On synchronous robotic networks – Part I: Models, tasks, and complexity

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***Abstract*— This paper proposes a formal model for a network of robotic agents that move and communicate. Building on concepts from distributed computation, robotics and control theory, we define notions of robotic network, control and communication law, coordination task, and time and communication complexity. We illustrate our model and compute the proposed complexity measures in the example of a network of locally connected agents on a circle that agree upon a direction of motion and pursue their immediate neighbors.**

I. Introduction

Problem motivation: The study of networked mobile systems presents new challenges that lie at the confluence of communication, computing, and control. In this paper, we consider the problem of designing joint communication protocols and control algorithms for groups of agents with controlled mobility. For such groups of agents, we define the notion of communication and control law by extending the classic notion of distributed algorithm in synchronous networks. Decentralized control strategies are appealing for networks of robots because they can be scalable and they provide robustness to vehicle and communication failures.

One of our key objectives is to develop a theory of time and communication complexity for motion coordination algorithms. Hopefully, our formal model will be suitable to analyze objectively the performance of various coordination algorithms. It is our contention that such a theory is required to assess the complex trade-offs between computation, communication, and motion control or, in other words, to establish what algorithms are *scalable* and implementable in large networks of mobile autonomous agents. The need for modern models of computation in wireless and sensor network applications is discussed in the well-known reports [1], [2].

Literature review: The literature on multirobot systems is very extensive. Examples include the survey in [3] and the recent special issue [4] of the IEEE Transaction on Robotics and Automation. Together with this literature our starting points are the standard notions of *synchronous and asynchronous networks* in distributed [5], [6] and parallel [7] computation,. This established body of knowledge on networks is, however, not applicable to the robotic network setting because of the agents' mobility and the ensuing dynamic communication topology.

An important contribution towards a network model of mobile interacting robots is introduced in [8]. This model consists of a group of identical "distributed anonymous mobile robots" characterized as follows: no explicit communication takes place between them, and at each time instant of an "activation schedule," each robot senses the relative position of all other robots and moves according to a pre-specified algorithm. A related model is presented in [9], where as few capabilities as possible are assumed on the agents, with the objective of understanding the limitations of multi-agent networks. A brief survey of models, algorithms, and the need for appropriate complexity notions is presented in [10]. Recently, a notion of communication complexity for control and communication algorithms in multi-robot systems is analyzed in [11], see also [12]. The general modeling paradigms discussed in [13], [14] do not take into account the specific features of robotic networks. The time complexity of a class of coordinated motion planning problems is computed in [15]. The convergence rate and communication overhead of two cyclic pursuit algorithms is examined in [16].

Statement of contributions: A key contribution of this paper is a model for robotic networks, which properly takes into account some important dynamical, communication and computational aspects of these systems. Our model is meaningful and tractable, it describes feasible operations and their costs, and it allows us to study tradeoffs in control and communication problems. We summarize our approach as follows. A *robotic network* is a group of robotic agents moving in space and endowed with communication capabilities. The agents' positions obey a differential equation and the communication topology is a function of the agents' relative positions. Each agent repeatedly performs communication, computation and physical motion in the following way. At predetermined time instants, the agents exchange information according to the communication graph and update their internal state. Between successive communication instants, the agents move according to a motion control law, computed as a function of the agent location and of the internal state. In short, a *control and communication law* for a robotic network consists of a message-generation function (what do the agents communicate?), a state-transition function (how do the agents update their internal state with the received information?), and a motion control law (how do the agents move between communication

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rounds?). The *time complexity* of a control and communication law (aimed at solving a given coordination task) is the minimum number of communication rounds required by the agents to achieve the task. We also provide similar definitions for mean and total communication complexity. We show that our notions of complexity satisfy a basic well-posedness property that we refer to as "invariance under reschedulings." To the best of our knowledge, the proposal of studying the complexity of coordination algorithms for synchronous robotic networks under a comprehensive modeling framework presented here is a novel contribution on its own.

Next, we illustrate the proposed framework with the example of a network of agents moving on the unit circle under the action of a novel agree-and-pursue control and communication law. Despite the apparent simplicity, this example is remarkable in that it combines a leader election task (in the internal states) with a uniform deployment task (in the agents positions), i.e., it combines two of the most basic tasks in distributed algorithms and cooperative control, respectively. We prove that the agree-and-pursue law achieves consensus on the agents' direction of motion and equidistance between the agents' positions. Furthermore, we provide upper and lower bounds on the time and total communication complexity of the proposed law. These complexity estimates build on known and novel results on the convergence rates of discrete-time dynamical systems defined by tridiagonal Toeplitz and circulant matrices presented in the appendix. The companion paper [17] builds on this framework to establish complexity estimates for motion coordination algorithms that achieve rendezvous and deployment.

Organization: Section II presents a general approach to the modeling of robotic networks by formally introducing notions such as communication graph, control and communication law, and network evolution. Section III defines the notions of task and of time and communication complexity. We also study the invariance properties of the complexity notions under rescheduling. Section IV provides bounds on the time and communication complexity of the agree-and-pursue law. We gather our conclusions in Section V. The appendix contains the results on discrete-time dynamical systems defined by tridiagonal Toeplitz and circulant matrices.

Notation: We let $\texttt{BooleSet} = \{\texttt{true}, \texttt{false}\}$. We let $\prod_{i\in\{1,\dots,n\}} S_i$ denote the Cartesian product of sets $S_1,\dots,S_n$. We let $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the strictly positive and non-negative real numbers, respectively. We let $\mathbb{N}$ and $\mathbb{N}_0$ denote the natural numbers and the non-negative integers, respectively. For $x \in \mathbb{R}^d$, we let $\|x\|_2$ and $\|x\|_\infty$ denote the Euclidean and the ∞-norm of x, respectively (we also recall $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{d}\|x\|_\infty$). We define the vectors $\mathbf{0} = (0,\dots,0)$ and $\mathbf{1} = (1,\dots,1)$ in $\mathbb{R}^d$. For $f, g\colon \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ (respectively, $f \in \Omega(g)$) if there exist $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}_{>0}$ such that $|f(n)| \leq c|g(n)|$ for all $n \geq n_0$ (respectively, $|f(n)| \geq c|g(n)|$ for all $n \geq n_0$). If $f \in O(g)$ and $f \in \Omega(g)$, then we use the notation $f \in \Theta(g)$.

II. A FORMAL MODEL FOR SYNCHRONOUS ROBOTIC NETWORKS

Here we introduce a notion of robotic network as a group of robotic agents with the ability to move and communicate according to a specified communication topology. Our model is inspired by the synchronous network model in [5] and has connections with the hybrid systems models in [13], [14].

A. The physical components of a robotic network

Here we introduce our basic definition of physical quantities such as the agents and such as the ability of agents to communicate. We begin by providing a basic model for how each robotic agent moves in space. A *control system* is a tuple (X, U, X_0, f), where

(i) X is a differentiable manifold, called the *state space*;
(ii) U is a subset of $\mathbb{R}^m$ containing 0, called the *input space*;
(iii) X_0 is a subset of X, called the *set of allowable initial states*;
(iv) $f\colon X \times U \to TX$ is a C^∞-map with $f(x,u) \in T_xX$ for all $(x,u) \in X \times U$.

We refer to $x \in X$ and $u \in U$ as a *state* and an *input* of the control system, respectively. We will often consider control-affine systems, i.e., control systems with $f(x,u) = f_0(x) + \sum_{a=1}^m f_a(x)\,u_a$. In such a case, we represent f as the ordered family of C^∞-vector fields $(f_0, f_1, \dots, f_m)$ on X.

Definition II.1 (Network of robotic agents) *A* network of robotic agents *(or* robotic network*)* $\mathcal{S}$ *is a tuple* $(I, \mathcal{A}, E_{\text{cmm}})$ *consisting of*

(i) $I = \{1,\dots,n\}$*, called the* set of unique identifiers (UIDs)*;*
(ii) $\mathcal{A} = \{A^{[i]}\}_{i\in I} = \{(X^{[i]}, U^{[i]}, X_0^{[i]}, f^{[i]})\}_{i\in I}$ *is a set of control systems, called the* set of physical agents*;*
(iii) E_{cmm} *is a map from* $\prod_{i\in I} X^{[i]}$ *to the subsets of* $I \times I$*, called the* communication edge map.

If $A^{[i]} = (X, U, X_0, f)$ *for all* $i \in I$*, then the robotic network is called* uniform. •

Remarks II.2 (i) By convention, we let the superscript $[i]$ denote the variables and spaces corresponding to the agent with unique identifier i; for instance, $x^{[i]} \in X^{[i]}$ and $x_0^{[i]} \in X_0^{[i]}$ denote the state and the initial state of agent $A^{[i]}$, respectively. We refer to $x = (x^{[1]},\dots,x^{[n]}) \in \prod_{i\in I} X^{[i]}$ as a *state* of the network.

(ii) The map E_{cmm} models the topology of the communication service among the agents: at a network state $x = (x^{[1]},\dots,x^{[n]})$, agent $x^{[i]}$ can send a message to agent $x^{[j]}$ if the pair (i,j) is an edge in $E_{\text{cmm}}(x^{[1]},\dots,x^{[n]})$. Accordingly, we refer to $(I, E_{\text{cmm}}(x^{[1]},\dots,x^{[n]}))$ as the *communication graph* at x. When and what agents communicate is discussed in Section II-B. Maps from $\prod_{i\in I} X^{[i]}$ to the subsets of $I \times I$ are called *proximity edge maps* and arise in wireless networks and computational geometry, e.g., see [18]. •

To make things concrete, let us present an example of robotic network. Let $\mathbb{S}^1$ be the unit circle, and measure positions on $\mathbb{S}^1$ counterclockwise from the positive horizontal axis. For $x, y \in \mathbb{S}^1$, we let $\mathrm{dist}(x, y)$ be the geodesic distance between x and y defined by $\mathrm{dist}(x, y) = \min\{\mathrm{dist}_{\mathtt{c}}(x, y), \mathrm{dist}_{\mathtt{cc}}(x, y)\}$, where $\mathrm{dist}_{\mathtt{c}}(x, y) = (x - y) \pmod{2\pi}$ and $\mathrm{dist}_{\mathtt{cc}}(x, y) = (y - x) \pmod{2\pi}$ are the path lengths from x to y traveling clockwise and counterclockwise, respectively. Here $x \pmod{2\pi}$ is the remainder of the division of x by 2π.

Example II.3 (Locally-connected first-order agents on the circle) For $r \in \mathbb{R}_{>0}$, consider the uniform robotic network $\mathcal{S}_{\text{circle}} = (I, \mathcal{A}, E_{r\text{-disk}})$ composed of identical agents of the form $(\mathbb{S}^1, \mathbb{R}, \mathbb{S}^1, (0, \mathbf{e}))$. Here $\mathbf{e}$ is the vector field on $\mathbb{S}^1$ describing unit-speed counterclockwise rotation. We define the r-disk proximity edge map $E_{r\text{-disk}}$ on the circle by setting $(i, j) \in E_{r\text{-disk}}(\theta^{[1]}, \dots, \theta^{[n]})$ if and only if $i \neq j$ and

$$\mathrm{dist}(\theta^{[i]}, \theta^{[j]}) \leq r\,. \qquad \bullet$$

B. Control and communication laws for robotic networks

Here we present a discrete-time communication, continuous-time motion model for the evolution of a robotic network. In our model, the robotic agents evolve in the physical domain in continuous-time and have the ability to exchange information (position and other variables) at discrete-time instants.

Definition II.4 (Control and communication law) *Let $\mathcal{S}$ be a robotic network. A* control and communication law $\mathcal{CC}$ *for $\mathcal{S}$ consists of the sets:*

(i) $\mathbb{T} = \{t_\ell\}_{\ell \in \mathbb{N}_0} \subset \mathbb{R}_{\geq 0}$, *an increasing sequence of time instants with no accumulation points, called* communication schedule*;*
(ii) $\mathcal{L}$, *a set containing the* `null` *element, called the* communication alphabet*; elements of $\mathcal{L}$ are called* messages*;*
(iii) $W^{[i]}$, $i \in I$, *sets of values of some* logic variables $w^{[i]}$, $i \in I$*;*
(iv) $W_0^{[i]} \subseteq W^{[i]}$, $i \in I$, *subsets of* allowable initial values *for the logic variables;*

and of the maps:

(i) $\mathrm{msg}^{[i]} \colon \mathbb{T} \times X^{[i]} \times W^{[i]} \times I \to \mathcal{L}$, $i \in I$, *called* message-generation functions*;*
(ii) $\mathrm{stf}^{[i]} \colon \mathbb{T} \times X^{[i]} \times W^{[i]} \times \mathcal{L}^n \to W^{[i]}$, $i \in I$, *called* state-transition functions*;*
(iii) $\mathrm{ctl}^{[i]} \colon \mathbb{R}_{\geq 0} \times X^{[i]} \times X^{[i]} \times W^{[i]} \times \mathcal{L}^n \to U^{[i]}$, $i \in I$, *called* control functions.

