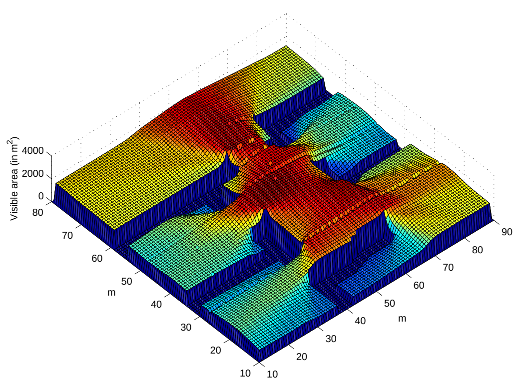

Fig. 12. Example of visible area function over a polygon in the shape of a floor plan of a building.

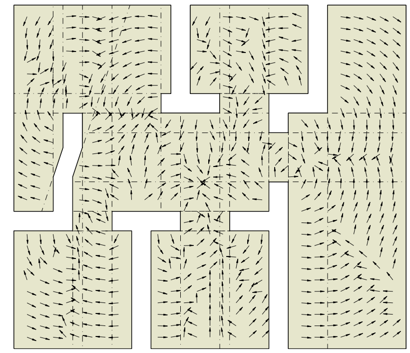

Fig. 13. Example of vector field over a polygon in the shape of a floor plan of a building.

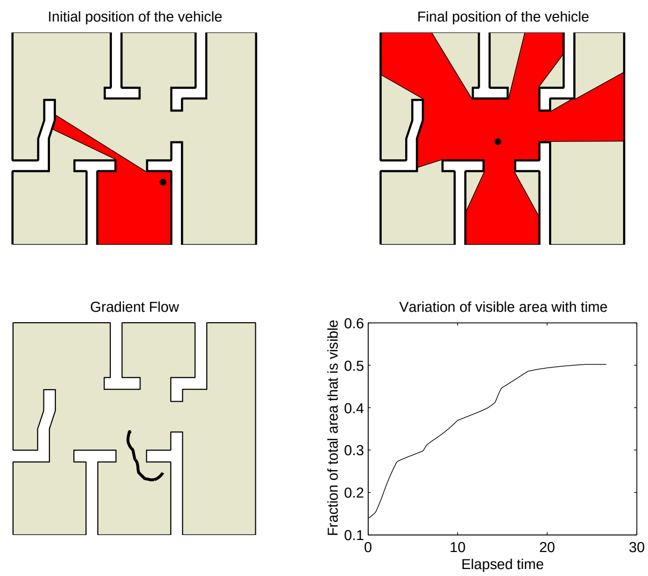

Fig. 14. Simulation results of the gradient algorithm. The observer arrives, in finite time, at a local maximum.

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