NONHOLONOMIC LAGRANGIAN SYSTEMS ON LIE ALGEBROIDS

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Abstract. This paper presents a geometric description on Lie algebroids of Lagrangian systems subject to nonholonomic constraints. The Lie algebroid framework provides a natural generalization of classical tangent bundle geometry. We define the notion of nonholonomically constrained system, and characterize regularity conditions that guarantee the dynamics of the system can be obtained as a suitable projection of the unconstrained dynamics. The proposed novel formalism provides new insights into the geometry of nonholonomic systems, and allows us to treat in a unified way a variety of situations, including systems with symmetry, morphisms and reduction, and nonlinearly constrained systems. Various examples illustrate the results.

Contents

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1. Introduction

The category of Lie algebroids has proved useful to formulate problems in applied mathematics, algebraic topology, and algebraic and differential geometry. In the context of Mechanics, an ambitious program was proposed by A. Weinstein [60] in order to develop formulations of the dynamical behavior of Lagrangian and Hamiltonian systems on Lie algebroids and discrete mechanics on Lie groupoids. In the last years, this program has been actively developed by many authors, and as a result, a powerful and complete mathematical structure is emerging.

One of the main features of the Lie algebroid framework is its inclusive nature. Under the same umbrella, one can consider such disparate situations as systems with symmetry, systems evolving on semidirect products, Lagrangian and Hamiltonian systems on Lie algebras, and field theory equations (see [17, 37] for recent topical reviews illustrating this). The Lie algebroid approach to Mechanics builds on the particular structure of the tangent bundle to develop a geometric treatment of Lagrangian systems parallel to Klein's formalism [20, 30]. At the same time, the attention devoted to Lie algebroids from a purely geometrical viewpoint has led to an spectacular development of the field, e.g. see [5, 12, 42, 53] and references therein. The merging of both perspectives has already provided mutual benefit, and will undoubtedly lead to important developments in the future.

The other main theme of this paper are nonholonomically constrained Lagrangian systems. Since the seminal paper by J. Koiller [31], there has been a renewed interest in the study of nonholonomic mechanical systems, i.e., mechanical systems subject to constraints involving velocities. In the last years, several authors have extended the ideas and techniques of the geometrical treatment of unconstrained systems to the study of nonholonomic mechanical systems, see for instance [2, 4, 7, 9, 11, 16, 21, 32, 33, 34, 35, 41, 57] and the recent monographs [3, 15].

In this paper we develop a comprehensive treatment of nonholonomic systems on Lie algebroids. This class of systems was introduced in [18] when studying mechanical control systems (see also [49] for a recent approach to mechanical systems on Lie algebroids subject to linear constraints). Here, we build on the geometry of Lie algebroids to identify suitable regularity conditions guaranteeing that the nonholonomic system admits a unique solution. We also develop a projection procedure to obtain the constrained dynamics as a modification of the unconstrained one. We show that many of the properties that standard nonholonomic systems enjoy have their counterpart in the proposed setup. From a methodological point of view, this approach has enormous advantages. As important examples, we highlight that the analysis clearly explains the use of pseudo-coordinates techniques and naturally lends itself to the treatment of constrained systems with symmetry, following the ideas developed in [18, 46]. We carefully examine the reduction procedure for this class of systems, and provide with various examples illustrating the results. Developments of a similar caliber are expected in the future as indicated in the conclusions of the paper. In the course of the preparation of this manuscript, the recent research efforts [13, 48] were brought to our attention. Both references, similar in spirit to the present work, deal with nonholonomic Lagrangian systems and focus on the reduction of Lie algebroid structures under symmetry.

The paper is organized as follows. In Section 2 we collect some preliminary notions and geometric objects on Lie algebroids, including differential calculus, morphisms and prolongations. We also describe classical Lagrangian systems within the formalism of Lie algebroids. In Section 3, we introduce the class of nonholonomic Lagrangian systems subject to linear constraints, given by a regular Lagrangian $L : E \longrightarrow \mathbb{R}$ on the Lie algebroid $\tau : E \longrightarrow M$ and a constraint subbundle D of E. We show that the known results in Mechanics for these systems also hold in the context of Lie algebroids. In particular, drawing analogies with d'Alembert principle, we derive the Lagrange-d'Alembert equations of motion, prove the conservation of energy and state a Noether's theorem. We also derive local expressions for the dynamics of nonholonomic Lagrangian systems, which are further simplified by the choice of a convenient basis of D. As an illustration, we consider the class of nonholonomic mechanical systems. For such systems, the Lagrangian L is the polar form of a bundle metric on E minus a potential function on M. In Section 4, we perform the analysis of the existence and uniqueness of solutions of constrained systems on general Lie algebroids, and extend the results in [2, 8, 9, 16, 34] for constrained systems evolving on tangent bundles. We obtain several characterizations for the regularity of a nonholonomic system, and prove that a nonholonomic system of mechanical type is always regular. The constrained dynamics can be obtained by projecting the unconstrained dynamics in two different ways. Under the first projection, we develop a distributional approach analogous to that in [2], see also [49]. Using the second projection, we introduce the nonholonomic bracket. The evolution of any observable can be measured by computing its bracket with the energy of the system. Section 5 is devoted to studying the reduction of the dynamics under symmetry. Our approach follows the ideas developed by H. Cendra, J.E. Marsden and T. Ratiu [14], who defined a minimal subcategory of the category of Lie algebroids which is stable under Lagrangian reduction. We study the behavior of the different geometric objects introduced under morphisms of Lie algebroids, and show that fiberwise surjective morphisms induce consistent reductions of the dynamics. This result covers, but does not reduce to, the usual case of reduction of the dynamics by a symmetry group. In accordance with the philosophy of the paper, we study first the unconstrained dynamics case, and obtain later the results for the constrained dynamics using projections. A (Poisson) reduction by stages procedure can also be developed within this formalism. It should be noticed that the reduction under the presence of a Lie group of symmetries G is performed in two steps: first we reduce by a normal subgroup N of G, and then by the residual group. In Section 6, we prove a general version of the momentum equation introduced in [4]; this equation shows the important role played by the kernel of the anchor map. In Section 7, we show some interesting examples and in Section 8, we extend some of the results previously obtained for linear constraints to the case of nonlinear constraints. The paper ends with our conclusions and a description of future research directions.

2. Preliminaries

In this section we recall some well-known facts concerning the geometry of Lie algebroids. We refer the reader to [6, 28, 42] for details about Lie groupoids, Lie algebroids and their role in differential geometry.

Lie algebroids. Let M be an n-dimensional manifold and let $\tau\colon E \to M$ be a vector bundle. A vector bundle map $\rho\colon E \to TM$ over the identity is called an anchor map. The vector bundle E together with an anchor map ρ is said to be an ***anchored vector bundle*** (see [52]). A structure of ***Lie algebroid*** on E is given by a Lie algebra structure on the $C^\infty(M)$-module of sections of the bundle, $(\mathrm{Sec}(E), [\cdot\,,\cdot])$, together with an anchor map, satisfying the compatibility condition

$$[\sigma, f\eta] = f[\sigma,\eta] + \big(\rho(\sigma)f\big)\eta.$$

Here f is a smooth function on M, σ, η are sections of E and $\rho(\sigma)$ denotes the vector field on M given by $\rho(\sigma)(m) = \rho(\sigma(m))$. From the compatibility condition and the Jacobi identity, it follows that the map $\sigma \mapsto \rho(\sigma)$ is a Lie algebra homomorphism from the set of sections of E, $\mathrm{Sec}(E)$, to the set of vector fields on M, $\mathfrak{X}(M)$.

In what concerns Mechanics, it is convenient to think of a Lie algebroid $\rho\colon E \to TM$, and more generally an anchored vector bundle, as a substitute of the tangent bundle of M. In this way, one regards an element a of E as a generalized velocity, and the actual velocity v is obtained when applying the anchor to a, i.e., $v = \rho(a)$. A curve $a\colon [t_0, t_1] \to E$ is said to be ***admissible*** if $\dot{m}(t) = \rho(a(t))$, where $m(t) = \tau(a(t))$ is the base curve. We will denote by $\mathrm{Adm}(E)$ the space of admissible curves on E.

Given local coordinates (x^i) in the base manifold M and a local basis $\{e_\alpha\}$ of sections of E, we have local coordinates (x^i, y^α) in E. If $a \in E$ is an element in the fiber over $m \in M$, then we can write $a = y^\alpha e_\alpha(m)$ and thus the coordinates of a are (m^i, y^α), where m^i are the coordinates of the point m. The anchor map is locally determined by the local functions ρ^i_α on M defined by $\rho(e_\alpha) = \rho^i_\alpha(\partial/\partial x^i)$. In addition, for a Lie algebroid, the Lie bracket is determined by the functions $C^\gamma_{\alpha\beta}$ defined by $[e_\alpha, e_\beta] = C^\gamma_{\alpha\beta} e_\gamma$. The functions ρ^i_α and $C^\gamma_{\alpha\beta}$ are called the structure functions of the Lie algebroid in this coordinate system. They satisfy the following relations

$$\rho^j_\alpha \frac{\partial \rho^i_\beta}{\partial x^j} - \rho^j_\beta \frac{\partial \rho^i_\alpha}{\partial x^j} = \rho^i_\gamma C^\gamma_{\alpha\beta} \quad \text{and} \quad \sum_{\mathrm{cyclic}(\alpha,\beta,\gamma)} \left[\rho^i_\alpha \frac{\partial C^\nu_{\beta\gamma}}{\partial x^i} + C^\mu_{\beta\gamma} C^\nu_{\alpha\mu} \right] = 0,$$

which are called the structure equations of the Lie algebroid.

Exterior differential. The anchor ρ allows to define the differential of a function on the base manifold with respect to an element $a \in E$. It is given by

$$df(a) = \rho(a) f.$$

It follows that the differential of f at the point $m \in M$ is an element of E^*_m. Moreover, a structure of Lie algebroid on E allows to extend the differential to sections of the bundle $\bigwedge^p E$, which will be called p-sections or just p-forms. If $\omega \in \mathrm{Sec}(\bigwedge^p E)$, then $d\omega \in \mathrm{Sec}(\bigwedge^{p+1} E)$ is defined by

$$\begin{aligned} d\omega(\sigma_0, \sigma_1, \dots, \sigma_p) = \sum_i (-1)^i \rho(\sigma_i)(\omega(\sigma_0, \dots, \widehat{\sigma}_i, \dots, \sigma_p)) \\ + \sum_{i<j} (-1)^{i+j} \omega([\sigma_i, \sigma_j], \sigma_0, \dots, \widehat{\sigma}_i, \dots, \widehat{\sigma}_j, \dots, \sigma_p). \end{aligned}$$

It follows that d is a cohomology operator, that is, $d^2 = 0$. Locally the exterior differential is determined by

$$dx^i = \rho^i_\alpha e^\alpha \qquad \text{and} \qquad de^\gamma = -\frac{1}{2} C^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta.$$

Throughout this paper, the symbol d will refer to the exterior differential on the Lie algebroid E and not to the ordinary exterior differential on a manifold. Of course, if $E = TM$, then both exterior differentials coincide.

The usual Cartan calculus extends to the case of Lie algebroids (see [42, 51]). For every section σ of E we have a derivation i_σ (contraction) of degree -1 and a derivation $d_\sigma = i_\sigma \circ d + d \circ i_\sigma$ (Lie derivative) of degree 0. Since $d^2 = 0$, we have that $d_\sigma \circ d = d \circ d_\sigma$.

Morphisms. Let $\tau\colon E \to M$ and $\tau'\colon E' \to M'$ be two anchored vector bundles, with anchor maps $\rho\colon E \to TM$ and $\rho'\colon E' \to TM'$. A vector bundle map $\Phi\colon E \to E'$ over a map $\varphi\colon M \to M'$ is said to be ***admissible*** if it maps admissible curves onto admissible curves, or equivalently $T\varphi \circ \rho = \rho' \circ \Phi$. If E and E' are Lie algebroids, then we say that Φ is a ***morphism*** if $\Phi^\star d\theta = d\Phi^\star\theta$ for every $\theta \in \mathrm{Sec}(\bigwedge E')$. It is easy to see that morphisms are admissible maps.

In the above expression, the pullback $\Phi^\star\beta$ of a p-form β is defined by

$$(\Phi^\star\beta)_m(a_1,a_2,\dots,a_p)=\beta_{\varphi(m)}\big(\Phi(a_1),\Phi(a_2),\dots,\Phi(a_p)\big),$$

for every $a_1,\dots,a_p\in E_m$. For a function $f\in C^\infty(M')$ (i.e., for $p=0$), we just set $\Phi^\star f=f\circ\varphi$.

Let (x^i) and (x'^i) be local coordinate systems on M and M', respectively. Let $\{e_\alpha\}$ and $\{e'_\alpha\}$ be local basis of sections of E and E', respectively, and $\{e^\alpha\}$ and $\{e'^\alpha\}$ the corresponding dual basis. The bundle map Φ is determined by the relations $\Phi^\star x'^i=\phi^i(x)$ and $\Phi^\star e'^\alpha=\phi^\alpha_\beta e^\beta$ for certain local functions ϕ^i and ϕ^α_β on M. Then, Φ is admissible if and only if

$$\rho^j_\alpha\frac{\partial\phi^i}{\partial x^j}=\rho'^i_\beta\phi^\beta_\alpha.$$

The map Φ is a morphism of Lie algebroids if and only if, in addition to the admissibility condition above, one has

$$\phi^\beta_\gamma C^\gamma_{\alpha\delta}=\left(\rho^i_\alpha\frac{\partial\phi^\beta_\delta}{\partial x^i}-\rho^i_\delta\frac{\partial\phi^\beta_\alpha}{\partial x^i}\right)+C'^\beta_{\theta\sigma}\phi^\theta_\alpha\phi^\sigma_\delta.$$

In these expressions, ρ^i_α, $C^\alpha_{\beta\gamma}$are the local structure functions on E and ρ'^i_α, $C'^\alpha_{\beta\gamma}$ are the local structure functions on E'.

Prolongation of a fibered manifold with respect to a Lie algebroid. Let $\pi\colon P\to M$ be a fibered manifold with base manifold M. Thinking of E as a substitute of the tangent bundle of M, the tangent bundle of P is not the appropriate space to describe dynamical systems on P. This is clear if we note that the projection to M of a vector tangent to P is a vector tangent to M, and what one would like instead is an element of E, the 'new' tangent bundle of M.

A space which takes into account this restriction is the ***E-tangent bundle*** of P, also called the ***prolongation*** of P with respect to E, which we denote by $\mathcal{T}^EP$ (see [37, 43, 45, 52]). It is defined as the vector bundle $\tau^E_P\colon\mathcal{T}^EP\to P$ whose fiber at a point $p\in P_m$ is the vector space

$$\mathcal{T}^E_pP=\{\,(b,v)\in E_m\times T_pP\mid\rho(b)=T_p\pi(v)\,\}.$$

We will frequently use the redundant notation (p,b,v) to denote the element $(b,v)\in\mathcal{T}^E_pP$. In this way, the map τ^E_P is just the projection onto the first factor. The anchor of $\mathcal{T}^EP$ is the projection onto the third factor, that is, the map $\rho^1\colon\mathcal{T}^EP\to TP$ given by $\rho^1(p,b,v)=v$. The projection onto the second factor will be denoted by $\mathcal{T}\pi\colon\mathcal{T}^EP\to E$, and it is a vector bundle map over π. Explicitly $\mathcal{T}\pi(p,b,v)=b$.

An element $z\in\mathcal{T}^EP$ is said to be vertical if it projects to zero, that is $\mathcal{T}\pi(z)=0$. Therefore it is of the form $(p,0,v)$, with v a vertical vector tangent to P at p.

Given local coordinates (x^i,u^A) on P and a local basis $\{e_\alpha\}$ of sections of E, we can define a local basis $\{\mathcal{X}_\alpha,\mathcal{V}_A\}$ of sections of $\mathcal{T}^EP$ by

$$\mathcal{X}_\alpha(p)=\Big(p,e_\alpha(\pi(p)),\rho^i_\alpha\frac{\partial}{\partial x^i}\Big|_p\Big)\qquad\text{and}\qquad\mathcal{V}_A(p)=\Big(p,0,\frac{\partial}{\partial u^A}\Big|_p\Big).$$

If $z=(p,b,v)$ is an element of $\mathcal{T}^EP$, with $b=z^\alpha e_\alpha$, then v is of the form $v=\rho^i_\alpha z^\alpha\frac{\partial}{\partial x^i}+v^A\frac{\partial}{\partial u^A}$, and we can write

$$z=z^\alpha\mathcal{X}_\alpha(p)+v^A\mathcal{V}_A(p).$$

Vertical elements are linear combinations of $\{\mathcal{V}_A\}$.

The anchor map ρ^1 applied to a section Z of $\mathcal{T}^EP$ with local expression $Z=Z^\alpha\mathcal{X}_\alpha+V^A\mathcal{V}_A$ is the vector field on P whose coordinate expression is

$$\rho^1(Z)=\rho^i_\alpha Z^\alpha\frac{\partial}{\partial x^i}+V^A\frac{\partial}{\partial u^A}.$$

If E carries a Lie algebroid structure, then so does $\mathcal{T}^E P$. The associated Lie bracket can be easily defined in terms of projectable sections, so that $\mathcal{T}\pi$ is a morphism of Lie algebroids. A section Z of $\mathcal{T}^E P$ is said to be projectable if there exists a section σ of E such that $\mathcal{T}\pi \circ Z = \sigma \circ \pi$. Equivalently, a section Z is projectable if and only if it is of the form $Z(p) = (p, \sigma(\pi(p)), X(p))$, for some section σ of E and some vector field X on E (which projects to $\rho(\sigma)$). The Lie bracket of two projectable sections Z_1 and Z_2 is then given by

$$[Z_1, Z_2](p) = (p, [\sigma_1, \sigma_2](m), [X_1, X_2](p)), \qquad p \in P, \quad m = \pi(p).$$

It is easy to see that $[Z_1, Z_2](p)$ is an element of $\mathcal{T}^E_p P$ for every $p \in P$. Since any section of $\mathcal{T}^E P$ can be locally written as a linear combination of projectable sections, the definition of the Lie bracket for arbitrary sections of $\mathcal{T}^E P$ follows.

The Lie brackets of the elements of the basis are

$$[\mathcal{X}_\alpha, \mathcal{X}_\beta] = C^\gamma_{\alpha\beta}\, \mathcal{X}_\gamma, \qquad [\mathcal{X}_\alpha, \mathcal{V}_B] = 0 \qquad \text{and} \qquad [\mathcal{V}_A, \mathcal{V}_B] = 0,$$

and the exterior differential is determined by

$$\begin{aligned} dx^i &= \rho^i_\alpha \mathcal{X}^\alpha, & du^A &= \mathcal{V}^A, \\ d\mathcal{X}^\gamma &= -\frac{1}{2} C^\gamma_{\alpha\beta} \mathcal{X}^\alpha \wedge \mathcal{X}^\beta, & d\mathcal{V}^A &= 0, \end{aligned}$$

where $\{\mathcal{X}^\alpha, \mathcal{V}^A\}$ is the dual basis corresponding to $\{\mathcal{X}_\alpha, \mathcal{V}_A\}$.

Prolongation of a map. Let $\Psi\colon P \to P'$ be a fibered map from the fibered manifold $\pi\colon P \to M$ to the fibered manifold $\pi'\colon P' \to M'$ over a map $\varphi\colon M \to M'$. Let $\Phi\colon E \to E'$ be an admissible map from $\tau\colon E \to M$ to $\tau'\colon E' \to M'$ over the same map φ. The prolongation of Φ with respect to Ψ is the mapping $\mathcal{T}^\Phi\Psi\colon \mathcal{T}^E P \to \mathcal{T}^{E'} P'$ defined by

$$\mathcal{T}^\Phi\Psi(p, b, v) = (\Psi(p), \Phi(b), (T_p\Psi)(v)).$$

It is clear from the definition that $\mathcal{T}^\Phi\Psi$ is a vector bundle map from $\tau^E_P\colon \mathcal{T}^E P \to P$ to $\tau^{E'}_{P'}\colon \mathcal{T}^{E'} P' \to P'$ over Ψ. Moreover, in [47] it is proved the following result.

Proposition 2.1. *The map $\mathcal{T}^\Phi\Psi$ is an admissible map. Moreover, $\mathcal{T}^\Phi\Psi$ is a morphism of Lie algebroids if and only if Φ is a morphism of Lie algebroids.*

Given local coordinate systems (x^i) on M and (x'^i) on M', local adapted coordinates (x^i, u^A) on P and (x'^i, u'^A) on P' and a local basis of sections $\{e_\alpha\}$ of E and $\{e'_\alpha\}$ of E', the maps Φ and Ψ are determined by $\Phi^\star e'^\alpha = \Phi^\alpha_\beta e^\beta$ and $\Psi(x, u) = (\phi^i(x), \psi^A(x, u))$. Then the action of $\mathcal{T}^\Phi\Psi$ is given by

$$\begin{aligned} (\mathcal{T}^\Phi\Psi)^\star \mathcal{X}'^\alpha &= \Phi^\alpha_\beta \mathcal{X}^\beta, \\ (\mathcal{T}^\Phi\Psi)^\star \mathcal{V}'^A &= \rho^i_\alpha \frac{\partial \psi^A}{\partial x^i} \mathcal{X}^\alpha + \frac{\partial \psi^A}{\partial u^B} \mathcal{V}^B. \end{aligned}$$

We finally mention that the composition of prolongation maps is the prolongation of the composition. Indeed, let Ψ' be another bundle map from $\pi'\colon P' \to M'$ to another bundle $\pi''\colon P'' \to M''$ and Φ' be another admissible map from $\tau'\colon E' \to M'$ to $\tau''\colon E'' \to M''$ both over the same base map. Since Φ and Φ' are admissible maps then so is $\Phi' \circ \Phi$, and thus we can define the prolongation of $\Psi' \circ \Psi$ with respect to $\Phi' \circ \Phi$. We have that $\mathcal{T}^{\Phi' \circ \Phi}(\Psi' \circ \Psi) = (\mathcal{T}^{\Phi'}\Psi') \circ (\mathcal{T}^\Phi\Psi)$.