If $\mathcal{S}$ is uniform and if $W^{[i]} = W$, $\mathrm{msg}^{[i]} = \mathrm{msg}$, $\mathrm{stf}^{[i]} = \mathrm{stf}$, $\mathrm{ctl}^{[i]} = \mathrm{ctl}$, for all $i \in I$, then $\mathcal{CC}$ is said to be uniform *and is described by a tuple* $(\mathbb{T}, \mathcal{L}, W, \{W_0^{[i]}\}_{i \in I}, \mathrm{msg}, \mathrm{stf}, \mathrm{ctl})$. •

We sometimes refer to a control and communication law as a *motion coordination algorithm*. Roughly speaking, the rationale behind Definition II.4 is the following: for all $i \in I$, to the ith physical agent corresponds a logic process, labeled i, that performs the following actions. First, at each time instant $t_\ell \in \mathbb{T}$, the ith logic process sends to each of its neighbors in the communication graph a message (possibly the `null` message) computed by applying the message-generation function to the current values of $x^{[i]}$ and $w^{[i]}$. After a negligible period of time (therefore, still at time instant $t_\ell \in \mathbb{T}$), the ith logic process updates the value of its logic variables $w^{[i]}$ by applying the state-transition function to the current value of $x^{[i]}$ and $w^{[i]}$ and to the messages received at time t_ℓ. Between communication instants, i.e., for $t \in [t_\ell, t_{\ell+1})$, the motion of the ith agent is determined by applying the control function to the current value of $x^{[i]}$, the value of $x^{[i]}$ at time t_ℓ, and the current value of $w^{[i]}$. This idea is formalized as follows.

Definition II.5 (Evolution of a robotic network) *Let $\mathcal{S}$ be a robotic network and $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$. The* evolution *of $(\mathcal{S}, \mathcal{CC})$ from initial conditions $x_0^{[i]} \in X_0^{[i]}$ and $w_0^{[i]} \in W_0^{[i]}$, $i \in I$, is the collection of curves $x^{[i]} \colon [t_0, +\infty) \to X^{[i]}$ and $w^{[i]} \colon \mathbb{T} \to W^{[i]}$, $i \in I$, satisfying*

$$\dot{x}^{[i]}(t) = f\big(x^{[i]}(t),\, u^{[i]}(t)\big),$$
$$u^{[i]}(t) = \mathrm{ctl}^{[i]}\big(t, x^{[i]}(t), x^{[i]}(\lfloor t \rfloor_{\mathbb{T}}), w^{[i]}(\lfloor t \rfloor_{\mathbb{T}}), y^{[i]}(\lfloor t \rfloor_{\mathbb{T}})\big),$$

where $\lfloor t \rfloor_{\mathbb{T}} = \max\{t_\ell \in \mathbb{T} \mid t_\ell < t\}$, and

$$w^{[i]}(t_\ell) = \mathrm{stf}^{[i]}(t_\ell, x^{[i]}(t_\ell), w^{[i]}(t_{\ell-1}), y^{[i]}(t_\ell))\,,$$

with $x^{[i]}(t_0) = x_0^{[i]}$ and $w^{[i]}(t_{-1}) = w_0^{[i]}$, $i \in I$. In the previous equations, the curve $y^{[i]} \colon \mathbb{T} \to \mathcal{L}^n$ (describing the messages received by agent i) has components $y_j^{[i]}(t_\ell)$, $j \in I$, given by

$$y_j^{[i]}(t_\ell) = \mathrm{msg}^{[j]}(t_\ell, x^{[j]}(t_\ell), w^{[j]}(t_{\ell-1}), i)$$

if $(i, j) \in E_{\text{cmm}}\big(x^{[1]}(t_\ell), \dots, x^{[n]}(t_\ell)\big)$, and $y_j^{[i]}(t_\ell) =$ `null` otherwise. •

With slight abuse of notation, we let $t \mapsto (x(t), w(t))$ denote the curves $x^{[i]}$ and $w^{[i]}$, for $i \in \{1, \dots, n\}$.

Remark II.6 (Properties of control and communication laws) A control and communication law $\mathcal{CC}$ is:

(i) *time-independent* if all message-generation, state-transition and control functions are time-independent; in this case $\mathcal{CC}$ can be described by maps of the form $\mathrm{msg}^{[i]} \colon X^{[i]} \times W^{[i]} \times I \to \mathcal{L}$, $\mathrm{stf}^{[i]} \colon X^{[i]} \times W^{[i]} \times \mathcal{L}^n \to W^{[i]}$, and $\mathrm{ctl}^{[i]} \colon X^{[i]} \times X^{[i]} \times W^{[i]} \times \mathcal{L}^n \to U^{[i]}$, for $i \in I$;
(ii) *static* if $W^{[i]}$ is a singleton for all $i \in I$; in this case $\mathcal{CC}$ can be described by a tuple $(\mathbb{T}, \mathcal{L}, \{\mathrm{msg}^{[i]}\}_{i \in I}, \{\mathrm{ctl}^{[i]}\}_{i \in I})$, with $\mathrm{msg}^{[i]} \colon \mathbb{T} \times X^{[i]} \times I \to \mathcal{L}$, and $\mathrm{ctl}^{[i]} \colon \mathbb{R}_{\geq 0} \times X^{[i]} \times X^{[i]} \times \mathcal{L}^n \to U^{[i]}$, for $i \in I$;
(iii) *data-sampled* if the control functions have the following property: given a time t, a logic state $w^{[i]} \in W^{[i]}$, an array of messages $y^{[i]} \in \mathcal{L}^n$, a current state $x^{[i]}$, and a state at last sample time $x_{\text{smpld}}^{[i]}$, the control input $\mathrm{ctl}^{[i]}(t, x^{[i]}, x_{\text{smpld}}^{[i]}, w^{[i]}, y^{[i]})$ is independent of $x^{[i]}$. In this case the control functions in $\mathcal{CC}$ can be described by maps of the form $\mathrm{ctl}^{[i]} \colon \mathbb{R}_{\geq 0} \times X^{[i]} \times W^{[i]} \times \mathcal{L}^n \to U^{[i]}$, for $i \in I$. •

Remark II.7 (Idealized aspects of communication model) We refer to $\mathcal{CC}$ as a *synchronous* control and communication law because the communications between all agents takes always place at the same time for all agents.

The set $\mathcal{L}$ is used to exchange information between two robotic agents; the message $\texttt{null}$ indicates no communication. We assume that the messages in the communication alphabet $\mathcal{L}$ allow us to encode logical expressions such as $\texttt{true}$ and $\texttt{false}$, integers, and real numbers. A realistic assumption on $\mathcal{L}$ would be to adopt a finite-precision representation for integers and real numbers in the messages. Instead, in what follows, we neglect any inaccuracies due to quantization.

In many uniform control and communication laws, the messages interchanged among the network agents are (quantized representations of) the agents' states and logic states. We will identify the corresponding communication alphabet with $\mathcal{L} = (X \times W) \cup \{\texttt{null}\}$; the message generation function $\text{msg}_{\text{std}}(t, x, w, j) = (x, w)$ is referred to as the *standard message-generation function*. •

Remark II.8 (Groups of robotic agents with relative-position sensing) Although we focus on robots with communication capabilities, at the cost of additional notation it is possible to include sensors in our treatment. A control and communication law can be implemented on a group of robots that can sense each other's relative position if the law (1) is static and uniform, (2) relies on communicating only the agents' positions (e.g., the message-generation function is the standard one), and (3) entails a control function that only depends on relative positions (as opposed to absolute positions). •

Remark II.9 (Congestion models) Two types of congestion problems affect a robotic network. First, *wireless transmissions can interfere*: node i receives a message transmitted by node j only if all other neighbors of i are silent, i.e., the transmission medium is shared among the agents. As the density of agents increases, so does wireless communication congestion. For n uniformly randomly placed nodes in a compact environment, the maximum-throughput communication range $r(n)$ of each node decreases [19] with the number of nodes; in a d-dimensional environment the appropriate scaling law is $r(n) \in \Theta\big(\sqrt[d]{\log(n)/n}\big)$. This is referred to as the connectivity regime in percolation theory and statistical mechanics. Second, *agents can collide*: as the number of agents increases, so should the area available for their motion or, vice-versa, their size should shrink. In the approach proposed by [20] robots' safety zones decrease with decreasing robots' speed. In other words, in a d-dimensional environment, individual nodes of a large ensemble have to move at a speed decreasing with n, and in particular, at a speed proportional to $1/\sqrt[d]{n}$. In summary, one way to incorporate congestion effects into the robotic network model is to assume that the parameters of the physical components of the network depend upon the number of robots. •

C. The agree-and-pursue control and communication law

Here we present an example of a dynamic control and communication law with the aim of illustrating the proposed framework. The following coordination law is related to leader election algorithms as studied in the distributed algorithms literature, e.g., see [5] (more will be said about this analogy in Remark IV.3), and to cyclic pursuit algorithms as studied in the control literature, e.g., see [21], [16]. Despite the apparent simplicity, this example is remarkable in that it combines a leader election task (in the logic variables) with a uniform agent deployment task (in the state variables), arguably two of the most basic tasks in distributed algorithms and cooperative control, respectively. Another advantage of the agree-and-pursue law is that its correctness and performance can be characterized as we will show in Section IV.

We consider the uniform network $\mathcal{S}_{\text{circle}}$ of locally-connected first-order agents in $\mathbb{S}^1$ introduced in Example II.3. We now define the agree-and-pursue law, denoted by $\mathcal{CC}_{\text{agr-pursuit}}$, as the *uniform*, *time-independent* and *data-sampled* law loosely described as follows:

> *[Informal description]* The logic variables are $\texttt{drctn}$ (the agent's direction of motion) taking values in $\{\texttt{c}, \texttt{cc}\}$ (meaning clockwise and counterclockwise) and $\texttt{prior}$ (the largest UID received by the agent, initially set to the agent's UID) taking values in I. At each communication round, each agent transmits its position and its logic variables. Among the messages received from agents moving towards its position, each agent picks the message with the largest value of $\texttt{prior}$. If this value is larger than its own value, the agent resets its logic variable with the selected message. Between communication rounds, each agent moves in the counterclockwise or clockwise direction depending on whether its logic variable $\texttt{drctn}$ is $\texttt{cc}$ or $\texttt{c}$. For $k_{\text{prop}} \in \,]0, \frac{1}{2}[$, each agent moves k_{prop} times the distance to the immediately next neighbor in the chosen direction, or, if no neighbors are detected, k_{prop} times the communication range r.

Next, we define the law formally. Each agent has logic variables $w = (w_1, w_2)$, where $w_1 = \texttt{drctn} \in \{\texttt{cc}, \texttt{c}\}$, with arbitrary initial value, and $w_2 = \texttt{prior} \in I$, with initial value set equal to the agent's identifier i. In other words, we define $W = \{\texttt{cc}, \texttt{c}\} \times I$, and we set $W_0^{[i]} = \{\texttt{cc}, \texttt{c}\} \times \{i\}$. Each agent $i \in I$ operates with the standard message-generation function, i.e., we set $\mathcal{L} = (\mathbb{S}^1 \times W) \cup \{\texttt{null}\}$ and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, where $\text{msg}_{\text{std}}(\theta, w, j) = (\theta, w)$. Define an ordering in the logic set W by saying that $(\texttt{drctn}_1, \texttt{prior}_1) > (\texttt{drctn}_2, \texttt{prior}_2)$ if $\texttt{prior}_1 > \texttt{prior}_2$. Given a physical state $\theta \in \mathbb{S}^1$, a logic state $w \in W$ and an array of messages $y \in \mathcal{L}^n$, the state-transition function is defined by

$$\text{stf}(\theta, w, y) = \begin{cases} w_{\max}, & \text{if } w_{\max} > w, \\ w, & \text{otherwise}, \end{cases}$$

where

$$\begin{aligned} w_{\max} = \max\{w_{\text{rcvd}} \in W \mid (\theta_{\text{rcvd}}, w_{\text{rcvd}}) \in y \text{ such that} \\ (\text{dist}_{\texttt{cc}}(\theta, \theta_{\text{rcvd}}) \leq r \text{ and } (w_{\text{rcvd}})_1 = \texttt{c}) \quad \text{or} \\ (\text{dist}_{\texttt{c}}(\theta, \theta_{\text{rcvd}}) \leq r \text{ and } (w_{\text{rcvd}})_1 = \texttt{cc})\}. \end{aligned}$$

For $k_{\text{prop}} \in \mathbb{R}_{>0}$, given a logic state $w \in W$, an array of messages $y \in \mathcal{L}^n$, and a state at last sample time θ_{smpld}, the data-sampled control function $\mathrm{ctl}(\theta_{\text{smpld}}, w, y)$ is

$$k_{\text{prop}} \min(\{r\} \cup \{\mathrm{dist}_{\texttt{cc}}(\theta_{\text{smpld}}, \theta_{\text{rcvd}}) \mid (\theta_{\text{rcvd}}, w_{\text{rcvd}}) \in y\}),$$

if $\texttt{drctn} = \texttt{cc}$, and

$$-k_{\text{prop}} \min(\{r\} \cup \{\mathrm{dist}_{\texttt{c}}(\theta_{\text{smpld}}, \theta_{\text{rcvd}}) \mid (\theta_{\text{rcvd}}, w_{\text{rcvd}}) \in y\}),$$

if $\texttt{drctn} = \texttt{c}$. An implementation of this control and communication law is shown in Figure 1. As we will show later, along the evolution, all agents agree upon a common direction of motion and, after suitable time, they reach a uniform distribution.

