On synchronous robotic networks
Part II: Time complexity of rendezvous and deployment algorithms

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Abstract— **This paper analyzes a number of basic coordination algorithms running on synchronous robotic networks. We provide upper and lower bounds on the time complexity of the move-toward average and circumcenter laws, both achieving rendezvous, and of the centroid law, achieving deployment over a region of interest. The results are derived via novel analysis methods, including a set of results on the convergence rates of linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices.**

I. Introduction

Problem motivation: Although recent years have witnessed the emergence of numerous coordination algorithms for networked mobile systems, the fundamental limits in terms of achievable performance, energy consumption and operational time remain largely unknown. This is in part explained by the inherent difficulty in integrating the various sensing, computing and communication aspects of problems involving groups of mobile agents. In this paper, we analyze the performance of several coordination algorithms achieving rendezvous and deployment (see [3], [4] for a discussion on the practical motivation of these tasks). To achieve this goal, we rely on the general framework proposed in the companion paper [2] to formally model the behavior of robotic networks. Our research effort aims at developing tools and results to assess to what extent coordination algorithms are scalable and implementable in large mobile networks. Ultimately, we aim at characterizing the minimum amount of communication, sensing and control necessary to perform a desired task, and at designing algorithms that achieve those limits.

Literature review: A survey on cooperative mobile robotics is presented in [5] and an overview of control and communication issues is contained in [6]. Specific topics related to the present treatment include rendezvous [3], [7], [8], [9], [10], cyclic pursuit [11], deployment [4], flocking [12] and consensus [13], [14]. The papers [15], [16], [17] discuss convergence rates of various coordination algorithms. See the aforementioned works for references on other cooperative strategies designed to perform spatially-distributed tasks.

Statement of contributions: The companion paper [2] proposes a general framework to model robotic networks and formally analyze their behavior. In particular, this work defines notions of time and communication complexity aimed at capturing the performance and cost of coordination algorithms. Building on these notions, we establish here complexity estimates for various algorithms that achieve rendezvous and deployment. First, we analyze a simple averaging law for a network of locally-connected agents moving on a line, related to the widely known Vicsek's model, see [12], [18]. We show that this law achieves rendezvous (without preserving connectivity) with time complexity belonging to $\Omega(N)$ and $O(N^5)$. Second, for a network of locally-connected agents moving on a line or on a segment, we show that the circumcenter algorithm in [3] has time complexity of order $\Theta(N)$. (This algorithm achieves rendezvous while preserving connectivity with a communication graph with $O(N^2)$ links.) We then consider a network based on a different communication graph, called the limited Delaunay graph, that arises naturally in computational geometry and in wireless communication. For this less dense graph with $O(N)$ links, we show that the time complexity of the circumcenter algorithm grows to $\Theta(N^2 \log N)$. For a network of agents moving on $\mathbb{R}^d$ we introduce a novel "parallel-circumcenter algorithm" and establish its time complexity of order $\Theta(N)$. Third and last, for a network of agents in a one-dimensional environment, we show that the time complexity of the deployment algorithm in [4] is $O(N^3 \log N)$. To obtain these complexity estimates, we develop novel analysis methods, particularly a set of results on linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices, and characterize their convergence rates. The interested reader is referred to [1] for a complete discussion of the proofs of all results presented here.

Organization: Section II develops some facts about convergence rates of dynamical systems defined by tridiagonal Toeplitz and circulant matrices. Section III reviews the general approach to the modeling of robotic networks proposed in [2]. Sections IV and V define the rendezvous and deployment tasks, respectively, and present coordination algorithms that achieve them. We establish their asymptotic correctness and characterize their time complexity. We present our conclusions in Section VI. We refer the reader to [1] for the definition of various basic geometric concepts used in the paper.

Notation: Let $\texttt{BooleSet} = \{\texttt{true}, \texttt{false}\}$. We let $\prod_{i \in \{1,\dots,N\}} S_i$ denote the Cartesian product of sets $S_1, \dots, S_N$. We let $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ denote the set of strictly positive and non-negative real numbers, respectively. The set of of positive natural numbers is denoted by $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of non-negative integers. If S is a set, then

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$\mathrm{diag}(S \times S) = \{(s, s) \in S \times S \mid s \in S\}$. For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the floor of x. For $x \in \mathbb{R}^d$, we denote by $\|x\|_2$ and $\|x\|_\infty$ the Euclidean and the ∞-norm of x, respectively. For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_+$, $\overline{B}(x, r)$ denotes the closed ball in $\mathbb{R}^d$ centered at x of radius r. We let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard orthonormal basis of $\mathbb{R}^d$. Also, we define the vectors $\mathbf{0} = (0, \dots, 0)^T$ and $\mathbf{1} = (1, \dots, 1)^T$ in $\mathbb{R}^d$. For $f, g \colon \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ (respectively, $f \in \Omega(g)$) if there exist $N_0 \in \mathbb{N}$ and $k \in \mathbb{R}_+$ such that $|f(N)| \leq k|g(N)|$ for all $N \geq N_0$ (respectively, $|f(N)| \geq k|g(N)|$ for all $N \geq N_0$). If $f \in O(g)$ and $f \in \Omega(g)$, then we use the notation $f \in \Theta(g)$.