In the particular case when the bundles P and P' are just $P = E$ and $P' = E'$, whenever we have an admissible map $\Phi\colon E \to E'$ we can define the prolongation of Φ along Φ itself, by $\mathcal{T}^\Phi\Phi(a, b, v) = (\Phi(a), \Phi(b), T\Phi(v))$. From the result above, we have that $\mathcal{T}^\Phi\Phi$ is a Lie algebroid morphism if and only if Φ is a Lie algebroid morphism. In coordinates we obtain

$$(\mathcal{T}^\Phi\Phi)^\star\mathcal{X}'^\alpha = \Phi^\alpha_\beta\mathcal{X}^\beta,$$
$$(\mathcal{T}^\Phi\Phi)^\star\mathcal{V}'^\alpha = \rho^i_\beta\frac{\partial\Phi^\alpha_\gamma}{\partial x^i}y^\gamma\mathcal{X}^\beta + \Phi^\alpha_\beta\mathcal{V}^\beta,$$

where (x^i, y^γ) are the corresponding fibred coordinates on E. From this expression it is clear that $\mathcal{T}^\Phi\Phi$ is fiberwise surjective if and only if Φ is fiberwise surjective.

Lagrangian Mechanics. In [43] a geometric formalism for Lagrangian Mechanics on Lie algebroids was defined. It is developed in the prolongation $\mathcal{T}^E E$ of a Lie algebroid E over itself. The canonical geometrical structures defined on $\mathcal{T}^E E$ are the following:

- The ***vertical lift*** $\xi^V : \tau^* E \to \mathcal{T}^E E$ given by $\xi^V(a, b) = (a, 0, b^V_a)$, where b^V_a is the vector tangent to the curve $a + tb$ at $t = 0$,
- The ***vertical endomorphism*** $S\colon \mathcal{T}^E E \to \mathcal{T}^E E$ defined as follows:
$$S(a, b, v) = \xi^V(a, b) = (a, 0, b^V_a),$$
- The ***Liouville section*** which is the vertical section corresponding to the Liouville dilation vector field:
$$\Delta(a) = \xi^V(a, a) = (a, 0, a^V_a).$$

Given a Lagrangian function $L \in C^\infty(E)$ we define the ***Cartan 1-form*** θ_L and the ***Cartan 2-form*** ω_L as the forms on $\mathcal{T}^E E$ given by

$$\theta_L = S^*(dL) \qquad \text{and} \qquad \omega_L = -d\theta_L. \tag{2.1}$$

The function L is said to be a ***regular Lagrangian*** if ω_L is regular at every point as a bilinear map. Finally, we define the ***energy function*** by $E_L = d_\Delta L - L$, in terms of which we set the ***Euler-Lagrange equations*** as

$$i_\Gamma\omega_L - dE_L = 0. \tag{2.2}$$

The local expressions for the vertical endomorphism, the Liouville section, the Cartan 2-form and the Lagrangian energy are

$$S\mathcal{X}_\alpha = \mathcal{V}_\alpha, \quad S\mathcal{V}_\alpha = 0, \quad \text{for all } \alpha, \tag{2.3}$$
$$\Delta = y^\alpha\mathcal{V}_\alpha, \tag{2.4}$$
$$\omega_L = \frac{\partial^2 L}{\partial y^\alpha\partial y^\beta}\mathcal{X}^\alpha\wedge\mathcal{V}^\beta + \frac{1}{2}\left(\frac{\partial^2 L}{\partial x^i\partial y^\alpha}\rho^i_\beta - \frac{\partial^2 L}{\partial x^i\partial y^\beta}\rho^i_\alpha + \frac{\partial L}{\partial y^\gamma}C^\gamma_{\alpha\beta}\right)\mathcal{X}^\alpha\wedge\mathcal{X}^\beta, \tag{2.5}$$
$$E_L = \frac{\partial L}{\partial y^\alpha}y^\alpha - L. \tag{2.6}$$

¿From (2.3), (2.4), (2.5) and (2.6), it follows that

$$i_{SX}\omega_L = -S^*(i_X\omega_L), \quad i_\Delta\omega_L = -S^*(dE_L), \tag{2.7}$$

for $X \in \mathrm{Sec}(\mathcal{T}^E E)$.

On the other hand, the vertical distribution is isotropic with respect to ω_L, see [37]. This fact implies that the contraction of ω_L with a vertical vector is a semibasic form. This property allows us to define a symmetric 2-tensor G^L along τ by

$$G^L_a(b, c) = \omega_L(\tilde{b}, c^V_a), \tag{2.8}$$

where $\tilde{b}$ is any element in $\mathcal{T}^E_a E$ which projects to b, i.e., $\mathcal{T}\tau(\tilde{b}) = b$, and $a \in E$. In coordinates $G^L = W_{\alpha\beta}e^\alpha\otimes e^\beta$, where the matrix $W_{\alpha\beta}$ is given by

$$W_{\alpha\beta} = \frac{\partial^2 L}{\partial y^\alpha\partial y^\beta}. \tag{2.9}$$

It is easy to see that the Lagrangian L is regular if and only the matrix W is regular at every point, that is if the tensor G^L is regular at every point. By the kernel of G^L at a point a we mean the vector space

$$\operatorname{Ker} G^L_a = \{\, b \in E_{\tau(a)} \,|\, G^L_a(b,c) = 0 \text{ for all } c \in E_{\tau(a)} \,\}.$$

In the case of a regular Lagrangian, the Cartan 2-section ω_L is symplectic (non-degenerate and d-closed) and the vertical subbundle is Lagrangian. It follows that a 1-form is semi-basic if and only if it is the contraction of ω_L with a vertical element.

In addition, if Γ_L is the solution of the dynamical equations (2.2) then, using (2.7), we deduce that

$$i_{S\Gamma_L}\omega_L = i_\Delta \omega_L$$

which implies that Γ_L is a SODE section, that is, it satisfies $S(\Gamma_L) = \Delta$, or alternatively $\mathcal{T}\tau(\Gamma_L(a)) = a$ for every $a \in E$.

In coordinates, a SODE Γ is a section of the form

$$\Gamma = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha.$$

The SODE Γ is a solution of the Euler-Lagrange equations if and only if the functions f^α satisfy the linear equations

$$\frac{\partial^2 L}{\partial y^\beta \partial y^\alpha} f^\beta + \frac{\partial^2 L}{\partial x^i \partial y^\alpha}\rho^i_\beta y^\beta + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta - \rho^i_\alpha \frac{\partial L}{\partial x^i} = 0, \text{ for all } \alpha. \tag{2.10}$$

The ***Euler-Lagrange differential equations*** are the differential equations for the integral curves of the vector field $\rho^1(\Gamma)$, where the section Γ is the solution of the Euler-Lagrange equations. Thus, these equations may be written as

$$\dot{x}^i = \rho^i_\alpha y^\alpha, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) - \rho^i_\alpha \frac{\partial L}{\partial x^i} + \frac{\partial L}{\partial y^\gamma} C^\gamma_{\alpha\beta} y^\beta = 0.$$

In other words, if $\delta L : \mathrm{Adm}(E) \to E^*$ is the ***Euler-Lagrange operator***, which locally reads

$$\delta L = \left(\frac{d}{dt}\left(\frac{\partial L}{\partial y^\alpha}\right) + C^\gamma_{\alpha\beta} y^\beta \frac{\partial L}{\partial y^\gamma} - \rho^i_\alpha \frac{\partial L}{\partial x^i}\right) e^\alpha,$$

where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$, then the Euler-Lagrange differential equations read

$$\delta L = 0.$$

Finally, we mention that the ***complete lift*** σ^C of a section $\sigma \in \mathrm{Sec}(E)$ is the section of $\mathcal{T}^E E$ characterized by the two following properties:

1. projects to σ, i.e., $\mathcal{T}\tau \circ \sigma^C = \sigma \circ \tau$,
2. $d_{\sigma^C}\hat{\mu} = \widehat{d_\sigma \mu}$,

where by $\hat{\alpha} \in C^\infty(E)$ we denote the linear function associated to a 1-section $\alpha \in \mathrm{Sec}(E^*)$.

3. Linearly constrained Lagrangian systems

Nonholonomic systems on Lie algebroids were introduced in [18]. This class of systems includes, as particular cases, standard nonholonomic systems defined on the tangent bundle of a manifold and systems obtained by the reduction of the action of a symmetry group. The situation is similar to the non-constrained case, where the general equation $\delta L = 0$ comprises as particular cases the standard Lagrangian Mechanics, Lagrangian Mechanics with symmetry, Lagrangian systems with holonomic constraints, systems on semi-direct products and systems evolving on Lie algebras, see e.g. [43].

We start with a free Lagrangian system on a Lie algebroid E. As mentioned above, these two objects can describe a wide class of systems. Now, we plug in some nonholonomic linear constraints described by a subbundle D of the bundle

E of admissible directions. If we impose to the solution curves $a(t)$ the condition to stay on the manifold D, we arrive at the equations $\delta L_{a(t)} = \lambda(t)$ and $a(t) \in D$, where the constraint force $\lambda(t) \in E^*_{\tau(a(t))}$ is to be determined. In the tangent bundle geometry case ($E = TM$), the d'Alembert principle establishes that the mechanical work done by the constraint forces vanishes, which implies that λ takes values in the annihilator of the constraint manifold D. Therefore, in the case of a general Lie algebroid, the natural equations one should pose are (see [18])

$$\delta L_{a(t)} \in D^\circ_{\tau(a(t))} \qquad \text{and} \qquad a(t) \in D.$$

In more explicit terms, we look for curves $a(t)$ on E such that

– they are admissible, $\rho(a(t)) = \dot{m}(t)$, where $m = \tau \circ a$,
– they stay in D, $a(t) \in D_{m(t)}$,
– there exists $\lambda(t) \in D^\circ_{m(t)}$ such that $\delta L_{a(t)} = \lambda(t)$.

If $a(t)$ is one of such curves, then $(a(t), \dot{a}(t))$ is a curve in $\mathcal{T}^E E$. Moreover, since $a(t)$ is in D, we have $\dot{a}(t)$ is tangent to D, that is, $(a(t), \dot{a}(t)) \in \mathcal{T}^D D$. Under some regularity conditions (to be made precise later on), we may assume that the above curves are integral curves of a section Γ, which as a consequence will be a SODE section taking values in $\mathcal{T}^D D$. Based on these arguments, we may reformulate geometrically our problem as the search for a SODE Γ (defined at least on a neighborhood of D) satisfying $(i_\Gamma \omega_L - dE_L)_a \in \widetilde{D^\circ_{\tau(a)}}$ and $\Gamma(a) \in \mathcal{T}^D D$, at every point $a \in D$. In the above expression $\widetilde{D^\circ}$ is the pullback of D° to $\mathcal{T}^E E$, that is, $\alpha \in \widetilde{D^\circ_a}$ if and only if there exists $\lambda \in D^\circ_{\tau(a)}$ such that $\alpha = \lambda \circ \mathcal{T}_a \tau$.

Definition 3.1. *A nonholonomically* ***constrained Lagrangian system*** *on a Lie algebroid E is a pair (L, D), where L is a smooth function on E,* ***the Lagrangian****, and $i\colon D \hookrightarrow E$ is a smooth subbundle of E, known as the* ***constraint subbundle****. By a solution of the nonholonomically constrained Lagrangian system (L, D) we mean a section $\Gamma \in \mathcal{T}^E E$ which satisfies the* ***Lagrange-d'Alembert equations***

$$\begin{aligned} &(i_\Gamma \omega_L - dE_L)|_D \in \operatorname{Sec}(\widetilde{D^\circ}), \\ &\Gamma|_D \in \operatorname{Sec}(\mathcal{T}^D D). \end{aligned} \tag{3.1}$$

With a slight abuse of language, we will interchangeably refer to a solution of the constrained Lagrangian system as a section or the collection of its corresponding integral curves. The restriction of the projection $\tau\colon E \to M$ to D will be denoted by π, that is, $\pi = \tau|_D \colon D \to M$.

Remark 3.2. We want to stress that a solution of the Lagrange-d'Alembert equations needs to be defined only over D, but for practical purposes we consider it extended to E (or just to a neighborhood of D in E). We will not make any notational distinction between a solution on D and any of its extensions. Solutions which coincide on D will be considered as equal. See [26, 34] for a more in-depth discussion. In accordance with this convention, by a SODE on D we mean a section of $\mathcal{T}^D D$ which is the restriction to D of some SODE defined in a neighborhood of D. Alternatively, a SODE on D is a section Γ of $\mathcal{T}^D D$ such that $\mathcal{T}\tau(\Gamma(a)) = a$ for every $a \in D$. $\diamond$

The different spaces we will consider are shown in the following commutative diagram

$$
\begin{tikzcd}
& TM \arrow[r, equal] & TM & \\
& D \arrow[u, "\rho_D"] \arrow[r, "i"] & E \arrow[u, "\rho"'] & \\
TD \arrow[uur, "T\pi"] \arrow[dr, "\tau_D"'] & \mathcal{T}^D D \arrow[l, "\rho^1"'] \arrow[u, "\mathcal{T}\pi"] \arrow[r, "\mathcal{I}"] \arrow[d, "\pi^D_D"] & \mathcal{T}^E E \arrow[u, "\mathcal{T}\tau"'] \arrow[r, "\rho^1"] \arrow[d, "\tau^E_E"'] & TE \arrow[uul, "T\tau"'] \arrow[dl, "\tau_E"] \\
& D \arrow[r, "i"] \arrow[d, "\pi"'] & E \arrow[d, "\tau"] & \\
& M \arrow[r, equal] & M &
\end{tikzcd}
$$

where $\mathcal{I} = \mathcal{T}^i i$ is the canonical inclusion of $\mathcal{T}^D D$ into $\mathcal{T}^E E$.

As an intermediate space in our analysis of the regularity of the constrained systems, we will also consider $\mathcal{T}^E D$, the E-tangent to D. The main difference between $\mathcal{T}^E D$ and $\mathcal{T}^D D$ is that the former has a natural Lie algebroid structure while the later does not.

Example 3.3. Let $\mathcal{G} : E \times_M E \to \mathbb{R}$ be a bundle metric on E. The ***Levi-Civita connection*** $\nabla^{\mathcal{G}}$ is determined by the formula

$$
\begin{aligned}
2\mathcal{G}(\nabla^{\mathcal{G}}_\sigma \eta, \zeta) = {}& \rho(\sigma)(\mathcal{G}(\eta,\zeta)) + \rho(\eta)(\mathcal{G}(\sigma,\zeta)) - \rho(\zeta)(\mathcal{G}(\eta,\sigma)) \\
& + \mathcal{G}(\sigma,[\zeta,\eta]) + \mathcal{G}(\eta,[\zeta,\sigma]) - \mathcal{G}(\zeta,[\eta,\sigma]),
\end{aligned}
$$

for $\sigma, \eta, \zeta \in \mathrm{Sec}(E)$. The coefficients of the connection $\nabla^{\mathcal{G}}$ are given by

$$
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}\mathcal{G}^{\alpha\nu}([\nu,\beta;\gamma] + [\nu,\gamma;\beta] + [\beta,\gamma;\nu]),
$$

where $\mathcal{G}_{\alpha\nu}$ are the coefficients of the metric $\mathcal{G}$, $(\mathcal{G}^{\alpha\nu})$ is the inverse matrix of $(\mathcal{G}_{\alpha\nu})$ and

$$
[\alpha,\beta;\gamma] = \frac{\partial \mathcal{G}_{\alpha\beta}}{\partial x^i}\rho^i_\gamma + C^\mu_{\alpha\beta}\mathcal{G}_{\mu\gamma}.
$$

Using the covariant derivate induced by $\nabla^{\mathcal{G}}$, one may introduce the notion of a geodesic of $\nabla^{\mathcal{G}}$ as follows. An admissible curve $a : I \to E$ is said to be a ***geodesic*** if $\nabla^{\mathcal{G}}_{a(t)} a(t) = 0$, for all $t \in I$. In local coordinates, the conditions for being a geodesic read

$$
\frac{da^\gamma}{dt} + \frac{1}{2}(\Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha})a^\alpha a^\beta = 0, \quad \text{for all } \gamma.
$$

The geodesics are the integral curves of a SODE section $\Gamma_{\nabla^{\mathcal{G}}}$ of $\mathcal{T}^E E$, which is locally given by

$$
\Gamma_{\nabla^{\mathcal{G}}} = y^\gamma \mathcal{X}_\gamma - \frac{1}{2}(\Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\beta\alpha})y^\alpha y^\beta \mathcal{V}_\gamma.
$$

$\Gamma_{\nabla^{\mathcal{G}}}$ is called the ***geodesic flow*** (for more details, see [18]).

The class of systems that were considered in detail in [18] is that of ***mechanical systems with nonholonomic constraints***[1]. The Lagrangian function L is of mechanical type, i.e., it is of the form

$$
L(a) = \frac{1}{2}\mathcal{G}(a,a) - V(\tau(a)), \quad a \in E,
$$

[1]In fact, in [18], we considered controlled mechanical systems with nonholonomic constraints, that is, mechanical systems evolving on Lie algebroids and subject to some external control forces.

with V a function on M.

The Euler-Lagrange section for the unconstrained system can be written as

$$\Gamma_L = \Gamma_{\nabla^{\mathcal{G}}} - (\operatorname{grad}_{\mathcal{G}} V)^V.$$

In this expression, by $\operatorname{grad}_{\mathcal{G}} V$ we mean the section of E such that $\langle dV(m), a\rangle = \mathcal{G}(\operatorname{grad}_{\mathcal{G}} V(m), a)$, for all $m \in M$ and all $a \in E_m$, and where we remind that d is the differential in the Lie algebroid. The Euler-Lagrange differential equations can be written as

$$\begin{aligned} \dot{x}^i &= \rho^i_\alpha y^\alpha, \\ \dot{y}^\alpha &= -\frac{1}{2}\left(\Gamma^\alpha_{\beta\gamma} + \Gamma^\alpha_{\gamma\beta}\right) y^\beta y^\gamma - \mathcal{G}^{\alpha\beta}\rho^i_\beta \frac{\partial V}{\partial x^i}. \end{aligned} \tag{3.2}$$

Alternatively, one can describe the dynamical behavior of the mechanical control system by means of an equation on E via the covariant derivative. An admissible curve $a : t \mapsto a(t)$ with base curve $t \mapsto m(t)$ is a solution of the system (3.2) if and only if

$$\nabla^{\mathcal{G}}_{a(t)} a(t) + \operatorname{grad}_{\mathcal{G}} V(m(t)) = 0. \tag{3.3}$$

Note that

$$\mathcal{G}(m(t))(\nabla^{\mathcal{G}}_{a(t)} a(t) + \operatorname{grad}_{\mathcal{G}} V(m(t)), b) = \delta L(a(t))(b), \qquad \text{for } b \in E_{m(t)}.$$

If this mechanical control system is subject to the constraints determined by a subbundle D of E, we can do the following. Consider the orthogonal decomposition $E = D \oplus D^\perp$, and the associated orthogonal projectors $P\colon E \to D$, $Q\colon E \to D^\perp$. Using the fact that $\mathcal{G}(P\cdot, \cdot) = \mathcal{G}(\cdot, P\cdot)$, one can write the Lagrange-d'Alembert equations in the form

$$P(\nabla^{\mathcal{G}}_{a(t)} a(t)) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = 0, \qquad Q(a) = 0.$$

A specially well-suited form of these equations makes use of the ***constrained connection*** $\check{\nabla}$ defined by $\check{\nabla}_\sigma \eta = P(\nabla^{\mathcal{G}}_\sigma \eta) + \nabla^{\mathcal{G}}_\sigma (Q\eta)$. In terms of $\check{\nabla}$, we can rewrite this equation as $\check{\nabla}_{a(t)} a(t) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = 0$, $Q(a) = 0$, where we have used the fact that the connection $\check{\nabla}$ restricts to the subbundle D.

Moreover, following the ideas in [39], we proved in [18] that the subbundle D is geodesically invariant for the connection $\check{\nabla}$, that is, any integral curve of the spray $\Gamma_{\check{\nabla}}$ associated with $\check{\nabla}$ starting from a point in D is entirely contained in D. Since the terms coming from the potential V also belongs to D, we have that the constrained equations of motion can be simply stated as

$$\check{\nabla}_{a(t)} a(t) + P(\operatorname{grad}_{\mathcal{G}} V(m(t))) = 0, \qquad a(0) \in D. \tag{3.4}$$

Note that one can write the constrained equations of the motion as follows

$$\dot{a}(t) = \rho^1(\Gamma_{\check{\nabla}}(a(t)) - P(\operatorname{grad}_{\mathcal{G}} V)^v(a(t)))$$

and that the restriction to D of the vector field $\rho^1(\Gamma_{\check{\nabla}} - P(\operatorname{grad}_{\mathcal{G}} V)^v)$ is tangent to D.

