Distributed algorithms for reaching consensus on general functions ⋆

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Abstract

This paper presents analysis and design results for distributed consensus algorithms in multi-agent networks. We consider continuous consensus functions of the initial state of the network agents. Under mild smoothness assumptions, we obtain necessary and sufficient conditions characterizing any algorithm that asymptotically achieves consensus. This characterization is the building block to obtain various design results for networks with weighted, directed interconnection topologies. We first identify a class of smooth functions for which one can synthesize in a systematic way distributed algorithms that achieve consensus. We apply this result to the family of weighted power mean functions, and characterize the exponential convergence properties of the resulting algorithms. We establish the validity of these results for scenarios with switching interconnection topologies. Finally, we conclude with two discontinuous distributed algorithms that achieve, respectively, max and min consensus in finite time.

Key words: Cooperative control; network consensus; distributed algorithms; balanced directed graphs; weighted power means.

1 Introduction

Arguably, the ability to reach consensus, or agreement, upon some (a priori unknown) quantity is critical for any multi-agent system. Network coordination problems involving self-organization, formation pattern, distributed estimation, or parallel processing, to name a few, require individual agents to agree on the identity of a leader, jointly synchronize their operation, decide which specific pattern to form, balance the computational load, or fuse the information gathered on some spatial process.

In this paper, we address the problem of designing (continuous-time) coordination algorithms that make a networked system asymptotically agree upon the value of a desired function of the initial state of the individual agents. The emphasis on general continuous functions is motivated by data fusion problems. In spatially-distributed scenarios, mobile sensor networks can implement the results developed here to compute, for instance, sample statistical moments of arbitrary order, weighted-least squares estimates of noisy signals, or posterior probabilities for multi-hypothesis testing via the product of conditional independent probabilities. The network topology is modeled by a weighted, directed graph. In practical scenarios, the network topology might be directed because of packet losses, obstacles in the environment, or interference, and weighted because of agents with different bandwidth capabilities or varying signal strengths in particular network configurations.

Literature review. Distributed consensus algorithms have a long-standing tradition in computer science, e.g., [20]. Agreement under limited communication is also an established topic of research in network games, e.g., [11]. Within the literature on cooperative control and multi-agent systems, recent years have witnessed the introduction of distributed strategies that achieve various forms of agreement. This interest is reflected in the recent surveys [23, 27]. A growing body of work focuses in algorithms that make individual network agents agree upon the value of some function of their initial states. These include average consensus [25, 26, 30], average-max-min consensus [8], geometric-mean consensus [24], and power-mean consensus [3]. The state variables associated to the individual agents might be, for instance, sensor measurements of some signal or spatial process, or probability distributions about the likelihood of some event, and do not necessarily correspond to physical variables, such as spatial coordinates or velocities. Coordination problems that focus, instead, on "spatial versions" of consensus include rendezvous [1, 9, 18, 19], flocking [4, 14, 21, 22], and synchronization [15, 29].

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Applications of consensus algorithms to data fusion problems and distributed filtering include [28, 31].

Statement of contributions. The contributions of this paper pertain both analysis and design of cooperative strategies for consensus. Regarding analysis, we identify (cf. Theorem 8 and Corollary 9) a set of conditions that characterize when a coordination algorithm makes the network agents asymptotically agree upon the value of a function of their individual states. This characterization holds under mild assumptions on the smoothness of the consensus function and the coordination algorithm, and constrains the class of allowable weighted digraphs for *distributed* coordination algorithms to be weakly connected and weight-balanced. This turns out to have remarkable implications on the connectivity properties of the directed graph (cf. Propositions 1 and 2): for instance, a digraph with unit weights must be Eulerian, i.e., it must have a cycle that visits all the graph edges exactly once.

Regarding design, we identify (cf. Proposition 10 and Corollary 11) a class of smooth consensus functions for which one can synthesize in a systematic way distributed coordination algorithms over weighted directed graphs. The property common to these functions is that the computation of their gradients enjoys some special distributed features. Building on this result, we characterize (cf. Proposition 14) the exponential rate of convergence of a class of distributed algorithms that achieve weighted power mean consensus, originally introduced in [3] for undirected graphs. We also establish the validity of these results for scenarios with switching interconnection topologies (cf. Corollary 12 and Remark 16). The maximum and the minimum functions do not belong to the special class of functions mentioned above. The last contribution of the paper is the introduction of two discontinuous distributed algorithms that achieve max and min consensus in finite time (cf. Proposition 17).