II. Tridiagonal Toeplitz and circulant dynamical systems

This section presents some general results on certain classes of Toeplitz matrices, see [19]. These are later employed to obtain complexity estimates in Sections IV and V. For $N \geq 2$ and $a, b, c \in \mathbb{R}$, let the Toeplitz $N \times N$-matrices $\mathrm{Trid}_N(a, b, c)$ and $\mathrm{Circ}_N(a, b, c)$ be

$$\mathrm{Trid}_N(a, b, c) = \begin{bmatrix} b & c & 0 & \dots & 0 \\ a & b & c & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a & b & c \\ 0 & \dots & 0 & a & b \end{bmatrix},$$

and

$$\mathrm{Circ}_N(a, b, c) = \mathrm{Trid}_N(a, b, c) + \begin{bmatrix} 0 & \dots & \dots & 0 & a \\ 0 & \dots & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ c & 0 & \dots & 0 & 0 \end{bmatrix}.$$

Trid_N and Circ_N are tridiagonal and circulant, respectively. They only differ in their $(1, N)$ and $(N, 1)$ entries.

Theorem II.1 (Tridiagonal Toeplitz and circulant dynamical systems) *Let $N \geq 2$, $\varepsilon \in]0, 1[$, and $a, b, c \in \mathbb{R}$. Let $x \colon \mathbb{N}_0 \to \mathbb{R}^N$, $y \colon \mathbb{N}_0 \to \mathbb{R}^N$ be solutions to*

$$x(\ell+1) = \mathrm{Trid}_N(a, b, c)\, x(\ell), \quad y(\ell+1) = \mathrm{Circ}_N(a, b, c)\, y(\ell),$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$, respectively. The following statements hold:

(i) *if $a = c \neq 0$ and $|b| + 2|a| = 1$, then $\lim_{\ell \to +\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \leq \varepsilon \|x_0\|_2$ (over all initial $x_0 \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$;*

(ii) *if $a \neq 0$, $c = 0$ and $0 < |b| < 1$, then $\lim_{\ell \to +\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \leq \varepsilon \|x_0\|_2$ (over all initial $x_0 \in \mathbb{R}^N$) is $O\big(N \log N + \log \varepsilon^{-1}\big)$;*

(iii) *if $a \geq 0$, $c \geq 0$, $b > 0$, and $a + b + c = 1$, then $\lim_{\ell \to +\infty} y(\ell) = y_{\mathrm{ave}} \mathbf{1}$, where $y_{\mathrm{ave}} = \frac{1}{N} \mathbf{1}^T y_0$, and the maximum time required for $\|y(\ell) - y_{\mathrm{ave}} \mathbf{1}\|_2 \leq \varepsilon \|y_0 - y_{\mathrm{ave}} \mathbf{1}\|_2$ (over all initial $y_0 \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$.* •

For $N \geq 2$ and $a, b \in \mathbb{R}$, define the $N \times N$ matrices $\mathrm{ATrid}_N^+(a, b)$ and $\mathrm{ATrid}_N^-(a, b)$ by

$$\mathrm{ATrid}_N^{\pm}(a, b) = \mathrm{Trid}_N(a, b, a) \pm \begin{bmatrix} a & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & a \end{bmatrix}.$$

One can show [1] that these matrices are, respectively, similar (in the algebraic sense) to the block diagonal matrices

$$\begin{bmatrix} b \pm 2a & 0 \\ 0 & \mathrm{Trid}_{N-1}(a, b, a) \end{bmatrix}.$$

To state the convergence properties of the dynamical systems determined by $\mathrm{ATrid}_N^+(a, b)$ and $\mathrm{ATrid}_N^-(a, b)$, we define $\mathbf{1}_- = (1, -1, 1, \dots, (-1)^{N-2}, (-1)^{N-1})^T \in \mathbb{R}^N$.

Theorem II.2 *Let $N \geq 2$, $\varepsilon \in]0, 1[$, $0 \neq a, b \in \mathbb{R}$ with $|b| + 2|a| = 1$. Let $x \colon \mathbb{N}_0 \to \mathbb{R}^N$, $z \colon \mathbb{N}_0 \to \mathbb{R}^N$ be solutions to*

$$x(\ell+1) = \mathrm{ATrid}_N^+(a, b)\, x(\ell), \quad z(\ell+1) = \mathrm{ATrid}_N^-(a, b)\, z(\ell),$$

with initial conditions $x(0) = x_0$ and $z(0) = z_0$, respectively. The following statements hold:

(i) *$\lim_{\ell \to +\infty} \big(x(\ell) - x_{\mathrm{ave}}(\ell) \mathbf{1}\big) = \mathbf{0}$, where $x_{\mathrm{ave}}(\ell) = (\frac{1}{N} \mathbf{1}^T x_0)(b + 2a)^\ell$, and the maximum time required for $\|x(\ell) - x_{\mathrm{ave}}(\ell) \mathbf{1}\|_2 \leq \varepsilon \|x_0 - x_{\mathrm{ave}}(0) \mathbf{1}\|_2$ (over all initial $x_0 \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$;*

(ii) *$\lim_{\ell \to +\infty} \big(z(\ell) - z_{\mathrm{ave}}(\ell) \mathbf{1}_-\big) = \mathbf{0}$, where $z_{\mathrm{ave}}(\ell) = (\frac{1}{N} \mathbf{1}_-^T z_0)(b - 2a)^\ell$, and the maximum time required for $\|z(\ell) - z_{\mathrm{ave}}(\ell) \mathbf{1}_-\|_2 \leq \varepsilon \|z_0 - z_{\mathrm{ave}}(0) \mathbf{1}_-\|_2$ (over all initial $z_0 \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$.* •

III. Synchronous robotic networks

The companion paper [2] proposes a formal model for robotic networks, and defines the notions of control and communication laws, tasks, and time and communication complexity. We present here simplified versions of them.

Definition III.1 *A* uniform network of robotic agents *(or* robotic network*) $\mathcal{S}$ is a tuple $(I, \mathcal{A}, E_{\mathrm{cmm}})$ consisting of*

(i) *$I = \{1, \dots, N\}$; I is the* set of unique identifiers*;*

(ii) *$\mathcal{A} = \{A^{[i]}\}_{i \in I}$, with $A^{[i]} = (X, U, X_0, f)$, is a set of identical control systems (the* set of physical agents*);*

(iii) *E_{cmm} is a map from $\prod_{i \in I} X$ to the subsets of $I \times I \setminus \mathrm{diag}(I \times I)$ called the* communication edge map. •

Definition III.2 *A* (synchronous, static, uniform, feedback) control and communication law *$\mathcal{CC}$ for $\mathcal{S}$ consists of the sets: (i) $\mathbb{T} = \{t_\ell\}_{\ell \in \mathbb{N}_0} \subset \overline{\mathbb{R}}_+$, an increasing sequence of time instants, called* communication schedule*; and (ii) L, a set containing the* `null` *element, called the* communication language*; elements of L are called* messages*; and of the maps: (i)* $\mathrm{msg} \colon \mathbb{T} \times X \times I \to L$, *the* message-generation function*; (ii)* $\mathrm{ctl} \colon \overline{\mathbb{R}}_+ \times X \times X \times L^N \to U$, *the* control function. •

The law $\mathcal{CC}$ is said to be *time-independent* if the message-generation and control functions are of the form $\mathrm{msg} \colon X \times I \to L$ and $\mathrm{ctl} \colon X \times X \times L^N \to U$, respectively.

Definition III.3 *The* evolution *of* $(\mathcal{S}, \mathcal{CC})$ *from initial conditions* $x_0^{[i]} \in X_0{}^{[i]}$, $i \in I$, *is the set of curves* $x^{[i],\ell}\colon [t_\ell, t_{\ell+1}] \to X$, $i \in I$, $\ell \in \mathbb{N}_0$, *satisfying*

$$\dot{x}^{[i],\ell}(t) = f\big(x^{[i],\ell}(t), \operatorname{ctl}(t, x^{[i],\ell}(t), x^{[i],\ell}(t_\ell), y^{[i]}(t_\ell))\big),$$

where $x^{[i],\ell}(t_\ell) = x^{[i],\ell-1}(t_\ell)$ *(with* $x^{[i],-1}(t_0) = x_0^{[i]}$*) for* $\ell \in \mathbb{N}_0$. $y^{[i]}\colon \mathbb{T} \to L^N$ *describes the messages received by agent* i*, with components* $y_j^{[i]}(t_\ell) = \operatorname{msg}(t_\ell, x^{[j],\ell-1}(t_\ell), i)$*, if* $(i,j) \in E_{\text{cmm}}(x^{[1],\ell-1}(t_\ell), \dots, x^{[N],\ell-1}(t_\ell))$ *and* $y_j^{[i]}(t_\ell) =$ `null` *otherwise.* •

Remarks III.4 (Notation) The projection $\pi_L : L^N \to 2^L$ maps an array of messages y to the subset of L containing only the non-`null` messages in y. In many uniform control and communication laws, the messages interchanged among the agents are (quantized representations of) the agents' states. The corresponding communication language is $L = X$, and $\operatorname{msg}_{\text{std}}\colon \mathbb{T}\times X \times I \to X$ is referred to as the *standard message-generation function*, $\operatorname{msg}_{\text{std}}(t,x,j) = x$. •

Let us introduce some examples of robotic networks. We start with a basic example and define some variations of it.