The coordinate expression of equations (3.4) is greatly simplified if we take a basis $\{e_\alpha\} = \{e_a, e_A\}$ of E adapted to the orthogonal decomposition $E = D \oplus D^\perp$, i.e., $D = \operatorname{span}\{e_a\}$, $D^\perp = \operatorname{span}\{e_A\}$. Denoting by $(y^\alpha) = (y^a, y^A)$ the induced coordinates, the constraint equations $Q(a) = 0$ just read $y^A = 0$. The differential equations of the motion are then

$$\begin{aligned} \dot{x}^i &= \rho^i_a y^a, \\ \dot{y}^a &= -\frac{1}{2}\left(\check{\Gamma}^a_{bc} + \check{\Gamma}^a_{cb}\right) y^b y^c - \mathcal{G}^{ab}\rho^i_b \frac{\partial V}{\partial x^i}, \\ y^A &= 0, \end{aligned}$$

where $\check{\Gamma}^\alpha_{\beta\gamma}$ are the connection coefficients of the constrained connection $\check{\nabla}$. ◁

In the above example the dynamics exists and is completely determined whatever the (linear) constraints are. As we will see later on, this property is lost in the general case. Let us start by analyzing the form of the Lagrange-d'Alembert equations in local coordinates. Following the example above, we will choose a special coordinate system adapted to the structure of the problem as follows. We consider local coordinates (x^i) on an open set $\mathcal{U}$ of M and we take a basis $\{e_a\}$ of local sections of D and complete it to a basis $\{e_a, e_A\}$ of local sections of E (both defined on the open $\mathcal{U}$). In this way, we have coordinates (x^i, y^a, y^A) on E. In this set of coordinates, the constraints imposed by the submanifold $D \subset E$ are simply $y^A = 0$. If $\{e^a, e^A\}$ is the dual basis of $\{e_a, e_A\}$, then a basis for the annihilator D° of D is $\{e^A\}$ and a basis for $\widetilde{D^\circ}$ is $\mathcal{X}^A$.

An element z of $\mathcal{T}^E D$ is of the form $z = u^\alpha \mathcal{X}_\alpha + z^a \mathcal{V}_a = u^a \mathcal{X}_a + u^A \mathcal{X}_A + z^a \mathcal{V}_a$, that is, the component $\mathcal{V}_A$ vanishes since $\rho^1(z)$ is a vector tangent to the manifold D with equations $y^A = 0$. The projection of z to E is $\mathcal{T}\tau(z) = u^a e_a + u^A e_A$, so that the element z is in $\mathcal{T}^D D$ if and only if $u^A = 0$. In other words, an element in $\mathcal{T}^D D$ is of the form $z = u^a \mathcal{X}_a + z^a \mathcal{V}_a$.

Let us find the local expression of the Lagrange-d'Alembert equations in these coordinates. We consider a section Γ such that $\Gamma|D \in \operatorname{Sec}(\mathcal{T}^D D)$, which is therefore of the form $\Gamma = g^a \mathcal{X}_a + f^a \mathcal{V}_a$. From the local expression (2.5) of the Cartan 2-form and the local expression (2.6) of the energy function, we get

$$0 = \langle i_\Gamma \omega_L - dE_L, \mathcal{V}_\alpha \rangle = -y^B \frac{\partial^2 L}{\partial y^\alpha \partial y^B} - (y^b - g^b) \frac{\partial^2 L}{\partial y^\alpha \partial y^b}.$$

If we assume that the Lagrangian L is regular, when we evaluate at $y^A = 0$, we have that $g^a = y^a$ and thus Γ is a SODE. Moreover, contracting with $\mathcal{X}_a$, after a few calculations we get

$$0 = \langle i_\Gamma \omega_L - dE_L, \mathcal{X}_a \rangle = -\left\{ d_\Gamma \left(\frac{\partial L}{\partial y^a} \right) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{a\beta} y^\beta - \rho^i_a \frac{\partial L}{\partial x^i} \right\},$$

so that (again after evaluation at $y^A = 0$), the functions f^a are solution of the linear equations

$$\frac{\partial^2 L}{\partial y^b \partial y^a} f^b + \frac{\partial^2 L}{\partial x^i \partial y^a} \rho^i_b y^b + \frac{\partial L}{\partial y^\gamma} C^\gamma_{ab} y^b - \rho^i_a \frac{\partial L}{\partial x^i} = 0, \tag{3.5}$$

where all the partial derivatives of the Lagrangian are to be evaluated on $y^A = 0$.

As a consequence, we get that there exists a unique solution of the Lagrange-d'Alembert equations if and only if the matrix

$$\mathcal{C}_{ab}(x^i, y^c) = \frac{\partial^2 L}{\partial y^a \partial y^b}(x^i, y^c, 0) \tag{3.6}$$

is regular. Notice that $\mathcal{C}_{ab}$ is a submatrix of $W_{\alpha\beta}$, evaluated at $y^A = 0$. The differential equations for the integral curves of the vector field $\rho^1(\Gamma)$ are the Lagrange-d'Alembert differential equations, which read

$$\begin{aligned}
&\dot{x}^i = \rho^i_a y^a, \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial y^a} \right) + \frac{\partial L}{\partial y^\gamma} C^\gamma_{ab} y^b - \rho^i_a \frac{\partial L}{\partial x^i} = 0, \\
&y^A = 0.
\end{aligned} \tag{3.7}$$

Finally, notice that the contraction with $\mathcal{X}_A$ just gives the components $\lambda_A = \langle i_\Gamma \omega_L - dE_L, \mathcal{X}_A \rangle|_{y^A=0}$ of the constraint forces $\lambda = \lambda_A e^A$.

Remark 3.4. In some occasions, it is useful to write the equations in the form

$$\begin{aligned} &\dot{x}^i = \rho^i_a y^a, \\ &\frac{d}{dt}\left(\frac{\partial L}{\partial y^a}\right) + \frac{\partial L}{\partial y^c} C^c_{ab} y^b - \rho^i_a \frac{\partial L}{\partial x^i} = -\frac{\partial L}{\partial y^A} C^A_{ab} y^b, \\ &y^A = 0, \end{aligned} \tag{3.8}$$

where, on the left-hand side of the second equation, all the derivatives can be calculated from the value of the Lagrangian on the constraint submanifold D. In other words, we can substitute L by the constrained Lagrangian L_c defined by $L_c(x^i, y^a) = L(x^i, y^a, 0)$. $\diamond$

Remark 3.5. A particular case of this construction is given by constrained systems defined in the standard Lie algebroid $\tau_M \colon TM \to M$. In this case, the equations (3.7) are the Lagrange-d'Alembert equations written in quasicoordinates, where $C^\alpha_{\beta\gamma}$ are the so-called Hamel's transpositional symbols, which obviously are nothing but the structure coefficients (in the Cartan's sense) of the moving frame $\{e_\alpha\}$, see e.g. [23, 27]. $\diamond$

The following two results are immediate consequences of the above form of the Lagrange-d'Alembert equations.

Theorem 3.6 (Conservation of energy)**.** *If (L, D) is a constrained Lagrangian system and Γ is a solution of the dynamics, then $d_\Gamma E_L = 0$ (on D).*

Proof. Indeed, for every $a \in D$, we have $\Gamma(a) \in \mathcal{T}^D_a D$, so that $\mathcal{T}\tau(\Gamma(a)) \in D$. Therefore $i_\Gamma \widetilde{D^\circ} = 0$ and contracting $0 = i_\Gamma(i_\Gamma \omega_L - dE_L) = -d_\Gamma E_L$ at every point in D. $\square$

Theorem 3.7 (Noether's theorem)**.** *Let (L, D) be a constrained Lagrangian system which admits a unique* SODE *Γ solution of the dynamics. If σ is a section of D such that there exists a function $f \in C^\infty(M)$ satisfying*

$$d_{\sigma^C} L = \dot{f},$$

then the function $F = \langle \theta_L, \sigma^C \rangle - f$ is a constant of the motion, that is, $d_\Gamma F = 0$ (on D).

Proof. Using that $\Theta_L(\Gamma) = d_\Delta(L)$, we obtain $i_{\sigma^C}(i_\Gamma \omega_L - dE_L) = i_{\sigma^C}(-d_\Gamma \Theta_L + dL) = d_{\sigma^C} L - d_\Gamma \langle \theta_L, \sigma^C \rangle + \Theta_L[\Gamma, \sigma^C]$ and, since $[\Gamma, \sigma^C]$ is vertical, we deduce

$$i_{\sigma^c}(i_\Gamma \omega_L - dE_L) = d_{\sigma^c} L - d_\Gamma \langle \theta_L, \sigma^C \rangle .$$

Thus, taking into account that $i_{\sigma^C} \widetilde{D^\circ} = 0$, we get $0 = d_\Gamma(\langle \theta_L, \sigma^C \rangle - f) = -d_\Gamma F$. $\square$

4. Solution of Lagrange-d'Alembert equations

Assumption 4.1. *In what follows, we will assume that the Lagrangian L is regular at least in a neighborhood of D.*

Let us now perform a precise global analysis of the existence and uniqueness of the solution of Lagrange-d'Alembert equations.

Definition 4.2. *A constrained Lagrangian system (L, D) is said to be* ***regular*** *if the Lagrange-d'Alembert equations have a unique solution.*

In order to characterize geometrically those nonholonomic systems which are regular, we define the tensor $G^{L,D}$ as the restriction of G^L to D, that is, $G^{L,D}_a(b, c) = G^L_a(b, c)$ for every $a \in D$ and every $b, c \in D_{\tau(a)}$. In coordinates adapted to D, we have that the local expression of $G^{L,D}$ is $G^{L,D} = \mathcal{C}_{ab} e^a \otimes e^b$ where the matrix $\mathcal{C}_{ab}$ is given by equation (3.6).

A second important geometric object is the subbundle $F \subset \mathcal{T}^E E|_D \to D$ whose fiber at the point $a \in D$ is $F_a = \omega_L^{-1}(\widetilde{D^\circ_{\tau(a)}})$. More explicitly,

$$F_a = \{\, z \in \mathcal{T}^E_a E \,|\, \text{exists } \zeta \in D^\circ_{\tau(a)} \text{ s.t. } \omega_L(z,u) = \langle \zeta, \mathcal{T}\tau(u)\rangle \text{ for all } u \in \mathcal{T}^E_a E \,\}.$$

¿From the definition, it is clear that the rank of F is $\operatorname{rank}(F) = \operatorname{rank}(D^\circ) = \operatorname{rank}(E) - \operatorname{rank}(D)$.

Finally, we also consider the subbundle $(\mathcal{T}^D D)^\perp \subset \mathcal{T}^E E|_D \to D$, the orthogonal to $\mathcal{T}^D D$ with respect to the symplectic form ω_L. The rank of $(\mathcal{T}^D D)^\perp$ is $\operatorname{rank}(\mathcal{T}^D D)^\perp = \operatorname{rank}(\mathcal{T}^E E) - \operatorname{rank}(\mathcal{T}^D D) = 2(\operatorname{rank}(E) - \operatorname{rank}(D)) = 2\operatorname{rank}(D^\circ)$.

The relation among these three objects is described by the following result.

Lemma 4.3. *The following properties are satisfied:*

1. *The elements in F are vertical. An element $\xi^V(a,b) \in F_a$ if and only if $G^L_a(b,c) = 0$ for all $c \in D_{\tau(a)}$.*
2. *$(\mathcal{T}^D D)^\perp \cap \operatorname{Ver}(\mathcal{T}^E E) = F$.*

Proof. (1) The elements in F are vertical because the elements in $\widetilde{D^\circ}$ are semi-basic. If $\xi^V(a,b) \in F_a$ then there exists $\zeta \in D^\circ_{\tau(a)}$ such that $\omega_L(\xi^V(a,b),u) = \langle \zeta, \mathcal{T}\tau(u)\rangle$ for all $u \in \mathcal{T}^E_a E$. In terms of G^L and writing $c = \mathcal{T}\tau(u)$, the above equation reads $-G^L_a(b,c) = \langle \zeta, c\rangle$. By taking $u \in \mathcal{T}\tau^{-1}(D)$, then c is in D and therefore $G^L_a(b,c) = 0$ for all $c \in D_{\tau(a)}$. Conversely, if $G^L_a(b,c) = 0$ for all $c \in D_{\tau(a)}$, then the 1-form $\zeta = -G^L_a(b,\ )$ is in $D^\circ_{\tau(a)}$. Therefore $\omega_L(\xi^V(a,b),u) = -G^L_a(b,\mathcal{T}\tau(u)) = \langle \zeta, \mathcal{T}\tau(u)\rangle)$, which is the condition for $\xi^V(a,b) \in F_a$.
(2) The condition for a vertical element $\xi^V(a,b)$ to be in $(\mathcal{T}^D D)^\perp$ is $\omega_L(\xi^V(a,b),w) = 0$ for all $w \in \mathcal{T}^D_a D$, or equivalently, $G^L_a(b,\mathcal{T}\tau(w)) = 0$. The vector $c = \mathcal{T}\tau(w)$ is an arbitrary element of $D_{\tau(a)}$, so that the above condition reads $G^L_a(b,c) = 0$, for all $c \in D_{\tau(a)}$, which is precisely the condition for $\xi^V(a,b)$ to be in F_a. □

Theorem 4.4. *The following properties are equivalent:*

1. *The constrained Lagrangian system (L,D) is regular,*
2. $\operatorname{Ker} G^{L,D} = \{0\}$,
3. $\mathcal{T}^E D \cap F = \{0\}$,
4. $\mathcal{T}^D D \cap (\mathcal{T}^D D)^\perp = \{0\}$.

Proof. [(1)⇔(2)] The equivalence between the first two conditions is clear from the local form of the Lagrange-d'Alembert equations (3.5), since the coefficients of the unknowns f^a are precisely the components (3.6) of $G^{L,D}$.

[(2)⇔(3)] (⇒) Let $a \in D$ and consider an element $z \in \mathcal{T}^E_a D \cap F_a$. Since the elements of F are vertical, we have $z = \xi^V(a,b)$ for some $b \in E_{\tau(a)}$. Moreover, $z \in \mathcal{T}^E_a D$ implies that b is an element in $D_{\tau(a)}$. On the other hand, if $z = \xi^V(a,b)$ is in F_a, then Lemma 4.3 implies that $G^L_a(b,c) = 0$ for all $c \in D_{\tau(a)}$. Thus $G^{L,D}_a(b,c) = 0$ for all $c \in D_{\tau(a)}$, from where $b = 0$, and hence $z = 0$.

(⇐) Conversely, if for some $a \in D$, there exists $b \in \operatorname{Ker} G^{L,D}_a$ with $b \neq 0$ then, using Lemma 4.3, we deduce that $z = \xi^V(a,b) \in \mathcal{T}^E_a D \cap F_a$ and $z \neq 0$.

[(2)⇔(4)] (⇒) Let $a \in D$ and consider an element $v \in \mathcal{T}^D_a D \cap (\mathcal{T}^D_a D)^\perp$, that is, $\omega_L(v,w) = 0$ for all $w \in \mathcal{T}^D_a D$. If we take $w = \xi^V(a,b)$ for $b \in D_{\tau(a)}$ arbitrary, then we have $\omega_L(v,\xi^V(a,b)) = G^{L,D}_a(\mathcal{T}\tau(v),b) = 0$ for all $b \in D_{\tau(a)}$, from where it follows that $\mathcal{T}\tau(v) = 0$. Thus v is vertical, $v = \xi^V(a,c)$, for some $c \in D$ and then $\omega_L(\xi^V(a,c),w) = -G^{L,D}_a(c,\mathcal{T}\tau(w)) = 0$ for all $w \in \mathcal{T}^D_a D$. Therefore $c = 0$ and hence $v = 0$.

(⇐) Conversely, if for some $a \in D$, there exists $b \in \operatorname{Ker} G^{L,D}_a$ with $b \neq 0$, then $0 \neq \xi^V(a,b) \in \mathcal{T}^D_a D \cap (\mathcal{T}^D_a D)^\perp$, because $\omega_L(\xi^V(a,b),w) = G^{L,D}(b,\mathcal{T}\tau(w)) = 0$ for all $w \in \mathcal{T}^D_a D$. □

In the case of a constrained mechanical system, the tensor G^L is given by $G^L_a(b,c) = \mathcal{G}_{\tau(a)}(b,c)$, so that it is positive definite at every point. Thus the restriction to any subbundle D is also positive definite and hence regular. Thus, nonholonomic mechanical systems are always regular.

Proposition 4.5. *Conditions (3) and (4) in Theorem 4.4 are equivalent, respectively, to*

(3') $\mathcal{T}^E E|_D = \mathcal{T}^E D \oplus F$,
(4') $\mathcal{T}^E E|_D = \mathcal{T}^D D \oplus (\mathcal{T}^D D)^\perp$.

Proof. The equivalence between (4) and (4') is obvious, since we are assuming that the free Lagrangian is regular, i.e., ω_L is symplectic. The equivalence of (3) and (3') follows by computing the dimension of the corresponding spaces. The ranks of $\mathcal{T}^E E$, $\mathcal{T}^E D$ and F are

$$\begin{aligned}
&\operatorname{rank}(\mathcal{T}^E E) = 2\operatorname{rank}(E),\\
&\operatorname{rank}(\mathcal{T}^E D) = \operatorname{rank}(E) + \operatorname{rank}(D),\\
&\operatorname{rank}(F) = \operatorname{rank}(D^\circ) = \operatorname{rank}(E) - \operatorname{rank}(D).
\end{aligned}$$

Thus $\operatorname{rank}(\mathcal{T}^E E) = \operatorname{rank}(\mathcal{T}^E D) + \operatorname{rank}(F)$, and the result follows. □

4.1. **Projectors.** We can express the constrained dynamical section in terms of the free dynamical section by projecting to the adequate space, either $\mathcal{T}^E D$ or $\mathcal{T}^D D$, according to each of the above decompositions of $\mathcal{T}^E E|_D$. Of course, both procedures give the same result.

Projection to $\mathcal{T}^E D$. Assuming that the constrained system is regular, we have a direct sum decomposition

$$\mathcal{T}^E_a E = \mathcal{T}^E_a D \oplus F_a,$$

for every $a \in D$, where we recall that the subbundle $F \subset \mathcal{T}^E D$ is defined by $F = \omega_L^{-1}(\widetilde{D^\circ})$, or equivalently $\widetilde{D^\circ} = \omega_L(F)$.

Let us denote by P and Q the complementary projectors defined by this decomposition, that is,

$$P_a\colon \mathcal{T}^E_a E \to \mathcal{T}^E_a D \quad\text{and}\quad Q_a\colon \mathcal{T}^E_a E \to F_a,\ \text{for all } a \in D.$$

Then we have,

Theorem 4.6. *Let (L,D) be a regular constrained Lagrangian system and let Γ_L be the solution of the free dynamics, i.e., $i_{\Gamma_L}\omega_L = dE_L$. Then the solution of the constrained dynamics is the* SODE *Γ obtained by projection $\Gamma = P(\Gamma_L|_D)$.*

Proof. Indeed, if we write $\Gamma(a) = \Gamma_L(a) - Q(\Gamma_L(a))$ for $a \in D$, then we have

$$i_{\Gamma(a)}\omega_L - dE_L(a) = i_{\Gamma_L(a)}\omega_L - i_{Q(\Gamma_L(a))}\omega_L - dE_L(a) = -i_{Q(\Gamma_L(a))}\omega_L \in \widetilde{D^\circ_{\tau(a)}},$$

which is an element of $\widetilde{D^\circ_{\tau(a)}}$ because $Q(\Gamma_L(a))$ is in F_a. Moreover, since Γ_L is a SODE and $Q(\Gamma_L)$ is vertical (since it is in F), we have that Γ is also a SODE. □

We consider adapted local coordinates (x^i, y^a, y^A) corresponding to the choice of an adapted basis of sections $\{e_a, e_A\}$, where $\{e_a\}$ generate D. The annihilator D° of D is generated by $\{e^A\}$, and thus $\widetilde{D^\circ}$ is generated by $\{\mathcal{X}^A\}$. A simple calculation shows that a basis $\{Z_A\}$ of local sections of F is given by

$$Z_A = \mathcal{V}_A - Q^a_A \mathcal{V}_a, \tag{4.1}$$

where $Q^a_A = W_{Ab}\mathcal{C}^{ab}$ and $\mathcal{C}^{ab}$ are the components of the inverse of the matrix $\mathcal{C}_{ab}$ given by equation (3.6). The local expression of the projector over F is then

$$Q = Z_A \otimes \mathcal{V}^A.$$

If the expression of the free dynamical section Γ_L in this local coordinates is

$$\Gamma_L = y^\alpha \mathcal{X}_\alpha + f^\alpha \mathcal{V}_\alpha,$$

(where f^α are given by equation (2.10)), then the expression of the constrained dynamical section is

$$\Gamma = y^a \mathcal{X}_a + (f^a + f^A Q^a_A)\mathcal{V}_a,$$

where all the functions f^α are evaluated at $y^A = 0$.

