GENERAL SYMMETRIES IN OPTIMAL CONTROL

Manuel de León* Jorge Cortés† David Martín de Diego‡
Sonia Martínez§

Laboratory of Dynamical Systems, Mechanics and Control
Instituto de Matemáticas y Física Fundamental
Consejo Superior de Investigaciones Científicas
Serrano 123, 28006 Madrid, SPAIN

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Abstract

Making use of the geometric setting proposed in [7] to formulate optimal control problems, we address here the treatment of general symmetries. This viewpoint allows us to reduce the number of equations associated with optimal control problems with symmetry and compare the solutions of the original system with the solutions of the reduced one. The reconstruction of the optimal controls starting from the reduced problem is also explored.

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1 Introduction

As is well known, symmetries are a valuable tool for integrating the equations of motion in Classical Mechanics. Indeed, Noether's theorem asserts that each symmetry gives rise to

*mdeleon@imaff.cfmac.csic.es
†Current address: Coordinated Science Laboratory, University of Illinois at Urbana-Champaign, 1308 W Main St, Urbana IL 61801, USA, jcortes@uiuc.edu
‡d.martin@imaff.cfmac.csic.es
§Current address: Departamento de Matemática Aplicada IV, Universidad Politécnica de Cataluña, Av. V. Balaguer s/n, Vilanova i la Geltrú, 08800, Spain, soniam@mat.upc.es

a first integral of the equations of motion, which allows us to reduce the total number of differential equations to be integrated. In case the symmetries form a Lie group, the reduction procedure, known as symplectic reduction, is performed via the momentum mapping [21, 22, 25]. Usually, the symplectic reduction procedure is presented in a Hamiltonian context, tough it can also be developed in a Lagrangian setting. Indeed, several extensions of it have been proposed to deal with more general situations, such as systems with singular Lagrangians, or presymplectic systems [10, 12, 18].

As in Classical Mechanics, the existence of symmetries may simplify the equations associated with optimal control problems (see [15, 27, 32, 33]). The usual formulation of optimal control problems is Hamiltonian and is based on Pontryagin's Maximum Principle. If $\mathbb{L} = \mathbb{L}(x^i, u^a)$, $1 \leq i \leq m$, $1 \leq a \leq k$, is the cost function of the problem and $\dot{x}^i = \Gamma(x, u)$ is the control equation, one introduces a set of co-states p_i, and defines a Hamiltonian function as $H(x, p, u) = p_i \Gamma^i(x, u) - \mathbb{L}(x, u)$. Then the Maximum Principle leads to the following system of differential equations,

$$\dot{x}^i = \frac{\partial H}{\partial p_i}(x, p, u)\,, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}(x, p, u)\,, \quad y_a = \frac{\partial H}{\partial u_a}(x, p, u)\,, \tag{1}$$

where y_a are considered as the outputs of the system. A necessary condition for an admissible control u^* to be optimal is that the outputs corresponding to u^* are constantly zero, that is, $y_a \equiv 0$, $1 \leq a \leq k$.

A possible set of symmetries for the above optimal control problem could be given by a Lie group acting on the space of states and controls, while leaving invariant the cost function and the control equation. Applying then a reduction procedure to the (regular) optimal control problem leads to a new optimal control problem, with the same controls and a lower dimensional state space. These are essentially the results in [27, 32]; more general symmetries are considered for instance in [33] (see also [5, 6, 9, 11, 28, 31]). A recent approach to the study of symmetries in optimal control problems is due to Blankenstein and van der Schaft in the framework of implicit Hamiltonian systems and Dirac structures [3, 4]. They show that the application of the Maximum Principle gives rise to an implicit Hamiltonian system, and that symmetries of the optimal control problem naturally lift to symmetries of the corresponding implicit Hamiltonian system. The reduction is investigated and constrained optimal control problems are also considered.

In this paper we present a different approach to study the symmetry properties of optimal control problems. We start by describing an optimal control problem as a *vakonomic system*. In a few words, a vakonomic system is given by a Lagrangian function $L = L(q^A, \dot{q}^A)$, $1 \leq A \leq n$, subject to some non-holonomic constraints $\Phi^\alpha(q, \dot{q}) = 0$, $1 \leq \alpha \leq m$. The vakonomic problem consists of finding the curves $q(t)$ that extremize the functional $\int_0^T L(q, \dot{q})\, dt$ among all the curves satisfying the constraints. Using techniques from Constrained Variational Calculus, one can obtain the *normal* solutions of the vakonomic problem as the extremals of the functional defined by the extended (singular)

Lagrangian $\mathcal{L}(q^A, \lambda_\alpha, \dot{q}^A, \dot{\lambda}_\alpha) = L(q^A, \dot{q}^A) + \lambda_\alpha \Phi^\alpha(q^A, \dot{q}^A)$, where λ_α are arbitrary Lagrange multipliers [1] (see [23, 24] for a recent exploration of this setting).

On the other hand, inspired by the so-called Skinner and Rusk formalism [30] for singular Lagrangian systems, we have developed in [7] an intrinsic geometric setting for vakonomic dynamics. This description is formalized by means of a presymplectic system in the fibered product $W_0 = T^*Q \times_Q M$, where $M \subset TQ$ represents the constraint submanifold, locally described by the vanishing of the functions Φ^α. The presymplectic system is established by taking the pullback of the canonical symplectic form on T^*Q as the presymplectic 2-form, and $H_0 = \langle \, , \, \rangle - \tilde{L}$, as Hamiltonian function, where $\langle \, , \, \rangle$ is the canonical pairing between tangent vectors and covectors, and $\tilde{L}$ is the restriction of the Lagrangian function to M.

One of the main advantages of this framework is that it provides an intrinsic and coordinate-free setting for the description of optimal control problems, allowing us to develop a theory of general symmetries. In this setting, previous results by Grizzle, Marcus and van der Schaft [15, 32, 33] can be naturally recovered. Other problems can also be undertaken, as for instance the issue of the consistency of the optimal equations and the treatment of singular optimal control problems. This is done here by means of a constraint algorithm yielding a final submanifold $W_f \subset W_0$ where a well-defined dynamics exist. Equations (1) are readily obtained, and the outputs appear as the functions defining the secondary constraint submanifold. For regular control problems, $W_f = W_3$ is symplectic, and therefore there is a unique dynamics defined on it. When considering symmetries, we identify the notions of infinitesimal, Noether and Cartan symmetries in this context, and provide the corresponding Noether's theorems. We also hint on how non-autonomous optimal control problems can be casted within this framework, and consequently how the results on symmetries can be incorporated in their analysis. Finally, we describe the reduction and reconstruction processes, not necessarily restricting our attention to the zero-momentum case as is the case for instance in [3, 4]. One of the further possibilities of this framework that will be explored in the future is the extension to optimal control problems whose evolution equations are given by partial differential equations.

The paper is organized as follows. In Section 2, we recall the geometric formulation of vakonomic dynamics given in [7], within the framework of presymplectic geometry. A classification of infinitesimal symmetries in this context is analyzed then in Section 3. Section 4 is devoted to discuss optimal control problems in the above framework, and a constraint algorithm is developed providing a well-defined dynamics at the final step. The classification of infinitesimal symmetries is the subject of Section 5. Finally, in Section 6 we give a reduction procedure for optimal control systems with a Lie group of symmetries. The reconstruction problem is also considered (see Grizzle [14] for a related approach). Along the paper, several examples are worked out in order to illustrate the results.

2 Geometric formulation of constrained variational problems

In this section, we present the presymplectic description of variational problems given by a Lagrangian function subject to nonholonomic constraints (see [7]). This description is strongly inspired by Skinner and Rusk's formulation of singular Lagrangian systems [30].

Consider a vakonomic problem given by a Lagrangian function $L : TQ \longrightarrow \mathbb{R}$ and a constraint submanifold M of TQ, where Q is an n-dimensional manifold. Take the Whitney sum $T^*Q \oplus TQ$ with its canonical projections

$$pr_1 : T^*Q \oplus TQ \longrightarrow T^*Q \ , \quad pr_2 : T^*Q \oplus TQ \longrightarrow TQ \ .$$

We assume that $(\tau_Q)_{|M} : M \longrightarrow Q$ is a fiber bundle, but not necessarily a vector subbundle of $\tau_Q : TQ \longrightarrow Q$. In other words, the constraints may be linear or not.

Now, consider the fiber product $W_0 = T^*Q \times_Q M$ with canonical projections $\pi_1 = (pr_1)_{|W_0} : W_0 \longrightarrow T^*Q$ and $\pi_2 = (pr_2)_{|W_0} : W_0 \longrightarrow M$. Notice that $W_0 = (pr_2)^{-1}(M)$. We define a presymplectic form on W_0 by $\omega = \pi_1^*\omega_Q$, where ω_Q is the canonical symplectic form on T^*Q, and introduce a Hamiltonian function $H_0 = \langle \pi_1, \pi_2 \rangle - \tilde{L}$, where $\langle \pi_1, \pi_2 \rangle$ is the restriction of the natural pairing $\langle pr_1, pr_2 \rangle$ on $T^*Q \oplus TQ$, and $\tilde{L} : M \longrightarrow \mathbb{R}$ is the restriction of L to M, i.e. $\tilde{L} = L_{|M}$.