Example III.5 (Locally-connected first-order agents in $\mathbb{R}^d$) Consider N points $x^{[1]}, \dots, x^{[N]}$ in the Euclidean space $\mathbb{R}^d$, $d \geq 1$, obeying a first-order dynamics $\dot{x}^{[i]}(t) = u^{[i]}(t)$. These are identical agents of the form $A = (\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d, (\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d))$. Assume that each agent can communicate to any other agent within Euclidean distance r, that is, adopt as communication edge map the r-disk proximity edge map $E_{r\text{-disk}}$ defined by $(i,j) \in E_{r\text{-disk}}(x^{[1]}, \dots, x^{[N]})$ iff $\|x^{[i]} - x^{[j]}\|_2 \leq r$. These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}} = (I, \mathcal{A}, E_{r\text{-disk}})$. •

Example III.6 (LD-connected first-order agents in $\mathbb{R}^d$) Consider the set of physical agents defined in Example III.5. For $r \in \mathbb{R}_+$, adopt as communication graph the r-limited Delaunay map $E_{r\text{-LD}}$ defined by $(i,j) \in E_{r\text{-LD}}(x^{[1]}, \dots, x^{[N]})$ iff $\big(V_i \cap \overline{B}(x^{[i]}, \frac{r}{2})\big) \cap \big(V_j \cap \overline{B}(x^{[j]}, \frac{r}{2})\big) \neq \emptyset$,, where $\{V_1, \dots, V_N\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x^{[1]}, \dots, x^{[N]}\}$ (cf. [20]). These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}} = (I, \mathcal{A}, E_{r\text{-LD}})$. •

Example III.7 (Locally-∞-connected first-order agents in $\mathbb{R}^d$) Consider the set of physical agents defined in the previous two examples. For $r \in \mathbb{R}_+$, define the proximity edge map $E_{r\text{-}\infty\text{-disk}}$ by $(i,j) \in E_{r\text{-}\infty\text{-disk}}(x^{[1]}, \dots, x^{[N]})$ iff $\|x^{[i]} - x^{[j]}\|_\infty \leq r$, $i \neq j$ These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}} = (I, \mathcal{A}, E_{r\text{-}\infty\text{-disk}})$. •

In order to analyze the performance of a communication and control law, we first define the notion of coordination task, and of task achievement by a robotic network.

Definition III.8 (Coordination task) *Let* $\mathcal{S}$ *be a robotic network. A* (static) coordination task *for* $\mathcal{S}$ *is a map* $\mathcal{T}\colon \prod_{i \in I} X^{[i]} \to$ `BooleSet`. *Additionally, let* $\mathcal{CC}$ *a control and communication law for* $\mathcal{S}$. *The law* $\mathcal{CC}$ achieves *the task* $\mathcal{T}$ *if for all initial conditions* $x_0^{[i]} \in X_0^{[i]}$, $i \in I$, *the network evolution* $t \mapsto x(t)$ *has the property that there exists* $T \in \mathbb{R}_+$ *such that* $\mathcal{T}(x(t)) =$ `true` *for all* $t \geq T$. •

The notions of time and communication complexity (cf. [2]) describe the performance and cost of a law achieving a coordination task. Here, we focus on time complexity.

Definition III.9 (Time complexity) *Let* $\mathcal{S}$ *be a robotic network,* $\mathcal{T}$ *a coordination task for* $\mathcal{S}$ *and* $\mathcal{CC}$ *a law for* $\mathcal{S}$.

(i) *The* time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ from $x_0 \in \prod_{i \in I} X_0^{[i]}$ *is* $\operatorname{TC}(\mathcal{T}, \mathcal{CC}, x_0) = \inf\{\ell \mid \mathcal{T}(x(t_k)) =$ `true`, *for all* $k \geq \ell\}$, *where* $t \mapsto (x(t))$ *is the evolution of* $(\mathcal{S}, \mathcal{CC})$ *from* x_0.

(ii) *The* time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ *is*
$$\operatorname{TC}(\mathcal{T}, \mathcal{CC}) = \sup\Big\{\operatorname{TC}(\mathcal{T}, \mathcal{CC}, x_0) \mid x_0 \in \prod_{i \in I} X_0^{[i]}\Big\}.$$

(iii) *The* time complexity of $\mathcal{T}$ *is* $\operatorname{TC}(\mathcal{T}) = \inf\{\operatorname{TC}(\mathcal{T}, \mathcal{CC}) \mid \mathcal{CC}$ *achieves* $\mathcal{T}\}$. •

IV. Rendezvous

In this section, we introduce rendezvous coordination tasks and analyze various coordination algorithms that achieve them, providing upper and lower bounds on their time complexity. We will consider the networks $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$ presented in Example III.5 and Example III.6, respectively.