Projection to $\mathcal{T}^D D$. We have seen that the regularity condition for the constrained system (L, D) can be equivalently expressed by requiring that the subbundle $\mathcal{T}^D D$ is a symplectic subbundle of $(\mathcal{T}^E E, \omega_L)$. It follows that, for every $a \in D$, we have a direct sum decomposition

$$\mathcal{T}^E_a E = \mathcal{T}^D_a D \oplus (\mathcal{T}^D_a D)^\perp.$$

Let us denote by $\bar{P}$ and $\bar{Q}$ the complementary projectors defined by this decomposition, that is,

$$\bar{P}_a \colon \mathcal{T}^E_a E \to \mathcal{T}^D_a D \qquad \text{and} \qquad \bar{Q}_a \colon \mathcal{T}^E_a E \to (\mathcal{T}^D_a D)^\perp, \quad \text{for all } a \in D.$$

Then, we have the following result:

Theorem 4.7. *Let (L, D) be a regular constrained Lagrangian system and let Γ_L be the solution of the free dynamics, i.e., $i_{\Gamma_L}\omega_L = dE_L$. Then the solution of the constrained dynamics is the* SODE *Γ obtained by projection $\Gamma = \bar{P}(\Gamma_L|_D)$.*

Proof. From Theorem 4.6 we have that the solution Γ of the constrained dynamics is related to the free dynamics by $\Gamma_L|_D = \Gamma + Q(\Gamma_L|_D)$. Let us prove that $Q(\Gamma_L|_D)$ takes values in $(\mathcal{T}^D D)^\perp$. Indeed, $Q(\Gamma_L|_D)$ takes values in $F = (\mathcal{T}^D D)^\perp \cap \operatorname{Ver}(\mathcal{T}^E E)$, so that, in particular, it takes values in $(\mathcal{T}^D D)^\perp$. Thus, since Γ is a section of $\mathcal{T}^D D$, it follows that $\Gamma_L|_D = \Gamma + Q(\Gamma_L|_D)$ is a decomposition of $\Gamma_L|_D$ according to $\mathcal{T}^E E|_D = \mathcal{T}^D D \oplus (\mathcal{T}^D D)^\perp$, which implies $\Gamma = \bar{P}(\Gamma_L|_D)$. $\square$

In adapted coordinates, a local basis of sections of $(\mathcal{T}^D D)^\perp$ is $\{Y_A, Z_A\}$, where the sections Z_A are given by (4.1) and the sections Y_A are

$$Y_A = \mathcal{X}_A - Q^a_A \mathcal{X}_a + \mathcal{C}^{bc}(M_{Ab} - M_{ab}Q^a_A)\mathcal{V}_c,$$

with $M_{\alpha\beta} = \omega_L(\mathcal{X}_\alpha, \mathcal{X}_\beta)$. Therefore the expression of the projector onto $(\mathcal{T}^D D)^\perp$ is

$$\bar{Q} = Z_A \otimes \mathcal{V}^A + Y_A \otimes \mathcal{X}^A.$$

Note that $S(Y_A) = Z_A$.

4.2. **The distributional approach.** The equations for the Lagrange-d'Alembert section Γ can be entirely written in terms of objects in the manifold $\mathcal{T}^D D$. Recall that $\mathcal{T}^D D$ is not a Lie algebroid. In order to do this, define the 2-section $\omega^{L,D}$ as the restriction of ω_L to $\mathcal{T}^D D$. If (L, D) is regular, then $\mathcal{T}^D D$ is a symplectic subbundle of $(\mathcal{T}^E E, \omega_L)$. From this, it follows that $\omega^{L,D}$ is a symplectic section on that bundle. We also define $\varepsilon^{L,D}$ to be the restriction of dE_L to $\mathcal{T}^D D$. Then, taking the restriction of Lagrange-d'Alembert equations to $\mathcal{T}^D D$, we get the following equation

$$i_\Gamma \omega^{L,D} = \varepsilon^{L,D}, \tag{4.2}$$

which uniquely determines the section Γ. Indeed, the unique solution Γ of the above equations is the solution of Lagrange-d'Alembert equations: if we denote by λ the constraint force, we have for every $u \in \mathcal{T}^D_a D$ that

$$\omega_L(\Gamma(a), u) - \langle dE_L(a), u\rangle = \langle \lambda(a), \mathcal{T}\tau(u)\rangle = 0,$$

where we have taken into account that $\mathcal{T}\tau(u) \in D$ and $\lambda(a) \in D^\circ$.

This approach, the so called ***distributional approach***, was initiated by Bocharov and Vinogradov (see [59]) and further developed by Śniatycki and coworkers [2, 22, 55]. Similar equations, within the framework of Lie algebroids, are the base of the theory proposed in [49].

Remark 4.8. One can also consider the restriction to $\mathcal{T}^E D$, which is a Lie algebroid, but no further simplification is achieved by this. If $\bar{\omega}$ is the restriction of ω_L to $\mathcal{T}^E D$ and $\bar{\varepsilon}$ is the restriction of dE_L to $\mathcal{T}^E D$, then the Lagrange-d'Alembert equations can be written in the form $i_\Gamma \bar{\omega} - \bar{\varepsilon} = \bar{\lambda}$, where $\bar{\lambda}$ is the restriction of the constraint force to $\mathcal{T}^E D$, which, in general, does not vanish. Also notice that the 2-form $\bar{\omega}$ is closed but, in general, degenerated. $\diamond$

4.3. **The nonholonomic bracket.** Let f, g be two smooth functions on D and take arbitrary extensions to E denoted by the same letters (if there is no possibility of confusion). Suppose that X_f and X_g are the Hamiltonian sections on $\mathcal{T}^E E$ given respectively by

$$i_{X_f}\,\omega_L = df \qquad \text{and} \qquad i_{X_g}\,\omega_L = dg.$$

We define the ***nonholonomic bracket*** of f and g as follows:

$$\{f, g\}_{nh} = \omega_L(\bar{P}(X_f), \bar{P}(X_g)). \tag{4.3}$$

Note that if f' is another extension of f, then $(X_f - X_{f'})_{|D}$ is a section of $(\mathcal{T}^D D)^\perp$ and, thus, we deduce that (4.3) does not depend on the chosen extensions. The nonholonomic bracket is an almost-Poisson bracket, i.e., it is skew-symmetric, a derivation in each argument with respect to the usual product of functions and does not satisfy the Jacobi identity.

In addition, one can prove the following formula

$$\dot{f} = \{f, E_L\}_{nh}. \tag{4.4}$$

Indeed, we have

$$\begin{aligned}\dot{f} &= d_\Gamma f = i_\Gamma df = i_\Gamma i_{X_f}\omega_L \\ &= \omega_L(X_f, \Gamma) = \omega_L(X_f, \bar{P}(\Gamma_L)) \\ &= \omega_L(\bar{P}(X_f), \bar{P}(\Gamma_L)) = \{f, E_L\}_{nh}.\end{aligned}$$

Equation (4.4) implies once more the conservation of the energy (by the skew-symmetric character of the nonholonomic bracket).

Alternatively, since $\mathcal{T}^D D$ is an anchored vector bundle, one can take the function $f \in C^\infty(D)$ and its differential $\bar{d}f \in \operatorname{Sec}((\mathcal{T}^D D)^*)$. Since $\omega^{L,D}$ is regular, we have a unique section $\bar{X}_f \in \operatorname{Sec}(\mathcal{T}^D D)$ defined by $i_{\bar{X}_f}\omega^{L,D} = \bar{d}f$. Then the nonholonomic bracket of two functions f and g is $\{f, g\}_{nh} = \omega^{L,D}(\bar{X}_f, \bar{X}_g)$. Note that if $\tilde{f} \in C^\infty(E)$ (resp. $\tilde{g} \in C^\infty(E)$) is an extension to E of f (resp., g), then $\bar{X}_f = \bar{P}(X_{\tilde{f}})_{|D}$ (resp., $\bar{X}_g = \bar{P}(X_{\tilde{g}})_{|D}$).

5. Morphisms and reduction

One important advantage of dealing with Lagrangian systems evolving on Lie algebroids is that the reduction procedure can be naturally handled by considering morphisms of Lie algebroids, as it was already observed by Weinstein [60]. We study in this section the transformation laws of the different geometric objects in our theory and we apply these results to the study of the reduction theory.

Proposition 5.1. *Let $\Phi\colon E \to E'$ be a morphism of Lie algebroids, and consider the Φ-tangent prolongation of Φ, i.e $\mathcal{T}^\Phi\Phi\colon \mathcal{T}^E E \to \mathcal{T}^{E'} E'$. Let $\xi^{\rm V}$ and $\xi'^{\rm V}$, S and S', and Δ and Δ', be the vertical liftings, the vertical endomorphisms and the Liouville sections on E and E', respectively. Then,*

(1) $\mathcal{T}^\Phi\Phi(\xi^V(a,b)) = \xi'^V(\Phi(a),\Phi(b))$, *for all* $(a,b)\in E\times_M E$,
(2) $\mathcal{T}^\Phi\Phi\circ\Delta = \Delta'\circ\Phi$,
(3) $\mathcal{T}^\Phi\Phi\circ S = S'\circ\mathcal{T}^\Phi\Phi$.

Proof. For the first property, we notice that both terms are vertical, so that we just have to show that their action on functions coincide. For every function $f'\in C^\infty(E')$, we have

$$\begin{aligned}\rho'^1(\mathcal{T}^\Phi\Phi(\xi^V(a,b)))f' &= T\Phi(\rho^1(\xi^V(a,b)))f' = T\Phi(b_a^V)f' = b_a^V(f'\circ\Phi)\\ &= \frac{d}{dt}f'(\Phi(a+tb))\Big|_{t=0} = \frac{d}{dt}f'(\Phi(a)+t\Phi(b))\Big|_{t=0}\\ &= \Phi(b)^V_{\Phi(a)}(f') = \rho'^1(\xi'^V(\Phi(a),\Phi(b)))f'.\end{aligned}$$

For the second property, we have $\Delta(a)=\xi^V(a,a)$ so that applying the first property we have

$$\mathcal{T}^\Phi\Phi(\Delta(a)) = \mathcal{T}^\Phi\Phi(\xi^V(a,a)) = \xi'^V(\Phi(a),\Phi(a)) = \Delta'(\Phi(a)).$$

Finally, for any $z=(a,b,V)\in\mathcal{T}^EE$, we have

$$\begin{aligned}\mathcal{T}^\Phi\Phi(S(z)) &= \mathcal{T}^\Phi\Phi(\xi^V(a,b)) = \xi'^V(\Phi(a),\Phi(b))\\ &= S'(\Phi(a),\Phi(b),T\Phi(V)) = S'(\mathcal{T}^\Phi\Phi(z)),\end{aligned}$$

which concludes the proof. □

Proposition 5.2. *Let $L\in C^\infty(E)$ be a Lagrangian function, θ_L the Cartan form and $\omega_L=-d\theta_L$. Let $\Phi\colon E\to E'$ be a Lie algebroid morphism and suppose that $L=L'\circ\Phi$, with $L'\in C^\infty(E')$ a Lagrangian function. Then, we have*
(1) $(\mathcal{T}^\Phi\Phi)^\star\theta_{L'}=\theta_L$,
(2) $(\mathcal{T}^\Phi\Phi)^\star\omega_{L'}=\omega_L$,
(3) $(\mathcal{T}^\Phi\Phi)^\star E_{L'}=E_L$,
(4) $G^{L'}_{\phi(a)}(\Phi(b),\Phi(c)) = G^L_a(b,c)$, *for every $a\in E$ and every $b,c\in E_{\tau(a)}$.*

Proof. Indeed, for every $Z\in\mathcal{T}^EE$ we have

$$\begin{aligned}\big\langle(\mathcal{T}^\Phi\Phi)^\star\theta_{L'},Z\big\rangle &= \big\langle\theta_{L'},\mathcal{T}^\Phi\Phi(Z)\big\rangle = \big\langle dL',S'(\mathcal{T}^\Phi\Phi(Z))\big\rangle = \big\langle dL',\mathcal{T}^\Phi\Phi(S(Z))\big\rangle\\ &= \big\langle(\mathcal{T}^\Phi\Phi)^\star dL',S(Z)\big\rangle = \big\langle d(\mathcal{T}^\Phi\Phi)^\star L',S(Z)\big\rangle = \langle d(L'\circ\Phi),S(Z)\rangle\\ &= \langle dL,S(Z)\rangle = \langle\theta_L,Z\rangle,\end{aligned}$$

where we have used the transformation rule for the vertical endomorphism. The second property follows from the fact that $\mathcal{T}^\Phi\Phi$ is a morphism, so that $(\mathcal{T}^\Phi\Phi)^\star d = d(\mathcal{T}^\Phi\Phi)^\star$. The third one follows similarly and the fourth is a consequence of the second property and the definitions of the tensors G^L and $G^{L'}$. □

Let Γ be a SODE and $L\in C^\infty(E)$ be a Lagrangian. For convenience, we define the 1-form $\delta_\Gamma L\in\operatorname{Sec}((\mathcal{T}^EE)^*)$ by

$$\langle\delta_\Gamma L,Z\rangle = \langle dE_L - i_\Gamma\omega_L,Z\rangle = \langle dE_L,Z\rangle - \omega_L(\Gamma,Z),$$

for every section Z of $\mathcal{T}^EE$. We notice that Γ is the solution of the free dynamics if and only if $\delta_\Gamma L=0$. On the other hand, notice that the 1-form $\delta_\Gamma L$ is semibasic, because Γ is a SODE.

Proposition 5.3. *Let Γ be a SODE in E and Γ' a SODE in E'. Let $L\in C^\infty(E)$ and $L'\in C^\infty(E')$ be Lagrangian functions defined on E and E', respectively, such that $L=L'\circ\Phi$. Then,*

$$\big\langle\delta_\Gamma L-(\mathcal{T}^\Phi\Phi)^\star\delta_{\Gamma'}L',Z\big\rangle = \omega_{L'}(\Gamma'\circ\Phi-\mathcal{T}^\Phi\Phi\circ\Gamma,\mathcal{T}^\Phi\Phi(Z)), \tag{5.1}$$

for every section Z of $\mathcal{T}^EE$.

Proof. Indeed, from $(\mathcal{T}^\Phi\Phi)^\star dE_{L'} = dE_L$, we have that

$$\begin{aligned}\left\langle \delta_\Gamma L - (\mathcal{T}^\Phi\Phi)^\star \delta_{\Gamma'} L', Z \right\rangle &= \left\langle (\mathcal{T}^\Phi\Phi)^\star i_{\Gamma'}\omega_{L'} - i_\Gamma \omega_L, Z \right\rangle \\ &= \left\langle (\mathcal{T}^\Phi\Phi)^\star i_{\Gamma'}\omega_{L'} - i_\Gamma (\mathcal{T}^\Phi\Phi)^\star \omega_{L'}, Z \right\rangle \\ &= \omega_{L'}(\Gamma' \circ \Phi - \mathcal{T}^\Phi\Phi \circ \Gamma, \mathcal{T}^\Phi\Phi(Z)),\end{aligned}$$

which concludes the proof. □

5.1. **Reduction of the free dynamics.** Here, we build on Propositions 5.2 and 5.3 to identify conditions under which the dynamics can be reduced under a morphism of Lie algebroids. We first notice that, from Proposition 5.2, if Φ is fiberwise surjective morphism and L is a regular Lagrangian on E, then L' is a regular Lagrangian on E' (note that $\mathcal{T}^\Phi\Phi : \mathcal{T}^E E \to \mathcal{T}^{E'} E'$ is a fiberwise surjective morphism). Thus, the dynamics of both systems is uniquely defined.

Theorem 5.4 (Reduction of the free dynamics)**.** *Suppose that the Lagrangian functions L and L' are Φ-related, that is, $L = L' \circ \Phi$. If Φ is a fiberwise surjective morphism and L is a regular Lagrangian then L' is also a regular Lagrangian. Moreover, if Γ_L and $\Gamma_{L'}$ are the solutions of the free dynamics defined by L and L' then*

$$\mathcal{T}^\Phi\Phi \circ \Gamma_L = \Gamma_{L'} \circ \Phi.$$

Therefore, if $a(t)$ is a solution of the free dynamics defined by L, then $\Phi(a(t))$ is a solution of the free dynamics defined by L'.

Proof. If Γ_L and $\Gamma_{L'}$ are the solutions of the dynamics, then $\delta_{\Gamma_L} L = 0$ and $\delta_{\Gamma_{L'}} L' = 0$ so that the left-hand side in equation (5.1) vanishes. Thus

$$\omega_{L'}(\Gamma_{L'} \circ \Phi - \mathcal{T}^\Phi\Phi \circ \Gamma_L, \mathcal{T}^\Phi\Phi(Z)) = 0,$$

for every $Z \in \mathrm{Sec}(\mathcal{T}^E E)$. Therefore, using that L' is regular and the fact that $\mathcal{T}^\Phi\Phi$ is a fiberwise surjective morphism, we conclude the result. □

We will say that the unconstrained dynamics $\Gamma_{L'}$ is the ***reduction of the unconstrained dynamics*** Γ_L by the morphism Φ.

5.2. **Reduction of the constrained dynamics.** The above results about reduction of unconstrained Lagrangian systems can be easily generalized to nonholonomic constrained Lagrangian systems whenever the constraints of one system are mapped by the morphism to the constraints of the second system. Let us elaborate on this.

Let (L, D) be a constrained Lagrangian system on a Lie algebroid E and let (L', D') be another constrained Lagrangian system on a second Lie algebroid E'. Along this section, we assume that there is a fiberwise surjective morphism of Lie algebroids $\Phi\colon E \to E'$ such that $L = L' \circ \Phi$ and $\Phi(D) = D'$. The latter condition implies that the base map is also surjective, so that we will assume that Φ is an epimorphism (i.e., in addition to being fiberwise surjective, the base map φ is a submersion).

As a first consequence, we have $G^{L',D'}_{\Phi(a)}(\Phi(b), \Phi(c)) = G^{L,D}_a(b, c)$, for every $a \in D$ and every $b, c \in D_{\pi(a)}$, and therefore, if (L, D) is regular, then so is (L', D').

Lemma 5.5. *With respect to the decompositions $\mathcal{T}^E E|_D = \mathcal{T}^E D \oplus F$ and $\mathcal{T}^{E'} E'|_{D'} = \mathcal{T}^{E'} D' \oplus F'$, we have the following properties:*

1. *$\mathcal{T}^\Phi\Phi(\mathcal{T}^E D) = \mathcal{T}^{E'} D'$,*
2. *$\mathcal{T}^\Phi\Phi(F) = F'$,*
3. *If P, Q and P', Q' are the projectors associated with (L, D) and (L', D'), respectively, then $P' \circ \mathcal{T}^\Phi\Phi = \mathcal{T}^\Phi\Phi \circ P$ and $Q' \circ \mathcal{T}^\Phi\Phi = \mathcal{T}^\Phi\Phi \circ Q$.*

With respect to the decompositions $\mathcal{T}^E E|_D = \mathcal{T}^D D \oplus (\mathcal{T}^D D)^\perp$ and $\mathcal{T}^{E'} E'|_{D'} = \mathcal{T}^{D'} D' \oplus (\mathcal{T}^{D'} D')^\perp$ we have the following properties:

(4) *$\mathcal{T}^\Phi\Phi(\mathcal{T}^D D) = \mathcal{T}^{D'}D'$,*
(5) *$\mathcal{T}^\Phi\Phi\big((\mathcal{T}^D D)^\perp\big) = (\mathcal{T}^{D'}D')^\perp$,*
(6) *If $\bar{P}, \bar{Q}$ and $\bar{P}', \bar{Q}'$ are the projectors associated with (L, D) and (L', D'), respectively, then $\bar{P}' \circ \mathcal{T}^\Phi\Phi = \mathcal{T}^\Phi\Phi \circ \bar{P}$ and $\bar{Q}' \circ \mathcal{T}^\Phi\Phi = \mathcal{T}^\Phi\Phi \circ \bar{Q}$.*

Proof. From the definition of $\mathcal{T}^\Phi\Phi$, it follows that

$$(\mathcal{T}^\Phi\Phi)(\mathcal{T}^E D) \subseteq \mathcal{T}^{E'}D', \quad (\mathcal{T}^\Phi\Phi)(\mathcal{T}^D D) \subseteq \mathcal{T}^{D'}D'.$$

Thus, one may consider the vector bundle morphisms

$$\mathcal{T}^\Phi\Phi : \mathcal{T}^E D \to \mathcal{T}^{E'}D', \quad \mathcal{T}^\Phi\Phi : \mathcal{T}^D D \to \mathcal{T}^{D'}D'.$$

Moreover, using that Φ is fiberwise surjective and that φ is a submersion, we deduce that the rank of the above morphisms is maximum. This proves (1) and (4).