Take local coordinates (q^A) in Q such that (q^A, p_A) and $(q^A, \dot{q}^A)$ are bundle coordinates in T^*Q and TQ, respectively. The constraint submanifold M is locally defined by some independent constraint functions $\Phi^\alpha = \Phi^\alpha(q^A, \dot{q}^A)$, i.e. M is characterized by the equations $\Phi^\alpha = 0$, where $1 \leq \alpha \leq m$, and $\dim M = 2n - m$. It is usually assumed the following *admissibility condition* [7]. The matrix

$$\frac{\partial(\Phi^1, \ldots, \Phi^m)}{\partial(\dot{q}^1, \ldots, \dot{q}^n)}$$

has rank m for any choice of bundle coordinates in TQ. Then, by the implicit function theorem, we can locally express the constraints (reordering coordinates if necessary) as

$$\dot{q}^\alpha = \Psi^\alpha(q^A, \dot{q}^a),$$

where $1 \leq \alpha \leq m$, $1 \leq a \leq n - m$, and $1 \leq A \leq n$. In this way, we can take $(q^A, \dot{q}^a)$ as coordinates for M.

In [7] it was shown that the vakonomic problem given by L and M is equivalent to the presymplectic system (W_0, ω, H_0) in the sense that both give rise to the same solutions. Thus, in order to obtain the vakonomic solutions, one is to apply a presymplectic constraint algorithm, which is just the geometrization and extension of Dirac's constraint algorithm for degenerate Lagrangian systems (see [13] for details). We do this in the following.

In first place, consider the points of $T^*Q \times_Q M$ where the equation

$$i_X \omega = dH_0, \tag{2}$$

has a solution. This determines a submanifold W_1 as,

$$W_1 = \{x \in T^*Q \times_Q M \mid dH_0(x)(V) = 0,\ \forall V \in \ker \omega(x)\} .$$

However, the solutions on W_1 may not be tangent to W_1. In such a case, we have to restrict W_1 to the submanifold W_2 where the solutions are tangent to W_1. Proceeding further, we obtain a sequence of subsets

$$\cdots \hookrightarrow W_k \hookrightarrow \cdots \hookrightarrow W_2 \hookrightarrow W_1 \hookrightarrow W_0 = T^*Q \times_Q M \ ,$$

which will be assumed to be submanifolds of W_0. Algebraically, these constraint submanifolds may be described as

$$W_i = \{x \in T^*Q \times_Q M \mid dH_0(x)(v) = 0\ ,\ \forall v \in T_x W_{i-1}^{\perp}\ \}\ , \quad i \geq 1\ ,$$

where $T_x W_{i-1}^{\perp} = \{v \in T_x(T^*Q \times_Q M) \mid \omega(x)(u,v) = 0\ ,\ \forall u \in T_x W_{i-1}\ \}$. If this constraint algorithm stabilizes, i.e., if there exists a positive integer $k \in \mathbb{N}$ such that $W_{k+1} = W_k$ and $\dim W_k \neq 0$, then we will have obtained a final constraint submanifold $W_f = W_k$ on which a vector field X exists such that

$$(i_X \omega = dH_0)_{|W_f} \ .$$

In local coordinates the algorithm can be described as follows. Notice that we can introduce coordinates $(q^a, q^\alpha, p_a, p_\alpha, \dot{q}^a)$ on W_0 so that

$$H_0 = \dot{q}^a p_a + \Psi^\alpha p_\alpha - \tilde{L}(q^a, q^\alpha, \dot{q}^a)\ , \quad \omega = dq^a \wedge dp_a + dq^\alpha \wedge dp_\alpha \ .$$

A direct computation shows that a solution X of equation $i_X\omega = dH_0$, is of the form

$$X = \dot{q}^a \frac{\partial}{\partial q^a} + \Psi^\alpha \frac{\partial}{\partial q^\alpha} + (\frac{\partial \tilde{L}}{\partial q^a} - p_\alpha \frac{\partial \Psi^\alpha}{\partial q^a})\frac{\partial}{\partial p_a} + (\frac{\partial \tilde{L}}{\partial q^\alpha} - p_\beta \frac{\partial \Psi^\beta}{\partial q^\alpha})\frac{\partial}{\partial p_\alpha} + Z^a \frac{\partial}{\partial \dot{q}^a} \tag{3}$$

for some arbitrary functions $Z^a = Z^a(q^A, p_A, \dot{q}^b)$. In addition, W_1 is locally defined by the constraint functions,

$$p_a = \frac{\partial \tilde{L}}{\partial \dot{q}^a} - p_\alpha \frac{\partial \Psi^\alpha}{\partial \dot{q}^a}, \quad 1 \leq a \leq n-m.$$

In this way, we can introduce local coordinates $(q^a, q^\alpha, p_\alpha, \dot{q}^a)$ on W_1.

As we have said before, the vector field given in (3) is a solution of the vakonomic problem which may not be

tangent to W_1. Imposing this tangency condition leads to new constraints defining W_2. The algorithm goes on until the stabilization is reached. On W_f we will have an explicit and well-defined solution of the vakonomic problem. Notice that this solution is not unique due to the gauge freedom arising from the presymplectic character of ω. Indeed, given a particular solution X on W_f, we obtain the whole family of solutions as $X + (\ker \omega \cap TW_f)$.

A special case representative of many examples is that of $W_f = W_1$. A sufficient condition for this to hold is that

$$\det\left(\frac{\partial^2 \tilde{L}}{\partial \dot{q}^a \partial \dot{q}^b} - p_\alpha \frac{\partial^2 \Psi^\alpha}{\partial \dot{q}^a \partial \dot{q}^b}\right) \neq 0 \,. \tag{4}$$

This condition allows to determine Z^a in (3) such that $X_{|W_2} \in TW_1$. Denote by ω_1 the restriction of the presymplectic 2-form ω to W_1. We have $\omega_1 = dq^a \wedge dp_a + dq^\alpha \wedge dp_\alpha$.

Proposition 2.1 ([7]). *(W_1, ω_1) is a symplectic manifold if and only if, for any choice of coordinates $(q^A, p_A, \dot{q}^a)$ on $T^*Q \times_Q M$, equation (4) is satisfied on W_1.*

In case the constraints are linear on the velocities, we can write $\dot{q}^\alpha = \Psi^\alpha_a(q)\dot{q}^a$. Then, from Proposition 2.1, we can decide about the symplecticity of ω_1 by checking the condition

$$\det\left(\frac{\partial^2 \tilde{L}}{\partial \dot{q}^a \partial \dot{q}^b}\right) \neq 0 \,.$$

This condition can be loosely stated as asking for the restriction of the cost function to the constraint submanifold to be non-degenerate.

3 Symmetries

Now, we make use of the intrinsic formulation described in the previous section to study the symmetry properties of vakonomic systems. To do so, we will build on the results of [18], where symmetries of presymplectic systems were considered (see also [2, 8, 12]). The following notation will be useful [19]. Let $\phi : Q \longrightarrow Q$ be a diffeomorphism on Q. Consider the mapping $T^*\phi^{-1} \times T\phi$, where $T\phi : TQ \longrightarrow TQ$ and $T^*\phi^{-1} : T^*Q \longrightarrow TQ$ are the induced diffeomorphisms by tangent and cotangent prolongations, respectively. Then we can define the following.

Definition 3.1 ([1, 24]). *A* vakonomic symmetry *for the vakonomic problem given by L and M is a diffeomorphism $\phi : Q \longrightarrow Q$ such that*

(i) the induced diffeomorphism $T\phi$ leaves M invariant, i.e. $T\phi(M) \subset M$;

(ii) $L_{|M} \circ T\phi_{|M} = L_{|M}$.

From this definition, it follows that $T^*\phi^{-1} \times T\phi$ is a symmetry of the presymplectic system (W_0, ω, H_0). That is,

(i) $T^*\phi^{-1} \times T\phi$ leaves W_0 invariant, i.e. $(T^*\phi^{-1} \times T\phi)(W_0) \subset W_0$,

(ii) $\tilde{\phi}^*\omega = \omega$, and

(iii) $\tilde{\phi}^* H_0 = H_0$,

where $\tilde{\phi}$ denotes de restriction of $T^*\phi^{-1} \times T\phi$ to W_0. From [18], we have that these symmetries are preserved by the constraint algorithm or, equivalently, the constraint submanifolds are preserved by $\tilde{\phi}$. If W_f denotes the final constraint submanifold, then $\tilde{\phi}(W_f) = W_f$.