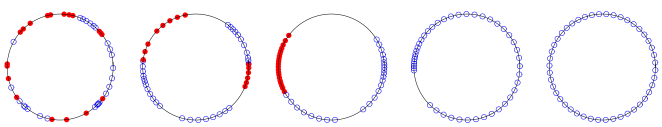

Fig. 1. The agree-and-pursue control and communication law in Section II-C with $N = 45$, $r = 2\pi/40$, and $k_{\text{prop}} = 7/16$. Disks and circles correspond to agents moving counterclockwise and clockwise, respectively. The initial positions and the initial directions of motion are randomly generated. The five pictures depict the network state at times $0, 9, 20, 100, 800$.

III. Coordination tasks and complexity measures

In this section we introduce concepts and tools useful to analyze a control and communication law. We address the following questions: What is a coordination task for a robotic network? When does a control and communication law achieve a task? And with what time and communication complexity?

A. Coordination tasks

Our first analysis step is to characterize the correctness properties of a control and communication law. We do so by defining the notion of task and of task achievement by a robotic network.

Definition III.1 (Coordination task) *Let $\mathcal{S}$ be a robotic network and let $\mathcal{W}$ be a set.*

(i) *A* coordination task *for $\mathcal{S}$ is a map $\mathcal{T}\colon \prod_{i\in I} X^{[i]} \times \mathcal{W}^n \to \texttt{BooleSet}$.*

(ii) *If $\mathcal{W}$ is a singleton, then the coordination task is said to be* static *and can be described by a map $\mathcal{T}\colon \prod_{i\in I} X^{[i]} \to \texttt{BooleSet}$.*

Additionally, let $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$.

(i) *The law $\mathcal{CC}$ is* compatible *with the task $\mathcal{T}\colon \prod_{i\in I} X^{[i]} \times \mathcal{W}^n \to \texttt{BooleSet}$ if its logic variables take values in $\mathcal{W}$, that is, if $W^{[i]} = \mathcal{W}$, for all $i \in I$.*

(ii) *The law $\mathcal{CC}$* achieves *the task $\mathcal{T}$ if it is compatible with $\mathcal{T}$ and if, for all initial conditions $x_0^{[i]} \in X_0^{[i]}$ and $w_0^{[i]} \in W_0^{[i]}$, $i \in I$, the corresponding network evolution $t \mapsto (x(t), w(t))$ has the property that there exists $T \in \mathbb{R}_{>0}$ such that $\mathcal{T}(x(t), w(t)) = \texttt{true}$ for all $t \geq T$.* •

Remark III.2 (Temporal logic) Loosely speaking, achieving a task means obtaining and maintaining a specified pattern in the agents' positions or in their logic variables. In other words, the task is achieved if *at some time* and *for all subsequent times* the predicate evaluates to true along system trajectories. It is possible to consider more general tasks through more expressive predicates on trajectories. Such predicates can be defined through various forms of temporal and propositional logic, e.g., see [22]. •

Example III.3 (Direction-agreement and equidistance tasks) Consider the uniform network $\mathcal{S}_{\text{circle}}$ of locally-connected first-order agents in $\mathbb{S}^1$ and the agree-and-pursue control and communication law $\mathcal{CC}_{\text{agr-pursuit}}$ with logic variables taking values in $W = \{\texttt{cc}, \texttt{c}\} \times I$. This network and this law were introduced in Example II.3 and Example II-C, respectively. . There are two tasks of interest. First, we define the *direction-agreement task* $\mathcal{T}_{\text{agrmnt}}\colon (\mathbb{S}^1)^n \times W^n \to \texttt{BooleSet}$ by $\mathcal{T}_{\text{agrmnt}}(\theta, w) = \texttt{true}$ if and only if

$$\texttt{drctn}^{[1]} = \cdots = \texttt{drctn}^{[n]},$$

where $\theta = (\theta^{[1]}, \ldots, \theta^{[n]})$, $w = (w^{[1]}, \ldots, w^{[n]})$, and $w^{[i]} = (\texttt{drctn}^{[i]}, \texttt{prior}^{[i]})$, $i \in I$. Second, for $\varepsilon \in \mathbb{R}_{>0}$, we define the static *equidistance task* $\mathcal{T}_{\varepsilon\text{-eqdstnc}}\colon (\mathbb{S}^1)^n \to \texttt{BooleSet}$ by $\mathcal{T}_{\varepsilon\text{-eqdstnc}}(\theta) = \texttt{true}$ if and only if, for all $i \in I$,

$$\Big| \min_{j\neq i} \mathrm{dist}_{\texttt{c}}(\theta^{[i]}, \theta^{[j]}) - \min_{j\neq i} \mathrm{dist}_{\texttt{cc}}(\theta^{[i]}, \theta^{[j]}) \Big| < \varepsilon.$$

In other words, $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$ is true when, for every agent, the distance to the closest clockwise neighbor and to the closest counterclockwise neighbor are approximately equal. •

B. Complexity notions for control and communication laws and for coordination tasks

We are finally ready to define the key notions of time and communication complexity. These notions describe the cost that a certain control and communication law incurs while completing a certain coordination task.

Definition III.4 (Time complexity) *Let $\mathcal{S}$ be a robotic network and let $\mathcal{T}$ be a coordination task for $\mathcal{S}$. Let $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$ compatible with $\mathcal{T}$.*

(i) *The* (worst-case) time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ from $(x_0, w_0) \in \prod_{i\in I} X_0^{[i]} \times \prod_{i\in I} W_0^{[i]}$ *is*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) = \inf \{\ell \mid \mathcal{T}(x(t_k), w(t_k)) = \texttt{true}\,, \text{ for all } k \geq \ell\}\,,$$

where $t \mapsto (x(t), w(t))$ is the evolution of $(\mathcal{S}, \mathcal{CC})$ from the initial condition (x_0, w_0).

(ii) *The* (worst-case) time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ *is*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}) = \sup \Big\{ \mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \mid (x_0, w_0) \in \prod_{i\in I} X_0^{[i]} \times \prod_{i\in I} W_0^{[i]} \Big\}\,.$$ •

The time complexity of a task can be also defined by taking the infimum among all compatible laws that achieve it.

Next, we define the notions of mean and total communication complexities for an algorithm. We begin by discussing the cost of realizing one communication round. At each communication round, each agent generates a certain number of messages, destined to neighboring agents as defined by the communication edge map. We indicate the set of all non-null messages generated during one communication round with

$$\mathcal{M}(t,x,w)=\{(i,j)\in E_{\text{cmm}}(x)\mid \text{msg}^{[i]}(t,x^{[i]},w^{[i]},j)\neq \texttt{null}\}.$$

To compute the cost of delivering all such messages to the intended recipients, we introduce the following function.

Definition III.5 (One-round cost) *A function* $\text{C}_{\text{rnd}}\colon 2^{I\times I}\to\mathbb{R}_{\geq 0}$ *is a* one-round cost function *if* $\text{C}_{\text{rnd}}(\emptyset)=0$*, and* $S_1\subset S_2\subset I\times I$ *implies* $\text{C}_{\text{rnd}}(S_1)\leq\text{C}_{\text{rnd}}(S_2)$*. A one-round cost function* C_{rnd} *is* additive *if, for all* $S_1, S_2\subset I\times I$*,* $S_1\cap S_2=\emptyset$ *implies* $\text{C}_{\text{rnd}}(S_1\cup S_2)=\text{C}_{\text{rnd}}(S_1)+\text{C}_{\text{rnd}}(S_2)$*.* •

More specific detail about the communication cost depends necessarily on the type of communication service (e.g., unidirectional versus omnidirectional) available between the agents. We postpone our discussion about specific functions C_{rnd} to the next subsection.

Definition III.6 (Communication complexity) *Let* $\mathcal{S}$ *be a robotic network,* $\mathcal{CC}$ *be a control and communication law that achieves the task* $\mathcal{T}$*, and* C_{rnd} *be a one-round cost function.*

(i) *The* (worst-case) mean communication complexity *and the* (worst-case) total communication complexity *to achieve* $\mathcal{T}$ *with* $\mathcal{CC}$ *from* $(x_0,w_0)\in\prod_{i\in I}X_0^{[i]}\times\prod_{i\in I}W_0^{[i]}$ *are, respectively,*

$$\text{MCC}(\mathcal{T},\mathcal{CC},x_0,w_0)=\frac{1}{\lambda}\sum_{\ell=0}^{\lambda-1}\text{C}_{\text{rnd}}\circ\mathcal{M}(t_\ell,x(t_\ell),w(t_\ell)),$$

$$\text{TCC}(\mathcal{T},\mathcal{CC},x_0,w_0)=\sum_{\ell=0}^{\lambda-1}\text{C}_{\text{rnd}}\circ\mathcal{M}(t_\ell,x(t_\ell),w(t_\ell)),$$

where $\lambda=\text{TC}(\mathcal{CC},\mathcal{T},x_0,w_0)$ *and* $t\mapsto(x(t),w(t))$ *is the evolution of* $(\mathcal{S},\mathcal{CC})$ *from the initial condition* (x_0,w_0)*. (Here* MCC *is defined only for* (x_0,w_0) *with the property that* $\mathcal{T}(x_0,w_0)=\texttt{false}$*.)*

(ii) *The* (worst-case) mean communication complexity *and the* (worst-case) total communication complexity *to achieve* $\mathcal{T}$ *with* $\mathcal{CC}$ *are the supremum of* $\{\text{MCC}(\mathcal{T},\mathcal{CC},x_0,w_0)\mid(x_0,w_0)\in\prod_{i\in I}X_0^{[i]}\times\prod_{i\in I}W_0^{[i]}\}$ *and* $\{\text{TCC}(\mathcal{T},\mathcal{CC},x_0,w_0)\mid(x_0,w_0)\in\prod_{i\in I}X_0^{[i]}\times\prod_{i\in I}W_0^{[i]}\}$*, respectively.* •

Note that by (worst-case) mean communication complexity we intend the worst-case over all initial conditions and the mean over the time required to achieve the task.

Remark III.7 (Infinite-horizon mean communication complexity) The mean communication complexity MCC measures the average cost of the communication rounds required to achieve a task over a finite time horizon; a similar statement holds for the total communication complexity TCC. One might be interested in a notion of mean communication complexity required to maintain true the task for all times. Accordingly, the infinite-horizon mean communication complexity of $\mathcal{CC}$ from initial condition (x_0,w_0) is

$$\text{IH-MCC}(\mathcal{CC},x_0,w_0)=\lim_{\lambda\to+\infty}\frac{1}{\lambda}\sum_{\ell=0}^{\lambda}\text{C}_{\text{rnd}}\circ\mathcal{M}(t_\ell,x(t_\ell),w(t_\ell))\,.$$

Note that a similar notion is presented in [11] for a different robotic network model. •

Remark III.8 (Communication costs in unidirectional and omnidirectional wireless channels) Here we discuss some modeling aspects of the one-round communication cost function described in Definition III.5. Broadly speaking, it is difficult to encompass with a single abstract model the cost of all possible communication technologies. In *unidirectional* models of communication (e.g., wireless networks with unidirectional antennas) messages are sent in a point-to-point fashion. For this model, we make the simplified convention that $\text{C}_{\text{rnd}}(\mathcal{M})$ is proportional to the number messages in $\mathcal{M}$, that is, $\text{C}_{\text{rnd}}(\mathcal{M})=c_0\cdot\text{cardinality}(\mathcal{M})$, where $c_0\in\mathbb{R}_{>0}$ is the cost of sending a single message. This one-round cost function is additive. This number is trivially upper bounded by twice the number of edges of the complete graph, which is $n(n-1)$. Therefore, we have $\text{MCC}_{\text{unidir}}(\mathcal{T})\in O(n^2)$.