Organization. Section 2 presents some preliminary notions on graph theory, distributed maps, and nonsmooth stability analysis. Section 3 formally introduces the consensus problem of interest. Section 4 identifies, under some conditions on the desired function, necessary and sufficient conditions for any coordination algorithm that asymptotically achieves consensus. Section 5 investigates the design of distributed coordination algorithms, paying special attention to weighted power mean, max, and min consensus. Finally, Section 6 presents our conclusions.

Notation. Let $\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{\geq 0}$, $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote, respectively, the set of integer, positive integer, non-negative integer, real, positive real, and non-negative real numbers. For a set X, $\mathfrak{P}(X)$ denotes the collection of all subsets of X and $\mathrm{Id}_X : X \to X$ denotes the identity map, $\mathrm{Id}_X(x) = x$, for $x \in X$. Let $\mathrm{diag}(\mathbb{R}^n) = \{(p, \dots, p) \in \mathbb{R}^n \mid p \in \mathbb{R}\}$. Let $i_{\mathbb{R}} : \mathbb{R} \to \mathrm{diag}(\mathbb{R}^n)$ denote the natural inclusion, and $\mathbf{1}$ denote the vector $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. Given $\chi : \mathcal{V} \subset \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$, where $\mathcal{V}$ is an open and connected set, let $\mathrm{Im}(\chi) = \{\chi(P) \in \mathbb{R}^{d_2} \mid P \in \mathcal{V}\}$ denote the range of χ. Note that $\mathrm{Im}(i_{\mathbb{R}}) = \mathrm{diag}(\mathbb{R}^n)$. Let $\partial\mathcal{V}$ and $\overline{\mathcal{V}}$ denote the boundary and the closure of $\mathcal{V}$, respectively. For a continuous function χ, its extension $\chi_e : \overline{\mathcal{V}} \subset \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is defined as $\chi_e(P) = \chi(P)$ for $P \in \mathcal{V}$, and $\chi_e(P) = \lim_{m \to +\infty} \chi(P_m)$ for $P \in \partial\mathcal{V}$ and $\mathcal{V} \ni P_m \to P$. For a matrix $A \in \mathbb{R}^{n \times n}$, we denote by $\ker(A) \subset \mathbb{R}^n$ its kernel and by $\mathrm{Sym}(A) = \frac{1}{2}(A + A^T)$ its symmetric part. Note that A and $\mathrm{Sym}(A)$ define the same quadratic form. For a positive semidefinite symmetric matrix A, let $\pi_{\ker(A)} : \mathbb{R}^n \to \ker(A)$ be the orthogonal projection. Let $\lambda_2(A)$ be the smallest non-zero eigenvalue of A, $\lambda_2(A) = \min\{\lambda \mid \lambda > 0,\ \lambda \text{ eigenvalue of } A\}$. For $u \in \mathbb{R}^n$,

$$\lambda_2(A)\,\|u - \pi_{\ker(A)}(u)\|_2^2 \leq u^T A\, u. \tag{1}$$

2 Preliminary developments

This section presents tools from graph theory, distributed maps, and nonsmooth stability analysis.

2.1 Weighted digraphs

A *directed graph* (or *digraph*) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ of order n consists of a *vertex set* $\mathcal{V}$ with n elements, and an *edge set* $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. For simplicity, we take $\mathcal{V} = \{1, \dots, n\}$. A digraph is *undirected* if $(j, i) \in \mathcal{E}$ anytime $(i, j) \in \mathcal{E}$. In a digraph $\mathcal{G}$ with an edge $(i, j) \in \mathcal{E}$, i is called an *in-neighbor* of j, and j is called an *out-neighbor* of i. We let $\mathcal{N}_{\mathrm{in}}(i)$ and $\mathcal{N}_{\mathrm{out}}(i)$ denote the sets of in-neighbors and out-neighbors of i, respectively. The *in-degree* and *out-degree* of i are the cardinality of $\mathcal{N}_{\mathrm{in}}(i)$ and $\mathcal{N}_{\mathrm{out}}(i)$, respectively. A digraph is *topologically balanced* if each vertex has the same in- and out-degrees.