Assume now that Y is a vector field on Q such that its flow ϕ_t consists of vakonomic symmetries. From Definition 3.1 it is straightforward to see that,

(i) the complete lift Y^C of Y to TQ is tangent to the submanifold M,

(ii) $(Y^C)_{|M}(L_{|M}) = 0$.

Definition 3.2 ([24]). *A vector field Y on Q whose flow consists of vakonomic symmetries will be called an* infinitesimal vakonomic symmetry*.*

Taking into account that the tangent flow $T\phi_t$ on TQ is generated by the complete lift Y^C, and that the cotangent flow $T^*(\phi_t)^{-1} = T\phi_{-t}$ on T^*Q is generated by the complete lift Y^{C^*}, we deduce that the induced flow $T^*(\phi_t)^{-1} \times T\phi_t$ is generated by the vector field $Y^{C^*,C} = (Y^{C^*}, Y^C)$ on $T^*Q \times_Q TQ$. Locally, if $Y = Y^A \dfrac{\partial}{\partial q^A}$, we have that

$$Y^{C^*} = Y^A \frac{\partial}{\partial q^A} - p_B \frac{\partial Y^B}{\partial q^A}\frac{\partial}{\partial p_A}, \qquad Y^C = Y^A \frac{\partial}{\partial q^A} + \dot{q}^B \frac{\partial Y^A}{\partial q^B}\frac{\partial}{\partial \dot{q}^A}$$
$$Y^{C^*,C} = Y^A \frac{\partial}{\partial q^A} - p_B \frac{\partial Y^B}{\partial q^A}\frac{\partial}{\partial p_A} + \dot{q}^B \frac{\partial Y^A}{\partial q^B}\frac{\partial}{\partial \dot{q}^A}.$$

With the above hypotheses the vector field $Y^{C^*,C}$ is tangent to W_0, since this manifold is preserved by its flow. We denote by $\tilde{Y}$ the restriction of $Y^{C^*,C}$ to W_0. In addition, $\tilde{Y}(H_0) = 0$ and $\mathcal{L}_{\tilde{Y}}\omega = 0$. Thus, $\tilde{Y}$ is an infinitesimal symmetry of the presymplectic system (W_0, ω, H_0) in the sense studied in [18]. A direct computation shows that

$$i_{Y^{C^*,C}}\,(pr_1)^*\omega_Q = d((pr_1)^*\iota Y),$$

where $\iota Y : T^*Q \longrightarrow \mathbb{R}$ is the *evaluation function*, i.e. $\iota Y(\alpha_q) = \langle \alpha_q, Y_q \rangle$, for all $\alpha_q \in T^*Q$. Therefore, if we denote by $\widetilde{\iota Y}$ the restriction of $(pr_1)^*\iota Y$ to W_0, we have

$$i_{\tilde{Y}}\,\omega = d(\widetilde{\iota Y}). \tag{5}$$

Let us now recall a general result obtained in [18] for infinitesimal symmetries of presymplectic systems. Consider a presymplectic system given by a presymplectic manifold (U, ω) and a Hamiltonian function $H : U \to \mathbb{R}$. After applying the constraint algorithm, we obtain a final constraint submanifold U_f where a solution X exists. In this case, we have the following result.

Proposition 3.1 (Noether's theorem [18]). *Let Z be a vector field on U preserving the Hamiltonian function; $Z(H) = 0$. If Z is Hamiltonian for ω,*

$$i_Z\omega = dg$$

for some function g, then g restricted to U_f is a constant of the motion for any solution X.

Notice that a vector field Z satisfying the hypothesis of Proposition 3.1 is necessarily tangent to U_f.

Corollary 3.1. *Let Y be an infinitesimal vakonomic symmetry, then the restriction of $\widetilde{\iota Y}$ to the final constraint submanifold W_f is a conserved quantity for the vakonomic problem.*

Locally, if $Y = Y^A \dfrac{\partial}{\partial q^A}$, then we have $\widetilde{\iota Y} = p_A Y^A$. An improvement of the above result can be given if we consider a more general kind of symmetries.

Definition 3.3 ([24]). *A* Noether symmetry *of the vakonomic problem given by L and M is a vector field Y on Q such that,*

(i) the complete lift Y^C of Y to TQ is tangent to the submanifold M,

(ii) $(Y^C)_{|M}(L_{|M}) = F^C_{|M}$, for some function F on Q.

Note that an infinitesimal vakonomic symmetry is a Noether symmetry.

Corollary 3.2. *Let Y be a Noether symmetry of the vakonomic problem given by L and M. Then, the restriction to W_f of the function $\widetilde{\iota Y} - F$ is a constant of the motion (here, we denote by the same letter the function F and its pull-back to W_0).*

Proof. Let X_F be the Hamiltonian vector field on T^*Q of the function $\pi_Q^*(F)$ with respect to the canonical symplectic form ω_Q, where $\pi_Q : T^*Q \longrightarrow Q$ denotes the canonical projection. In local coordinates, we have

$$X_F = -\frac{\partial F}{\partial q^A}\frac{\partial}{\partial p_A}.$$

Take now the vector field $\tilde{Y} = Y^{C^*,C} - X_F$. Then, $\tilde{Y}(H_0) = 0$ and $i_{\tilde{Y}}\omega = d(\widetilde{\iota Y} - F)$. From Proposition 3.1 we conclude the result. □

The infinitesimal symmetries discussed above are usually called *point-wise* or *geometric*, since they are vector fields on the configuration manifold Q. However, Proposition 3.1 suggests that a more general kind of infinitesimal symmetries may be considered.

Definition 3.4. *A Hamiltonian vector field on W_0 which is tangent to W_f and preserves H_0 along W_f is called a* Cartan symmetry *of the vakonomic problem given by L and M.*

Proposition 3.1 implies that if $i_Z\omega = dg$ and $Z(H_0) = 0$ along W_f, then g, restricted to W_f, is a constant of the motion. Notice that the lift to W_0 of a Noether symmetry is a Cartan symmetry.

3.1 Vakonomic Lie group actions

Assume that a Lie group G acts on the configuration manifold Q of a vakonomic system determined by a Lagrangian function $L : TQ \longrightarrow \mathbb{R}$ and a constraint submanifold M of TQ.

Definition 3.5 ([24]). *An action $\phi : G \times Q \longrightarrow Q$ of G on Q is called* vakonomic *if L and M are invariant by G, that is, we have*

(i) $T\phi_g(M) \subset M$*;*

(ii) $L_{|M} \circ (T\phi_g)_{|M} = L_{|M}$

for all $g \in G$, where $\phi_g : Q \longrightarrow Q$ is defined by $\phi_g(q) = \phi(g, q)$.

In other words, the transformations ϕ_g are vakonomic symmetries. Also, the fundamental vector fields ξ_Q determined by the action are infinitesimal vakonomic symmetries for all $\xi \in \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G.

Proposition 3.2. *Let Φ be a vakonomic action of a Lie group G on Q, then it lifts to a presymplectic action on the presymplectic system $(W_0 = T^*Q \times_Q M, \omega_0, H_0)$, and this action admits a momentum mapping.*

Proof. The first assertion follows from Section 3. For the second assertion, define a mapping $J : W_0 \longrightarrow \mathfrak{g}^*$ by

$$\langle J(\alpha_q, X_q), \xi\rangle = \langle \alpha_q, \xi_Q(q)\rangle\,, \quad \text{for all } (\alpha_q, X_q) \in W_0 \text{ and } \xi \in \mathfrak{g}\,.$$

A direct computation in local coordinates shows that $\xi_{W_0} = (\xi_Q)^{C^*,C}$ from which we obtain

$$i_{\xi_{W_0}}\omega_0 = d(\widehat{J\xi}),$$

where $\widehat{J\xi} : W_0 \longrightarrow \mathbb{R}$ is defined by $\widehat{J\xi}(\alpha_q, X_q) = \langle J(\alpha_q, X_q), \xi\rangle$. Thus, the induced vector field ξ_{W_0} is Hamiltonian for the presymplectic form which completes the proof. □

Since the algorithm is preserved by the group action one deduces the following consequence [8].

Corollary 3.3. *The restriction of functions $\widehat{J\xi}$, $\xi \in \mathfrak{g}$ to the final constraint submanifold W_f are constants of the motion.*

In local coordinates, we have $\widehat{J\xi}(q^A, p_A, \dot{q}^A) = p_A \mathcal{A}_i^A \xi^i$, where $\xi = \xi^i E_i$, $\{E_i\}$ is a basis of $\mathfrak{g}$, $(E_i)_Q = \mathcal{A}_i^A \dfrac{\partial}{\partial q^A}$ and $\{E^i\}$ is the dual basis of $\{E_i\}$.