In *omnidirectional* models of communication (e.g., wireless networks equipped with omnidirectional antennas), a single transmission made by a node can be heard by several other nodes simultaneously. For this model, we make the simplified convention that $\text{C}_{\text{rnd}}(\mathcal{M})$ is proportional to the number of turns employed to complete a communication round without interference between the agents (this choice is related to the well-studied media access control problem in wireless communications). This number is trivially upper bounded by n. Therefore, we have $\text{MCC}_{\text{omnidir}}(\mathcal{T})\in O(n)$. •

C. Law rescheduling for driftless agents

In this section, we discuss the invariance properties of the notions of time and communication complexity under the *rescheduling* of a control and communication law. The idea behind rescheduling is to "spread" the execution of the law over time without affecting the trajectories described by the robotic agents. Our objective is to formalize this idea and to examine the effect on the notions of complexity introduced earlier. For simplicity we consider the setting of static laws; similar results can be obtained for the general setting.

Let $\mathcal{S}=(I,\mathcal{A},E_{\text{cmm}})$ be a robotic network where each physical agent is a driftless control system. Let $\mathcal{CC}=(\mathbb{N}_0,\mathcal{L},\{\text{msg}^{[i]}\}_{i\in I},\{\text{ctl}^{[i]}\}_{i\in I})$ be a static control and communication law. Next, we define a new control and communication law by modifying $\mathcal{CC}$; to do so we introduce some notation. Let $s\in\mathbb{N}$, with $s\leq n$, and let $\mathcal{P}_I=\{I_0,\dots,I_{s-1}\}$ be an s-*partition* of I, that is, $I_0,\dots,I_{s-1}\subset I$ are disjoint and nonempty and $I=\cup_{k=0}^{s-1}I_k$. For $i\in I$, define the message-generation functions $\text{msg}^{[i]}_{\mathcal{P}_I}\colon\mathbb{N}_0\times X^{[i]}\times I\to\mathcal{L}$ by

$$\text{msg}^{[i]}_{\mathcal{P}_I}(t_\ell,x,j)=\text{msg}^{[i]}(t_{\lfloor\ell/s\rfloor},x,j),\qquad(1)$$

if $i\in I_k$ and $k=\ell(\text{mod}\,s)$, and $\text{msg}^{[i]}_{\mathcal{P}_I}(t_\ell,x,j)=\texttt{null}$ otherwise. According to this message-generation function, only the agents with unique identifier in I_k will send messages at time t_ℓ, with $\ell \in \{k + as\}_{a \in \mathbb{N}_0}$. Equivalently, this can be stated as follows: according to (1), the messages originally sent at the time instant t_ℓ are now rescheduled to be sent at the time instants $t_{F(\ell)-s+1}, \dots, t_{F(\ell)}$, where $F\colon \mathbb{N}_0 \to \mathbb{N}_0$ is defined by $F(\ell) = s(\ell+1) - 1$. Figure 2 illustrates this idea.

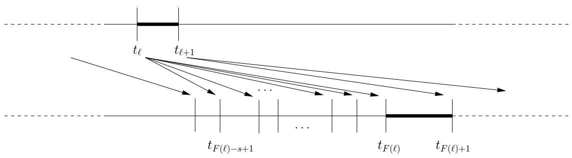

Fig. 2. Under the rescheduling, the messages that are sent at the time instant t_ℓ under the control and communication law $\mathcal{CC}$ are rescheduled to be sent over the time instants $t_{F(\ell)-s+1}, \dots, t_{F(\ell)}$ under the control and communication law $\mathcal{CC}_{(s,\mathcal{P}_I)}$.

For $i \in I$, define the control functions $\mathrm{ctl}^{[i]}\colon \mathbb{R}_{\geq 0} \times X^{[i]} \times X^{[i]} \times \mathcal{L}^n \to U^{[i]}$ by

$$\begin{aligned}\mathrm{ctl}^{[i]}_{\mathcal{P}_I}&(t, x, x_{\mathrm{smpld}}, y)\\ &= \frac{t_{F^{-1}(\ell)+1} - t_{F^{-1}(\ell)}}{t_{\ell+1} - t_\ell} \cdot \mathrm{ctl}^{[i]}\left(h_\ell(t), x, x_{\mathrm{smpld}}, y\right), \quad (2)\end{aligned}$$

if $t \in [t_\ell, t_{\ell+1}]$ and $\ell = -1 (\operatorname{mod} s)$, and $\mathrm{ctl}^{[i]}_{\mathcal{P}_I}(t, x, x_{\mathrm{smpld}}, y) = 0$ otherwise. Here $F^{-1}\colon \mathbb{N}_0 \to \mathbb{N}_0$ is the inverse of F, defined by $F^{-1}(\ell) = (\ell+1)/s - 1$, and for $\ell = -1 (\operatorname{mod} s)$, the function $h_\ell\colon [t_\ell, t_{\ell+1}] \to [t_{F^{-1}(\ell)}, t_{F^{-1}(\ell)+1}]$ is the unique linear map between the two time intervals. Roughly speaking, the control law $\mathrm{ctl}^{[i]}_{\mathcal{P}_I}$ makes the agent i wait for the time intervals $[t_\ell, t_{\ell+1}]$, with $\ell \in \{as - 1\}_{a \in \mathbb{N}}$, to execute any motion. Accordingly, the evolution of the robotic network under the original law $\mathcal{CC}$ during the time interval $[t_\ell, t_{\ell+1}]$ now takes place when all the corresponding messages have been transmitted, i.e., along the time interval $[t_{F(\ell)}, t_{F(\ell)+1}]$. The following definition summarizes this construction.

Definition III.9 (Rescheduling of control and communication laws) *Let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a robotic network with driftless physical agents, and let $\mathcal{CC} = (\mathbb{N}_0, \mathcal{L}, \{\mathrm{msg}^{[i]}\}_{i \in I}, \{\mathrm{ctl}^{[i]}\}_{i \in I})$ be a static control and communication law. Let $s \in \mathbb{N}$, with $s \leq n$, and let $\mathcal{P}_I$ be an s-partition of I. The control and communication law $\mathcal{CC}_{(s,\mathcal{P}_I)} = (\mathbb{N}_0, \mathcal{L}, \{\mathrm{msg}^{[i]}_{\mathcal{P}_I}\}_{i \in I}, \{\mathrm{ctl}^{[i]}_{\mathcal{P}_I}\}_{i \in I})$ defined by equations* (1) *and* (2) *is called a $\mathcal{P}_I$-rescheduling of $\mathcal{CC}$.* •

The following result shows that the total communication complexity of $\mathcal{CC}$ remains invariant under rescheduling.

Proposition III.10 (Invariance under rescheduling) *With the assumptions of Definition III.9, let $\mathcal{T}\colon \prod_{i \in I} X^{[i]} \to$ `BooleSet` be a coordination task for $\mathcal{S}$. Then, for all $x_0 \in \prod_{i \in I} X_0^{[i]}$,*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}_{(s,\mathcal{P}_I)}, x_0) = s \cdot \mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0).$$

Moreover, if $\mathrm{C}_{\mathrm{rnd}}$ is additive, then, for all $x_0 \in \prod_{i \in I} X_0^{[i]}$

$$\mathrm{MCC}(\mathcal{T}, \mathcal{CC}_{(s,\mathcal{P}_I)}, x_0) = \frac{1}{s} \cdot \mathrm{MCC}(\mathcal{T}, \mathcal{CC}, x_0),$$

and, therefore, $\mathrm{TCC}(\mathcal{T}, \mathcal{CC}_{(s,\mathcal{P}_I)}, x_0) = \mathrm{TCC}(\mathcal{T}, \mathcal{CC}, x_0)$, i.e., the total communication complexity of $\mathcal{CC}$ is invariant under rescheduling. •

Proof: Let $t \mapsto x(t)$ and $t \mapsto \tilde{x}(t)$ denote the network evolutions starting from $x_0 \in \prod_{i \in I} X_0^{[i]}$ under $\mathcal{CC}$ and $\mathcal{CC}_{(s,\mathcal{P}_I)}$, respectively. From the definition of rescheduling, one can verify that, for all $k \in \mathbb{N}_0$,

$$\tilde{x}^{[i]}(t) = \begin{cases} \tilde{x}^{[i]}(t_{F(k-1)+1}), & \text{for } t \in \bigcup_{\ell=F(k-1)+1}^{F(k)-1} [t_\ell, t_{\ell+1}], \\ x^{[i]}(h_{F(k)}(t)), & \text{for } t \in [t_{F(k)}, t_{F(k)+1}]. \end{cases} \tag{3}$$

By definition of $\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0)$, we have $\mathcal{T}(x(t_k)) = \texttt{true}$, for all $k \geq \mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0)$, and $\mathcal{T}(x(t_{\mathrm{TC}(\mathcal{T},\mathcal{CC},x_0)-1})) = \texttt{false}$. Let us rewrite these equalities in terms of the trajectories of $\mathcal{CC}_{(s,\mathcal{P}_I)}$. From equation (3), one can write $x^{[i]}(t_k) = x^{[i]}(h_{F(k)}(t_{F(k)})) = \tilde{x}^{[i]}(t_{F(k)})$, for all $i \in I$ and $k \in \mathbb{N}_0$. Therefore, we have

$$\mathcal{T}(\tilde{x}(t_{F(k)})) = \mathcal{T}(x(t_k)) = \texttt{true},$$

for all $F(k) \geq F(\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0))$, and

$$\mathcal{T}(\tilde{x}(t_{F(\mathrm{TC}(\mathcal{T},\mathcal{CC},x_0)-1)})) = \mathcal{T}(x(t_{\mathrm{TC}(\mathcal{T},\mathcal{CC},x_0)-1})) = \texttt{false},$$

where we have used the rescheduled message-generation function in (1). Now, note that by equation (3), $\tilde{x}^{[i]}(t_\ell) = \tilde{x}^{[i]}(t_{F(\lfloor \ell/s \rfloor - 1)+1})$, for all $\ell \in \mathbb{N}_0$ and all $i \in I$. Therefore, $\mathcal{T}(\tilde{x}(t_{F(\mathrm{TC}(\mathcal{T},\mathcal{CC},x_0)-1)+1})) = \mathcal{T}(\tilde{x}(t_{F(\mathrm{TC}(\mathcal{T},\mathcal{CC},x_0))}))$ and we can rewrite the previous identities as

$$\mathcal{T}(\tilde{x}(t_k)) = \texttt{true},$$

for all $k \geq F(\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0) - 1) + 1$, and

$$\mathcal{T}(\tilde{x}(t_{F(\mathrm{TC}(\mathcal{T},\mathcal{CC},x_0)-1)})) = \texttt{false},$$

which imply that $\mathrm{TC}(\mathcal{T}, \mathcal{CC}_{(s,\mathcal{P}_I)}, x_0) = F(\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0) - 1) + 1 = s\,\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0)$. As for the mean communication complexity, additivity of $\mathrm{C}_{\mathrm{rnd}}$ implies

$$\begin{aligned}\mathrm{C}_{\mathrm{rnd}} \circ \mathcal{M}(t_\ell, x(t_\ell)) = \mathrm{C}_{\mathrm{rnd}} \circ \mathcal{M}(t_{F(\ell)-s+1}, \tilde{x}(t_{F(\ell)-s+1})) +& \\ \dots + \mathrm{C}_{\mathrm{rnd}} \circ \mathcal{M}(t_{F(\ell)}, \tilde{x}(t_{F(\ell)})),&\end{aligned}$$

where we have used $F(\ell - 1) + 1 = F(\ell) - s + 1$. We conclude the proof by computing

$$\begin{aligned}\sum_{\ell=0}^{\lambda_1 - 1} \mathrm{C}_{\mathrm{rnd}} \circ \mathcal{M}(t_\ell, \tilde{x}(t_\ell)) &= \sum_{\ell=0}^{F(\lambda_2 - 1)} \mathrm{C}_{\mathrm{rnd}} \circ \mathcal{M}(t_\ell, \tilde{x}(t_\ell)) \\ &= \sum_{\ell=0}^{\lambda_2 - 1} \sum_{k=F(\ell)-s+1}^{F(\ell)} \mathrm{C}_{\mathrm{rnd}} \circ \mathcal{M}(t_k, \tilde{x}(t_k)) \\ &= \sum_{\ell=0}^{\lambda_2 - 1} \mathrm{C}_{\mathrm{rnd}} \circ \mathcal{M}(t_\ell, x(t_\ell)),\end{aligned}$$

where $\lambda_1 = \mathrm{TC}(\mathcal{T}, \mathcal{CC}_{(s,\mathcal{P}_I)}, x_0)$ and $\lambda_2 = \mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0)$. ■

Remark III.11 (Appropriate complexity notions for driftless agents) Given the results in the previous theorem, one should be careful in choosing what notion of communication complexity to evaluate control and communication laws. For driftless physical agents, rather than the *mean* communication complexity MCC, one should really consider the *total* communication complexity TCC, since the latter is invariant with respect to rescheduling. Note that the notion of infinite-horizon mean communication complexity IH-MCC defined in Remark III.7 satisfies the same relationship as MCC, that is, $\text{IH-MCC}(\mathcal{CC}_{(s,\mathcal{P}_I)}, x_0) = \frac{1}{s}\,\text{IH-MCC}(\mathcal{CC}, x_0)$ for any s-partition $\mathcal{P}_I$ of I. •

IV. Direction-agreement and equidistance

As introduced in Examples II.3, II-C and III.3, consider the uniform network $\mathcal{S}_{\text{circle}}$ of locally-connected first-order agents in $\mathbb{S}^1$, the agree-and-pursue control and communication law $\mathcal{CC}_{\text{agr-pursuit}}$, and the two coordination tasks $\mathcal{T}_{\text{agrmnt}}$ and $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$. The following result characterizes the complexity to achieve these coordination tasks with $\mathcal{CC}_{\text{agr-pursuit}}$.