A *weighted digraph* is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $(\mathcal{V}, \mathcal{E})$ is a digraph and where $\mathcal{A}$ is an $n \times n$ *weighted adjacency matrix* with the following properties: for $i, j \in \{1, \dots, n\}$, the entry $a_{ij} > 0$ if (i, j) is an edge of $\mathcal{G}$, and $a_{ij} = 0$ otherwise. In other words, the scalars a_{ij} are a set of weights for the edges of $\mathcal{G}$. A weighted digraph is *undirected* if $a_{ij} = a_{ji}$ for all $i, j \in \{1, \dots, n\}$. When convenient, we write $\mathcal{A}(\mathcal{G})$ to make clear the explicit dependence on the graph. Note that a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ can be naturally thought of as a weighted digraph by defining the weighted adjacency matrix $\mathcal{A}$ with nonnegative entries a_{ij}, $i, j \in \{1, \dots, n\}$ given by $a_{ij} = 1$ if (i, j) is an edge of $\mathcal{G}$, and $a_{ij} = 0$ otherwise. Reciprocally, one can define the *unweighted version* of a weighted digraph $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ by simply considering the digraph $(\mathcal{V}, \mathcal{E})$.

In a weighted digraph, the *weighted out-degree* and the *weighted in-degree* of vertex i are defined by, respectively,

$$d_{\mathrm{out}}(i) = \sum_{j=1}^{n} a_{ij} \quad \text{and} \quad d_{\mathrm{in}}(i) = \sum_{j=1}^{n} a_{ji}.$$

The *weighted out-degree matrix* $D_{\mathrm{out}}(\mathcal{G})$ and the *weighted in-degree matrix* $D_{\mathrm{in}}(\mathcal{G})$ are the diagonal matrices defined by $(D_{\mathrm{out}}(\mathcal{G}))_{ii} = d_{\mathrm{out}}(i)$ and

$(D_{\text{in}}(\mathcal{G}))_{ii} = d_{\text{in}}(i)$, respectively. The digraph $\mathcal{G}$ is *weight-balanced* if $D_{\text{out}}(\mathcal{G}) = D_{\text{in}}(\mathcal{G})$. The *graph Laplacian* of the weighted digraph $\mathcal{G}$ is

$$L(\mathcal{G}) = D_{\text{out}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}).$$

Note that $L(\mathcal{G})\mathbf{1} = 0$, and that $\mathcal{G}$ is undirected iff $L(\mathcal{G})$ is symmetric. For undirected graphs, the Laplacian is a symmetric, positive semidefinite matrix. Weight-balanced digraphs can be characterized in terms of the Laplacian matrix: $\mathcal{G}$ is weight-balanced iff $\mathbf{1}^T L(\mathcal{G}) = 0$ iff $\text{Sym}(L(\mathcal{G}))$ is positive semi-definite.

2.2 Graph connectivity notions

Let us review some basic connectivity notions. A *directed path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices are an edge of the digraph. A *cycle* in a digraph is a non-trivial directed path that starts and ends at the same vertex. An *undirected path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices are connected by an edge of the digraph. A node of a digraph is *globally reachable* if it can be reached from any other node by traversing a directed path. A digraph is *weakly connected* if every pair of nodes are connected by an undirected path. For a weight-balanced digraph $\mathcal{G}$, one can show that $\mathcal{G}$ is weakly connected if and only if $\ker(\text{Sym}(L(\mathcal{G}))) = \text{span}\{\mathbf{1}\}$. A digraph is *strongly connected* if every pair of nodes are connected by a directed path. Note that a digraph is strongly connected if and only if every node is globally reachable. A digraph is *strongly semiconnected* if the existence of a directed path from a vertex i to a vertex j implies the existence of a directed path from j to i. Strongly semiconnected digraphs might not be weakly connected. However, strongly semiconnected digraphs that are also weakly connected must be strongly connected. For undirected graphs, the notions of weakly connected and strongly connected are equivalent, and we will simply refer to connected undirected graphs.