A. Rendezvous tasks

Let $\mathcal{S} = (I, \mathcal{A}, E_{\text{cmm}})$ be a uniform robotic network. The *(exact) rendezvous task* $\mathcal{T}_{\text{rndzvs}}\colon X^N \to$ `BooleSet` for $\mathcal{S}$ is $\mathcal{T}_{\text{rndzvs}}(x^{[1]}, \dots, x^{[N]}) =$ `true` iff $x^{[i]} = x^{[j]}$, for all $(i,j) \in E_{\text{cmm}}(x^{[1]}, \dots, x^{[N]})$. Second, let $\mathcal{S} = (I, \mathcal{A}, E_{\text{cmm}})$ be a uniform robotic network with agents' state space $X \subset \mathbb{R}^d$. Examples networks of this form are $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$, see Examples III.5 and IV-B, and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$, see Examples III.6. For $\varepsilon > 0$, the ε*-rendezvous task* $\mathcal{T}_{\varepsilon\text{-rndzvs}}\colon (\mathbb{R}^d)^N \to$ `BooleSet` for $\mathcal{S}$ is defined by $\mathcal{T}_{\varepsilon\text{-rndzvs}}(x) =$ `true` iff

$$\Big\|x^{[i]} - \operatorname{avrg}\big(\{x^{[i]}\} \cup \{x^{[j]} \mid (i,j) \in E_{\text{cmm}}(x)\}\big)\Big\|_2 < \varepsilon,$$

for all $i \in I$. Here $x = (x^{[1]}, \dots, x^{[N]}) \in X^N \subset (\mathbb{R}^d)^N$. In other words, $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ is `true` at $x \in (\mathbb{R}^d)^N$ if, for all $i \in I$, $x^{[i]}$ is at distance less than ε from the average of its own position with the position of its E_{cmm}-neighbors.

B. Rendezvous without connectivity via the move-toward-average control and communication law

Consider the network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ (cf. Example III.5). Let us define the static, uniform and time-independent move-toward-average law, denoted $\mathcal{CC}_{\text{avrg}}$. We describe it as:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each round, agents transmit their position. Between rounds, each agent moves towards and reaches the average of its own position and its neighbors' positions.

Next, we define the law *formally*. First, we take $\mathbb{T} = \mathbb{N}_0$. Each agent operates with the standard message-generation function, i.e., $L = \mathbb{R}^d$ and $\operatorname{msg}_{\text{std}}(x,j) = x$. Second, we define the control function $\operatorname{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$

$$\operatorname{ctl}(x, x_{\text{smpld}}, y) = -k_{\text{prop}} \operatorname{vers}\big(x - \operatorname{avrg}(y \cup \{x_{\text{smpld}}\})\big),$$

where $k_{\text{prop}} \geq r$, vers: $\mathbb{R}^d \to \mathbb{R}^d$ is defined by $\text{vers}(0) = 0$ and $\text{vers}(v) = v/\|v\|_2$ for $v \neq 0$, and the map avrg computes the average of a finite point set in $\mathbb{R}^d$:

$$\text{avrg}(S) = \frac{1}{\sum_{p \in \pi_{\mathbb{R}}(S)} 1} \sum_{p \in \pi_{\mathbb{R}}(S)} p\,.$$

In summary we set $\mathcal{CC}_{\text{avrg}} = (\mathbb{N}_0, \mathbb{R}^d, \text{msg}_{\text{std}}, \text{ctl})$. Fig. 1 shows an execution for $d = 1$. Along the evolution, (1) several agents *rendezvous*, i.e., agree upon a common location, and (2) some agents are connected at the beginning and not connected at the end. This law is related to the Vicsek's model discussed in [12], [18]. The next result characterizes the complexity of this law. The proof can be found in [1].

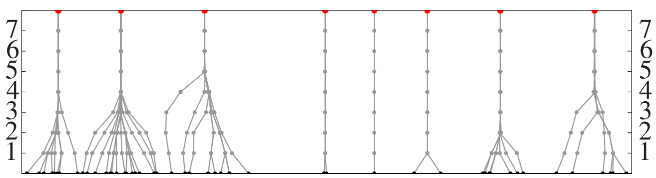

Fig. 1. Evolution of a robotic network under the move-toward-average control and communication law in Example IV-B implemented over the r-disk graph, with $r = 1.5$. The vertical axis corresponds to the elapsed time, and the horizontal axis to the positions of the agents in the real line. The 51 agents are initially randomly deployed over the interval $[-15, 15]$.

Theorem IV.1 *For $d = 1$, the network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$, the law $\mathcal{CC}_{\text{avrg}}$, and the task $\mathcal{T}_{\text{rndzvs}}$ satisfy $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) \in O(N^5)$ and $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{avrg}}) \in \Omega(N)$.*

C. Rendezvous with connectivity constraint via the circumcenter control and communication law

Here we define the *circumcenter* control and communication law $\mathcal{CC}_{\text{crcmcntr}}$ for both $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$. This is a uniform, static, time-independent law originally introduced by [3] and later studied in [8], [10]. Loosely speaking, the evolution of the network under this law is:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each round, each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the circumcenter of the point set comprised of its neighbors and of itself; and (iii) it moves toward this circumcenter while maintaining connectivity with its neighbors.