4 Optimal control problems

A control system of ordinary differential equations is usually given by

$$\dot{x}^i = \Gamma^i(x(t), u(t)), \quad 1 \le i \le m, \tag{6}$$

where (x^i) are called *state variables* and (u^a), $1 \le a \le k$ are the *control functions.*

The optimal control problem is the following. Given initial and final states x_0, x_f, the objective is to find a C^2-piecewise smooth curve $c(t) = (x(t), u(t))$ such that $x(t_0) = x_0$, $x(t_f) = x_f$, satisfying the control equations (6) and minimizing the functional

$$\mathcal{J}(c) = \int_{t_0}^{t_f} \mathbb{L}(x(t), u(t)) dt,$$

for some *cost function* $\mathbb{L} = \mathbb{L}(x, u)$.

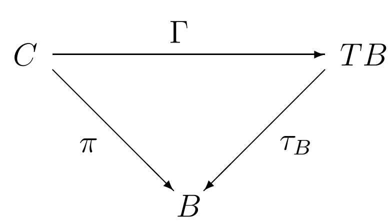
The usual formulation of optimal control problems is based on Pontryagin's Maximum Principle, which we will briefly describe below. Firstly, one introduces a Hamiltonian function $H(x, p, u) = p_i \Gamma^i(x, u) - \mathbb{L}(x, u)$, where p_i are the *co-states.* The Maximum Principle leads to the following system of differential equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}(x, p, u), \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}(x, p, u), \quad y_a = \frac{\partial H}{\partial u_a}(x, p, u), \tag{7}$$

where y_a are called the *outputs* of the system. A necessary condition for a control u^* to be optimal is that the outputs resulting from u^* are constantly zero.

Remark 4.1. It should be observed that here we focus our attention on *normal* curves. Otherwise, one has to consider a Hamiltonian of the form $H = p_i \Gamma^i$ (see [1, 20, 29]).

Now, we adopt a presymplectic approach which is equivalent to the Maximum Principle. In a global description, one assumes a fiber bundle structure $\pi : C \longrightarrow B$, where B is the configuration manifold with local coordinates x^i and C is the bundle of controls, with local coordinates (x^i, u^a). The ordinary differential equations (6) on B depending on the parameters u can be seen as a vector field Γ along the projection map π, that is, Γ is a smooth map $\Gamma : C \longrightarrow TB$ such that the following diagram



is commutative. In local coordinates, we have

$$\Gamma = \Gamma^i(x, u) \frac{\partial}{\partial x^i}.$$

This optimal control problem $(C, \mathbb{L}, \Gamma)$ can be described as a vakonomic problem associated with the Lagrangian function $L : TC \longrightarrow \mathbb{R}$, where $L = \mathbb{L} \circ \tau_C$, and the constraint submanifold,

$$M = \{v \in TC \mid \pi_*(v) = \Gamma(\tau_C(v))\} ,$$

which can be viewed as $M = T\pi^{-1}(\Gamma(C))$. Locally this submanifold is defined by the conditions $\dot{x}^i = \Gamma^i(x, u)$, $1 \leq i \leq m$, which are just equations (6). Therefore, we can introduce local coordinates $(x^i, u^a, \dot{u}^a)$ on M. The local expressions for ω and H_0 are

$$\omega = du^a \wedge dp_a + dx^i \wedge dp_i \,, \quad H_0 = \dot{u}^a p_a + \Gamma^i p_i - \mathbb{L}(x, u) \,.$$

In what follows, we discuss the behavior of the constraint algorithm in this case. In bundle coordinates (x^i, u^a), a solution X of the equation $i_X \omega = dH_0$ has the form

$$X = \Gamma^i \frac{\partial}{\partial x^i} + \dot{u}^a \frac{\partial}{\partial u^a} + (\frac{\partial \mathbb{L}}{\partial x^i} - p_j \frac{\partial \Gamma^j}{\partial x^i}) \frac{\partial}{\partial p_i} + (\frac{\partial \mathbb{L}}{\partial u^a} - p_i \frac{\partial \Gamma^i}{\partial u^a}) \frac{\partial}{\partial p_a} + Z^a \frac{\partial}{\partial \dot{u}^a}, \tag{8}$$

and W_1 is locally defined by the constraint functions

$$p_a = 0.$$

Therefore, we have coordinates $(x^i, u^a, p_i, \dot{u}^a)$ for W_1 such that ω_1 reads as $\omega_1 = dx^i \wedge dp_i$. This shows that (W_1, ω_1) is a presymplectic manifold of constant rank $2m$. Proceeding with the algorithm, we obtain new constraint functions

$$\psi_a = X(p_a) = \frac{\partial \mathbb{L}}{\partial u^a} - p_i \frac{\partial \Gamma^i}{\partial u^a},$$

whose vanishing guarantees the tangency of the solution. Thus, the next constraint submanifold W_2 is locally characterized by the constraints

$$p_a = 0, \quad \psi_a = 0.$$

Now we can assume that X, restricted to W_2, is tangent to W_1, but in order to ensure the tangency to W_2 we have to ask the new conditions $X_{|W_2}(\psi_a) = 0$, $1 \leq a \leq k$. But

$$X_{|W_2} = \Gamma^i \frac{\partial}{\partial x^i} + \dot{u}^a \frac{\partial}{\partial u^a} + \kappa_i \frac{\partial}{\partial p_i} + Z^a \frac{\partial}{\partial \dot{u}^a},$$

where

$$\kappa_i = \frac{\partial \mathbb{L}}{\partial x^i} - p_j \frac{\partial \Gamma^j}{\partial x^i},$$

so we obtain

$$\varphi_a = X_{|W_2}(\psi_a) = \Gamma^i \frac{\partial \psi_a}{\partial x^i} + \dot{u}^b \frac{\partial \psi_a}{\partial u^b} + \kappa_i \frac{\partial \psi_a}{\partial p_i} = 0. \tag{9}$$

The constraints $\varphi_a = 0$, along with $p_a = 0$ and $\psi_a = 0$, $1 \leq a \leq k$ define the submanifold W_3.

Now, consider the case when the matrix

$$\left(\frac{\partial \psi_a}{\partial u^b} = \frac{\partial^2 \mathbb{L}}{\partial u^a \partial u^b} - p_i \frac{\partial^2 \Gamma^i}{\partial u^a \partial u^b}\right) \tag{10}$$

is regular, which is just the condition defining the so-called *regular optimal control problems* (see [16, 17]). We can use the implicit function theorem on the equations $\psi_a = 0$ to get u^a as a function of x^i and p_i,

$$u^a = \zeta^a(x^i, p_i)\,, \quad \forall 1 \leq a \leq m, \tag{11}$$

and, in addition, to obtain explicitly the $\dot{u}^a$. Combining Eqs. (9) and (11),

$$\dot{u}^a = \xi^a(x^i, p_i)\,. \tag{12}$$

Therefore, for regular control problems, we can choose local coordinates (x^i, p_i) on W_3. Notice that, by applying $X_{|W_2}$ to the constraints $\dot{u}^a = \xi^a(x^i, p_i)$, we determine the remaining components Z^a's of X.

Summarizing, we have the following.

Proposition 4.1. *Assume that the optimal control problem is regular, and denote by ω_3 the restriction of ω to W_3. Then (W_3, ω_3) is a symplectic manifold, (x^i, p_i) are canonical coordinates, and the restriction of $X_{|W_2}$ to W_3 is the Hamiltonian vector field corresponding to the restriction of H_0 to W_3.*

If the control problem is not regular, then, in general, one has to continue the algorithm to obtain the final constraint submanifold W_f. Assume for instance that the matrix (10) has constant rank r on W_2. In this case, using the implicit function theorem, we can select some controls in terms of the others,

$$v^\alpha = \chi^\alpha(x^i, p_i, \bar{u}^l)\,, \quad 1 \leq \alpha \leq r\,,\ 1 \leq l \leq s\,,\ r + s = k.$$

Therefore, we introduce new r (local) constraints $\tilde{\chi}^\alpha = v^\alpha - \chi^\alpha$ and assume that these new constraints determine completely W_2. Then, we obtain

$$X(\tilde{\chi}^\alpha) = -\Gamma^i \frac{\partial \chi^\alpha}{\partial x^i} - \dot{\bar{u}}^l \frac{\partial \chi^\alpha}{\partial \bar{u}^l} + \dot{v}^\alpha - \kappa_i \frac{\partial \chi^\alpha}{\partial p_i} = 0. \tag{13}$$

From (13) one can get a well-defined dynamics (up to a gauge freedom) on the submanifolds characterized by the equations $\bar{u}^l = (\bar{u}^l)_0$, $\dot{\bar{u}}^l = 0$, where $(\bar{u}^l)_0$ are arbitrary constants.

Remark 4.2 (Interpretation of the results in terms of control theory). Note that

$$y_a = \frac{\partial H}{\partial u^a} = p_i \frac{\partial \Gamma^i}{\partial u^a} - \frac{\partial \mathbb{L}}{\partial u^a} = -\psi_a,$$

and hence the outputs are just the constraint functions defining W_2. If the problem is regular, then we can obtain the optimal control u^* using the implicit function theorem,

$$(u^*)^a = \zeta^a(x^i, p_i).$$

In addition, we have determined a unique dynamics on W_3 giving the optimal trajectories for such optimal control problem. If the problem is singular, but the matrix (10) has constant rank, say r, we can obtain r optimal controls for arbitrary choices of the remainder $k - r$.