The proof of (5) is as follows. For every $a' \in D'$, one can choose $a \in D$ such that $\Phi(a) = a'$, and one can write any element $w' \in \mathcal{T}^{D'}_{a'}D'$ as $w' = \mathcal{T}^\Phi\Phi(w)$ for some $w \in \mathcal{T}^D_a D$. Thus, if $z \in (\mathcal{T}^D_a D)^\perp$, for every $w' \in \mathcal{T}^{D'}_{a'}D'$ we have

$$\omega_{L'}(\mathcal{T}^\Phi\Phi(z), w') = \omega_{L'}(\mathcal{T}^\Phi\Phi(z), \mathcal{T}^\Phi\Phi(w)) = \omega_L(z, w) = 0,$$

from where it follows that $\mathcal{T}^\Phi\Phi(z) \in (\mathcal{T}^{D'}D')^\perp$. In a similar way, using that $\mathcal{T}^\Phi\Phi : (\mathcal{T}^E E)_{|D} \to (\mathcal{T}^{E'}E')_{|D'}$ is fiberwise surjective, (2) in Proposition 5.2 and (4), we obtain that $(\mathcal{T}^{D'}D')^\perp \subseteq (\mathcal{T}^\Phi\Phi)((\mathcal{T}^D D)^\perp)$.

For the proof of (2) we have that

$$\mathcal{T}^\Phi\Phi(F) = \mathcal{T}^\Phi\Phi((\mathcal{T}^D D)^\perp \cap \mathrm{Ver}(\mathcal{T}^E E)) \subseteq (\mathcal{T}^{D'}D')^\perp \cap \mathrm{Ver}(\mathcal{T}^{E'}E') = F'.$$

Thus, using that $\mathcal{T}^\Phi\Phi : (\mathcal{T}^E E)_{|D} \to (\mathcal{T}^{E'}E')_{|D'}$ is fiberwise surjective, the fact that $(\mathcal{T}^E E)_{|D} = \mathcal{T}^E D \oplus F$ and (1), it follows that

$$(\mathcal{T}^{E'}E')_{|D'} = \mathcal{T}^{E'}D' \oplus (\mathcal{T}^\Phi\Phi)(F).$$

Therefore, since $(\mathcal{T}^{E'}E')_{|D'} = \mathcal{T}^{E'}D' \oplus F'$, we conclude that (2) holds.

Finally, (3) is an immediate consequence of (1) and (2), and similarly, (6) is an immediate consequence of (4) and (5). □

¿From the properties above, we get the following result.

Theorem 5.6 (Reduction of the constrained dynamics)**.** *Let (L, D) be a regular constrained Lagrangian system on a Lie algebroid E and let (L', D') be a constrained Lagrangian system on a second Lie algebroid E'. Assume that a fiberwise surjective morphism of Lie algebroids $\Phi \colon E \to E'$ exists such that $L = L' \circ \Phi$ and $\Phi(D) = D'$. If Γ is the constrained dynamics for L and Γ' is the constrained dynamics for L', then $\mathcal{T}^\Phi\Phi \circ \Gamma = \Gamma' \circ \Phi$. If $a(t)$ is a solution of Lagrange-d'Alembert differential equations for L, then $\Phi(a(t))$ is a solution of Lagrange-d'Alembert differential equations for L'.*

Proof. For the free dynamics, we have that $\mathcal{T}^\Phi\Phi \circ \Gamma_L = \Gamma_{L'} \circ \Phi$. Moreover, from property (3) in Lemma 5.5, for every $a \in D$, we have that

$$\mathcal{T}^\Phi\Phi(\Gamma(a)) = \mathcal{T}^\Phi\Phi(P(\Gamma_L(a))) = P'(\mathcal{T}^\Phi\Phi(\Gamma_L(a))) = P'(\Gamma_{L'}(\Phi(a))) = \Gamma'(\Phi(a)),$$

which concludes the proof. □

We will say that the constrained dynamics Γ' is the ***reduction of the constrained dynamics*** Γ by the morphism Φ.

Theorem 5.7. *Under the same hypotheses as in Theorem 5.6, we have that*

$$\{f'\circ\Phi, g'\circ\Phi\}_{nh} = \{f', g'\}_{nh}\circ\Phi,$$

for $f', g' \in C^\infty(D')$, where $\{\cdot,\cdot\}_{nh}$ (respectively, $\{\cdot,\cdot\}'_{nh}$) is the nonholonomic bracket for the constrained system (L, D) (respectively, (L', D')). In other words, $\Phi : D \to D'$ is an almost-Poisson morphism.

Proof. Using (2) in Proposition 5.2 and the fact that Φ is a Lie algebroid morphism, we deduce that

$$(i_{X_{f'\circ\Phi}}(\mathcal{T}^\Phi\Phi)^*\omega_{L'}) = i_{X_{f'}}\omega_{L'}\circ\Phi.$$

Thus, since $\mathcal{T}^\Phi\Phi$ is fiberwise surjective, we obtain that

$$\mathcal{T}^\Phi\Phi\circ X_{f'\circ\Phi} = X_{f'}\circ\Phi.$$

Now, from (4.3) and Lemma 5.5, we conclude that

$$\{f'\circ\Phi, g'\circ\Phi\}_{nh} = \{f', g'\}'_{nh}\circ\Phi.$$

□

One of the most important cases in the theory of reduction is the case of reduction by a symmetry group. In this respect, we have the following result.

Theorem 5.8 ([37, 42])**.** *Let $q^Q_G\colon Q \to M$ be a principal G-bundle, let $\tau\colon E \to Q$ be a Lie algebroid, and assume that we have an action of G on E such that the quotient vector bundle E/G is well-defined. If the set $\mathrm{Sec}(E)^G$ of equivariant sections of E is a Lie subalgebra of $\mathrm{Sec}(E)$, then the quotient $E' = E/G$ has a canonical Lie algebroid structure over M such that the canonical projection $q^E_G\colon E \to E/G$, given by $a \mapsto [a]_G$, is a (fiberwise bijective) Lie algebroid morphism over q^G_Q.*

As a concrete example of application of the above theorem, we have the well-known case of the Atiyah or Gauge algebroid. In this case, the Lie algebroid E is the standard Lie algebroid $TQ \to Q$, the action is by tangent maps $gv \equiv T\psi_g(v)$, the reduction is the Atiyah Lie algebroid $TQ/G \to Q/G$ and the quotient map $q^{TQ}_G\colon TQ \to TQ/G$ is a Lie algebroid epimorphism. It follows that if L is a G-invariant regular Lagrangian on TQ then the unconstrained dynamics for L projects to the unconstrained dynamics for the reduced Lagrangian L'. Moreover, if the constraints D are also G-invariant, then the constrained dynamics for (L, D) reduces to the constrained dynamics for $(L', D/G)$.

On a final note, we mention that the pullback of the distributional equation $i_{\Gamma'}\omega^{L',D'} - \varepsilon^{L',D'} = 0$ by $\mathcal{T}^\Phi\Phi$ is precisely $(i_\Gamma\omega^{L,D} - \varepsilon^{L,D})\circ\mathcal{T}^\Phi\Phi = 0$.

5.3. **Reduction by stages.** As a direct consequence of the results exposed above, one can obtain a theory of reduction by stages. In Poisson geometry, reduction by stages is a straightforward procedure. Given the fact that the Lagrangian counterpart of Poisson reduction is Lagrangian reduction, it is not strange that reduction by stages in our framework becomes also straightforward.

The Lagrangian theory of reduction by stages is a consequence of the following basic observation:

> Let $\Phi_1\colon E_0 \to E_1$ and $\Phi_2\colon E_1 \to E_2$ be a fiberwise surjective morphisms of Lie algebroids and let $\Phi\colon E_0 \to E_2$ be the composition $\Phi = \Phi_2\circ\Phi_1$. The reduction of a Lagrangian system in E_0 by Φ can be obtained by first reducing by Φ_1 and then reducing the resulting Lagrangian system by Φ_2.

This result follows using that $\mathcal{T}^{\Phi}\Phi = \mathcal{T}^{\Phi_2}\Phi_2 \circ \mathcal{T}^{\Phi_1}\Phi_1$. Based on this fact, one can analyze one of the most interesting cases of reduction: the reduction by the action of a symmetry group. We consider a group G acting on a manifold Q and a closed normal subgroup N of G. The process of reduction by stages is illustrated in the following diagram

$$\begin{tikzcd}
\tau_Q\colon E_0 = TQ \to M_0 = Q \arrow[d, "\text{First reduction}"', "\cdot/N"] \arrow[dd, bend right=60, "\text{Total reduction}"', "\cdot/G"] \\
\tau_1\colon E_1 = TQ/N \to M_1 = Q/N \arrow[d, "\text{Second reduction}"', "\cdot/(G/N)"] \\
\tau_2\colon E_2 = (TQ/N)/(G/N) \to M_2 = (Q/N)/(G/N)
\end{tikzcd}$$

In order to prove our results about reduction by stages, we have to prove that E_0, E_1 and E_2 are Lie algebroids, that the quotient maps $\Phi_1\colon E_0 \to E_1$, $\Phi_2\colon E_1 \to E_2$ and $\Phi\colon E_0 \to E_2$ are Lie algebroids morphisms and that the composition $\Phi_1 \circ \Phi_2$ equals to Φ. Our proof is based on the following well-known result (see [14]), which contains most of the ingredients in the theory of reduction by stages.

Theorem 5.9 ([14])**.** *Let $q_G^Q\colon Q \to M$ be a principal G-bundle and N a closed normal subgroup of G. Then,*

1. *$q_N^Q\colon Q \to Q/N$ is a principal N-bundle,*
2. *G/N acts on Q/N by the rule $[g]_N[q]_N = [gq]_N$,*
3. *$q_{G/N}^{Q/N}\colon Q/N \to (Q/N)/(G/N)$ is a principal (G/N)-bundle.*
4. *The map $i\colon Q/G \to (Q/N)/(G/N)$ defined by $[q]_G \mapsto [[q]_N]_{G/N}$ is a diffeomorphism.*

Building on the previous results, one can deduce the following theorem, which states that the reduction of a Lie algebroid can be done by stages.

Theorem 5.10. *Let $q_G^Q\colon Q \to M$ be a G-principal bundle and N be a closed normal subgroup of G. Then,*

1. *$\tau_{TQ/G}\colon TQ/G \to Q/G$ is a Lie algebroid and $q_G^{TQ}\colon TQ \to TQ/G$ is a Lie algebroid epimorphism,*
2. *$\tau_{TQ/N}\colon TQ/N \to Q/N$ is a Lie algebroid and $q_N^{TQ}\colon TQ \to TQ/N$ is a Lie algebroid epimorphism,*
3. *G/N acts on TQ/N by the rule $[g]_N[v]_N = [gv]_N$,*
4. *$\tau_{(TQ/N)/(G/N)}\colon (TQ/N)/(G/N) \to (Q/N)/(G/N)$ is a Lie algebroid and $q_{G/N}^{TQ/N}\colon TQ/N \to (TQ/N)/(G/N)$ is a Lie algebroid epimorphism,*
5. *The map $I\colon TQ/G \to (TQ/N)/(G/N)$ defined by $[v]_G \mapsto [[v]_N]_{G/N}$ is an isomorphism of Lie algebroids over the map i.*

Proof. The vector bundle $\tau_{TQ/G} : TQ/G \to Q/G$ (respectively, $\tau_{TQ/N} : TQ/N \to Q/N$) is the Atiyah algebroid for the principal G-bundle $q_G^Q : Q \to Q/G$ (respectively, $q_N^Q : Q \to Q/N$), so that (1) and (2) are obvious. Condition (3) is just condition (2) of Theorem 5.9 applied to the principal N-bundle $TQ \to TQ/N$. To prove condition (4), we notice that the action of G/N on the Lie algebroid TQ/N is free and satisfies the conditions of Theorem 5.8. Finally, the Lie algebroid morphism $j\colon TQ \to TQ/N$ is equivariant with respect to the G-action on TQ and the (G/N)-action on TQ/N. Thus it induces a morphism of Lie algebroids in the quotient. It is an isomorphism since it is a diffeomorphism by Theorem 5.9. $\square$

The following diagram illustrates the above situation:

$$\begin{tikzcd}
TQ \arrow[r, "\Phi_1"'] \arrow[d, "\tau_Q"'] \arrow[rr, bend left, "\Phi"] & TQ/N \arrow[r, "\Phi_2"'] \arrow[d, "\tau_1"] & (TQ/N)/(G/N) \arrow[d, "\tau_2"] \\
Q \arrow[r, "q_N^Q"] \arrow[rr, bend right, "q_G^Q"'] & Q/N \arrow[r, "q_{G/N}^{Q/N}"] & (Q/N)/(G/N)
\end{tikzcd}$$

In particular, for the unconstrained case one has the following result.

Theorem 5.11 (Reduction by stages of the free dynamics)**.** *Let $q_G^Q\colon Q \to Q/G$ be a principal G-bundle, and N a closed normal subgroup of G. Let Q be a Lagrangian function on Q which is G-invariant. Then the reduction by the symmetry group G can be performed in two stages:*

1. *reduce by the normal subgroup N,*
2. *reduce the resulting dynamics from 1. by the residual symmetry group G/N.*

Since the dynamics of a constrained system is obtained by projection of the free dynamics, we also the following result.

Theorem 5.12 (Reduction by stages of the constrained dynamics)**.** *Let $q_G^Q\colon Q \to Q/G$ be a principal G-bundle and N a closed normal subgroup of G. Let (L, D) be a G-invariant constrained Lagrangian system. Then the reduction by the symmetry group G can be performed in two stages:*

1. *reduce by the normal subgroup N,*
2. *reduce the resulting dynamics from 1. by the residual symmetry group G/N.*

6. The momentum equation

In this section, we examine the evolution of the momentum map along the evolution of the constrained system. The kernel of the anchor map plays, in some sense that we will make clear in this section, the role of symmetry directions.

Let K be an ideal of E, that is, a Lie subalgebroid $i\colon K \hookrightarrow E$ such that $[\xi, \eta]$ is a section of K for every section ξ of E and every section η of K. It follows that K is a bundle of Lie algebras, i.e., $\rho|_K = 0$, since $[f\xi, \eta] = -(\rho(\eta)f)\xi + f[\xi, \eta]$ must be a section of K. For instance, if E is a regular Lie algebroid, $\operatorname{rank}(\rho) =$ constant, we can just take $K = \operatorname{Ker}(\rho)$.

On the bundle K, we can define a linear E-connection (see [25] for details about E-connections) by means of the E-covariant derivative

$$\nabla_\xi \eta = [\xi, \eta],$$

for ξ a section of E and η a section of K. Indeed, we have that $\nabla_\xi \eta$ is a section of K because K is an ideal and

$$\nabla_{(f\xi)}\eta = [f\xi, \eta] = -(\rho(\eta)f)\xi + f[\xi, \eta] = f[\xi, \eta] = f\nabla_\xi \eta,$$

because $\rho(\eta) = 0$. Finally,

$$\nabla_\xi(f\eta) = [\xi, f\eta] = (\rho(\xi)f)\eta + f[\xi, \eta] = (\rho(\xi)f)\eta + f\nabla_\xi \eta.$$

Note that K may be considered as an ideal of $\mathcal{T}^E E$ and thus ∇ induces a linear $\mathcal{T}^E E$-connection on K that we will also denote by ∇.

Now, let J be the restriction of θ_L (considered as a form along τ) to K. In other words, J is the bundle map $J\colon E \to K^*$ over the identity in M defined by

$$\langle J(a), k\rangle = \left\langle \theta_L\big|_a, i(k)\right\rangle = \frac{d}{ds} L(a+sk)\big|_{s=0}.$$

The map J is said to be the ***momentum map*** with respect to K.

Theorem 6.1. *The momentum map satisfies the* ***momentum equation****:*

$$\nabla_{\Gamma_L} J = 0.$$

Proof. For every section η of K, we have that $\delta_{\Gamma_L} L(\eta) = d_{\Gamma_L}\langle \theta_L, \eta^C\rangle - d_{\eta^C} L = 0$, where η^C is the complete lift of the section η, see [43]. Therefore

$$\delta_{\Gamma_L} L(\eta) = \nabla_{\Gamma_L}\langle J, \eta\rangle - d_{\eta^C} L = \langle \nabla_{\Gamma_L} J, \eta\rangle + \langle J, \nabla_{\Gamma_L}\eta\rangle - d_{\eta^C} L,$$

where we have used the definition of J, and taken into account that both η and $\nabla_{\Gamma_L}\eta$ are sections of K. On the other hand, we have $\rho^1(\eta^C)$ projects to $\rho(\eta) = 0$, so that it is vertical. Moreover, straightforward coordinate calculations show that $d_{\eta^C} L = d_{(\nabla_{\Gamma_L}\eta)^V} L = \langle J, \nabla_{\Gamma_L}\eta\rangle$. Therefore

$$\delta_{\Gamma_L} L(\eta) = \langle \nabla_{\Gamma_L} J, \eta\rangle,$$

which completes the proof. □

If we take local coordinates associated to a local basis of sections $\{e_\alpha\}$ where the first k elements $\{e_I\}$ are a local basis of K, then we have

$$\nabla_{e_\alpha} e_I = [e_\alpha, e_I] = C^J_{\alpha I} e_J,$$

i.e., the connection coefficients are just (some of) the structure functions. Thus, the covariant derivative of the momentum map is

$$\nabla_{\Gamma_L} J = \left\{ d_{\Gamma_L}\left(\frac{\partial L}{\partial y^J}\right) - \frac{\partial L}{\partial y^I} y^\alpha C^I_{\alpha J}\right\} e^J,$$

and the momentum equations reads

$$d_{\Gamma_L}\left(\frac{\partial L}{\partial y^J}\right) + \frac{\partial L}{\partial y^I} C^I_{J\alpha} y^\alpha = 0,$$

i.e., the first k Euler-Lagrange equations for L, where $k = \operatorname{rank}(K)$. (Notice that $\rho^i_I = 0$ and $C^\beta_{J\alpha} = 0$ for $\beta = k+1, \ldots, \operatorname{rank}(E)$).

In the case of a nonholonomically constrained system, to obtain a similar equation, K, in addition to being an ideal, must be a subbundle of the constraint distribution $K \subset D$. With this condition, we also get the momentum equation in the form $\nabla_\Gamma J = 0$.

In general, the momentum equation does not provide with constants of the motion. Only for a parallel section η of K, i.e., $\nabla\eta = 0$, we have $\nabla_\Gamma \eta = 0$ and, thus, $\langle J, \eta\rangle$ is a constant of the motion. In particular, if K is a trivial Lie algebroid $K \simeq M \times \mathfrak{g}$ for some Lie algebra $\mathfrak{g}$, we recover the standard case of Lagrangian Mechanics, and we have a constant of the motion for every element of $\mathfrak{g}$. In this case, the momentum map can be considered as a map from E to $\mathfrak{g}^*$.

7. Examples

As in the unconstrained case, Lagrangian constrained systems on Lie algebroids appear frequently. We show some examples next. In all examples, after the reduction by the group symmetry, there can still be some residual symmetry, and the result about reduction by stages applies.

Example 7.1. [Suslov system] The so-called Suslov problem [31, 56] consists of a rigid body to which we impose the constraint of having body angular velocity orthogonal to some fixed direction in the body frame. Taking $G = SO(3)$ and identifying $\mathfrak{so}(3) = \mathbb{R}^3$, the Lagrangian is $L(R, \dot{R}) = \frac{1}{2}\Omega\cdot\mathbb{I}\Omega$, where $\hat{\Omega} = R^{-1}\dot{R}$, and the constraint is $\Omega \cdot \Gamma = 0$, Γ being constant. The Lagrangian and the constraints are clearly left-invariant and thus we can reduce from $TSO(3) \to SO(3)$ to the Lie algebra $\mathfrak{so}(3) \to \{e\}$. The morphism is just $\Phi(R, \dot{R}) = R^{-1}\dot{R} \in \mathfrak{so}(3)$. The Lagrange-d'Alembert equations are

$$\mathbb{I}\dot{\Omega} + \Omega \times \mathbb{I}\Omega = \lambda\Gamma.$$

◁

Example 7.2. [LL-systems] The class of LL-systems generalizes the Suslov problem. These systems consists of a left-invariant Lagrangian on a Lie group with a left-invariant constraint distribution $D \subset TG$. The system then reduces to a nonholonomic system on the Lie algebra of the Lie group. Lagrange-d'Alembert equations are known in this case as Euler-Poincaré-Suslov equations. ◁

Example 7.3. [Veselova system] The so-called Veselova system [58] consists of a rigid body to which we impose the constraint of having body angular velocity orthogonal to some fixed direction in the spatial frame. Taking $G = SO(3)$ and identifying $\mathfrak{so}(3) = \mathbb{R}^3$, the Lagrangian is $L(R, \dot{R}) = \frac{1}{2}\Omega \cdot \mathbb{I}\Omega$ and the constraint is $\Omega \cdot \Gamma = 0$, where as before $\hat{\Omega} = R^{-1}\dot{R}$.

In this case we cannot reduce directly as in the Suslov problem since the constraints are expressed in terms of a constant vector in the spatial frame. Nevertheless, we can perform a reduction procedure following similar ideas as in the reduction of the heavy top [43, 14].