Next, we define the law *formally*. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, $i \in I$. To define the control function, we introduce some constructions. First, connectivity is maintained by restricting the allowable motion of agents as follows. If agents i and j are neighbors at $\ell \in \mathbb{N}_0$, then we require their subsequent positions to be in $\overline{B}(\frac{x^{[i]}(\ell)+x^{[j]}(\ell)}{2}, \frac{r}{2})$. If an agent i has its neighbors at locations $\{q_1, \ldots, q_l\}$ at ℓ, then its *constraint set* $\mathcal{D}_{x^{[i]}(\ell), r}(\{q_1, \ldots, q_l\})$ is

$$\mathcal{D}_{x^{[i]}(\ell), r}(\{q_1, \ldots, q_l\}) = \bigcap_{q \in \{q_1, \ldots, q_l\}} \overline{B}\Big(\frac{x^{[i]}(\ell) + q}{2}, \frac{r}{2}\Big).$$

Second, to maximize the displacement toward the circumcenter, each agent solves a convex optimization problem that can be stated in general as follows. For q_0 and q_1 in $\mathbb{R}^d$, and for a convex closed set $Q \subset \mathbb{R}^d$ with $q_0 \in Q$, let $\lambda(q_0, q_1, Q)$ denote the solution to the strictly convex problem:

$$\text{maximize } \lambda, \quad \text{subject to } \lambda \leq 1,\ (1-\lambda)q_0 + \lambda q_1 \in Q.$$

Under the stated assumptions the solution exists and is unique. Third, since the agents operate with msg_{std}, the projection $\pi_{\mathbb{R}^d}$ maps the messages $y^{[i]}(\ell)$ received at time $\ell \in \mathbb{N}_0$ by the agent $i \in I$ onto the positions of its neighbors. We are now ready to define the control function ctl: $\mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \lambda_* \cdot (\text{Circum}(\pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}) - x_{\text{smpld}})\,, \tag{1}$$

with $\lambda_* = \lambda(x_{\text{smpld}}, (\text{Circum}(\pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}), \mathcal{D}_{x_{\text{smpld}}, r}(\pi_{\mathbb{R}^d}(y)))$. Evolving under this control law, it is clear that, at time $\lfloor t \rfloor + 1$, each agent i reaches the point $(1-\lambda_*)x^{[i]}(\lfloor t \rfloor) + \lambda_* \,\text{Circum}(\pi_{\mathbb{R}^d}(y^{[i]}(\lfloor t \rfloor)) \cup \{x^{[i]}(\lfloor t \rfloor)\})$.

Next, we consider the network $\mathcal{S}_{r\text{-}\infty\text{-disk}}$, see Example III.7. For this network we define the *parallel circumcenter law*, $\mathcal{CC}_{\text{pll-crcmcntr}}$, by designing d decoupled circumcenter laws running in parallel on each coordinate axis of $\mathbb{R}^d$. As before, this law is uniform, static and time-independent. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, $i \in I$. We define the control function ctl: $\mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\begin{aligned} \text{ctl}(x, x_{\text{smpld}}, y) = \Big(\ & \text{Circum}(\tau_1(\mathcal{M})) - \tau_1(x_{\text{smpld}}), \\ & \ldots, \text{Circum}(\tau_d(\mathcal{M})) - \tau_d(x_{\text{smpld}})\Big), \end{aligned} \tag{2}$$

where $\mathcal{M} = \pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}$, and $\tau_1, \ldots, \tau_d : \mathbb{R}^d \to \mathbb{R}$ denote the canonical projections of $\mathbb{R}^d$ onto $\mathbb{R}$. See Fig. 2 for an illustration of this law in $\mathbb{R}^2$.

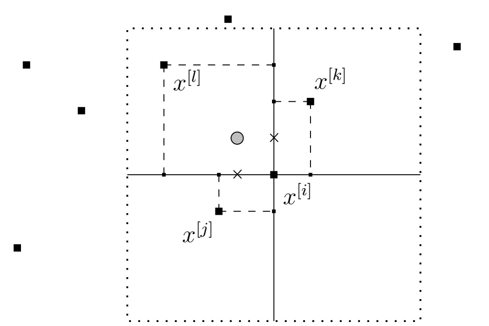

Fig. 2. Parallel circumcenter control and communication law in $\mathbb{R}^2$. The target point for the agent i is plotted in light gray and has coordinates $(\text{Circum}(\tau_1(\mathcal{M}^{[i]})), \text{Circum}(\tau_2(\mathcal{M}^{[i]})))$.

Asymptotic behavior and complexity analysis: The following theorem summarizes the results known in the literature about the asymptotic properties of the circumcenter law.

Theorem IV.2 (Correctness of the circumcenter law) *For $d \in \mathbb{N}$, $r \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, the following statements hold:*

(i) *on the network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;*
(ii) *on the network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$;*

(iii) *on the network* $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$*, the law* $\mathcal{CC}_{\text{pll-crcmcntr}}$ *achieves the exact rendezvous task* $\mathcal{T}_{\text{rndzvs}}$*;*

(iv) *the evolutions of* $(\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}, \mathcal{CC}_{\text{crcmcntr}})$*, of* $(\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}, \mathcal{CC}_{\text{crcmcntr}})$*, and of* $(\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}, \mathcal{CC}_{\text{pll-crcmcntr}})$ *have the property that, if two agents belong to the same connected component of the communication graph at* $\ell \in \mathbb{N}_0$*, then they belong to the same connected component of the graph for all subsequent* $k \geq \ell$*.* •

Next, we provide complete results for the time complexity of $\mathcal{CC}_{\text{crcmcntr}}$ when $d = 1$. As we see next, the complexity of $\mathcal{CC}_{\text{crcmcntr}}$ differs dramatically for the two robotic networks with different communication graphs. The proof of this result relies on Theorems II.1 and II.2 (see [1]).