Remark 4.3. The formulation of the optimal control problem by means of a constraint manifold leads us to consider additional variables, given by the momenta p_a and the time derivatives of the control functions, $\dot{u}^a$. The momentum coordinates are readily determined ($p_a = 0$) in the first step of the constraint algorithm. The determination of the time derivatives of the control functions relies on the specific nature of the problem under consideration. For regular problems, as stated in Proposition 4.1, they are obtained in the third step of the constraint algorithm. In general, however, the situations encountered in the determination of the optimal $\dot{u}^a$ can be quite diverse. The drawback of dealing with these additional variables gets compensated by the advantages that one obtains by formulating optimal control problems as vakonomic dynamical problems within the framework proposed in Section 2. Among them, we highlight the intrinsic and coordinate-free modeling with nontrivial control bundles, the identification in a systematic way of dynamically relevant geometric objects, and the study of the consistency problem of the optimal equations by means of the constraint algorithm. Some of these issues are not addressed in recent works such as [4, 11].

Remark 4.4. One can also consider non-autonomous optimal control problems within the above-developed framework. Consider the equations

$$\frac{dx^i}{dt} = \Gamma^i(t, x, u), \quad 1 \leq i \leq m,$$

a cost function $\mathbb{L}(t, x, u)$, and some boundary conditions. This time-dependent problem can be treated as an autonomous problem simply by enlarging the configuration space adding extra variables. To do that, one takes the control equations ($x^0 \equiv t$),

$$\frac{dx^0}{d\tau} = 1, \quad \frac{dx^i}{d\tau} = \Gamma^i(x^0, x, u), \quad 1 \leq i \leq m,$$

a cost function $\tilde{\mathbb{L}}(x^0, x, u) = \mathbb{L}(x^0, x, u)$, and the corresponding boundary conditions. An alternative formulation [26] consists of introducing a new extra control variable v and a new state x_0. In such a case, one considers as control equations ($x^0 \equiv t$),

$$\frac{dx^0}{d\tau} = v, \quad \frac{dx^i}{d\tau} = v\Gamma^i(x^0, x, u, v), \quad 1 \leq i \leq m,$$

a cost function $\tilde{\mathbb{L}}(x^0, x, u, v) = v\mathbb{L}(x^0, x, u)$, and the corresponding boundary conditions. We leave to the reader the development of the appropriate discussion in this setting.

Example 4.1. Consider the optimal control problem determined by the following system of differential equations

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u$$

and the cost function

$$\mathbb{L}(x, y, z, u) = \frac{1}{2}(x^2 - y^2 + z^2).$$

This problem corresponds to a slight variation of Hardy-Littlewood systems (see [16] for more details). We introduce local coordinates $(x, y, z, u, p_x, p_y, p_z, p_u, \dot{u})$ on W_0. The constraint algorithm provides the following family of submanifolds

$$\begin{aligned} W_1 &= W_0 \cap \{p_u = 0\}, & W_4 &= W_3 \cap \{u = -(y + p_x)\}, \\ W_2 &= W_1 \cap \{p_z = 0\}, & W_5 &= W_4 \cap \{p_u = 0\}, \\ W_3 &= W_2 \cap \{z = p_y\}, & W_6 &= W_5, \end{aligned}$$

and the dynamics on the final constraint submanifold $W_f = W_5$ is given by

$$X = y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - (y + p_x)\frac{\partial}{\partial z} - (x + z)\frac{\partial}{\partial u} + x\frac{\partial}{\partial p_x} + u\frac{\partial}{\partial p_y} - (y + u)\frac{\partial}{\partial \dot{u}}.$$

The optimal controls are $u = -(y + p_x)$, with $\dot{u} = -(x + z)$. Therefore, we can choose local coordinates (x, y, p_x, p_y) on W_f, and the dynamics is finally given by the following system of differential equations on W_f,

$$\dot{x} = y,\ \dot{y} = p_y,\ \dot{p}_x = x,\ \dot{p}_y = -(y + p_x).$$

5 Symmetries in optimal control problems

In order to study the symmetries for the vakonomic or optimal control problem $(W_0 = T^*C \times_C M, \omega, H_0)$ we will take into account the geometry of the fiber bundle $\pi : C \longrightarrow B$. Let $\phi_C : C \longrightarrow C$ be a fiber bundle isomorphism over a transformation $\phi_B : B \longrightarrow B$, locally written as

$$\phi_C(x^i, u^a) = (\phi^i(x), \phi^a(x, u)),$$

where $\phi_B(x^i) = (\phi^i(x))$.

Definition 5.1. *The fiber bundle morphism (ϕ_C, ϕ_B) preserves Γ if*

$$T\phi_B \circ \Gamma = \Gamma \circ \phi_C. \tag{14}$$

Lemma 5.1. *$T\phi_C$ preserves M if and only if (ϕ_C, ϕ_B) preserves Γ.*

Proof. $\Leftarrow$) Let $v \in M$. We have

$$\begin{aligned}\Gamma(\tau_C(T\phi_C(v))) &= \Gamma(\phi_C(\tau_C(v)))\\ &= T\phi_B \circ \Gamma(\tau_C(v)) \ \ \text{(since } (\phi_C, \phi_B) \text{ preserves } \Gamma)\\ &= T\phi_B \circ T\pi(v) \ \ \text{(since } v \in M)\\ &= T\pi \circ T\phi_C(v),\end{aligned}$$

which proves that $T\phi_C$ preserves M.

$\Rightarrow$) Assume that $T\phi_C(v) \in M$ for all $v \in M$. Then we have

$$\begin{aligned}\Gamma(\phi_C(\tau_C(v))) &= \Gamma(\tau_C(T\phi_C(v)))\\ &= T\pi \circ T\phi_C(v) = T\phi_B \circ T\pi(v) \ \ \text{(since } (\phi_C, \phi_B) \text{ is a fibered mapping)}\\ &= T\phi_B \circ \Gamma(\tau_C(v)) \ \ \text{(since } v \in M)\end{aligned}$$

Therefore, we have deduced that $\Gamma(\phi_C(\tau_C(v))) = T\phi_B \circ \Gamma(\tau_C(v))$, which implies that $\Gamma \circ \phi_C = T\phi_B \circ \Gamma$ since $\tau_C(M) = C$. □

Now, the preservation of the restriction of L to M by $T\phi_C$ is equivalent to the preservation of $\mathbb{L}$ by ϕ_C. Therefore, we can adapt Definition 3.1 as follows.

Definition 5.2. *A* symmetry of the optimal control problem $(\mathbb{L}, C)$ *consists of a fiber bundle isomorphism* (ϕ_C, ϕ_B) *preserving* Γ *and* $\mathbb{L}$.

Then, a symmetry for the optimal control problem $(C, \mathbb{L}, \Gamma)$ is always a symmetry for the vakonomic problem $(W_0 = T^*C \times_C M, \omega, H_0)$, but the converse does not hold in general (in fact, we can consider symmetries that are diffeomorphisms of C, but not necessarily bundle morphisms).

Consider now a vector field Y_C on C whose associated flow $((\phi_t)_C, (\phi_t)_B)$ consists of local symmetries. Such a vector field is always π-projectable to a vector field Y_B on B which generates the flow $(\phi_t)_B$, and, in addition, verifies

$$\mathcal{L}_{Y_C}\Gamma = 0, \ \ Y_C(\mathbb{L}) = 0.$$

The converse is also true for projectable vector fields and we thus introduce the next terminology.

Definition 5.3. *An* infinitesimal symmetry of the optimal control problem $(C, \mathbb{L}, \Gamma)$ *is a projectable vector field* Y_C *such that*

(i) $\mathcal{L}_{Y_C}\Gamma = 0$,

(ii) $Y_C(\mathbb{L}) = 0$.

The local expressions of Y_C and Y_B are the following

$$Y_C = Y^i(x)\,\frac{\partial}{\partial x^i} + U^a(x,u)\,\frac{\partial}{\partial u^a}\ ,\ Y_B = Y^i(x)\,\frac{\partial}{\partial x^i}.$$

Thus, an infinitesimal symmetry of the optimal control problem $(C,\mathbb{L},\Gamma)$ is always an infinitesimal symmetry of the vakonomic problem $(W_0 = T^*C \times_C M, \omega, H_0)$, but the converse does not hold in general.

Corollary 3.1 can be reformulated in this context to give an appropriate version of Noether's theorem for optimal control problems.

Corollary 5.1 (Noether's theorem). *Let Y be an infinitesimal symmetry for the optimal control problem $(C,\mathbb{L},\Gamma)$, then the function $\widetilde{\iota Y} : T^*C \times_C M \longrightarrow \mathbb{R}$ restricted to the final constraint submanifold W_f is a conserved quantity.*

If Y is locally expressed as $Y = Y^i\frac{\partial}{\partial x^i} + U^a\frac{\partial}{\partial u^a}$, then we have $\widetilde{\iota Y} = p_i Y^i$.