An *Euler tour* of a digraph $\mathcal{G}$ is a cycle that visits all edges of the digraph exactly once. A digraph is *Eulerian* if it has an Euler tour, see Figure 1(a). Clearly, weakly connected Eulerian graphs are strongly connected. The following result characterizes the family of Eulerian graphs, e.g., see [6, Theorem 4.6].

Proposition 1 (Topologically balanced digraph) *Let $\mathcal{G}$ be a weakly connected digraph. Then, $\mathcal{G}$ is topologically balanced iff $\mathcal{G}$ is Eulerian.*

In general, weight-balanced digraphs are not Eulerian, see Figure 1(b) and (c). Instead, weight-balanced digraphs are characterized as follows.

Proposition 2 (Weight-balanced digraph) *Given a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, there exists a weight-balanced digraph $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ iff $\mathcal{G}$ is strongly semiconnected.*

This proposition is a generalization of [13, Theorem 2], which establishes the same result for the case when all

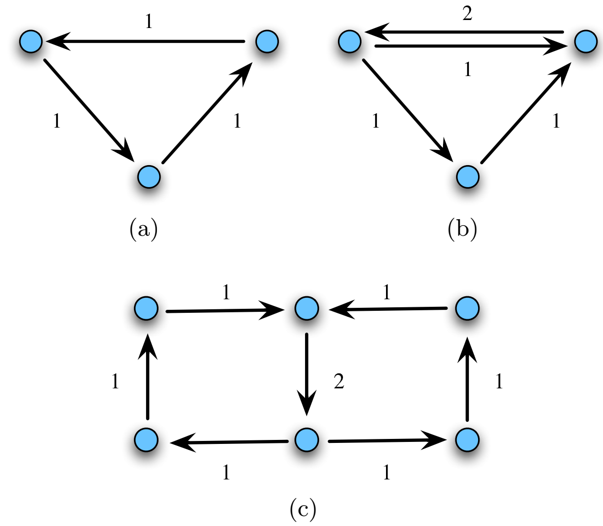


Fig. 1. Plot (a) shows an Eulerian graph. Plots (b) and (c) show weight-balanced digraphs that are not Eulerian.

weights are positive integer numbers, i.e., $\mathcal{A} \in \mathbb{Z}_{\geq 0}^{n \times n}$. The proof of Proposition 2 is given in Appendix 7.

2.3 Disagreement

Given a weighted digraph $\mathcal{G}$ of order n, let us associate a state $p_i \in \mathbb{R}$ to each vertex $i \in \{1, \ldots, n\}$. Two nodes are said to *agree* iff $p_i = p_j$. A meaningful function that quantifies the group disagreement in a network is the *disagreement function* $\Phi_{\mathcal{G}} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$,

$$\Phi_{\mathcal{G}}(P) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(p_j - p_i)^2, \tag{2}$$

with $P = (p_1, \ldots, p_n) \in \mathbb{R}^n$. The disagreement function $\Phi_{\mathcal{G}}$ is smooth. Clearly $\Phi_{\mathcal{G}}(P) = 0$ iff all neighboring nodes in the graph $\mathcal{G}$ agree. If the digraph $\mathcal{G}$ is weakly connected, then all nodes in the graph agree. Note that for weight-balanced digraphs, $\Phi_{\mathcal{G}}(P) = P^T L(\mathcal{G}) P$, for all $P \in \mathbb{R}^n$, and the gradient of $\Phi_{\mathcal{G}}$ is $\text{grad}(\Phi_{\mathcal{G}}) = 2P^T \,\text{Sym}(L(\mathcal{G}))$.