Motivated by Remark II.9, we model wireless communication congestion by assuming that the communication range is a monotone non-increasing function $r\colon \mathbb{N} \to \,]0, 2\pi[$ of the number of agents n. It is convenient to define the function $n \mapsto \delta(n) = nr(n) - 2\pi$ that compares the sum of the communication ranges of all the robots with the length of the unit circle.

Theorem IV.1 (Time complexity of agree-and-pursue law) *In the limit as $n \to +\infty$ and $\varepsilon \to 0^+$, the network $\mathcal{S}_{\text{circle}}$, the law $\mathcal{CC}_{\text{agr-pursuit}}$, and the tasks $\mathcal{T}_{\text{agrmnt}}$ and $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$ together satisfy:*

(i) $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Theta(r(n)^{-1})$;

(ii) *if $\delta(n)$ is lower bounded by a positive constant as $n \to +\infty$, then*

$$\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-eqdstnc}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Omega(n^2 \log(n\varepsilon)^{-1}),$$
$$\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-eqdstnc}}, \mathcal{CC}_{\text{agr-pursuit}}) \in O(n^2 \log(n\varepsilon^{-1})).$$

If $\delta(n)$ is lower bounded by a negative constant, then $\mathcal{CC}_{\text{agr-pursuit}}$ does not achieve $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$ in general. •

Proof: In the following four *STEPS* we prove the two upper bounds and the two lower bounds. *STEP 1:* We start by proving the upper bound in statement (i). We claim that $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \leq 2\pi/(k_{\text{prop}} r(n))$, and we reason by contradiction, i.e., we assume that there exists an initial condition which gives rise to an execution with time complexity strictly larger than $2\pi/(k_{\text{prop}} r(n))$. Without loss of generality, assume $\mathtt{drctn}^{[n]}(0) = \mathtt{c}$. For $\ell \leq 2\pi/(k_{\text{prop}} r(n))$, define

$$k(\ell) = \operatorname{argmin}\{\mathrm{dist}_{\mathtt{cc}}(\theta^{[n]}(0), \theta^{[i]}(\ell)) \mid \mathtt{drctn}^{[i]}(\ell) = \mathtt{cc},\, i \in I\}.$$

In other words, agent $k(\ell)$ is the agent moving counterclockwise that has smallest counterclockwise distance from the initial position of agent n. Note that $k(\ell)$ is well-defined since, by hypothesis of contradiction, $\mathcal{T}_{\text{agrmnt}}$ is $\mathtt{false}$ for $\ell \leq 2\pi/(k_{\text{prop}} r(n))$. According to the state-transition function of $\mathcal{CC}_{\text{agr-pursuit}}$ (cf. Section II-C), messages with $\mathtt{drctn} = \mathtt{cc}$ can only travel counterclockwise, while messages with $\mathtt{drctn} = \mathtt{c}$ can only travel clockwise. Therefore, the position of agent $k(\ell)$ at time ℓ can only belong to the counterclockwise interval from the position of agent $k(0)$ at time 0 to the position of agent n at time 0.

Let us examine how fast the message from agent n travels clockwise. To this end, for $\ell \leq 2\pi/(k_{\text{prop}} r(n))$, define

$$j(\ell) = \operatorname{argmax}\{\mathrm{dist}_{\mathtt{c}}(\theta^{[n]}(0), \theta^{[i]}(\ell)) \mid \mathtt{prior}^{[i]}(\ell) = n,\, i \in I\}.$$

In other words, agent $j(\ell)$ has $\mathtt{prior}$ equal to n, is moving clockwise, and is the agent furthest from the initial position of agent n in the clockwise direction with these two properties. Initially, $j(0) = n$. Additionally, for $\ell \leq 2\pi/(k_{\text{prop}} r(n))$, we claim that

$$\mathrm{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[j(\ell+1)]}(\ell+1)) \geq k_{\text{prop}} r(n).$$

This happens because either (1) there is no agent clockwise-ahead of $\theta^{[j(\ell)]}(\ell)$ within clockwise distance r and, therefore, the claim is obvious, or (2) there are such agents. In case (2), let m denote the agent whose clockwise distance to agent $j(\ell)$ is maximal within the set of agents with clockwise distance r from $\theta^{[j(\ell)]}(\ell)$. Then,

$$\begin{aligned}
&\mathrm{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[j(\ell+1)]}(\ell+1)) \\
&\quad = \mathrm{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell+1)) \\
&\quad = \mathrm{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell)) + \mathrm{dist}_{\mathtt{c}}(\theta^{[m]}(\ell), \theta^{[m]}(\ell+1)) \\
&\quad \geq \mathrm{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell)) \\
&\qquad\qquad + k_{\text{prop}}\big(r - \mathrm{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell))\big) \\
&\quad = k_{\text{prop}} r + (1 - k_{\text{prop}})\,\mathrm{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell)) \;\geq k_{\text{prop}} r,
\end{aligned}$$

where the first inequality follows from the fact that at time ℓ there can be no agent whose clockwise distance to agent m is less than $(r - \mathrm{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell)))$. Therefore, after $2\pi/(k_{\text{prop}} r(n))$ communication rounds, the message with $\mathtt{prior} = n$ has traveled the whole circle in the clockwise direction, and must therefore have reached agent $k(\ell)$. This is a contradiction.

STEP 2: We now prove the lower bound in statement (i). If $r(n) > \pi$ for all n, then $1/r(n) < 1/\pi$, and the upper bound reads $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \in O(1)$. Obviously, the time complexity of any evolution with an initial configuration where $\mathtt{drctn}^{[i]}(0) = \mathtt{cc}$ for $i \in \{1, \dots, n-1\}$, $\mathtt{drctn}^{[n]}(0) = \mathtt{c}$ and $E_{r\text{-disk}}(\theta^{[1]}(0), \dots, \theta^{[n]}(0))$ is the complete graph, is lower bounded by 1. Therefore, $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Omega(1)$. If $r(n) > \pi$ for all n, then we conclude $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Theta(r(n)^{-1})$. Assume now that $r(n) \leq \pi$ for sufficiently large n. Consider an initial configuration where $\mathtt{drctn}^{[i]}(0) = \mathtt{cc}$ for $i \in \{1, \dots, n-1\}$, $\mathtt{drctn}^{[n]}(0) = \mathtt{c}$, and the agents are placed as depicted in Figure 3. Note that, after each communication round, agent 1 has moved $k_{\text{prop}} r(n)$ in the counterclockwise direction, while agent n has moved $k_{\text{prop}} r(n)$ in the clockwise direction. These two agents keep moving at full speed towards each other until they become neighbors at a time lower bounded by

$$\frac{2\pi - r(n)}{2k_{\text{prop}} r(n)} > \frac{\pi}{k_{\text{prop}} r(n)} - 1.$$

We conclude $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Omega(r(n)^{-1})$.

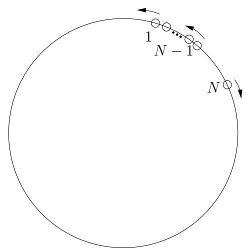

Fig. 3. Initial network configuration useful to establish the lower bound of $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}})$. We set $\mathrm{dist}_{\mathsf{c}}(\theta^{[n-1]}(0), \theta^{[n]}(0)) \in \,]r, r+\varepsilon'[$ and $\mathrm{dist}_{\mathsf{c}}(\theta^{[1]}(0), \theta^{[n-1]}(0)) \in \,]0, \varepsilon'[$, for some $\varepsilon' > 0$.

STEP 3: We now prove the upper bound in (ii). We begin by noting that the lower bound on δ implies $r(n)^{-1} \in O(n)$. Therefore, $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}})$ belongs to $O(n)$ and is negligible as compared with the claimed upper bound estimates for $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-eqdstnc}}, \mathcal{CC}_{\text{agr-pursuit}})$. In what follows, we therefore assume that $\mathcal{T}_{\text{agrmnt}}$ has been achieved and that, without loss of generality, all agents are moving clockwise. We now prove a fact regarding connectivity. At time $\ell \in \mathbb{N}_0$, let $H(\ell)$ be the union of all the empty "circular segments" of length at least r, that is, let

$$H(\ell) = \{x \in \mathbb{S}^1 \mid \min_{i \in I} \mathrm{dist}_{\mathsf{c}}(x, \theta^{[i]}(\ell)) + \min_{j \in I} \mathrm{dist}_{\mathsf{cc}}(x, \theta^{[j]}(\ell)) > r\}.$$

In other words, $H(\ell)$ does not contain any point between two agents separated by a distance less than r, and each connected component of $H(\ell)$ has length at least r. Let $n_H(\ell)$ be the number of connected components of $H(\ell)$, if $H(\ell)$ is empty, then we take the convention that $n_H(\ell) = 0$. Clearly, $n_H(\ell) \leq n$. We claim that, if $n_H(\ell) > 0$, then $t \mapsto n_H(\ell + t)$ is non-increasing. Let $d(\ell) < r$ be the distance between any two consecutive agents at time ℓ. Because both agents move in the same direction, a simple calculation shows that

$$\begin{aligned} d(\ell+1) &\leq d(\ell) + k_{\text{prop}}(r - d(\ell)) = (1 - k_{\text{prop}}) d(\ell) + k_{\text{prop}} r \\ &< (1 - k_{\text{prop}}) r + k_{\text{prop}} r = r. \end{aligned}$$