We can reduce from $TSO(3)\times S^2 \to SO(3)\times S^2$ (where $S^2 \subset \mathbb{R}^3$ is taken as the parameter manifold) to the transformation Lie algebroid $\mathfrak{so}(3) \times S^2 \to S^2$, where the action of $\mathfrak{so}(3)$ on S^2 is $\rho(\Omega, \Gamma) = (\Gamma, \Gamma \times \Omega)$. The reduction morphism is

$$(\Omega, \Gamma) = \Phi(R, \dot{R}, \gamma) = (R^{-1}\dot{R}, R^{-1}\gamma).$$

The Lagrange-d'Alembert equations are

$$\begin{cases} \mathbb{I}\dot{\Omega} + \Omega \times \mathbb{I}\Omega = \lambda\Gamma, \\ \dot{\Gamma} + \Omega \times \Gamma = 0. \end{cases}$$

◁

Example 7.4. [LR-systems] Generalizing the case of Veselova's problem, we get the so-called LR-systems. This systems are defined on the tangent bundle of the Lie group G, with a left-invariant Lagrangian L and a right-invariant constraint distribution $D \subset TG$. We set $\mathfrak{d} \subset \mathfrak{g}$ to be the value of the constraint distribution at the identity, so that $D_g = TR_g(\mathfrak{d})$.

The reduction of this kind of systems can be performed in a similar way to the Veselova problem. Given a Lagrangian $L_0 \in C^\infty(TG)$, we consider the Lie algebroid $\bar{\tau}\colon TG \times \mathfrak{d}^\circ \to \mathfrak{d}^\circ$, where the variables in $\mathfrak{d}^\circ$ are treated as parameters. Thus, the anchor $\bar{\rho}\colon TG \times \mathfrak{d}^\circ \to TG \times T\mathfrak{d}^\circ$ is given by $\bar{\rho}(v_g, a) = (v_g, 0_a)$, and the bracket is derived from the standard bracket of vector fields on G. The Lagrangian is extended[2] to $TG\times\mathfrak{d}^\circ$ by $L(v_g, a) = L_0(v_g)$. The free Euler-Lagrange equations for the Lagrangian L are the Euler-Lagrange equations for L_0 supplemented by the condition $\dot{a} = 0$. For the constrained system, the constraints read $\langle v_g, T^*R_{g^{-1}}(a)\rangle = 0$.

[2]It even may be that the original Lagrangian already depends on the variables on $\mathfrak{d}^\circ$ as parameters. In that case, we have to require to the Lagrangian to be fully invariant, i.e., $L(TL_hv_g, \mathrm{Ad}^*_{h^{-1}}\, a) = L(v_g, a)$.

In other words, the constraint functions are (the linear functions on TG associated to) right-invariant forms on $\mathfrak{g}$, $\theta(g) = T^*R_g(a)$.

We now consider the transformation Lie algebroid $\tau\colon \mathfrak{g}\times\mathfrak{d}^\circ \to \mathfrak{d}^\circ$, where the action of $\mathfrak{g}$ on $\mathfrak{d}^\circ$ is the coadjoint action. Thus the anchor is $\rho(\xi, A) = (A, -\operatorname{ad}^*_\xi A) \in \mathfrak{d}^\circ\times\mathfrak{d}^\circ \equiv T\mathfrak{d}^\circ$, in other words, it is the Lie algebroid of the transformation Lie groupoid $G\times\mathfrak{d}^\circ$ over $\mathfrak{d}^\circ$ with the coadjoint action $A\cdot g = \operatorname{Ad}^*_g A$. We consider the Lie algebroid morphism given by

$$\Phi\colon TG\times\mathfrak{d}^\circ \to \mathfrak{g}\times\mathfrak{d}^\circ \qquad (\Omega, A) = \Phi(v_g, a) = (TL_{g^{-1}}v_g, \operatorname{Ad}^*_g a).$$

Since the Lagrangian is left-invariant, it defines a reduced Lagrangian $l\in C^\infty(\mathfrak{g}\times\mathfrak{d}^\circ)$ by $l(\Omega, A) = L(TL_g\Omega, \operatorname{Ad}^*_{g^{-1}} A)$, where as usual $\Omega = TL_{g^{-1}}v_g$. Moreover the constraints read

$$\left\langle v_g, T^*R_{g^{-1}}(a)\right\rangle = \left\langle TL_g\Omega, T^*R_{g^{-1}}(a)\right\rangle = \left\langle \Omega, \operatorname{Ad}^*_g a\right\rangle = \langle\Omega, A\rangle = 0.$$

The Lagrange-d'Alembert equations are

$$\delta l(\Omega, A)\in\mathfrak{d}^\circ \qquad \text{and} \qquad \dot A = \operatorname{ad}^*_\Omega A.$$

As a concrete example, we can consider the Lagrangian given by the kinetic energy defined by an Ad_G-invariant metric, which gives $l(\Omega) = \frac{1}{2}\Omega\cdot\mathbb{I}\Omega$. Taking a basis Γ_i of $\mathfrak{d}^\perp$ (the orthogonal with respect to the metric), we can write the Lagrange-d'Alembert equations in the form

$$\mathbb{I}\dot\Omega + \operatorname{ad}_\Omega\mathbb{I}\Omega = \lambda^i\Gamma_i \qquad \text{and} \qquad \dot\Gamma_i + \operatorname{ad}_\Omega\Gamma_i = 0.$$

◁

Example 7.5. [Chaplygin-type systems] A frequent situation is the following. Consider a constrained Lagrangian system (L, D) on a Lie algebroid $\tau\colon E\to M$ such that the restriction of the anchor to the constraint distribution, $\rho|D\colon D\to TM$, is an isomorphism of vector bundles. Let $h\colon TM\to D\subset E$ be the right-inverse of $\rho|_D$, so that $\rho\circ h = \mathrm{id}_{TM}$. It follows that E is a transitive Lie algebroid and h is a splitting of the exact sequence

$$0 \longrightarrow \operatorname{Ker}(\rho) \longrightarrow E \xrightarrow{\ \rho\ } TM \longrightarrow 0\,.$$

Let us define the function $\bar L\in C^\infty(TM)$ by $\bar L = L\circ h$. The dynamics defined by L does not reduce to the dynamics defined by $\bar L$ because, while the map $\Phi = \rho$ is a morphism of Lie algebroids and $\Phi(D) = TM$, we have $\bar L\circ\Phi = L\circ h\circ\rho \neq L$. Nevertheless, we can use h to express the dynamics on TM, by finding relations between the dynamics defined by L and $\bar L$.

We need some auxiliary properties of the splitting h and its prolongation. We first notice that h is an admissible map over the identity in M, because $\rho_E\circ h = \mathrm{id}_{TM}$ and $T\,\mathrm{id}_M\circ\rho_{TM} = id_{TM}$, but in general h is not a morphism. We can define the tensor K, a $\ker(\rho)$-valued differential 2-form on M, by means of

$$K(X, Y) = [h\circ X, h\circ Y] - h\circ[X, Y]$$

for every $X, Y\in\mathfrak{X}(M)$. It is easy to see that h is a morphism if and only if $K = 0$. In coordinates (x^i) in M, (x^i, v^i) in TM, and linear coordinates (x^i, y^i, y^A) on E corresponding to a local basis $\{e^i, e^A\}$ of sections of E adapted to the splitting h, we have that

$$K = \frac{1}{2}\Omega^A_{ij}\,dx^i\wedge dx^j\otimes e_A,$$

where Ω^A_{ij} are defined by $[e_i, e_j] = \Omega^A_{ij}e_A$.

Since h is admissible, its prolongation $\mathcal{T}^h h$ is a well-defined map from $T(TM)$ to $\mathcal{T}^E E$. Moreover, it is an admissible map, which is a morphism if and only if h is a morphism. In what respect to the energy and the Cartan 1-form, we have that $(\mathcal{T}^h h)^\star E_L = E_{\bar L}$ and $(\mathcal{T}^h h)^\star \theta_L = \theta_{\bar L}$. Indeed, notice that by definition, $(\mathcal{T}^h h)^\star E_L = E_L \circ h$ and

$$\begin{aligned} E_L(h(v)) &= \frac{d}{dt} L(h(v) + t(h(v)))|_{t=0} - L(h(v)) = \frac{d}{dt} L(h(v+tv))|_{t=0} - L(h(v)) \\ &= \frac{d}{dt} \bar L(v+tv)|_{t=0} - \bar L(v) = E_{\bar L}(v). \end{aligned}$$

On the other hand, for every $V_v \equiv (v, w, V) \in T(TM) \equiv \mathcal{T}^{TM}(TM)$ where $w = T\tau(V)$, we have

$$\begin{aligned} \left\langle (\mathcal{T}^h h)^\star \theta_L, V \right\rangle &= \left\langle \theta_L, \mathcal{T}^h h(v,w,V) \right\rangle = \langle \theta_L, (h(v), h(w), Th(V)) \rangle \\ &= \frac{d}{dt} L(h(v) + t(h(w)))|_{t=0} = \frac{d}{dt} L(h(v+tw))|_{t=0} \\ &= \frac{d}{dt} \bar L(v+tw)|_{t=0} = \langle \theta_{\bar L}, V \rangle . \end{aligned}$$

Nevertheless, since h is not a morphism, and hence $(\mathcal{T}^h h)^\star \circ d \neq d \circ (\mathcal{T}^h h)^\star$, we have that $(\mathcal{T}^h h)^\star \omega_L \neq \omega_{\bar L}$. Let JK be the 2-form on TM defined by

$$JK_v(V, W) = \left\langle J_{h(v)}, K_{h(v)}(T\tau_M(V), T\tau_M(W)) \right\rangle$$

where J is the momentum map defined by L and $\operatorname{Ker}\rho$ and $V, W \in T_{h(v)}(TM)$. The notation resembles the contraction of the momentum map J with the curvature tensor K. Instead of being symplectic, the map $\mathcal{T}^h h$ satisfies

$$(\mathcal{T}^h h)^\star \omega_L = \omega_{\bar L} + JK.$$

Indeed, we have that

$$(\mathcal{T}^h h)^\star \omega_L - \omega_{\bar L} = [d \circ (\mathcal{T}^h h)^\star - (\mathcal{T}^h h)^\star \circ d]\, \theta_L$$

and on a pair of projectable vector fields U, V projecting onto X, Y respectively, one can easily prove that

$$[d \circ (\mathcal{T}^h h)^\star - (\mathcal{T}^h h)^\star \circ d]\, \theta_L(U, V) = \left\langle \theta_L, [\mathcal{T}^h h(U), \mathcal{T}^h h(V)] - \mathcal{T}^h h([U, V]) \right\rangle$$

from where the result follows by noticing that $\mathcal{T}^h h \circ U$ is a projectable section and projects to $h \circ X$, and similarly $\mathcal{T}^h h \circ V$ projects to $h \circ Y$. Hence $[\mathcal{T}^h h(U), \mathcal{T}^h h(V)] - \mathcal{T}^h h([U, V]$ is projectable and projects to $K(X, Y)$.

Let now Γ be the solution of the nonholonomic dynamics for (L, D), so that Γ satisfies the equation $i_\Gamma \omega_L - dE_L \in \widetilde{D^\circ}$ and the tangency condition $\Gamma\big|_D \in \mathcal{T}^D D$. From this second condition we deduce the existence of a vector field $\bar\Gamma \in \mathfrak{X}(TM)$ such that $\mathcal{T}^h h \circ \bar\Gamma = \Gamma \circ h$. Explicitly, the vector field $\bar\Gamma$ is defined by $\bar\Gamma = \mathcal{T}^\rho \rho \circ \Gamma \circ h$, from where it immediately follows that $\bar\Gamma$ is a SODE vector field on M.

Taking the pullback by $\mathcal{T}^h h$ of the first equation we get $(\mathcal{T}^h h)^\star \big(i_\Gamma \omega_L - dE_L \big) = 0$ since $(\mathcal{T}^h h)^\star \widetilde{D^\circ} = 0$. Therefore

$$\begin{aligned} 0 &= (\mathcal{T}^h h)^\star i_\Gamma \omega_L - (\mathcal{T}^h h)^\star dE_L \\ &= i_{\bar\Gamma} (\mathcal{T}^h h)^\star \omega_L - d(\mathcal{T}^h h)^\star E_L \\ &= i_{\bar\Gamma} \big(\omega_{\bar L} + JK \big) - dE_{\bar L} \\ &= i_{\bar\Gamma} \omega_{\bar L} - dE_{\bar L} + i_{\bar\Gamma} JK. \end{aligned}$$

Therefore, the vector field $\bar\Gamma$ is determined by the equations

$$i_{\bar\Gamma} \omega_{\bar L} - dE_{\bar L} = - \langle J, K(\mathbb{T}, \cdot) \rangle ,$$

where $\mathbb{T}$ is the identity in TM considered as a vector field along the tangent bundle projection τ_M (also known as the total time derivative operator). Equivalently we can write these equations in the form

$$d_{\bar\Gamma} \theta_{\bar L} - d\bar L = \langle J, K(\mathbb{T}, \cdot) \rangle .$$

The above equations determine the dynamics for the variables in TM. Taking into account the decomposition $E = h(TM) \oplus \ker\rho$, we need to determine the dynamics for the variables in $\operatorname{Ker}\rho$. Such complementary equation is but the momentum equation $\nabla_\Gamma J = 0$. Therefore, the Lagrange-D'Alembert equations are equivalent to the pair equations

$$i_{\bar\Gamma}\omega_{\bar L} - dE_{\bar L} = -\langle J, K(\mathbb{T}, \cdot)\rangle \qquad \text{and} \qquad \nabla_\Gamma J = 0.$$

Finally we mention that extension of the above decomposition for non transitive Lie algebroids is under development.

Chaplygin systems. A particular case of the above theory is that of ordinary Chaplygin systems(see [4, 7, 31] and references there in). In such case we have a principal G-bundle $\pi = q_G^Q \colon Q \to M = Q/G$ and the constraint distribution is the horizontal distribution of a principal connection Hor. If we set $E = TQ/G \to M$ and $\Phi = q_G^{TQ} \colon TQ \to E$ the quotient projection (which is a morphism of Lie algebroids), since the connection is a principal connection we have that Hor projects to a distribution $D = \Phi(\mathrm{Hor})$. Given a G-invariant Lagrangian $L_Q \in C^\infty(TQ)$ we have a reduced constrained system (E, D, L), with $L_Q = L \circ \Phi$. Moreover, the anchor in E, given by $\rho([v]) = T\pi(v)$ is surjective and the horizontal lifting of the original connection defines a map $h \colon TM \to E$ which is a splitting of the exact sequence $0 \to \ker(\rho) \to E \to TM \to 0$. The original Lagrangian and the function $\bar L$ are related by $\bar L(v) = L(h(v)) = L_Q(v^H)$ for every $v \in TM$ and where $v^H \in TQ$ is the horizontal lifting with respect to the original principal connection. $\triangleleft$

8. Nonlinearly constrained Lagrangian systems

We show in this section how the main results for linearly constrained Lagrangian systems can be extended to the case of Lagrangian systems with nonlinear nonholonomic constraints. This is true under the assumption that a suitable version of the classical Chetaev's principle in nonholonomic mechanics is valid (see e.g. [35] for the study of standard nonholonomic Lagrangian systems subject to nonlinear constraints).

Let $\tau : E \to M$ be a Lie algebroid and $\mathcal{M}$ be a submanifold of E such that $\pi = \tau|_{\mathcal{M}} \colon \mathcal{M} \to M$ is a fibration. $\mathcal{M}$ is the constraint submanifold. Since π is a fibration, the prolongation $\mathcal{T}^E\mathcal{M}$ is well-defined. We will denote by r the dimension of the fibers of $\pi \colon \mathcal{M} \to M$, that is, $r = \dim\mathcal{M} - \dim M$.

We define the bundle $\mathcal{V} \to \mathcal{M}$ of ***virtual displacements*** as the subbundle of τ^*E of rank r whose fiber at a point $a \in \mathcal{M}$ is

$$\mathcal{V}_a = \{\, b \in E_{\tau(a)} \,|\, b_a^V \in T_a\mathcal{M} \,\}.$$

In other words, the elements of $\mathcal{V}$ are pairs of elements $(a, b) \in E \oplus E$ such that

$$\frac{d}{dt}\phi(a + tb)\Big|_{t=0} = 0,$$

for every local constraint function ϕ.

We also define the bundle of ***constraint forces*** $\mathbf{\Psi}$ by $\mathbf{\Psi} = S^*((\mathcal{T}^E\mathcal{M})^\circ)$, in terms of which we set the Lagrange-d'Alembert equations for a regular Lagrangian function $L \in C^\infty(E)$ as follows:

$$\begin{aligned} &(i_\Gamma\omega_L - dE_L)|_{\mathcal{M}} \in \operatorname{Sec}(\mathbf{\Psi}), \\ &\Gamma|_{\mathcal{M}} \in \operatorname{Sec}(\mathcal{T}^E\mathcal{M}), \end{aligned} \tag{8.1}$$

the unknown being the section Γ. The above equations reproduce the corresponding ones for standard nonlinear constrained systems.

¿From (2.7) and (8.1), it follows that

$$(i_{S\Gamma}\omega_L - i_\Delta\omega_L)|_{\mathcal{M}} = 0,$$

which implies that a solution Γ of equations (8.1) is a SODE section along $\mathcal{M}$, that is, $(S\Gamma - \Delta)|\mathcal{M} = 0$.

Note that the rank of the vector bundle $(\mathcal{T}^E\mathcal{M})^\circ \to \mathcal{M}$ is $s = \mathrm{rank}(E) - r$ and, since π is a fibration, the transformation $S^* : (\mathcal{T}^E\mathcal{M})^0 \to \Psi$ defines an isomorphism between the vector bundles $(\mathcal{T}^E\mathcal{M})^0 \to \mathcal{M}$ and $\Psi \to \mathcal{M}$. Therefore, the rank of Ψ is also s. Moreover, if $a \in \mathcal{M}$ we have

$$\Psi_a = S^*((\mathcal{T}^E_a\mathcal{M})^\circ) = \{\, \zeta \circ \mathcal{T}\tau \,|\, \zeta \in \mathcal{V}_a^\circ \,\}. \tag{8.2}$$

In fact, if $\alpha_a \in (\mathcal{T}^E_a\mathcal{M})^\circ$, we may define $\zeta \in E^*_{\tau(a)}$ by

$$\zeta(b) = \alpha_a(\xi^V(a,b)), \text{ for } b \in E_{\tau(a)}.$$

Then, a direct computation proves that $\zeta \in \mathcal{V}_a^\circ$ and $S^*(\alpha_a) = \zeta \circ \mathcal{T}\tau$. Thus, we obtain

$$\Psi_a \subseteq \{\, \zeta \circ \mathcal{T}\tau \,|\, \zeta \in \mathcal{V}_a^\circ \,\}$$

and, using that the dimension of both spaces is s, we deduce (8.2) holds. Note that, in the particular case when the constraints are linear, we have $\mathcal{V} = \tau^*(D)$ and $\Psi = \widetilde{D^\circ}$.

Next, we consider the vector bundles F and $\mathcal{T}^{\mathcal{V}}\mathcal{M}$ over $\mathcal{M}$ whose fibers at the point $a \in \mathcal{M}$ are

$$F_a = \omega_L^{-1}(\mathbf{\Psi}_a), \quad \mathcal{T}^{\mathcal{V}}_a\mathcal{M} = \{\, (b,v) \in \mathcal{V}_a \times T_a\mathcal{M} \,|\, T\pi(v) = \rho(b) \,\}.$$

It follows that

$$F_a = \{\, z \in \mathcal{T}^E_a E \,|\, \text{ exists } \zeta \in \mathcal{V}_a^0 \text{ and } i_z\omega_L(a) = \zeta \circ \mathcal{T}\tau \,\}$$

and

$$\mathcal{T}^{\mathcal{V}}_a\mathcal{M} = \{\, z \in \mathcal{T}^E_a\mathcal{M} \,|\, \mathcal{T}\pi(z) \in \mathcal{V}_a \,\} = \{\, z \in \mathcal{T}^E_a\mathcal{M} \,|\, S(z) \in \mathcal{T}^E_a\mathcal{M} \,\}. \tag{8.3}$$

Note that the dimension of $\mathcal{T}^{\mathcal{V}}_a\mathcal{M}$ is $2r$ and, when the constraints are linear, i.e., $\mathcal{M}$ is a vector subbundle D of E, we obtain

$$\mathcal{T}^{\mathcal{V}}_a\mathcal{M} = \mathcal{T}^D_a D, \text{ for all } a \in \mathcal{M} = D.$$

Moreover, from (8.3), we deduce that the vertical lift of an element of $\mathcal{V}$ is an element of $\mathcal{T}^{\mathcal{V}}\mathcal{M}$. Thus we can define for $b, c \in \mathcal{V}_a$

$$G^{L,\mathcal{V}}_a(b,c) = \omega_L(a)(\tilde{b}, \xi^V(a,c)),$$

where $\tilde{b} \in \mathcal{T}^E_a E$ and $\mathcal{T}\tau(\tilde{b}) = b$.