2.4 Distributed maps over digraphs

The notion of spatially-distributed map was introduced in [9] for the class of (undirected) proximity graphs. Here, we introduce the notion of distributed map over a digraph $\mathcal{G}$. Given two sets X, Y, a function $T : X^n \to Y^n$ is *out-distributed over* $\mathcal{G}$ if there exist functions $\tilde{T}_1, \ldots, \tilde{T}_n : X \times \mathfrak{P}(X) \to Y$ such that

$$T_i(x_1, \ldots, x_n) = \tilde{T}_i(x_i, \{x_j \mid j \in \mathcal{N}_{\text{out}}(i)\}),$$

for all $(x_1, \ldots, x_n) \in X^n$ and all $i \in \{1, \ldots, n\}$. Roughly speaking, the ith component of a distributed map over $\mathcal{G}$ can be computed only with information about the state of node i and its out-neighbors in the digraph $\mathcal{G}$. The notion of in-distributed map over $\mathcal{G}$ can be defined analogously. If $\mathcal{G}$ is undirected, then we simply say that T is distributed over $\mathcal{G}$. In this paper, we have chosen to present our results for out-distributed maps, but analogous results can be also presented for in-distributed maps.

Remark 3 (Sensing versus communication capabilities) *It is natural to associate the notion of out-distributed maps with networks whose agents have sensing capabilities, and the notion of in-distributed maps with networks whose agents have communication capabilities. In the sensing scenario, the existence of an edge $(i,j) \in \mathcal{E}$ has the interpretation that agent i is able to* sense *the state of agent j. In the communication scenario, the existence of an edge $(i,j) \in \mathcal{E}$ has the interpretation that agent i is able to* send *information to agent j.* •

2.5 Nonsmooth stability analysis

This section introduces differential equations with discontinuous right-hand sides and presents various nonsmooth tools to analyze their stability properties. The presentation follows the exposition in [2]. For differential equations with discontinuous right-hand sides we understand the solutions in terms of differential inclusions following [10]. Let $F : \mathbb{R}^d \to \mathfrak{P}(\mathbb{R}^d)$, $d \in \mathbb{Z}_{>0}$, be a set-valued map. Consider the differential inclusion

$$\dot{x} \in F(x). \tag{3}$$

A solution to this equation on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as an absolutely continuous function $x : [t_0, t_1] \to \mathbb{R}^d$ such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in [t_0, t_1]$. Now, consider the differential equation

$$\dot{x}(t) = X(x(t)), \tag{4}$$

where $X : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and essentially locally bounded [10]. For each $x \in \mathbb{R}^d$, consider the set

$$K[X](x) = \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \mathrm{co}\{X(B(x,\delta) \setminus S)\}, \tag{5}$$

where μ denotes the Lebesgue measure in $\mathbb{R}^d$, $\mathrm{co}(A)$ denotes the convex closure of the set A, and $B(x,\delta)$ is the ball of center x and radius δ in $\mathbb{R}^d$. A Filippov solution of (4) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion

$$\dot{x} \in K[X](x). \tag{6}$$

A set M is *weakly invariant* (respectively *strongly invariant*) for (4) if for each $x_0 \in M$, M contains a maximal solution (respectively all maximal solutions) of (4).

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. From Rademacher's Theorem [7], locally Lipschitz functions are differentiable a.e. Let $\Omega_f \subset \mathbb{R}^d$ denote the set of points where f fails to be differentiable. The *generalized gradient of f at $x \in \mathbb{R}^d$* (cf. [7]) is defined by

$$\partial f(x) = \mathrm{co}\left\{ \lim_{i \to +\infty} df(x_i) \mid x_i \to x\,,\ x_i \notin S \cup \Omega_f \right\},$$

where S can be any set of zero measure. Note that if f is continuously differentiable, then $\partial f(x) = \{df(x)\}$.

Given a locally Lipschitz function f, the *set-valued Lie derivative of f with respect to X* at x is

$$\widetilde{\mathcal{L}}_X f(x) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ such that } \zeta \cdot v = a\,,\ \forall \zeta \in \partial f(x)\}.$$

The set-valued Lie derivative allows us to study the evolution of a function along the Filippov solutions. The reader is referred to [7] for the notion of regularity.