This means that the two agents remain within distance r and, therefore connected, at the following time instant. Because the number of connected components of $E_r(\theta^{[1]}, \dots, \theta^{[n]})$ does not increase, it follows that the number of connected components of H cannot increase. Next, we claim that, if $n_H(\ell) > 0$, then there exists $t > \ell$ such that $n_H(t) < n_H(\ell)$. By contradiction, assume $n_H(\ell) = n_H(t)$ for all $t > \ell$. Without loss of generality, let $\{1, \dots, m\}$ be a set of agents with the properties that $\mathrm{dist}_{\mathsf{cc}}\big(\theta^{[i]}(\ell), \theta^{[i+1]}(\ell)\big) \leq r$, for $i \in \{1, \dots, m\}$, that $\theta^{[1]}(\ell)$ and $\theta^{[m]}(\ell)$ belong to the boundary of $H(\ell)$, and that there is no other set with the same properties and more agents. (Note that this implies that the agents $1, \dots, m$ are in counterclockwise order.) One can show that, for $\tau \geq \ell$,

$$\begin{aligned} \theta^{[1]}(\tau+1) &= \theta^{[1]}(\tau) - k_{\text{prop}} r, \\ \theta^{[i]}(\tau+1) &= \theta^{[i]}(\tau) - k_{\text{prop}} \,\mathrm{dist}_{\mathsf{c}}(\theta^{[i]}(\tau), \theta^{[i-1]}(\tau)), \end{aligned}$$

for $i \in \{2, \dots, m\}$. If we define $d(\tau) = \big(\mathrm{dist}_{\mathsf{cc}}(\theta^{[1]}(\tau), \theta^{[2]}(\tau)), \dots, \ \mathrm{dist}_{\mathsf{cc}}(\theta^{[m-1]}(\tau), \theta^{[m]}(\tau))\big) \in \mathbb{R}^{m-1}_{>0}$, then the previous equations can be rewritten as

$$\begin{aligned} d(\tau+1) = \mathrm{Trid}_{m-1}&(k_{\text{prop}}, 1 - k_{\text{prop}}, 0)\, d(\tau) \\ &+ r[k_{\text{prop}}, \ 0, \ \cdots, \ 0]^T, \end{aligned}$$

where the linear map $(a, b, c) \mapsto \mathrm{Trid}_{m-1}(a, b, c) \in \mathbb{R}^{(m-1)\times(m-1)}$ is defined in Appendix A. This is a discrete-time affine time-invariant dynamical system with unique equilibrium point $r(1, \dots, 1)$. By Theorem A.3(ii) in Appendix A, for $\eta_1 \in \,]0, 1[$, the solution $\tau \mapsto d(\tau)$ to this system reaches a ball of radius η_1 centered at the equilibrium point in time $O(m \log m + \log \eta_1^{-1})$. (Here we used the fact that the initial condition of this system is bounded.) In turn, this implies that $\tau \mapsto \sum_{i=1}^{m} d_i(\tau)$ is larger than $(m-1)(r - \eta_1)$ in time $O(m \log m + \log \eta_1^{-1})$. We are now ready to find the contradiction and show that $n_H(\tau)$ cannot remain equal to $n_H(\ell)$ for all time τ. After time $O(m \log m + \log \eta_1^{-1}) = O(n \log n + \log \eta_1^{-1})$, we have:

$$\begin{aligned} 2\pi &\geq n_H(\ell) r + \sum_{j=1}^{n_H(\ell)} (r - \eta_1)(m_j - 1) \\ &= n_H(\ell) r + (n - n_H(\ell))(r - \eta_1) = n_H(\ell)\eta_1 + n(r - \eta_1). \end{aligned}$$

Here $m_1, \dots, m_{n_H(\ell)}$ are the number of agents in each isolated group, and each connected component of $H(\ell)$ has length at least r. Now, take $\eta_1 = (nr - 2\pi)n^{-1} = \delta(n)n^{-1}$, and the contradiction follows from

$$\begin{aligned} 2\pi &\geq n_H(\ell)\eta_1 + nr - n\eta_1 \\ &= n_H(\ell)\eta_1 + nr + 2\pi - nr = n_H(\ell)\eta_1 + 2\pi. \end{aligned}$$

In summary, this shows that the number of connected components of H decreases by one in time $O(n \log n + \log \eta_1^{-1}) = O(n \log n + \log(n\delta^{-1}(n)))$. Note that δ being lower bounded implies $n\delta^{-1}(n) = O(n)$ and, therefore, $O(n \log n + \log(n\delta^{-1}(n))) = O(n \log n)$. Iterating this argument n times, in time $O(n^2 \log n)$ the set H will become empty. At that time, the resulting network will obey the discrete-time linear time-invariant dynamical system:

$$d(\tau+1) = \mathrm{Circ}_n(k_{\text{prop}}, 1 - k_{\text{prop}}, 0)\, d(\tau), \tag{4}$$

where the linear map $(a, b, c) \mapsto \mathrm{Circ}_n(a, b, c) \in \mathbb{R}^{n \times n}$ is defined in Appendix A. Here $d(\tau) = \big(\mathrm{dist}_{\mathsf{cc}}(\theta^{[1]}(\tau), \theta^{[2]}(\tau)), \dots, \mathrm{dist}_{\mathsf{cc}}(\theta^{[n]}(\tau), \ \theta^{[n+1]}(\tau))\big) \in \mathbb{R}^n_{>0}$, with the convention $\theta^{[n+1]} = \theta^{[1]}$. By Theorem A.3(iii) in Appendix A, in time $O\big(n^2 \log \varepsilon^{-1}\big)$, the error 2-norm satisfies the contraction inequality $\|d(\tau) - d_*\|_2 \leq \varepsilon \|d(0) - d_*\|_2$, for $d_* = \frac{2\pi}{n}\mathbf{1}$. We convert this inequality on 2-norms into an appropriate inequality on ∞-norms as follows. Note that $\|d(0) - d_*\|_\infty = \max_{i \in I} |d^{[i]}(0) - d_*^{[i]}| \leq 2\pi$. For $\eta_2 \in \,]0, 1[$ and for τ of order $n^2 \log \eta_2^{-1}$,

$$\begin{aligned} \|d(\tau) - d_*\|_\infty &\leq \|d(\tau) - d_*\|_2 \leq \eta_2 \|d(0) - d_*\|_2 \\ &\leq \eta_2 \sqrt{n} \|d(0) - d_*\|_\infty \leq \eta_2 2\pi \sqrt{n}. \end{aligned}$$

This means that the desired configuration is achieved for $\eta_2 2\pi\sqrt{n} = \varepsilon$, that is, in time $O(n^2 \log \eta_2^{-1}) = O(n^2 \log(n\varepsilon^{-1}))$. In summary, the equidistance task is achieved in time $O(n^2 \log(n\varepsilon^{-1}))$.

STEP 4: Finally, we prove the lower bound in (ii). As before, $\mathrm{TC}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}})$ is negligible as compared with the claimed lower bound estimate for $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-eqdstnc}}, \mathcal{CC}_{\text{agr-pursuit}})$ and, therefore, we assume that $\mathcal{T}_{\text{agrmnt}}$ has been achieved. We consider an initial configuration with the properties that (i) agents are counterclockwise-ordered according to their unique identifier, (ii) the set H is empty, and (iii) the inter-agent distances $d(0) = \big(\mathrm{dist}_{\mathtt{cc}}(\theta^{[1]}(0), \theta^{[2]}(0)), \dots, \mathrm{dist}_{\mathtt{cc}}(\theta^{[n]}(0), \theta^{[1]}(0))\big)$ are given by

$$d(0) = \frac{2\pi}{n}\mathbf{1} + \frac{\pi - \varepsilon'}{n}(\mathbf{v}_n + \overline{\mathbf{v}}_n),$$

where $\varepsilon' \in \,]\pi, 0[$ and where $\mathbf{v}_n$ is the eigenvector of $\mathrm{Circ}_n(k_{\text{prop}}, 1-k_{\text{prop}}, 0)$ corresponding to the eigenvalue $1 - k_{\text{prop}} + k_{\text{prop}} \cos\left(\frac{2\pi}{n}\right) - k_{\text{prop}}\sqrt{-1}\sin\left(\frac{2\pi}{n}\right)$ (see equation (A.7) in Appendix A). One can verify that $\mathbf{v}_n + \overline{\mathbf{v}}_n = 2(1, \cos(2\pi/n), \dots, \cos((n-1)2\pi/n))$ and that $\|\mathbf{v}_n + \overline{\mathbf{v}}_n\|_2 = \sqrt{2n}$. In turn, this implies that $d(0) \in \mathbb{R}^n_{>0}$ and that $\|d(0) - \frac{2\pi}{n}\mathbf{1}\|_2 \in O(1/\sqrt{n})$. Take $\eta_3 \in \,]0,1[$. The argument described in the proof of Theorem A.3(iii) leads to the following statement: the 2-norm of the difference between $t \mapsto d(t)$ and the desired configuration $\frac{2\pi}{n}\mathbf{1}$ decreases by a factor η_3 in time of order $n^2 \log \eta_3^{-1}$. Given an initial error of order $O(1/\sqrt{n})$ and a final desired error of order ε, we set $\eta_3 = \varepsilon\sqrt{n}$ and obtain the desired result that it takes time of order $n^2 \log(n\varepsilon)^{-1}$ to reduce the 2-norm error, and therefore, the ∞-norm error to size ε. This concludes the proof. ∎

To conclude this section, we study the total communication complexity of the agree-and-pursue control and communication law. We consider the case of a unidirectional communication model with one-round cost function depending linearly on the cardinality of the communication graph.

Theorem IV.2 (Total communication complexity of agree-and-pursue law) *In the limit as $n \to +\infty$ and $\varepsilon \to 0^+$, the network $\mathcal{S}_{\text{circle}}$, the law $\mathcal{CC}_{\text{agr-pursuit}}$, and the tasks $\mathcal{T}_{\text{agrmnt}}$ and $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$ together satisfy:*

(i) *if $\delta(n) \geq \pi(1/k_{\text{prop}} - 2)$ as $n \to +\infty$, then*

$$\mathrm{TCC}_{\mathrm{unidir}}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Theta(n^2 r(n)^{-1}),$$

otherwise if $\delta(n) \leq \pi(1/k_{\text{prop}} - 2)$ as $n \to +\infty$, then

$$\mathrm{TCC}_{\mathrm{unidir}}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Omega(n^3 + nr(n)^{-1}),$$
$$\mathrm{TCC}_{\mathrm{unidir}}(\mathcal{T}_{\text{agrmnt}}, \mathcal{CC}_{\text{agr-pursuit}}) \in O(n^2 r(n)^{-1});$$

(ii) *if $\delta(n)$ is lower bounded by a positive constant as $n \to +\infty$, then*

$$\mathrm{TCC}_{\mathrm{unidir}}(\mathcal{T}_{\varepsilon\text{-eqdstnc}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Omega(n^3 \delta(n) \log(n\varepsilon)^{-1}),$$
$$\mathrm{TCC}_{\mathrm{unidir}}(\mathcal{T}_{\varepsilon\text{-eqdstnc}}, \mathcal{CC}_{\text{agr-pursuit}}) \in O(n^4 \log(n\varepsilon^{-1})).$$ •

Proof: The upper bounds in (i) and (ii) follow immediately from the inequality $\mathrm{TCC}(\mathcal{T}, \mathcal{CC}) \leq \mathrm{MCC}(\mathcal{T}, \mathcal{CC}) \cdot \mathrm{TC}(\mathcal{T}, \mathcal{CC})$ and from the fact that the number of edges in $E_{r\text{-disk}}$ is in $O(n^2)$. To prove the lower bounds we follow the steps and notation in the proof of Theorem IV.1. Regarding the lower bounds in (i), we examine the evolution of the initial configuration depicted in Figure 3. From *STEP 2:* in the proof of Theorem IV.1, recall that the time it takes agent 1 to receive the message with $\texttt{prior} = n$ is lower bounded by $\pi/(k_{\text{prop}} r(n)) - 1$. Our proof strategy is to lower bound the number of edges in the graph until this event happens. Note that, at initial time, there are $(n-1)^2$ edges in the communication graph of the network, and therefore, $(n-1)^2$ messages get transmitted. At the next communication round, agent 1 has moved $k_{\text{prop}} r(n)$ counterclockwise and, therefore, the number of edges is lower bounded by $(n-2)^2$. Iterating this reasoning, we see that after $i < \pi/(k_{\text{prop}} r(n))$ communication rounds, the number of edges is lower bounded by $(n-i)^2$. Now, if $\delta(n) > \pi(1/k_{\text{prop}} - 2)$, then $n > \pi/k_{\text{prop}} r(n))$, and therefore, the total communication complexity is lower bounded by

$$\sum_{i=1}^{\frac{\pi}{k_{\text{prop}} r(n)}} (n-i)^2 \in \Omega(n^2 r(n)^{-1}).$$

On the other hand, if $\delta(n) < \pi(1/k_{\text{prop}} - 2)$, then $n < \pi/k_{\text{prop}} r(n))$, and after n time steps, we lower bound the number of edges in the communication graph by the number of edges in a chain of length n, that is, $n-1$. Therefore, the total communication complexity is lower bounded by

$$\sum_{i=1}^{n} (n-i)^2 + (n-1)\Big(\frac{\pi}{k_{\text{prop}} r(n)} - n\Big) \in \Omega(n^3 + nr(n)^{-1}).$$

The two lower bounds match when $\delta(n) = \pi(1/k_{\text{prop}} - 2)$.