Concerning Noether symmetries, we make first the following remark. Assume that Y_C is a projectable vector field which is a Noether symmetry for the vakonomic problem associated with the control problem $(C,\mathbb{L},\Gamma)$. This means that

(i) the complete lift $(Y_C)^C$ of Y to TC is tangent to M;

(ii) $((Y_C)^C)_{|M}(L_{|M}) = (F^C)_{|M}$, for some function F on C.

If $Y_C = Y^i\frac{\partial}{\partial x^i} + Y^a\frac{\partial}{\partial u^a}$ then condition (ii) above implies that

$$Y^i\frac{\partial \mathbb{L}}{\partial x^i} + Y^a\frac{\partial \mathbb{L}}{\partial u^a} = \Gamma^i(x,u)\frac{\partial F}{\partial x^i} + \dot{u}^a\frac{\partial F}{\partial u^a}. \tag{15}$$

Since the first member in (15) does not depend on the derivatives of the controls, we deduce that $F = F(x^i)$, i.e. F is the pull-back via π of a function defined on B (and denoted by the same letter). Therefore (15) becomes

$$Y^i\frac{\partial \mathbb{L}}{\partial x^i} + Y^a\frac{\partial \mathbb{L}}{\partial u^a} = \Gamma^i(x,u)\frac{\partial F}{\partial x^i}. \tag{16}$$

This leads to the following definition.

Definition 5.4. *A* Noether symmetry *for the control problem $(C,\mathbb{L},\Gamma)$ is a projectable vector field Y_C on C such that*

(i) $\mathcal{L}_{Y_C}\Gamma = 0$;

(ii) $Y_C(\mathbb{L}) = \Gamma(F)$, for some function F on B.

Now, a direct application of Corollary 3.2 gives the following.

Corollary 5.2 (Noether's theorem). *Let Y_C be a Noether symmetry of the control problem $(C, \mathbb{L}, \Gamma)$ with associated function F. Then, the function $\iota\widetilde{Y_C} - F$ restricted to the final constraint submanifold W_f is a conserved quantity.*

Notice that since Γ is a vector field on C along π, then $\Gamma(F)$ is a well-defined function on C. In local coordinates, the conservation law reads

$$Y^i p_i - F(x^i)\,.$$

Remark 5.1. The Noether and infinitesimal symmetries of the optimal control problem described above were previously considered by Grizzle and Marcus [15].

Finally, we consider Cartan-like symmetries.

Definition 5.5. *A* Cartan symmetry *for the control problem $(C, \mathbb{L}, \Gamma)$ is a vector field Y on W_0 such that*

(i) Y is tangent to W_f and preserves H_0 along W_f;

(ii) Y is Hamiltonian for some function g, $i_Y\omega = dg$, along W_f.

Corollary 5.3. *Let Y be a Cartan symmetry as in Definition 5.5. Then, the function g restricted to the final constraint submanifold W_f is a conserved quantity.*

Example 5.1. We will revisit an example considered in [33]. Consider a mathematical pendulum in space ($Q = \mathbb{R}^3$) with mass $m = 1$ and length $l = 1$, and assume that there is horizontal force with components u_1 and u_2 in the x and y directions, respectively. The equations of motion are given by

$$\ddot{\phi} = -g\sin\phi + u_1\cos\theta\cos\phi + u_2\sin\theta\cos\phi\,, \quad \ddot{\theta} = -u_1\sin\theta + u_2\cos\theta \tag{17}$$

where (ϕ, θ) denotes coordinates in $\mathbb{S}^2$. We consider the cost function $\mathbb{L} : T\mathbb{S}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$\mathbb{L}(x_1, x_2, x_3, x_4, u_1, u_2) = \frac{1}{2}(u_1^2 + u_2^2).$$

The control bundle is $\pi = pr_1 : C = T\mathbb{S}^2 \times \mathbb{R}^2 \longrightarrow B = T\mathbb{S}^2$. We introduce coordinates $x_1 = \phi, x_2 = \theta, x_3 = \dot{\phi}, x_4 = \dot{\theta}$, and the controls are just the force components u_1, u_2. The control equations are given by equations (17), so that

$$\Gamma = x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2} + (-g\sin x_1 + u_1\cos x_2\cos x_1 + u_2\sin x_2\cos x_1)\frac{\partial}{\partial x_3} + (-u_1\sin x_2 + u_2\cos x_2)\frac{\partial}{\partial x_4}$$

Then, $W_0 = T^*(T\mathbb{S}^2 \times \mathbb{R}^2) \times_{T\mathbb{S}^2\times\mathbb{R}^2} M$, where M is the constraint submanifold of $T(T\mathbb{S}^2 \times \mathbb{R}^2)$ defined by Γ. The submanifold W_1 is defined by $p_{u_1} = 0$, $p_{u_2} = 0$. We also have

$$\psi_1 = u_1 - p_3 \cos x_2 \cos x_1 + p_4 \sin x_2 \,, \quad \psi_2 = u_2 - p_3 \sin x_2 \cos x_1 - p_4 \cos x_2 \,,$$

so that the submanifold W_2 is defined by the constraints $p_{u_1} = 0$, $p_{u_2} = 0$, $\psi_1 = 0$, $\psi_2 = 0$. Therefore, we deduce that this control problem is regular, and, consequently, the algorithm stabilizes at the third step, W_3. To determine W_3, we first compute

$$\begin{aligned} \kappa_1 &= p_3 g \cos x_1 + p_3 u_1 \cos x_2 \sin x_1 + p_3 u_2 \sin x_2 \sin x_1 \,, & \kappa_3 &= 0 \,, \\ \kappa_2 &= p_3 u_1 \sin x_2 \cos x_1 - p_3 u_2 \cos x_2 \cos x_1 + p_4 u_1 \cos x_2 + p_4 u_2 \sin x_2 \,, & \kappa_4 &= 0 \,. \end{aligned}$$

Thus, we have

$$\begin{aligned} \phi_1 &= \dot{u}_1 - x_3 p_3 \cos x_2 \sin x_1 + x_4 p_3 \sin x_2 \cos x_1 + x_4 p_4 \cos x_1 \\ \phi_2 &= \dot{u}_2 - x_3 p_3 \sin x_2 \sin x_1 - x_4 p_3 \cos x_2 \cos x_1 + x_4 p_4 \sin x_2 . \end{aligned}$$

A direct computation shows that $\dfrac{\partial}{\partial x_2}$ is a Cartan symmetry with associated function p_2, which is a conserved quantity on W_3.

6 Reducing control systems by a Lie group of symmetries

In this section we will discuss the reduction and reconstruction of optimal control problems with symmetries. This kind of problems was studied for instance in [14, 27, 32, 33] in a Hamiltonian setting. Here, we adopt a different point of view based on the vakonomic approach. In doing that, the treatment of the dynamics become more natural, and, in addition, we recover much of the spirit of the usual reduction and reconstruction schemes of Hamiltonian systems with symmetry. Our approach also allows to compare the constraint algorithms for the original optimal control problem and the corresponding one for the reduced problem.

Consider, as in the preceding section, an optimal control problem given by a fiber bundle $\pi : C \longrightarrow B$, with a control equation $\Gamma : C \longrightarrow TB$ and a cost function $\mathbb{L} : C \longrightarrow \mathbb{R}$. Suppose that a Lie group G acts on C by bundle morphisms, that is, the action $\Phi : G \times C \longrightarrow C$ covers an action $\phi : G \times B \longrightarrow B$ in the sense that $\pi \circ \Phi_g = \phi_g \circ \pi$, for all $g \in G$.

Definition 6.1. *Assume that*

(i) $\mathbb{L}$ *is* G*-invariant, so that we have* $\mathbb{L} \circ \Phi_g = \mathbb{L}$*;*

(ii) Γ *is* G*-invariant, that is, we have* $\Gamma \circ \Phi_g = T\phi_g \circ \Gamma$*,*

for all $g \in G$. Under these hypotheses we say that G is a Lie group of symmetries of the control problem $(C, \mathbb{L}, \Gamma)$.

This essentially means that each Φ_g is a symmetry in the sense of Definition 5.2. We also assume that the actions on C and B are free and proper so that $\rho_C : C \longrightarrow \bar{C} = C/G$ and $\rho_B : B \longrightarrow \bar{B} = C/G$ are principal bundles with structure group G. In [27] it is proved that $\bar{\pi} : \bar{C} \longrightarrow \bar{B}$ is again a fiber bundle. In addition, if $\pi : C \longrightarrow B$ is a vector (affine, principal) bundle the same happens with $\bar{\pi} : \bar{C} \longrightarrow \bar{B}$.