Theorem IV.3 (Time complexity of circumcenter law) *For* $r \in \mathbb{R}_+$ *and* $\varepsilon \in]0,1[$*, the following statements hold:*

(i) *for* $d = 1$*, on the network* $\mathcal{S}_{\mathbb{R},r\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N)$*;*

(ii) *for* $d = 1$*, on the network* $\mathcal{S}_{\mathbb{R},r\text{-LD}}$, $\mathrm{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N^2 \log(N\varepsilon^{-1}))$*;*

(iii) *for* $d \in \mathbb{N}$*, on the network* $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{pll-crcmcntr}}) \in \Theta(N)$. •

Remark IV.4 Theorem IV.3 induces a lower bound on the time communication complexity of $\mathcal{CC}_{\text{crcmcntr}}$ when $d \geq 2$. Indeed, as a consequence of this result, we have

(i) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R},r\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Omega(N)$;

(ii) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R},r\text{-LD}}$, $\mathrm{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Omega(N^2 \log(N\varepsilon^{-1}))$.

We have performed extensive numerical simulations for the case $d = 2$ and the network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$. Starting from generic initial configurations (where, in particular, agents' positions are not aligned) contained in a bounded region of $\mathbb{R}^2$m we have consistently obtained that the time complexity to achieve $\mathcal{T}_{\text{rndzvs}}$ with $\mathcal{CC}_{\text{crcmcntr}}$ is independent of the number of agents. This leads us to conjecture that, in fact, initial configurations where all agents are aligned (i.e., the 1-dimensional case) give rise to the worst possible algorithm performance. In more formal terms, we conjecture that, for $d \geq 2$, $\mathrm{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) = \Theta(N)$. •

V. Deployment

In this section, we introduce the deployment coordination task and analyze a coordination algorithm that achieves it, providing upper and lower bounds on its time complexity. Along the section, we consider the uniform robotic network $\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}$ presented in Example III.6 with parameter $r \in \mathbb{R}_+$. We assume that the initial positions of the agents belong to $Q \subset \mathbb{R}^d$, a convex simple polytope with an integrable density function $\phi\colon Q \to \mathbb{R}_+$. We intend to design a control law that keeps them in Q for subsequent times.

A. Deployment task

By optimal deployment on the convex simple polytope $Q \subset \mathbb{R}^d$ with density function $\phi\colon Q \to \mathbb{R}_+$, we mean the following objective: place the agents on Q so that the expected square Euclidean distance from any point in Q to one of the agents is minimized. Let us define this task formally. Consider the following network objective function $\mathcal{H}_{\text{deplmnt}}\colon Q^N \to \mathbb{R}$,

$$\mathcal{H}_{\text{deplmnt}}(x^{[1]}, \dots, x^{[N]}) = \int_Q \min_{i \in I} \|q - x^{[i]}\|_2^2 \, \phi(q) dq \,. \quad (3)$$

This function and variations of it are studied in the facility location and resource allocation research literature; see [20], [4]. It is convenient [4] to study a generalization of this function. For $r \in \mathbb{R}_+$, define the saturation function $\mathrm{sat}_r\colon \mathbb{R} \to \mathbb{R}$ by $\mathrm{sat}_r(x) = x$ if $x \leq r$ and $\mathrm{sat}_r(x) = r$ otherwise. For $r \in \mathbb{R}_+$, define the new objective function $\mathcal{H}_{r\text{-deplmnt}}\colon Q^N \to \mathbb{R}$ by

$$\mathcal{H}_{r\text{-deplmnt}}(x^{[1]}, \dots, x^{[N]}) = \int_Q \min_{i \in I} \mathrm{sat}_{\frac{r}{2}}(\|q - x^{[i]}\|_2^2) \, \phi(q) dq \,. \quad (4)$$

Note that if $r \geq 2\,\mathrm{diam}(Q)$, then $\mathcal{H}_{\text{deplmnt}} = \mathcal{H}_{r\text{-deplmnt}}$. Let $\{V^{[1]}, \dots, V^{[N]}\}$ be the Voronoi partition of Q associated with $\{x^{[1]}, \dots, x^{[N]}\}$. The partial derivative of the cost function takes the following meaningful form

$$\frac{\partial \mathcal{H}_{r\text{-deplmnt}}}{\partial x^{[i]}}(x^{[1]}, \dots, x^{[N]}) = 2\,\mathrm{Mass}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2}))\cdot \\ \cdot \big(\mathrm{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) - x^{[i]}\big), \quad i \in I \,.$$

(Here $\mathrm{Mass}(S)$ and $\mathrm{Centroid}(S)$ are, respectively, the mass and the centroid of $S \subset \mathbb{R}^d$, see [1].) Clearly, the critical points of $\mathcal{H}_{r\text{-deplmnt}}$ are network states where $x^{[i]} = \mathrm{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2}))$. We call such configurations $\frac{r}{2}$-centroidal Voronoi configurations. For $r \geq 2\,\mathrm{diam}(Q)$, they coincide with the standard centroidal Voronoi configurations on Q. Fig. 3 illustrates these notions.