Now, we will analyze the local nature of equations (8.1). We consider local coordinates (x^i) on an open subset U of M and take a basis $\{e_\alpha\}$ of local sections of E. In this way, we have local coordinates (x^i, y^α) on E. Suppose that the local equations defining $\mathcal{M}$ as a submanifold of E are

$$\phi^A = 0, \quad A = 1, \dots, s,$$

where ϕ^A are independent local constraint functions. Since $\pi : \mathcal{M} \to M$ is a fibration, it follows that the matrix $(\dfrac{\partial \phi^A}{\partial y^\alpha})$ is of rank s. Thus, if d is the differential of the Lie algebroid $\mathcal{T}^E E \to E$, we deduce that $\{d\phi^A|_{\mathcal{M}}\}_{A=1,\dots,s}$ is a local basis of sections of the vector bundle $(\mathcal{T}^E\mathcal{M})^0 \to \mathcal{M}$. Note that

$$d\phi^A = \rho^j_\alpha \frac{\partial \phi^A}{\partial x^j}\mathcal{X}^\alpha + \frac{\partial \phi^A}{\partial y^\alpha}\mathcal{V}^\alpha.$$

Moreover, $\{S^*(d\phi^A)|_{\mathcal{M}} = \dfrac{\partial \phi^A}{\partial y^\alpha}\mathcal{X}^\alpha|_{\mathcal{M}}\}_{A=1,\dots,s}$ is a local basis of sections of the vector bundle $\Psi \to \mathcal{M}$.

Next, we will introduce the local sections $\{Z_A\}_{A=1,\dots,s}$ of $\mathcal{T}^E E \to E$ defined by

$$i_{Z_A}\omega_L = S^*(d\phi^A) = \frac{\partial \phi^A}{\partial y^\alpha}\mathcal{X}^\alpha.$$

A direct computation, using (2.5), proves that

$$Z_A = -\frac{\partial \phi^A}{\partial y^\alpha} W^{\alpha\beta}\mathcal{V}_\beta, \qquad \text{for all } A, \tag{8.4}$$

where $(W^{\alpha\beta})$ is the inverse matrix of $(W_{\alpha\beta} = \dfrac{\partial^2 L}{\partial y^\alpha y^\beta})$. Furthermore, it is clear that $\{Z_A|_{\mathcal{M}}\}$ is a local basis of sections of the vector bundle $F \to \mathcal{M}$.

On the other hand, if Γ_L is the Euler-Lagrange section associated with the regular Lagrangian L, then a section Γ of $\mathcal{T}^E\mathcal{M} \to \mathcal{M}$ is a solution of equations (8.1) if and only if

$$\Gamma = (\Gamma_L + \lambda^A Z_A)|_{\mathcal{M}}$$

with λ^A local real functions on E satisfying

$$(\lambda^A d\phi^B(Z_A) + d\phi^B(\Gamma_L))|_{\mathcal{M}} = 0, \text{ for all } B = 1, \dots, s.$$

Therefore, using (8.4), we conclude that there exists a unique solution of the Lagrange-d'Alembert equations (8.1) if and only if the matrix

$$\big(\mathcal{C}^{AB} = \frac{\partial \phi^A}{\partial y^\alpha} W^{\alpha\beta} \frac{\partial \phi^B}{\partial y^\beta}\big)_{A,B=1,\dots,s}$$

is regular. In addition, we will prove the following result:

Theorem 8.1. *The following properties are equivalent:*

(1) *The constrained Lagrangian system $(L, \mathcal{M})$ is regular, that is, there exists a unique solution of the Lagrange-d'Alembert equations,*
(2) $\operatorname{Ker} G^{L,\mathcal{V}} = \{0\}$,
(3) $\mathcal{T}^E\mathcal{M} \cap F = \{0\}$,
(4) $\mathcal{T}^{\mathcal{V}}\mathcal{M} \cap (\mathcal{T}^{\mathcal{V}}\mathcal{M})^\perp = \{0\}$.

Proof. It is clear that the matrix $(\mathcal{C}^{AB})$ is regular if and only if $\mathcal{T}^E\mathcal{M} \cap F = \{0\}$. Thus, the properties (1) and (3) are equivalent. Moreover, proceeding as in the proof of Theorem 4.4, we deduce that the properties (2) and (3) (respectively, (2) and (4)) also are equivalent. □

Remark 8.2. If L is a Lagrangian function of mechanical type, then, using Theorem 8.1, we deduce (as in the case of linear constraints) that the constrained system $(L, \mathcal{M})$ is always regular. ⋄

Assume that the constrained Lagrangian system $(L, \mathcal{M})$ is regular. Then (3) in Theorem 8.1 is equivalent to $(\mathcal{T}^E E)|_{\mathcal{M}} = \mathcal{T}^E\mathcal{M} \oplus F$. Denote by P and Q the complementary projectors defined by this decomposition

$$P_a : \mathcal{T}^E_a E \to \mathcal{T}^E_a \mathcal{M}, \quad Q_a : \mathcal{T}^E_a E \to F_a, \quad \text{for all } a \in \mathcal{M}.$$

As in the case of linear constraints, we may prove the following.

Theorem 8.3. *Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system and let Γ_L be the solution of the free dynamics, i.e., $i_{\Gamma_L}\omega_L = dE_L$. Then, the solution of the constrained dynamics is the SODE Γ obtained as follows*

$$\Gamma = P(\Gamma_L|_{\mathcal{M}}).$$

On the other hand, (4) in Theorem 8.1 is equivalent to $(\mathcal{T}^E E)|_{\mathcal{M}} = \mathcal{T}^{\mathcal{V}}\mathcal{M} \oplus (\mathcal{T}^{\mathcal{V}}\mathcal{M})^{\perp}$ and we will denote by $\bar{P}$ and $\bar{Q}$ the corresponding projectors induced by this decomposition, that is,

$$\bar{P}_a : \mathcal{T}^E_a E \to \mathcal{T}^{\mathcal{V}}_a \mathcal{M}, \quad \bar{Q}_a = \mathcal{T}^E_a E \to (\mathcal{T}^{\mathcal{V}}_a \mathcal{M})^{\perp}, \text{ for all } a \in \mathcal{M}.$$

Theorem 8.4. *Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system, Γ_L (respectively, Γ) be the solution of the free (respectively, constrained) dynamics and Δ be the Liouville section of $\mathcal{T}^E E \to E$. Then, $\Gamma = \bar{P}(\Gamma_L|_{\mathcal{M}})$ if and only if the restriction to $\mathcal{M}$ of the vector field $\rho^1(\Delta)$ on E is tangent to $\mathcal{M}$.*

Proof. Proceeding as in the proof of Lemma 4.3, we obtain that

$$(\mathcal{T}^{\mathcal{V}}_a \mathcal{M})^{\perp} \cap \mathrm{Ver}(\mathcal{T}^E_a E) = F_a, \text{ for all } a \in \mathcal{M}.$$

Thus, it is clear that

$$Q(\Gamma_L(a)) \in F_a \subseteq (\mathcal{T}^{\mathcal{V}}_a \mathcal{M})^{\perp}, \text{ for all } a \in \mathcal{M}.$$

Moreover, from (8.3) and using the fact that the solution of the constrained dynamics is a SODE along $\mathcal{M}$, we deduce

$$\Gamma(a) = P(\Gamma_L(a)) \in \mathcal{T}^{\mathcal{V}}_a \mathcal{M}, \quad \text{for all } a \in \mathcal{M},$$

if and only if the restriction to $\mathcal{M}$ of the vector field $\rho^1(\Delta)$ on E is tangent to $\mathcal{M}$. This proves the result. □

Remark 8.5. Note that if $\mathcal{M}$ is a vector subbundle D of E, then the vector field $\rho^1(\Delta)$ is always tangent to $\mathcal{M} = D$. ◇

As in the case of linear constraints, one may develop the distributional approach in order to obtain the solution of the constrained dynamics. In fact, if $(L, \mathcal{M})$ is regular, then $\mathcal{T}^{\mathcal{V}}\mathcal{M} \to \mathcal{M}$ is a symplectic subbundle of $(\mathcal{T}^E E, \omega_L)$ and, thus, the restriction $\omega^{L,\mathcal{M}}$ of ω_L to $\mathcal{T}^{\mathcal{V}}\mathcal{M}$ is a symplectic section on that bundle. We may also define $\varepsilon^{L,\mathcal{M}}$ as the restriction of dE_L to $\mathcal{T}^{\mathcal{V}}\mathcal{M}$. Then, taking the restriction of Lagrange d'Alembert equations to $\mathcal{T}^{\mathcal{V}}\mathcal{M}$, we get the following equation

$$i_{\bar{\Gamma}}\omega^{L,\mathcal{M}} = \varepsilon^{L,\mathcal{M}}, \tag{8.5}$$

which uniquely determines a section $\bar{\Gamma}$ of $\mathcal{T}^{\mathcal{V}}\mathcal{M} \to \mathcal{M}$. It is not difficult to prove that $\bar{\Gamma} = \bar{P}(\Gamma_L|_{\mathcal{M}})$. Thus, the unique solution of equation (8.5) is the solution of the constrained dynamics if and only if the vector field $\rho^1(\Delta)$ is tangent to $\mathcal{M}$.

Let $(L, \mathcal{M})$ an arbitrary constrained Lagrangian system. Since $S^* : (\mathcal{T}^E\mathcal{M})^0 \to \Psi$ is a vector bundle isomorphism, it follows that there exists a unique section $\alpha_{(L,\mathcal{M})}$ of $(\mathcal{T}^E\mathcal{M})^0 \to \mathcal{M}$ such that

$$i_{Q(\Gamma_L|_{\mathcal{M}})}\omega_L = S^*(\alpha_{(L,\mathcal{M})}).$$

Moreover, we have the following result.

Theorem 8.6. *If $(L, \mathcal{M})$ is a regular constrained Lagrangian system and Γ is the solution of the dynamics, then $d_\Gamma(E_L|_{\mathcal{M}}) = 0$ if and only if $\alpha_{(L,\mathcal{M})}(\Delta|_{\mathcal{M}}) = 0$. In particular, if the vector field $\rho^1(\Delta)$ is tangent to M, then $d_\Gamma(E_L|_{\mathcal{M}}) = 0$.*

Proof. From Theorem 8.3, we deduce

$$(i_\Gamma\omega_L - dE_L)|_{\mathcal{M}} = -S^*(\alpha_{(L,\mathcal{M})}).$$

Therefore, using that Γ is a SODE along $\mathcal{M}$, we obtain

$$d_\Gamma(E_L|_{\mathcal{M}}) = \alpha_{(L,\mathcal{M})}(\Delta|_{\mathcal{M}}).$$

□

Now, let $(L, \mathcal{M})$ be a regular constrained Lagrangian system. In addition, suppose that f and g are two smooth functions on $\mathcal{M}$ and take arbitrary extensions to E denoted by the same letters. Then, as in Section 4.3, we may define ***the nonholonomic bracket*** of f and g as follows

$$\{f, g\}_{nh} = \omega_L(\bar{P}(X_f), \bar{P}(X_g))|_{\mathcal{M}},$$

where X_f and X_g are the Hamiltonian sections on $\mathcal{T}^E E$ associated with f and g, respectively.

Moreover, proceeding as in the case of linear constraints, one can prove that

$$\dot{f} = \rho^1(R_L)(f) + \{f, E_L\}_{nh}, \quad f \in C^\infty(\mathcal{M}),$$

where R_L is the section of $\mathcal{T}^E\mathcal{M} \to \mathcal{M}$ defined by $R_L = P(\Gamma_L|_{\mathcal{M}}) - \bar{P}(\Gamma_L|_{\mathcal{M}})$. Thus, in the particular case when the restriction to $\mathcal{M}$ of the vector field $\rho^1(\Delta)$ on E is tangent to $\mathcal{M}$, it follows that

$$\dot{f} = \{f, E_L\}_{nh}, \quad \text{for } f \in C^\infty(\mathcal{M}).$$

Alternatively, since $\mathcal{T}^{\mathcal{V}}\mathcal{M}$ is an anchored vector bundle, we may consider the differential $\bar{d}f \in \mathrm{Sec}((\mathcal{T}^{\mathcal{V}}\mathcal{M})^*)$ for a function $f \in C^\infty(\mathcal{M})$. Thus, since the restriction $\omega^{L,\mathcal{M}}$ of ω_L to $\mathcal{T}^{\mathcal{V}}\mathcal{M}$ is regular, we have a unique section $\bar{X}_f \in \mathrm{Sec}(\mathcal{T}^{\mathcal{V}}\mathcal{M})$ given by $i_{\bar{X}_f}\omega^{L,\mathcal{M}} = \bar{d}f$ and it follows that

$$\{f, g\}_{nh} = \omega^{L,\mathcal{M}}(\bar{X}_f, \bar{X}_g).$$

Next, let $(L, \mathcal{M})$ be a regular constrained Lagrangian system on a Lie algebroid $\tau : E \to M$ and let $(L', \mathcal{M}')$ be another constrained Lagrangian system on a second Lie algebroid $\tau' : E' \to M'$. Suppose also that we have a fiberwise surjective morphism of Lie algebroids $\Phi : E \to E'$ over a surjective submersion $\phi : M \to M'$ such that:

(i) $L = L' \circ \Phi$,
(ii) $\Phi|_{\mathcal{M}}\, \mathcal{M} \to \mathcal{M}'$ is a surjective submersion,
(iii) $\Phi(\mathcal{V}_a) = \mathcal{V}'_{\Phi(a)}$, for all $a \in \mathcal{M}$.

Remark 8.7. Condition (ii) implies that $\Phi(\mathcal{V}_a) \subseteq \mathcal{V}'_{\Phi(a)}$, for all $a \in \mathcal{M}$. Moreover, if $V(\Phi)$ is the vertical bundle of Φ and

$$V_a(\Phi) \subset T_a\mathcal{M}, \text{ for all } a \in \mathcal{M},$$

then condition (ii) also implies that $\mathcal{V}'_{\Phi(a)} \subseteq \Phi(\mathcal{V}_a)$, for all $a \in \mathcal{M}$. $\diamond$

On the other hand, using condition (iii) and Proposition 5.2, it follows that $\ker G^{L',\mathcal{V}'} = \{0\}$ and, thus, the constrained Lagrangian system $(L', \mathcal{M}')$ is regular. Moreover, proceeding as in the proof of Lemma 5.5 and Theorem 5.6, we deduce the following results.

Lemma 8.8. *With respect to the decompositions*

$$(\mathcal{T}^E E)|_{\mathcal{M}} = \mathcal{T}^E\mathcal{M} \oplus F \quad \textit{and} \quad (\mathcal{T}^{E'}E')|_{\mathcal{M}} = \mathcal{T}^{E'}\mathcal{M}' \oplus F'$$

we have the following properties

(1) $\mathcal{T}^\Phi\Phi(\mathcal{T}^E\mathcal{M}) = \mathcal{T}^{E'}\mathcal{M}'$,
(2) $\mathcal{T}^\Phi\Phi(F) = F'$,
(3) *If P, Q and P', Q' are the projectors associated with $(L, \mathcal{M})$ and $(L', \mathcal{M}')$, respectively, then $P' \circ \mathcal{T}^\Phi\Phi = \mathcal{T}^\Phi\Phi \circ P$ and $Q' \circ \mathcal{T}^\Phi\Phi = \mathcal{T}^\Phi\Phi \circ Q$.*

With respect to the decompositions

$$(\mathcal{T}^E E)|_{\mathcal{M}} = \mathcal{T}^{\mathcal{V}}\mathcal{M} \oplus (\mathcal{T}^{\mathcal{V}}\mathcal{M})^\perp \textit{ and } (\mathcal{T}^{E'}E')|_{\mathcal{M}'} = \mathcal{T}^{\mathcal{V}'}\mathcal{M}' \oplus (\mathcal{T}^{\mathcal{V}'}\mathcal{M}')^\perp$$

we have the following properties

(4) $(\mathcal{T}^\Phi\Phi)(\mathcal{T}^\mathcal{V}\mathcal{M}) = \mathcal{T}^{\mathcal{V}'}\mathcal{M}'$,
(5) $(\mathcal{T}^\Phi\Phi)((\mathcal{T}^\mathcal{V}\mathcal{M})^\perp) = (\mathcal{T}^{\mathcal{V}'}\mathcal{M}')^\perp$,
(6) *If $\bar{P}, \bar{Q}$ and $\bar{P}'$ and $\bar{Q}'$ are the projectors associated with $(L, \mathcal{M})$ and $(L', \mathcal{M}')$, respectively, then $\bar{P}' \circ \mathcal{T}^\Phi\Phi = \mathcal{T}^\Phi\Phi \circ \bar{P}$ and $\bar{Q}' \circ \mathcal{T}^\Phi\Phi = \mathcal{T}^\Phi\Phi \circ \bar{Q}$.*

Theorem 8.9 (Reduction of the constrained dynamics)**.** *Let $(L, \mathcal{M})$ be a regular constrained Lagrangian system on a Lie algebroid E and let $(L', \mathcal{M}')$ be a constrained Lagrangian system on a second Lie algebroid E'. Assume that we have a fiberwise surjective morphism of Lie algebroids $\Phi : E \to E'$ over $\phi : M \to M'$ such that conditions (i)-(iii) hold. If Γ is the constrained dynamics for L and Γ' is the constrained dynamics for L', respectively, then $\mathcal{T}^\Phi\Phi \circ \Gamma = \Gamma' \circ \Phi$. If $a(t)$ is a solution of Lagrange-d'Alembert differential equations for L, then $\Phi(a(t))$ is a solution of Lagrange-d'Alembert differential equations for L'.*

We will say that the constrained dynamics Γ' is ***the reduction of the constrained dynamics*** Γ by the morphism Φ.

As in the case of linear constraints (see Theorem 5.7), we also may prove the following result

Theorem 8.10. *Under the same hypotheses as in Theorem 8.9, we have that*

$$\{f' \circ \Phi, g' \circ \Phi\}_{nh} = \{f', g'\}'_{nh} \circ \Phi,$$

for $f', g' \in C^\infty(\mathcal{M}')$, where $\{\cdot, \cdot\}_{nh}$ (respectively, $\{\cdot, \cdot\}'_{nh}$) is the nonholonomic bracket for the constrained system $(L, \mathcal{M})$ (respectively, $(L', \mathcal{M}')$). In other words, $\Phi : \mathcal{M} \to \mathcal{M}'$ is an almost-Poisson morphism.

Now, let $\phi : Q \to M$ be a principal G-bundle and $\tau : E \to Q$ be a Lie algebroid over Q. In addition, assume that we have an action of G on E such that the quotient vector bundle E/G is defined and the set $\mathrm{Sec}(E)^G$ of equivariant sections of E is a Lie subalgebra of $\mathrm{Sec}(E)$. Then, $E' = E/G$ has a canonical Lie algebroid structure over M such that the canonical projection $\Phi : E \to E'$ is a fiberwise bijective Lie algebroid morphism over ϕ (see Theorem 5.8).

Next, suppose that $(L, \mathcal{M})$ is a G-invariant regular constrained Lagrangian system, that is, the Lagrangian function L and the constraint submanifold $\mathcal{M}$ are G-invariant. Then, one may define a Lagrangian function $L' : E' \to \mathbb{R}$ on E' such that

$$L = L' \circ \Phi.$$

Moreover, G acts on $\mathcal{M}$ and if the set of orbits $\mathcal{M}' = \mathcal{M}/G$ of this action is a quotient manifold, that is, $\mathcal{M}'$ is a smooth manifold and the canonical projection $\Phi_{|\mathcal{M}} : \mathcal{M} \to \mathcal{M}' = \mathcal{M}/G$ is a submersion, then one may consider the constrained Lagrangian system $(L', \mathcal{M}')$ on E'.

Remark 8.11. If $\mathcal{M}$ is a closed submanifold of E, then, using a well-known result (see [1, Theorem 4.1.20]), it follows that the set of orbits $\mathcal{M}' = \mathcal{M}/G$ is a quotient manifold. $\diamond$

Since the orbits of the action of G on E are the fibers of Φ and $\mathcal{M}$ is G-invariant, we deduce that

$$V_a(\Phi) \subseteq T_a\mathcal{M}, \text{ for all } a \in \mathcal{M},$$

which implies that $\Phi_{|\mathcal{V}_a} : \mathcal{V}_a \to \mathcal{V}'_{\Phi(a)}$ is a linear isomorphism, for all $a \in \mathcal{M}$.

Thus, from Theorem 8.9, we conclude that the constrained Lagrangian system $(L', \mathcal{M}')$ is regular and that

$$\mathcal{T}^\Phi\Phi \circ \Gamma = \Gamma' \circ \Phi,$$

where Γ (resp., Γ') is the constrained dynamics for L (resp., L'). In addition, using Theorem 8.10, we obtain that $\Phi : \mathcal{M} \to \mathcal{M}'$ is an almost-Poisson morphism when on

$\mathcal{M}$ and $\mathcal{M}'$ we consider the almost-Poisson structures induced by the corresponding nonholonomic brackets.

Let us illustrate the results above in a particular example.

Example 8.12. (A ball rolling without sliding on a rotating table with constant angular velocity [4, 10, 50]). A (homogeneous) sphere of radius $r > 0$, unit mass $m = 1$ and inertia about any axis k^2, rolls without sliding on a horizontal table which rotates with constant angular velocity Ω about a vertical axis through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere.