Regarding the lower bound in (ii), we consider first the case when $n_H(0) = 0$. In this case, the network obeys the discrete-time linear time-invariant dynamical system (4). Consider the initial condition $d(0)$ that we adopted for *STEP 4:*. We know it takes time of order $n^2 \log(n\varepsilon)^{-1}$ for the appropriate contraction property to hold. At $d(0)$, the maximal inter-agent distance is $(4\pi - \varepsilon')/n$ and it decreases during the evolution. Because each robot can communicate with any other robot within a distance $r(n)$, the number of agents within communication range of a given agent is of order $r(n)n/(4\pi - \varepsilon')$, that is, of order $\delta(n)$. From here we deduce that the total communication complexity belongs to $\Omega(n^3 \delta(n) \log(n\varepsilon)^{-1})$. ∎

Remark IV.3 (Comparison with leader election) Let us compare the agree-and-pursue control and communication law with the classical Lann-Chang-Roberts (LCR) algorithm for leader election (see [5, Chapter 3.3]). The leader election task consists of electing a unique agent among all agents in the network; it is therefore different from, but closely related to, the coordination task $\mathcal{T}_{\text{agrmnt}}$. The LCR algorithm operates on a static network with the ring communication topology, and achieves leader election with time and total communication complexity, respectively, $\Theta(n)$ and $\Theta(n^2)$. The agree-and-pursue law operates on a robotic network with the $r(n)$-disk communication topology, and achieves $\mathcal{T}_{\text{agrmnt}}$ with time and total communication complexity, respectively, $\Theta(r(n)^{-1})$ and $O(n^2 r(n)^{-1})$. If wireless communication congestion is modeled by $r(n)$ of order $1/n$ as in Remark II.9, then the two algorithms have identical time complexity and the LCR algorithm has better communication complexity. Note that computations on a possibly disconnected, dynamic network are more complex than on a static ring topology. •

V. Conclusions

We have introduced a formal model for the design and analysis of coordination algorithms executed by networks of robotic agents. In this framework motion coordination algorithms are formalized as feedback control and communication laws. Drawing analogies with the discipline of distributed algorithms, we have defined two measures of complexity for control and communication laws: the time and the communication complexity to achieve a specific task. We have defined the notion of re-scheduling of a control and communication law and analyzed the invariance of the proposed complexity measures under this operation. These concepts and results are illustrated in a network of locally connected agents on the circle executing a novel "agree-and-pursue" coordination algorithm that combines elements of the leader election and cyclic pursuit problems.

The proposed notions allow us to compare the scalability properties of different coordination algorithms with regards to performance and communication costs. Numerous avenues for future research appear open. An incomplete list include: (i) modeling of asynchronous networks (see however [23], [24], [9]); (ii) robustness analysis with respect to failures in the agents (arrivals/departures) and in the communication links (see however [18], [25], [26], [27]); (iii) probabilistic versions of the complexity measures that capture, for instance, the expected performance and cost of coordination algorithms (see however [11]); (iv) quantization and delays in the communication channels (see however [28] and the literature on quantized control); and (v) parallel, sequential, and hierarchical composition of control and communication laws. On the algorithmic side, the companion paper [17] provides time-complexity estimates for coordination algorithms that achieve rendezvous and deployment, and discusses other open questions.

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APPENDIX A
TRIDIAGONAL TOEPLITZ AND CIRCULANT DYNAMICAL SYSTEMS

This section presents some key facts about convergence rates of discrete-time dynamical systems defined by certain classes of Toeplitz matrices, see [29]. To the best of our knowledge, the results presented below in Theorem A.3 on tridiagonal Toeplitz matrices and in Theorem A.4 are novel contributions. The results on stochastic circulant matrices in Theorem A.3 are related to the literature on Markov chains [30], see also the recent developments in [31], [32]. For $n \geq 2$ and $a, b, c \in \mathbb{R}$, define the $n \times n$ Toeplitz matrices $\operatorname{Trid}_n(a,b,c)$ and $\operatorname{Circ}_n(a,b,c)$ by

$$\operatorname{Trid}_n(a,b,c) = \begin{bmatrix} b & c & 0 & \dots & 0 \\ a & b & c & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a & b & c \\ 0 & \dots & 0 & a & b \end{bmatrix},$$

and

$$\operatorname{Circ}_n(a,b,c) = \operatorname{Trid}_n(a,b,c) + \begin{bmatrix} 0 & \dots & \dots & 0 & a \\ 0 & \dots & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ c & 0 & \dots & 0 & 0 \end{bmatrix}.$$

The matrices $\operatorname{Trid}_n(a,b,c)$ and $\operatorname{Circ}_n(a,b,c)$ are tridiagonal and circulant, respectively, and only differ in their $(1,n)$ and $(n,1)$ entries. Note our convention that $C_2(a,b,c) = \begin{bmatrix} b & a+c \\ a+c & b \end{bmatrix}$. The following results are discussed, for example, in [29, Example 7.2.5 and Exercise 7.2.20].

Lemma A.1 (Eigenvalues of tridiagonal Toeplitz and circulant matrices) *For $n \geq 2$ and $a, b, c \in \mathbb{R}$, the following statements hold:*

(i) *for $ac \neq 0$, the eigenvalues and eigenvectors of $\operatorname{Trid}_n(a,b,c)$ are, for $i \in \{1,\dots,n\}$,*

$$b + 2c\sqrt{\frac{a}{c}}\cos\left(\frac{i\pi}{n+1}\right), \text{ and}$$
$$\Big[\left(\frac{a}{c}\right)^{1/2}\sin\left(\frac{i\pi}{n+1}\right), \dots, \left(\frac{a}{c}\right)^{n/2}\sin\left(\frac{ni\pi}{n+1}\right)\Big]^T;$$

(ii) *the eigenvalues and eigenvectors of $\operatorname{Circ}_n(a,b,c)$ are, for $\omega = \exp(\frac{2\pi\sqrt{-1}}{n})$ and for $i \in \{1,\dots,n\}$,*

$$b + (a+c)\cos\left(\frac{i2\pi}{n}\right) + \sqrt{-1}(c-a)\sin\left(\frac{i2\pi}{n}\right), \text{ and}$$
$$\big[1,\ \omega^i,\ \cdots,\ \omega^{(n-1)i}\big]^T.$$
•

Remarks A.2 (i) The set of eigenvalues of $\operatorname{Trid}_n(a,b,c)$ is contained in the real interval $[b - 2\sqrt{ac}, b + 2\sqrt{ac}]$, if $ac \geq 0$, and in the interval in the complex plane $[b - 2\sqrt{-1}\sqrt{|ac|}, b + 2\sqrt{-1}\sqrt{|ac|}]$, if $ac \leq 0$.

(ii) The set of eigenvalues of $\operatorname{Circ}_n(a,b,c)$ is contained in the ellipse on the complex plane with center b, horizontal axis $2|a+c|$ and vertical axis $2|c-a|$.

(iii) Recall from [29] that (1) a square matrix is normal if it has a complete orthonormal set of eigenvectors, (2) circulant matrices and real-symmetric matrices are normal, and (3) if a normal matrix has eigenvalues $\{\lambda_1,\dots,\lambda_n\}$, then its singular values are $\{|\lambda_1|,\dots,|\lambda_n|\}$. •

We can now state the main result of this section.

Theorem A.3 (Tridiagonal Toeplitz and circulant dynamical systems) *Let $n \geq 2$, $\varepsilon \in \,]0,1[$, and $a, b, c \in \mathbb{R}$. Let $x\colon \mathbb{N}_0 \to \mathbb{R}^n$ and $y\colon \mathbb{N}_0 \to \mathbb{R}^n$ be solutions to*

$$x(\ell+1) = \operatorname{Trid}_n(a,b,c)\,x(\ell),$$
$$y(\ell+1) = \operatorname{Circ}_n(a,b,c)\,y(\ell),$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$, respectively. The following statements hold:

(i) *if $a = c \neq 0$ and $|b| + 2|a| = 1$, then $\lim_{\ell\to+\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \leq \varepsilon\|x_0\|_2$ (over all initial conditions $x_0 \in \mathbb{R}^n$) is $\Theta\big(n^2 \log \varepsilon^{-1}\big)$;*

(ii) *if $a \neq 0$, $c = 0$ and $0 < |b| < 1$, then $\lim_{\ell\to+\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \leq \varepsilon\|x_0\|_2$ (over all initial conditions $x_0 \in \mathbb{R}^n$) is $O\big(n \log n + \log \varepsilon^{-1}\big)$;*

(iii) *if $a \geq 0$, $c \geq 0$, $b > 0$, and $a + b + c = 1$, then $\lim_{\ell\to+\infty} y(\ell) = y_{\text{ave}}\mathbf{1}$, where $y_{\text{ave}} = \frac{1}{n}\mathbf{1}^T y_0$, and the maximum time required for $\|y(\ell) - y_{\text{ave}}\mathbf{1}\|_2 \leq \varepsilon\|y_0 - y_{\text{ave}}\mathbf{1}\|_2$ (over all initial conditions $y_0 \in \mathbb{R}^n$) is $\Theta\big(n^2 \log \varepsilon^{-1}\big)$.* •

Proof: Let us prove fact (i). We start by bounding from above the eigenvalue with largest absolute value, that is, the largest singular value, of $\operatorname{Trid}_n(a,b,a)$

$$\max_{i\in\{1,\dots,n\}} \left| b + 2a\cos\left(\frac{i\pi}{n+1}\right)\right| \leq |b| + 2|a| \max_{i\in\{1,\dots,n\}} \left|\cos\left(\frac{i\pi}{n+1}\right)\right| \leq |b| + 2|a|\cos\left(\frac{\pi}{n+1}\right).$$

Because $\cos(\frac{\pi}{n+1}) < 1$ for any $n \geq 2$, the matrix $\operatorname{Trid}_n(a,b,a)$ is stable. Additionally, for $\ell > 0$, we bound from above the magnitude of the curve x as

$$\|x(\ell)\|_2 = \|\operatorname{Trid}_n(a,b,a)^\ell x_0\|_2 \leq \left(|b| + 2|a|\cos\left(\frac{\pi}{n+1}\right)\right)^\ell \|x_0\|_2.$$

In order to have $\|x(\ell)\|_2 < \varepsilon\|x_0\|_2$, it is sufficient that $\ell \log\left(|b| + 2|a|\cos\left(\frac{\pi}{n+1}\right)\right) < \log\varepsilon$, that is

$$\ell > \frac{\log \varepsilon^{-1}}{-\log\left(|b| + 2|a|\cos\left(\frac{\pi}{n+1}\right)\right)}. \tag{A.5}$$

To show the upper bound, note that as $t \to 0$ we have

$$-\frac{1}{\log(1-2|a|(1-\cos t))} = \frac{1}{|a|t^2} + O(1).$$

Now, assume without loss of generality that $ab > 0$ and consider the eigenvalue $b+2a\cos(\frac{\pi}{n+1})$ of $\mathrm{Trid}_n(a,b,a)$. Note that $|b+2a\cos(\frac{\pi}{n+1})| = |b| + 2|a|\cos(\frac{\pi}{n+1})$. (If $ab < 0$, then consider the eigenvalue $b+2a\cos(\frac{n\pi}{n+1})$.) For $n > 2$, define the unit-length vector

$$\mathbf{v}_n = \sqrt{\frac{2}{n+1}}\Big[\sin\frac{\pi}{n+1}, \ \cdots \ , \ \sin\frac{n\pi}{n+1}\Big]^T \in \mathbb{R}^n, \quad \text{(A.6)}$$

and note that, by Lemma A.1(i), $\mathbf{v}_n$ is an eigenvector of $\mathrm{Trid}_n(a,b,a)$ with eigenvalue $b + 2a\cos(\frac{\pi}{n+1})$. Note also that all components of $\mathbf{v}_n$ are positive. The trajectory x with initial condition $\mathbf{v}_n$ satisfies $\|x(\ell)\|_2 = \Big(|b| + 2|a|\cos\Big(\frac{\pi}{n+1}\Big)\Big)^\ell \|\mathbf{v}_n\|_2$ and, therefore, it will enter $B(\mathbf{0}, \varepsilon\|\mathbf{v}_n\|_2)$ only when ℓ satisfies (A.5). This completes the proof of fact (i).