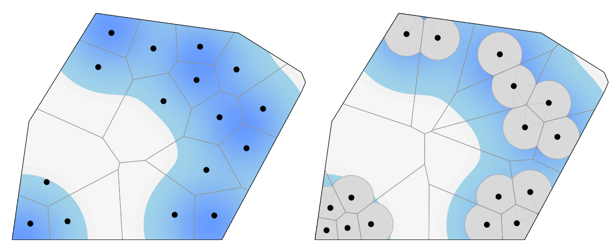

Fig. 3. Centroidal and $\frac{r}{2}$-centroidal Voronoi configurations. The density function ϕ is depicted by a contour plot. For each agent i, the set $V^{[i]} \cap \overline{B}(p_i, \frac{r}{2})$ is plotted in light gray.

Motivated by these observations, we define, for $r, \varepsilon \in \mathbb{R}_+$, the *$\varepsilon$-$r$-deployment task* $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}\colon Q^N \to \texttt{BooleSet}$ by $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}(x) = \texttt{true}$ iff

$$\big\|x^{[i]} - \mathrm{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2}))\big\|_2 \leq \varepsilon, \text{ for all } i \in I.$$

Roughly speaking, $\mathcal{T}_{\varepsilon\text{-}r\text{deplmnt}}$ is $\texttt{true}$ for those network configurations where each agent is sufficiently close to the centroid of an appropriate region $V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2})$.

B. Centroid law

To achieve the ε-r-deployment task discussed in Example V-A, we define the *centroid* control and communication law $\mathcal{CC}_{\text{centrd}}$. This is a uniform, static, time-independent law studied in [4]. Loosely speaking, the evolution of the network under this law is:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the centroid of the intersection between the agent's Voronoi cell and a closed ball centered at its position and of radius $\frac{r}{2}$, and (iii) it moves toward this centroid.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, $i \in I$. We define the control function $\text{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \text{Centroid}(\mathcal{X}) - x_{\text{smpld}}\,,$$

with $\mathcal{X} = Q \cap \overline{B}(x_{\text{smpld}}, \frac{r}{2}) \cap \big(\cap_{p \in \pi_L(y)} H_{x_{\text{smpld}},p}\big)$ and $H_{x_{\text{smpld}},p}$ is the half-space $\{q \in \mathbb{R}^d \mid \|q - x_{\text{smpld}}\|_2 \leq \|q - p\|_2\}$. One can show that Q^N is positively invariant for this law.

The following theorem on $\mathcal{CC}_{\text{centrd}}$ summarizes the known results about the asymptotic properties and the novel results on the complexity of this law. In characterizing complexity, we assume $\text{diam}(Q)$ is independent of N, r and ε, and we do not calculate how the bounds depend on r. As for the circumcenter law, we provide complete time-complexity results for the case $d = 1$. The proof of this result relies on Theorems II.1 and II.2 (see [1]).

Theorem V.1 (Time complexity of centroid law) *For $r \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, consider the network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$ with initial conditions in Q. The following statements hold:*

(i) *for $d \in \mathbb{N}$, the law $\mathcal{CC}_{\text{centrd}}$ achieves the ε-r-deployment task $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}$;*
(ii) *for $d = 1$ and $\phi = 1$, $\text{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\text{centrd}}) \in O(N^3 \log(N\varepsilon^{-1}))$.* •

VI. Conclusions

Building on the framework proposed in the companion paper [2] to model and analyze robotic networks, we have formalized various coordination algorithms: the move-toward-average and the circumcenter laws, achieving the rendezvous task, and the centroid law, achieving the deployment task. We have computed the time complexity of these algorithms, providing upper and lower bounds as the number of agents tends to infinity. To obtain these complexity estimates, we have developed some novel analysis methods involving linear dynamical systems defined by tridiagonal Toeplitz and circulant matrices. These results demonstrate the usefulness of the proposed formal model. We hope that they will help assess the complex trade-offs between computation, communication and motion control in robotic networks.

A number of research avenues look now promising and exciting. In this paper, our analysis results essentially consist of a time-complexity analysis of some basic algorithms, but many more open algorithmic questions remain unresolved including (i) analysis of the communication complexity for different models of communication; (ii) analysis of other known algorithms for flocking, cohesion, formation, motion planning and a long etcetera; and (iii) complexity analysis results for coordination tasks, as opposed to for algorithms.

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