A direct computation shows that Γ projects onto a vector field $\bar{\Gamma}$ on $\bar{C}$ along the projection $\bar{\pi}$. Indeed, $\bar{\Gamma}$ is defined by

$$\bar{\Gamma}(\bar{z}) = T\rho_C(\Gamma(z)) \tag{18}$$

for all $\bar{z} \in \bar{C}$, where z is an arbitrary point in the fiber over $\bar{z}$.

Remark 6.1. Equation (18) opens the possibility to consider more general kind of symmetries, named *partial symmetries* in [27]. Indeed, for reduction purposes we only need that (18) holds. But this property is ensured if we have

$$T\rho_B \circ \Gamma \circ \Phi_g = T\rho_B \circ T\phi_g \circ \Gamma$$

for all $g \in G$. This kind of symmetries was exploited in [27] to study feedback in control theory. To simplify the exposition, we will consider symmetries as in Definition 6.1, though our results are also verified by partial symmetries.

It is clear that the cost function $\mathbb{L}$ will also project onto a function $\bar{\mathbb{L}} : \bar{C} \longrightarrow \mathbb{R}$. Therefore, we obtain a new optimal control problem given by the fiber bundle $\bar{\pi} : \bar{C} \longrightarrow \bar{B}$, with a control equation $\bar{\Gamma} : \bar{C} \longrightarrow T\bar{B}$ and a cost function $\bar{\mathbb{L}} : \bar{C} \longrightarrow \mathbb{R}$. This new system $(\bar{C}, \bar{\mathbb{L}}, \bar{\Gamma})$ will be called the *reduced control system*. We now can develop the corresponding constraint algorithm for the presymplectic system $(\bar{W}_0, \bar{\omega}_0, \bar{H}_0)$. We notice that the reduced constraint submanifold $\bar{M}$ is just the quotient of M by G; in fact, $(\mathbb{T}\rho_C)_M : M \longrightarrow \bar{M}$ is a principal bundle with structure group G.

We are interested in comparing the dynamics of the original and the reduced system, the motivation for this being the possibility of reconstructing the original dynamics from the reduced one, in a similar way to the case of symplectic reduction. One interesting feature of this reduction procedure is that the number of controls is constant along the process.

In order to compare both dynamics it will be convenient to introduce local coordinates adapted to the reduced system. Since $\rho_C : C \longrightarrow \bar{C}$ is a principal G-bundle, we can choose bundle coordinates $(\bar{x}^I, y^\alpha, u^a)$ on C such that $(\bar{x}^I, u^a)$ are local coordinates on $\bar{C}$ and, simultaneously, $(\bar{x}^I, y^\alpha)$ are bundle coordinates on B. In this way, $(\bar{x}^I)$ are local coordinates on $\bar{B}$. Moreover, since Γ is projectable, it takes the local expression

$$\Gamma = \Gamma^I(\bar{x}, y, u)\frac{\partial}{\partial \bar{x}^I} + \Gamma^\alpha(\bar{x}, y, u)\frac{\partial}{\partial y^\alpha} \tag{19}$$

Since Γ is G-invariant, then the local components Γ^I actually do not depend on the coordinates y^a (they are constant along the fibers of ρ_C) so that the expression of $\bar{\Gamma}$ is

$$\bar{\Gamma} = \bar{\Gamma}^I(\bar{x}, u)\frac{\partial}{\partial \bar{x}^I} \tag{20}$$

where $\bar{\Gamma}^I(\bar{x}, u) = \Gamma^I(\bar{x}, y, u)$ for arbitrary coordinates y. In other words, the reduced control problem is given by a cost function $\bar{\mathbb{L}} = \bar{\mathbb{L}}(\bar{x}, u)$ and a control equation $\dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u)$.

The original constraint submanifold M is locally characterized by the equations

$$\dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u)\,, \quad \dot{y}^\alpha = \Gamma^\alpha(\bar{x}, y, u)\,.$$

(The above equations are referred as in normal form in [27]). Correspondingly, the constraint submanifold $\bar{M}$ is locally described as,

$$\dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u) \tag{21}$$

Since G acts on T^*C by lifting the action on C we have a quotient manifold T^*C/G. Next, we shall define the mappings

$$\mathcal{A} : T^*C \times_C M \longrightarrow T^*C/G \times_{\bar{C}} \bar{M}\,, \quad \mathcal{B} : T^*\bar{C} \times_{\bar{C}} \bar{M} \longrightarrow T^*C/G \times_{\bar{C}} \bar{M}$$

as follows,

- The mapping $\mathcal{A}$ by,

  $$\mathcal{A}(\gamma_z, X_z) = ([\gamma_z], [X_z])\,, \quad \text{with } \gamma_z \in T_z^*C,\ X_z \in M,\ z \in C\,,$$

  where $[\gamma_z], [X_z]$ denote the corresponding equivalence classes.

- The mapping $\mathcal{B}$ by,

  $$\mathcal{B}(\bar{\gamma}_{\bar{z}}, \bar{X}_{\bar{z}}) = ([\gamma_z], \bar{X}_{\bar{z}})$$

  where $\bar{\gamma}_{\bar{z}} \in T_{\bar{z}}^*\bar{C}$, γ_z is the pull-back of $\bar{\gamma}_{\bar{z}}$ to an arbitrary point z in the fiber over $\bar{z}$. A simple computation shows that $\mathcal{B}$ is well defined.

In local coordinates, we have

$$\mathcal{A}(\bar{x}, y, u, p_{\bar{x}}, p_y, p_u, \dot{u}) = (\bar{x}, u, p_{\bar{x}}, p_y, p_u, \dot{u}) \tag{22}$$

$$\mathcal{B}(\bar{x}, u, p_{\bar{x}}, p_u, \dot{u}) = (\bar{x}, u, p_{\bar{x}}, p_y = 0, p_u, \dot{u}) \tag{23}$$

where we have eliminated the indexes for simplicity. From (22) and (23) we deduce that $\mathcal{A}$ is a submersion, and $\mathcal{B}$ is an embedding.

Next, we shall investigate the behavior of both mappings with respect to the constraint algorithms

$$W_f \longrightarrow \cdots \longrightarrow W_2 \longrightarrow W_1 \longrightarrow W_0\,, \text{ and } \bar{W}_f \longrightarrow \cdots \longrightarrow \bar{W}_2 \longrightarrow \bar{W}_1 \longrightarrow \bar{W}_0\,.$$

In order to compare both algorithms it will be convenient to give the local characterizations of the corresponding constraint submanifolds. Before the reduction procedure, we have

$$
\begin{aligned}
&W_0 : (\bar{x}, y, u, p_{\bar{x}}, p_y, p_u, \dot{\bar{x}}, \dot{y}, \dot{u}),\ \dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u),\ \dot{y}^\alpha = \Gamma^\alpha(\bar{x}, y, u),\\
&W_1 : \dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u),\ \dot{y}^\alpha = \Gamma^\alpha(\bar{x}, y, u),\ p_u = 0,\\
&W_2 : \dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u),\ \dot{y}^\alpha = \Gamma^\alpha(\bar{x}, y, u),\ p_u = 0,\ \psi_a = \frac{\partial \mathbb{L}}{\partial u^a} - p_{\bar{x}^I}\frac{\partial \bar{\Gamma}^I}{\partial u^a} - p_{y^\alpha}\frac{\partial \Gamma^\alpha}{\partial u^a} = 0,
\end{aligned}
$$

and the further constraint submanifolds are determined by imposing the tangency condition to an arbitrary solution on W_2,

$$
X_{|W_2} = \Gamma^I \frac{\partial}{\partial \bar{x}^I} + \Gamma^\alpha \frac{\partial}{\partial y^\alpha} + \dot{u}^a \frac{\partial}{\partial u^a} + \kappa_I \frac{\partial}{\partial p_{\bar{x}^I}} + \kappa_\alpha \frac{\partial}{\partial p_{y^\alpha}} + Z^a \frac{\partial}{\partial u^a}\,, \tag{24}
$$

where

$$
\kappa_I = \frac{\partial \mathbb{L}}{\partial \bar{x}^I} - p_{\bar{x}^J}\frac{\partial \Gamma^J}{\partial \bar{x}^I} - p_{y^\alpha}\frac{\partial \Gamma^\alpha}{\partial \bar{x}^I}\,, \quad \kappa_\alpha = -p_{y^\beta}\frac{\partial \Gamma^\beta}{\partial y^\alpha}\,,
$$

since $\dfrac{\partial \mathbb{L}}{\partial y^\alpha} = \dfrac{\partial \Gamma^I}{\partial y^\alpha} = 0$. Here, Z^a are arbitrary functions.