Next, we consider statement (ii). Clearly, $\mathrm{Trid}_n(a,b,0)$ is stable. For $\ell > 0$, we compute

$$\begin{aligned}\mathrm{Trid}_n(a,b,0)^\ell &= b^\ell\Big(I_n + \frac{a}{b}\,\mathrm{Trid}_n(1,0,0)\Big)^\ell \\ &= b^\ell \sum_{j=0}^{n-1} \frac{\ell!}{j!(\ell-j)!}\Big(\frac{a}{b}\Big)^j \mathrm{Trid}_n(1,0,0)^j\end{aligned}$$

because of the nilpotency of $\mathrm{Trid}_n(1,0,0)$. Now we can bound from above the magnitude of the curve x as

$$\begin{aligned}\|x(\ell)\|_2 &= \|\,\mathrm{Trid}_n(a,b,0)^\ell x_0\|_2 \\ &\le |b|^\ell \sum_{j=0}^{n-1}\frac{\ell!}{j!(\ell-j)!}\Big(\frac{a}{b}\Big)^j \big\|\,\mathrm{Trid}_n(1,0,0)^j x_0\big\|_2 \\ &\le \mathrm{e}^{a/b}\ell^{n-1}\,|b|^\ell\,\|x_0\|_2.\end{aligned}$$

Here we used $\|\,\mathrm{Trid}_n(1,0,0)^j x_0\|_2 \le \|x_0\|_2$ and $\max\{\frac{\ell!}{(\ell-j)!} \mid j \in \{0,\dots,n-1\}\} \le \ell^{n-1}$. Therefore, in order to have $\|x(\ell)\|_2 < \varepsilon\|x_0\|_2$, it suffices that $\log(\mathrm{e}^{a/b}) + (n-1)\log\ell + \ell\log|b| \le \log\varepsilon$, that is

$$\ell - \frac{n-1}{-\log|b|}\log\ell > \frac{\frac{a}{b} - \log\varepsilon}{-\log|b|}.$$

A sufficient condition for $\ell - \alpha\log\ell > \beta$, for $\alpha, \beta > 0$, is that $\ell \ge 2\beta + 2\alpha\max\{1, \log\alpha\}$. For, if $\ell \ge 2\alpha$, then $\log\ell$ is bounded from above by the line $\ell/2\alpha + \log\alpha$. Furthermore, the line $\ell/2\alpha + \log\alpha$ is a lower bound for the line $(\ell-\beta)/\alpha$ if $\ell \ge 2\beta + 2\alpha\log\alpha$. In summary, it is true that $\|x(\ell)\|_2 \le \varepsilon\|x(0)\|_2$ whenever

$$\ell \ge 2\frac{\frac{a}{b} - \log\varepsilon}{-\log|b|} + 2\frac{n-1}{-\log|b|}\max\Big\{1, \log\frac{n-1}{-\log|b|}\Big\}.$$

This completes the proof of fact (ii).

The proof of fact (iii) is similar to that of fact (i). We analyze the singular values of $\mathrm{Circ}_n(a,b,c)$. It is clear that the eigenvalue corresponding to $i = n$ is equal to 1; this is the largest singular value of $\mathrm{Circ}_n(a,b,c)$ and the corresponding eigenvector is $\mathbf{1}$. In the orthogonal decomposition induced by the eigenvectors of $\mathrm{Circ}_n(a,b,c)$, the vector y_0 has a component y_{ave} along the eigenvector $\mathbf{1}$. The second largest singular value is

$$\Big|1-(a+c)\Big(1-\cos\big(\frac{2\pi}{n}\big)\Big) + \sqrt{-1}(c-a)\sin\Big(\frac{2\pi}{n}\Big)\Big|.$$

Here $|\cdot|$ is the norm in $\mathbb{C}$. Because of the assumptions on a, b, c, the second largest singular value is strictly less than 1. For $\ell > 0$, we bound the distance of $y(\ell)$ from $y_{\text{ave}}\mathbf{1}$ as

$$\begin{aligned}\|y(\ell) - y_{\text{ave}}\mathbf{1}\|_2 &= \|\,\mathrm{Circ}_n(a,b,c)^\ell y_0 - y_{\text{ave}}\mathbf{1}\|_2 \\ &= \|\,\mathrm{Circ}_n(a,b,c)^\ell\big(y_0 - y_{\text{ave}}\mathbf{1}\big)\|_2 \\ &\le \Big|1-(a+c)\Big(1-\cos\big(\frac{2\pi}{n}\big)\Big) + \sqrt{-1}(c-a)\sin\Big(\frac{2\pi}{n}\Big)\Big|^\ell \\ &\quad\cdot \|y_0 - y_{\text{ave}}\mathbf{1}\|_2.\end{aligned}$$

This proves that $\lim_{\ell\to+\infty} y(\ell) = y_{\text{ave}}\mathbf{1}$. Also, for $\alpha = a + c, \beta = c - a$ and as $t \to 0$, we have

$$\frac{-1}{\log\Big(\big(1-\alpha(1-\cos t)\big)^2 + \beta^2\sin^2 t\Big)^{\frac{1}{2}}} = \frac{2}{(\alpha-\beta^2)t^2} + O(1).$$

Here $\beta^2 < \alpha$ because $a, c \in \,]0,1[$. From this, one deduces the upper bound in (iii).

Now, consider the eigenvalues $\lambda_n = b + (a+c)\cos\big(\frac{2\pi}{n}\big) + \sqrt{-1}(c-a)\sin\big(\frac{2\pi}{n}\big)$ and $\overline{\lambda}_n = b + (a+c)\cos\Big(\frac{(n-1)2\pi}{n}\Big) + \sqrt{-1}(c-a)\sin\Big(\frac{(n-1)2\pi}{n}\Big)$ of $\mathrm{Circ}_n(a,b,c)$, and its associated eigenvectors (cf. Lemma A.1(ii))

$$\begin{aligned}\mathbf{v}_n &= \Big[1, \ \omega, \ \cdots \ , \ \omega^{n-1}\Big]^T \in \mathbb{C}^n, \\ \overline{\mathbf{v}}_n &= \Big[1, \ \omega^{n-1}, \cdots \ , \ \omega\Big]^T \in \mathbb{C}^n. \quad \text{(A.7)}\end{aligned}$$

Note that the vector $\mathbf{v}_n + \overline{\mathbf{v}}_n$ belongs to $\mathbb{R}^n$. Moreover, its component y_{ave} along the eigenvector $\mathbf{1}$ is 0. The trajectory y with initial condition $\mathbf{v}_n + \overline{\mathbf{v}}_n$ satisfies $\|y(\ell)\|_2 = \|\lambda_n^\ell\mathbf{v}_n + \overline{\lambda}_n^\ell\overline{\mathbf{v}}_n\|_2 = |\lambda_n|^\ell\|\mathbf{v}_n + \overline{\mathbf{v}}_n\|_2$ and, therefore, it will enter $B(\mathbf{0}, \varepsilon\|\mathbf{v}_n + \overline{\mathbf{v}}_n\|_2)$ only when

$$\ell > \frac{-\log\varepsilon^{-1}}{\log\Big|1-(a+c)\Big(1-\cos\big(\frac{2\pi}{n}\big)\Big) + \sqrt{-1}(c-a)\sin\big(\frac{2\pi}{n}\big)\Big|}.$$

This completes the proof of fact (iii). ■

Next, we extend these results to another interesting set of matrices. For $n \ge 2$ and $a, b \in \mathbb{R}$, define the $n \times n$ *augmented tridiagonal matrices* $\mathrm{ATrid}_n^+(a,b)$ and $\mathrm{ATrid}_n^-(a,b)$ by

$$\mathrm{ATrid}_n^\pm(a,b) = \mathrm{Trid}_n(a,b,a) \pm \begin{bmatrix} a & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & a \end{bmatrix}.$$

If we define

$$P_+ = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 1 & 0 & \dots & 0 \\ 1 & 0 & -1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 1 & 0 & \dots & 0 & -1 & 1 \\ 1 & 0 & \dots & 0 & 0 & -1 \end{bmatrix},$$

and

$$P_- = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{n-2} & 0 & \dots & 0 & 1 & 1 \\ (-1)^{n-1} & 0 & \dots & 0 & 0 & 1 \end{bmatrix},$$

then the following similarity transforms are satisfied:

$$\mathrm{ATrid}_n^{\pm}(a,b) = P_{\pm} \begin{bmatrix} b \pm 2a & 0 \\ 0 & \mathrm{Trid}_{n-1}(a,b,a) \end{bmatrix} P_{\pm}^{-1}. \quad \text{(A.8)}$$

To analyze the convergence properties of the dynamical systems determined by $\mathrm{ATrid}_n^+(a,b)$ and $\mathrm{ATrid}_n^-(a,b)$, we recall that $\mathbf{1}^T = (1,\dots,1) \in \mathbb{R}^n$, and we define $\mathbf{1}_- = (1,-1,1,\dots,(-1)^{n-2},(-1)^{n-1})^T \in \mathbb{R}^n$.

Theorem A.4 (Augmented tridiagonal Toeplitz dynamical systems) *Let $n \geq 2$, $\varepsilon \in]0,1[$, and $a,b \in \mathbb{R}$ with $a \neq 0$ and $|b| + 2|a| = 1$. Let $x\colon \mathbb{N}_0 \to \mathbb{R}^n$ and $z\colon \mathbb{N}_0 \to \mathbb{R}^n$ be solutions to*

$$x(\ell+1) = \mathrm{ATrid}_n^+(a,b)\, x(\ell),$$
$$z(\ell+1) = \mathrm{ATrid}_n^-(a,b)\, z(\ell),$$

with initial conditions $x(0) = x_0$ and $z(0) = z_0$, respectively. The following statements hold:

(i) *$\lim_{\ell\to+\infty} \big(x(\ell) - x_{\text{ave}}(\ell)\mathbf{1}\big) = \mathbf{0}$, where $x_{\text{ave}}(\ell) = (\frac{1}{n}\mathbf{1}^T x_0)(b+2a)^\ell$, and the maximum time required for $\|x(\ell) - x_{\text{ave}}(\ell)\mathbf{1}\|_2 \leq \varepsilon \|x_0 - x_{\text{ave}}(0)\mathbf{1}\|_2$ (over all initial conditions $x_0 \in \mathbb{R}^n$) is $\Theta\big(n^2 \log \varepsilon^{-1}\big)$;*

(ii) *$\lim_{\ell\to+\infty} \big(z(\ell) - z_{\text{ave}}(\ell)\mathbf{1}_-\big) = \mathbf{0}$, where $z_{\text{ave}}(\ell) = (\frac{1}{n}\mathbf{1}_-^T z_0)(b-2a)^\ell$, and the maximum time required for $\|z(\ell) - z_{\text{ave}}(\ell)\mathbf{1}_-\|_2 \leq \varepsilon \|z_0 - z_{\text{ave}}(0)\mathbf{1}_-\|_2$ (over all initial conditions $z_0 \in \mathbb{R}^n$) is $\Theta\big(n^2 \log \varepsilon^{-1}\big)$.* •

Proof: We prove fact (i) and observe that the proof of fact (ii) is analogous. Consider the change of coordinates

$$x(\ell) = P_+ \begin{bmatrix} x'_{\text{ave}}(\ell) \\ y(\ell) \end{bmatrix} = x'_{\text{ave}}(\ell)\mathbf{1} + P_+ \begin{bmatrix} 0 \\ y(\ell) \end{bmatrix},$$

where $x'_{\text{ave}}(\ell) \in \mathbb{R}$ and $y(\ell) \in \mathbb{R}^{n-1}$. A quick calculation shows that $x'_{\text{ave}}(\ell) = \frac{1}{n}\mathbf{1}^T x(\ell)$, and the similarity transformation described in equation (A.8) implies

$$y(\ell+1) = \mathrm{Trid}_{n-1}(a,b,a)\, y(\ell),$$
$$x'_{\text{ave}}(\ell+1) = (b+2a) x'_{\text{ave}}(\ell).$$

Therefore, $x_{\text{ave}} = x'_{\text{ave}}$. It is also clear that

$$x(\ell+1) - x_{\text{ave}}(\ell+1)\mathbf{1} = P_+ \begin{bmatrix} 0 \\ y(\ell+1) \end{bmatrix}$$
$$= \left(P_+ \begin{bmatrix} 0 & 0 \\ 0 & \mathrm{Trid}_{n-1}(a,b,a) \end{bmatrix} P_+^{-1} \right) (x(\ell) - x_{\text{ave}}(\ell)\mathbf{1}).$$

Consider the matrix in parenthesis determining the trajectory $\ell \mapsto (x(\ell) - x_{\text{ave}}(\ell)\mathbf{1})$. This matrix is symmetric, its eigenvalues are 0 and the eigenvalues of $\mathrm{Trid}_{n-1}(a,b,a)$, and its eigenvectors are $P_+(1,0,\dots,0) \in \mathbb{R}^n$ and the eigenvectors of $\mathrm{Trid}_{n-1}(a,b,a)$, padded with an extra zero and premultiplied by P_+. These facts are sufficient to duplicate, step by step, the proof of fact (i) in Theorem A.3. Therefore, fact (i) follows. ■

We conclude this appendix with some useful bounds whose proof is straightforward.

Lemma A.5 *Assume $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{n-1}$ and $z \in \mathbb{R}^{n-1}$ jointly satisfy*

$$x = P_+ \begin{bmatrix} 0 \\ y \end{bmatrix}, \qquad x = P_- \begin{bmatrix} 0 \\ z \end{bmatrix}.$$

Then $\frac{1}{2}\|x\|_2 \leq \|y\|_2 \leq (n-1)\|x\|_2$ and $\frac{1}{2}\|x\|_2 \leq \|z\|_2 \leq (n-1)\|x\|_2$. •