Choose a Cartesian reference frame with origin at the center of rotation of the table and z-axis along the rotation axis. Let (x, y) denote the position of the point of contact of the sphere with the table. The configuration space for the sphere on the table is $Q = \mathbb{R}^2 \times SO(3)$, where $SO(3)$ may be parameterized by the Eulerian angles θ, φ and ψ. The kinetic energy of the sphere is then given by

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + k^2(\dot{\theta}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta)),$$

and with the potential energy being constant, we may put $V = 0$. The constraint equations are

$$\begin{aligned}
&\dot{x} - r\dot{\theta}\sin\psi + r\dot{\varphi}\sin\theta\cos\psi = -\Omega y,\\
&\dot{y} + r\dot{\theta}\cos\psi + r\dot{\varphi}\sin\theta\sin\psi = \Omega x.
\end{aligned}$$

Since the Lagrangian function is of mechanical type, the constrained system is regular. Note that the constraints are not linear and that the restriction to the constraint submanifold $\mathcal{M}$ of the Liouville vector field on TQ is not tangent to $\mathcal{M}$. Indeed, the constraints are linear if and only if $\Omega = 0$.

Now, we can proceed from here to construct to equations of motion of the sphere, following the general theory. However, the use of the Eulerian angles as part of the coordinates leads to very complicated expressions. Instead, one may choose to exploit the symmetry of the problem, and one way to do this is by the use of appropriate ***quasi-coordinates*** (see [50]). First of all, observe that the kinetic energy may be expressed as

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2)),$$

where

$$\begin{aligned}
\omega_x &= \dot{\theta}\cos\psi + \dot{\varphi}\sin\theta\sin\psi,\\
\omega_y &= \dot{\theta}\sin\psi - \dot{\varphi}\sin\theta\cos\psi,\\
\omega_z &= \dot{\varphi}\cos\theta + \dot{\psi},
\end{aligned}$$

are the components of the angular velocity of the sphere. The constraint equations expressing the rolling conditions can be rewritten as

$$\begin{aligned}
&\dot{x} - rw_y = -\Omega y,\\
&\dot{y} + r\omega_x = \Omega x.
\end{aligned}$$

Next, following [10], we will consider local coordinates $(\bar{x}, \bar{y}, \bar{\theta}, \bar{\varphi}, \bar{\psi}; \pi_i)_{i=1,\ldots,5}$ on $TQ = T\mathbb{R}^2 \times T(SO(3))$, where

$$\bar{x} = x, \quad \bar{y} = y, \quad \bar{\theta} = \theta, \quad \bar{\varphi} = \varphi, \quad \bar{\psi} = \psi,$$

$$\pi_1 = r\dot{x} + k^2\dot{q}_2, \quad \pi_2 = r\dot{y} - k^2\dot{q}_1, \quad \pi_3 = k^2\dot{q}_3,$$

$$\pi_4 = \frac{k^2}{(k^2 + r^2)}(\dot{x} - r\dot{q}_2 + \Omega y), \quad \pi_5 = \frac{k^2}{(k^2 + r^2)}(\dot{y} + r\dot{q}_1 - \Omega x),$$

and $(\dot{q}_1, \dot{q}_2, \dot{q}_3)$ are the quasi-coordinates defined by

$$\dot{q}_1 = \omega_x, \quad \dot{q}_2 = \omega_y, \quad \dot{q}_3 = \omega_z.$$

As is well-known, the coordinates q_i only have a symbolic meaning. In fact, $\{\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_3}\}$ is the basis of left-invariant vector fields on $SO(3)$ given by

$$\begin{aligned} \frac{\partial}{\partial q_1} &= (\cos\psi)\frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta}(\frac{\partial}{\partial\varphi} - \cos\theta\frac{\partial}{\partial\psi}), \\ \frac{\partial}{\partial q_2} &= (\sin\psi)\frac{\partial}{\partial\theta} - \frac{\cos\psi}{\sin\theta}(\frac{\partial}{\partial\varphi} - \cos\theta\frac{\partial}{\partial\psi}), \\ \frac{\partial}{\partial q_3} &= \frac{\partial}{\partial\psi}, \end{aligned}$$

and we have that

$$[\frac{\partial}{\partial q_2}, \frac{\partial}{\partial q_1}] = \frac{\partial}{\partial q_3}, \quad [\frac{\partial}{\partial q_1}, \frac{\partial}{\partial q_3}] = \frac{\partial}{\partial q_2}, \quad [\frac{\partial}{\partial q_3}, \frac{\partial}{\partial q_2}] = \frac{\partial}{\partial q_1}.$$

Note that in the new coordinates the local equations defining the constraint submanifold $\mathcal{M}$ are $\pi_4 = 0$, $\pi_5 = 0$. On the other hand, if $P : (\mathcal{T}^{TQ}TQ)|_{\mathcal{M}} = T_{\mathcal{M}}(TQ) \to \mathcal{T}^{TQ}\mathcal{M} = T\mathcal{M}$ and $Q : T_{\mathcal{M}}(TQ) \to F$ are the projectors associated with the decomposition $T_{\mathcal{M}}(TQ) = T\mathcal{M} \oplus F$, then we have that (see [10])

$$\begin{aligned} Q &= \frac{\partial}{\partial\pi_4} \otimes d\pi_4 + \frac{\partial}{\partial\pi_5} \otimes d\pi_5, \\ P &= \mathrm{Id} - \frac{\partial}{\partial\pi_4} \otimes d\pi_4 - \frac{\partial}{\partial\pi_5} \otimes d\pi_5. \end{aligned}$$

Moreover, using that the unconstrained dynamics Γ_L is given by

$$\begin{aligned} \Gamma_L &= \dot{x}\frac{\partial}{\partial\bar{x}} + \dot{y}\frac{\partial}{\partial\bar{y}} + \dot{\theta}\frac{\partial}{\partial\bar{\theta}} + \dot{\varphi}\frac{\partial}{\partial\bar{\varphi}} + \dot{\psi}\frac{\partial}{\partial\bar{\psi}} + \frac{k^2\Omega}{(k^2+r^2)}\dot{y}\frac{\partial}{\partial\pi_4} + \frac{k^2\Omega}{(k^2+r^2)}\dot{x}\frac{\partial}{\partial\pi_5} \\ &= \dot{x}\frac{\partial}{\partial\bar{x}} + \dot{y}\frac{\partial}{\partial\bar{y}} + \dot{q}_1\frac{\partial}{\partial q_1} + \dot{q}_2\frac{\partial}{\partial q_2} + \dot{q}_3\frac{\partial}{\partial q_3} + \frac{k^2\Omega}{(k^2+r^2)}\dot{y}\frac{\partial}{\partial\pi_4} + \frac{k^2\Omega}{(k^2+r^2)}\dot{x}\frac{\partial}{\partial\pi_5}, \end{aligned}$$

we deduce that the constrained dynamics is the SODE Γ along $\mathcal{M}$ defined by

$$\begin{aligned} \Gamma = (P\Gamma_L|_{\mathcal{M}}) &= (\dot{x}\frac{\partial}{\partial\bar{x}} + \dot{y}\frac{\partial}{\partial\bar{y}} + \dot{\theta}\frac{\partial}{\partial\bar{\theta}} + \dot{\varphi}\frac{\partial}{\partial\bar{\varphi}} + \dot{\psi}\frac{\partial}{\partial\bar{\psi}})|_{\mathcal{M}} \\ &= (\dot{x}\frac{\partial}{\partial\bar{x}} + \dot{y}\frac{\partial}{\partial\bar{y}} + \dot{q}_1\frac{\partial}{\partial q_1} + \dot{q}_2\frac{\partial}{\partial q_2} + \dot{q}_3\frac{\partial}{\partial q_3})|_{\mathcal{M}}. \end{aligned} \tag{8.6}$$

This implies that

$$d_\Gamma(E_L|_{\mathcal{M}}) = d_\Gamma(L|_{\mathcal{M}}) = \frac{\Omega^2 k^2}{(k^2+r^2)}(x\dot{x} + y\dot{y})|_{\mathcal{M}}.$$

Consequently, the Lagrangian energy is a constant of the motion if and only if $\Omega = 0$.

When constructing the nonholonomic bracket on $\mathcal{M}$, we find that the only non-zero fundamental brackets are

$$\begin{array}{lll} \{x, \pi_1\}_{nh} = r, & \{y, \pi_2\}_{nh} = r, & \\ \{q_1, \pi_2\}_{nh} = -1, & \{q_2, \pi_1\}_{nh} = 1, & \{q_3, \pi_3\}_{nh} = 1, \\ \{\pi_1, \pi_2\}_{nh} = \pi_3, & \{\pi_2, \pi_3\}_{nh} = \frac{k^2}{(k^2+r^2)}\pi_1 + \frac{rk^2\Omega}{(k^2+r^2)}y, & \\ \{\pi_3, \pi_1\}_{nh} = \frac{k^2}{(k^2+r^2)}\pi_2 - \frac{rk^2\Omega}{(k^2+r^2)}x, & & \end{array} \tag{8.7}$$

in which the "appropriate operational" meaning has to be attached to the quasi-coordinates q_i. As a result, we have

$$\dot{f} = R_L(f) + \{f, L\}_{nh}, \text{ for } f \in C^\infty(\mathcal{M})$$

where R_L is the vector field on $\mathcal{M}$ given by

$$R_L = \Big(\frac{k^2\Omega}{(k^2+r^2)}\big(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\big) + \frac{r\Omega}{(k^2+r^2)}\big(x\frac{\partial}{\partial q_1} + y\frac{\partial}{\partial q_2}$$
$$+ x(\pi_3 - k^2\Omega)\frac{\partial}{\partial \pi_1} + y(\pi_3 - k^2\Omega)\frac{\partial}{\partial \pi_2} - k^2(\pi_1 x + \pi_2 y)\frac{\partial}{\partial \pi_3}\big)\Big)\Big|_{\mathcal{M}}.$$

Note that $R_L = 0$ if and only if $\Omega = 0$.

Now, it is clear that $Q = \mathbb{R}^2 \times SO(3)$ is the total space of a trivial principal $SO(3)$-bundle over $\mathbb{R}^2$ and the bundle projection $\phi : Q \to M = \mathbb{R}^2$ is just the canonical projection on the first factor. Therefore, we may consider the corresponding Atiyah algebroid $E' = TQ/SO(3)$ over $M = \mathbb{R}^2$. Next, we describe this Lie algebroid.

Using the left-translations in $SO(3)$, one may define a diffeomorphism λ between the tangent bundle to $SO(3)$ and the product manifold $SO(3) \times \mathbb{R}^3$ (see [1]). In fact, in terms of the Euler angles, the diffeomorphism λ is given by

$$\lambda(\theta, \varphi, \psi; \dot\theta, \dot\varphi, \dot\psi) = (\theta, \varphi, \psi; \omega_x, \omega_y, \omega_z). \tag{8.8}$$

Under this identification between $T(SO(3))$ and $SO(3) \times \mathbb{R}^3$, the tangent action of $SO(3)$ on $T(SO(3)) \cong SO(3) \times \mathbb{R}^3$ is the trivial action

$$SO(3) \times (SO(3) \times \mathbb{R}^3) \to SO(3) \times \mathbb{R}^3, \quad (g, (h, \omega)) \mapsto (gh, \omega). \tag{8.9}$$

Thus, the Atiyah algebroid $TQ/SO(3)$ is isomorphic to the product manifold $T\mathbb{R}^2 \times \mathbb{R}^3$, and the vector bundle projection is $\tau_{\mathbb{R}^2} \circ pr_1$, where $pr_1 : T\mathbb{R}^2 \times \mathbb{R}^3 \to T\mathbb{R}^2$ and $\tau_{\mathbb{R}^2} : T\mathbb{R}^2 \to \mathbb{R}^2$ are the canonical projections.

A section of $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$ is a pair (X, u), where X is a vector field on $\mathbb{R}^2$ and $u : \mathbb{R}^2 \to \mathbb{R}^3$ is a smooth map. Therefore, a global basis of sections of $T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^2$ is

$$\begin{aligned} e'_1 &= (\frac{\partial}{\partial x}, 0), \quad e'_2 = (\frac{\partial}{\partial y}, 0), \\ e'_3 &= (0, u_1), \quad e'_4 = (0, u_2), \quad e'_5 = (0, u_3), \end{aligned}$$

where $u_1, u_2, u_3 : \mathbb{R}^2 \to \mathbb{R}^3$ are the constant maps

$$u_1(x, y) = (1, 0, 0), \quad u_2(x, y) = (0, 1, 0), \quad u_3(x, y) = (0, 0, 1).$$

In other words, there exists a one-to-one correspondence between the space $\mathrm{Sec}(E' = TQ/SO(3))$ and the G-invariant vector fields on Q. Under this bijection, the sections e'_1 and e'_2 correspond with the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ and the sections e'_3, e'_4 and e'_5 correspond with the vertical vector fields $\frac{\partial}{\partial q_1}$, $\frac{\partial}{\partial q_2}$ and $\frac{\partial}{\partial q_3}$, respectively.

The anchor map $\rho' : E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3 \to T\mathbb{R}^2$ is the projection over the first factor and, if $[\![\cdot, \cdot]\!]'$ is the Lie bracket on the space $\mathrm{Sec}(E' = TQ/SO(3))$, then the only non-zero fundamental Lie brackets are

$$[\![e'_4, e'_3]\!]' = e'_5, \quad [\![e'_5, e'_4]\!]' = e'_3, \quad [\![e'_3, e'_5]\!]' = e'_4.$$

From (8.8) and (8.9), it follows that the Lagrangian function $L = T$ and the constraint submanifold $\mathcal{M}$ are $SO(3)$-invariant. Consequently, L induces a Lagrangian function L' on $E' = TQ/SO(3)$ and, since $\mathcal{M}$ is closed on TQ, the set of orbits $\mathcal{M}' = \mathcal{M}/SO(3)$ is a submanifold of $E' = TQ/SO(3)$ in such a way that the canonical projection $\Phi|_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}' = \mathcal{M}/SO(3)$ is a surjective submersion.

Under the identification between $E' = TQ/SO(3)$ and $T\mathbb{R}^2 \times \mathbb{R}^3$, L' is given by

$$L'(x, y, \dot{x}, \dot{y}; \omega_1, \omega_2, \omega_3) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{k^2}{2}(\omega_1^2 + \omega_2^2 + \omega_3^2),$$

where $(x, y, \dot{x}, \dot{y})$ and $(\omega_1, \omega_2, \omega_3)$ are the standard coordinates on $T\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Moreover, the equations defining $\mathcal{M}'$ as a submanifold of $T\mathbb{R}^2 \times \mathbb{R}^3$ are

$$\dot{x} - r\omega_2 + \Omega y = 0, \quad \dot{y} + r\omega_1 - \Omega x = 0.$$

So, we have the constrained Lagrangian system $(L', \mathcal{M}')$ on the Atiyah algebroid $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$. Note that the constraints are not linear, and that, if Δ' is the Liouville section of the prolongation $\mathcal{T}^{E'}E'$, then the restriction to $\mathcal{M}'$ of the vector field $(\rho')^1(\Delta')$ is not tangent to $\mathcal{M}'$.

Now, it is clear that the tangent bundle $TQ = T\mathbb{R}^2 \times T(SO(3)) \cong T\mathbb{R}^2 \times (SO(3) \times \mathbb{R}^3)$ is the total space of a trivial principal $SO(3)$- bundle over $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$ and, in addition (see [37, Theorem 9.1]), the prolongation $\mathcal{T}^{E'}E'$ is isomorphic to the Atiyah algebroid associated with this principal $SO(3)$-bundle. Therefore, the sections of the prolongation $\mathcal{T}^{E'}E' \to E'$ may be identified with the $SO(3)$-invariant vector fields on $TQ \cong T\mathbb{R}^2 \times (SO(3) \times \mathbb{R}^3)$. Under this identification, the constrained dynamics Γ' for the system $(L', \mathcal{M}')$ is just the $SO(3)$-invariant vector field $\Gamma = P(\Gamma_L|_{\mathcal{M}})$. We recall that if $\Phi : TQ \to TQ/SO(3)$ is the canonical projection, then

$$\mathcal{T}^\Phi\Phi \circ \Gamma = \Gamma' \circ \Phi. \tag{8.10}$$

Next, we will give a local description of the vector field $(\rho')^1(\Gamma')$ on $E' = TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$ and the nonholonomic bracket $\{\cdot, \cdot\}'_{nh}$ for the constrained system $(L', \mathcal{M}')$. For this purpose, we will consider a suitable system of local coordinates on $TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$. If we set

$$\begin{array}{lll} x' = x, & y' = y, & \\ \pi'_1 = r\dot{x} + k^2\omega_2, & \pi'_2 = r\dot{y} - k^2\omega_1, & \pi'_3 = k^2\omega_3, \\ \pi_4 = \frac{k^2}{(k^2+r^2)}(\dot{x} - r\omega_2 + \Omega y), & \pi'_5 = \frac{k^2}{(k^2+r^2)}(\dot{y} + r\omega_1 - \Omega x), & \end{array}$$

then $(x', y', \pi'_1, \pi'_2, \pi'_3, \pi'_4, \pi'_5)$ is a system of local coordinates on $TQ/SO(3) \cong T\mathbb{R}^2 \times \mathbb{R}^3$. In these coordinates the equations defining the submanifold $\mathcal{M}'$ are $\pi'_4 = 0$ and $\pi'_5 = 0$, and the canonical projection $\Phi : TQ \to TQ/SO(3)$ is given by

$$\Phi(\bar{x}, \bar{y}, \bar{\theta}, \bar{\varphi}, \bar{\psi}; \pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (\bar{x}, \bar{y}; \pi_1, \pi_2, \pi_3, \pi_4, \pi_5). \tag{8.11}$$

Thus, from (8.6) and (8.10), it follows that

$$(\rho')^1(\Gamma') = (\dot{x}'\frac{\partial}{\partial x'} + \dot{y}'\frac{\partial}{\partial y'})|_{\mathcal{M}'},$$

or, in the standard coordinates $(x, y, \dot{x}, \dot{y}; \omega_1, \omega_2, \omega_3)$ on $T\mathbb{R}^2 \times \mathbb{R}^3$,

$$\begin{aligned}(\rho')^1(\Gamma') = \{&\dot{x}(\frac{\partial}{\partial x} + \frac{\Omega k^2}{(k^2+r^2)}\frac{\partial}{\partial \dot{y}} + \frac{\Omega r}{(k^2+r^2)}\frac{\partial}{\partial \omega_1}) \\ &+ \dot{y}(\frac{\partial}{\partial y} - \frac{\Omega k^2}{(k^2+r^2)}\frac{\partial}{\partial \dot{x}} + \frac{\Omega r}{(k^2+r^2)}\frac{\partial}{\partial \omega_2})\}|_{\mathcal{M}'}.\end{aligned}$$

On the other hand, from (8.7), (8.11) and Theorem 8.10, we deduce that the only non-zero fundamental nonholonomic brackets for the system $(L', \mathcal{M}')$ are

$$\begin{array}{ll} \{x', \pi'_1\}'_{nh} = r, & \{y', \pi'_2\}'_{nh} = r, \\ \{\pi'_1, \pi'_2\}'_{nh} = \pi'_3, & \{\pi'_2, \pi'_3\}'_{nh} = \dfrac{k^2}{(k^2+r^2)}\pi'_1 + \dfrac{rk^2\Omega}{(k^2+r^2)}y', \\ \{\pi'_3, \pi'_1\}'_{nh} = \dfrac{k^2}{(k^2+r^2)}\pi'_2 - \dfrac{rk^2\Omega}{(k^2+r^2)}x'. & \end{array}$$

Therefore, we have that

$$\dot{f}' = (\rho')^1(R_{L'})(f') + \{f', L'\}'_{nh}, \text{ for } f' \in C^\infty(\mathcal{M}'),$$

where $(\rho')^1(R_{L'})$ is the vector field on $\mathcal{M}'$ given by

$$(\rho')^1(R_{L'}) = \{\frac{k^2\Omega}{k^2+r^2}(x'\frac{\partial}{\partial y'} - y'\frac{\partial}{\partial x'}) + \frac{r\Omega}{(k^2+r^2)}(x'(\pi'_3 - k^2\Omega)\frac{\partial}{\partial \pi'_1}$$

$$+ y'(\pi'_3 - k^2\Omega)\frac{\partial}{\partial \pi'_2} - k^2(\pi'_1 x' + \pi'_2 y')\frac{\partial}{\partial \pi'_3})\}|_{\mathcal{M}'}.$$

◁

9. Conclusions and outlook

We have developed a geometrical description of nonholonomic mechanical systems in the context of Lie algebroids. This formalism is the natural extension of the standard treatment on the tangent bundle of the configuration space. The proposed approach also allows to deal with nonholonomic mechanical systems with symmetry, and perform the reduction procedure in a unified way. The main results obtained in the paper are summarized as follows:

- we have identified the notion of regularity of a nonholonomic mechanical system with linear constraints on a Lie algebroid, and we have characterized it in geometrical terms;
- we have obtained the constrained dynamics by projecting the unconstrained one using two different decompositions of the prolongation of the Lie algebroid along the constraint subbundle;
- we have developed a reduction procedure by stages and applied it to nonholonomic mechanical systems with symmetry. These results have allowed us to get new insights in the technique of quasicoordinates;
- we have defined the operation of nonholonomic bracket to measure the evolution of observables along the solutions of the system;
- we have examined the setup of nonlinearly constrained systems;
- we have illustrated the main results of the paper in several examples.

Current and future directions of research will include the in-depth study of the reduction procedure following the steps of [4, 8] for the standard case; the synthesis of so-called nonholonomic integrators [15, 19, 36] for systems evolving on Lie algebroids, and the development of a comprehensive treatment of classical field theories within the Lie algebroid formalism following the ideas by E. Martínez [47].

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