After the reduction procedure, we obtain

$$
\begin{aligned}
&\bar{W}_0 : (\bar{x}, u, p_{\bar{x}}, p_u, \dot{\bar{x}}, \dot{u}),\ \dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u),\\
&\bar{W}_1 : \dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u),\ p_u = 0,\\
&\bar{W}_2 : \dot{\bar{x}}^I = \bar{\Gamma}^I(\bar{x}, u),\ p_u = 0,\ \bar{\psi}_a = \frac{\partial \bar{\mathbb{L}}}{\partial u^a} - p_{\bar{x}^I}\frac{\partial \bar{\Gamma}^I}{\partial u^a},
\end{aligned}
$$

and the further constraint submanifolds are determined by imposing the tangency condition to an arbitrary solution on $\bar{W}_2$,

$$
\bar{X}_{|\bar{W}_2} = \bar{\Gamma}^I \frac{\partial}{\partial \bar{x}^I} + \dot{u}^a \frac{\partial}{\partial u^a} + \bar{\kappa}_I \frac{\partial}{\partial p_{\bar{x}^I}} + \bar{Z}^a \frac{\partial}{\partial u^a} \tag{25}
$$

where

$$
\bar{\kappa}_I = \frac{\partial \bar{\mathbb{L}}}{\partial \bar{x}^I} - p_{\bar{x}^J}\frac{\partial \bar{\Gamma}^I}{\partial \bar{x}^J}
$$

is the projection of κ_I, and $\bar{Z}^a$ are arbitrary functions. From the above expressions we deduce the following result.

Lemma 6.1. *We have*

(i) $\mathcal{B}(\bar{W}_1) \subset \mathcal{A}(W_1)$,

(ii) $\mathcal{B}(\bar{W}_2) \subset \mathcal{A}(W_2)$,

Moreover, $\mathcal{B}(\bar{W}_1)$ *and* $\mathcal{B}(\bar{W}_2)$ *are submanifolds of* $\mathcal{A}(W_1)$ *and* $\mathcal{A}(W_1)$*, respectively.*

Notice that Lemma 6.1 implies that $\mathcal{A}$ and $\mathcal{B}$ induce mappings

$$\mathcal{A}_i : W_i \longrightarrow T^*C/G \times \bar{M}\,, \quad \mathcal{B}_i : \bar{W}_i \longrightarrow T^*C/G \times \bar{M}\,, \quad \text{for } i = 1, 2\,.$$

Take $X_{|W_2}$ with projectable components Z^a, and denote by $\bar{Z}^a$ the corresponding projections used to construct the vector field $\bar{X}_{|\bar{W}_2}$. Using Lemma 6.1 we can prove the next result.

Theorem 6.1. *For the choices* (24) *and* (25) *of vector field solutions, we have*

$$(T\mathcal{A}_2(X_{|W_2}))_{|\mathcal{B}_2(\bar{W}_2)} = T\mathcal{B}_2(\bar{X}_{|\bar{W}_2})\,.$$

Therefore, we obtain, by restriction, mappings $\mathcal{A}_i : W_i \longrightarrow T^*C/G \times \bar{M}$ and $\mathcal{B}_i : \bar{W}_i \longrightarrow T^*C/G \times \bar{M}$, for $i \geq 3$, such that the relation $\mathcal{B}(\bar{W}_i) \subset \mathcal{A}(W_i)$ holds for all positive integer i. Moreover, we have

$$(T\mathcal{A}_i(X_{|W_i}))_{|\mathcal{B}_i(\bar{W}_i)} = T\mathcal{B}_i(\bar{X}_{|\bar{W}_i})\,, \quad \textit{for all } i\,.$$

Reconstruction of the dynamics

Here, we discuss the reconstruction of the dynamics process. For a related treatment, see [14]. Based on Theorem 6.1, one can compare the original and the reduced dynamics. The first important fact is that $\mathcal{A} : W_0 \to T^*C/G \times_{\bar{C}} \bar{M}$ is a principal G-bundle. As a consequence, we have the following result.

Proposition 6.1. *W_i is a principal G-bundle over $\mathcal{A}_i(W_i)$. In particular, W_f is a principal G-bundle over $\mathcal{A}_f(W_f)$.*

Since $\mathcal{B}_f(\bar{W}_f)$ is a submanifold of $\mathcal{A}_f(W_f)$ we have a principal G-bundle $(W_f)_{|\mathcal{B}(\bar{W}_f)} \to \mathcal{B}(\bar{W}_f)$. Now we can proceed in the following manner in order to reconstruct the original dynamics:

- First, take a solution $\bar{X}_f$ of the reduced dynamics on $\bar{W}_f$.
- Secondly, that solution is transported to $\mathcal{A}_f(W_f)$ by means of $\mathcal{B}_f$.
- Finally, the solution is lifted to W_f using the canonical section $\beta : \mathcal{B}(\bar{W}_f) \longrightarrow (W_f)_{|\mathcal{B}(\bar{W}_f)}$ induced by the pull-back $\rho_C^* : T^*\bar{C} \longrightarrow T^*C$.

Remark 6.2 (Interpretation in terms of control theory). Assume for simplicity that our control problem is regular. Then, both algorithms stop at the same level, say at W_3 and $\bar{W}_3$, respectively. Notice that the controls u^a are the same for the non-reduced and reduced systems. The advantage here is that we have reduced the number of states to consider. Indeed, for the reduced problem, one has to apply the implicit function theorem to the function $\bar{\psi}_a$, so that we get

$$u^a = \bar{\xi}^a(\bar{x}^I, p_{\bar{x}^I})$$

which contains less variables than the functions ξ^a in the non-reduced problem.

In the singular case, the problem is more involved. The reduction procedure leads to a reduced control problem which consists of integrating the dynamics on $\bar{W}_f$, and then recovering the original one. To do that, we have to integrate the control equations

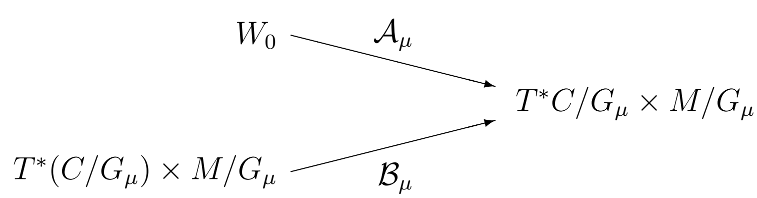
$$\dot{y}^\alpha = \Gamma^\alpha(\bar{x}, y, u)$$

with fixed momenta $p_{y^\alpha} = 0$, for all α.

Remark 6.3. The above described reduction and reconstruction process is a particular case of the following, more general, situation. Consider the momentum mapping $J : W_0 = T^*C \times M \to \mathfrak{g}^*$, defined in Section 3.1. Fix a regular value $\mu \in \mathfrak{g}^*$ and let G_μ be its isotropy group with respect to the coadjoint action. Now, we can perform a standard symplectic reduction scheme [21] to obtain a reduced dynamics on $J^{-1}(\mu)/G_\mu$. Note that this space can be identified with $J_1^{-1}(\mu)/G_\mu \times M/G_\mu$, where $J_1 : T^*C \to \mathfrak{g}^*$ is defined by $< J_1(\alpha), \xi >=\, < \alpha, \xi_C >$. In case $\mu = 0$, we have $G_\mu = G$ and $J_1^{-1}(0)/G \simeq T^*\bar{C}$, so we recover the former reduced space $T^*\bar{C} \times \bar{M}$.

Making use of the cotangent bundle reduction theorem of Satzer, Marsden and Kummer [21], we can embed $J_1^{-1}(\mu)/G_\mu$ in $T^*(C/G_\mu)$. To do so, we first select a principal connection A_μ on the principal fiber bundle $C \to C/G_\mu$. Let B be the pullback by $\pi_{C/G_\mu} : T^*(C/G_\mu) \to C/G_\mu$ of the two-form on C/G_μ induced by $\mu' \circ \mathrm{curv}(A_\mu)$, where $\mu' = \mu_{|\mathfrak{g}_\mu} \in \mathfrak{g}_\mu^*$ and $\mathrm{curv}(A_\mu)$ is the curvature of A_μ. Then there exists a symplectic embbeding $\varphi_\mu : \big(J_1^{-1}(\mu)/G_\mu, \omega_\mu\big) \to \big(T^*(C/G_\mu), \omega_{C/G_\mu} - B\big)$. In case $\mu = 0$, this embedding is a symplectomorphism.

Consequently, we can regard the reduced system as living in $T^*(C/G_\mu) \times M/G_\mu$. In this way, we can compare it with the original dynamics living in W_0 by means of the next diagram,



where $\mathcal{A}_\mu(\alpha, v) = (\rho_\mu(\alpha), [v])$ and $\mathcal{B}_\mu(\bar{\alpha}, [v]) = (\rho_\mu(\pi_\mu^*\bar{\alpha} + \mu' \circ A_\mu), [v])$,with $\rho_\mu : T^*C \to T^*C/G_\mu$ and $\pi_\mu : C \to C/G_\mu$ the canonical projections. Proceeding as above, we can recover the original control problem by integrating the control equations $\dot{y}_\mu = \Gamma(\bar{x}_\mu, y_\mu, u)$ for fixed values $p_{y_\mu} = -\mu$. Finally, note that if $\mu = 0$, we precisely recover the former scheme.

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