Theorem 4 (Evolution along Filippov solutions) *Let $x : [t_0, t_1] \to \mathbb{R}^d$ be a Filippov solution of (4). Let f be a locally Lipschitz and regular function. Then $\frac{d}{dt}\big(f(x(t))\big)$ exists a.e. and $\frac{d}{dt}\left(f(x(t))\right) \in \widetilde{\mathcal{L}}_X f(x(t))$ a.e.*

The following result is a generalization of LaSalle principle for discontinuous differential equations with nonsmooth Lyapunov functions.

Theorem 5 (LaSalle Invariance Principle) *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz and regular function. Let $x_0 \in S \subset \mathbb{R}^d$, with S compact and strongly invariant for (4). Assume that either $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ or $\widetilde{\mathcal{L}}_X f(x) = \emptyset$ for all $x \in S$. Let $Z_{X,f} = \{x \in \mathbb{R}^d \mid 0 \in \widetilde{\mathcal{L}}_X f(x)\}$. Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^d$ of (4) starting from x_0 converges to the largest weakly invariant set M contained in $\overline{Z}_{X,f} \cap S$.*

3 Problem statement

Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Consider a network of agents whose individual dynamics is given by

$$\dot{p}_i = u_i, \quad i \in \{1, \ldots, n\}. \tag{7}$$

We say that a coordination algorithm $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ *asymptotically achieves χ-consensus* if u is essentially locally bounded, and for any $(p_1(0), \ldots, p_n(0)) \in \mathcal{V}$, any solution of the dynamics (7) starting at $(p_1(0), \ldots, p_n(0))$ stays in $\mathcal{V}$ and verifies, for all $i \in \{1, \ldots, n\}$,

$$p_i(t) \longrightarrow \chi(p_1(0), \ldots, p_n(0)), \quad t \to +\infty.$$

Because the trajectories stay in $\mathcal{V}$, for consistency, $\mathrm{Im}(i_{\mathbb{R}} \circ \chi) \subset \mathcal{V}$ must hold. Note that we do not require u to be continuous. If u is discontinuous, then solutions are understood in the Filippov sense (cf. Section 2.5). We usually refer to χ as the *consensus function.*

Now, assume the network interconnection topology is described by a weighted digraph $\mathcal{G}$. Our objective is to design coordination algorithms that verify,

(P.1): u is out-distributed over $\mathcal{G}$, and
(P.2): u asymptotically achieves χ-consensus.

Property **(P.1)** guarantees that the control law u is implementable over the network (7), and property **(P.2)** guarantees that individual agents asymptotically agree on the value of χ.

4 Necessary and sufficient conditions for χ-consensus

In this section, we obtain, under some conditions on the desired function, necessary and sufficient conditions for any coordination algorithm that asymptotically achieve consensus (i.e., satisfies property **(P.2)** in Section 3). We undertake this study as a necessary step previous to the synthesis of coordination algorithms with properties **(P.1)** and **(P.2)**. Section 5 builds on this discussion to design distributed algorithms for χ-consensus.

We start by showing that the function χ must be constant along the trajectories of a coordination algorithm that asymptotically achieves χ-consensus. The statement here is a generalization to continuous functions of a result in [3].

Lemma 6 (χ is preserved) *Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Assume $\operatorname{Im}(i_{\mathbb{R}} \circ \chi) \subset \mathcal{V}$ and let $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a coordination algorithm asymptotically achieving χ-consensus. Then χ is constant along the trajectories of (7).*

PROOF. Let $P_0 \in \mathcal{V}$ and consider a trajectory $\mathbb{R}_{\geq 0} \ni t \mapsto P(t)$ of (7) that starts at P_0. Since u asymptotically achieves χ-consensus, then $P(t) \to (\chi(P_0), \dots, \chi(P_0))$. Let $t^* \in \mathbb{R}_{>0}$. The curve $\mathbb{R}_{\geq 0} \ni t \mapsto P(t + t^*)$ starts at $P(t^*) \in \mathcal{V}$ and it is a trajectory of (7). Since u asymptotically achieves χ-consensus, then $P(t + t^*) \to (\chi(P(t^*)), \dots, \chi(P(t^*)))$. Since $P(t) \to (\chi(P_0), \dots, \chi(P_0))$, we conclude $\chi(P(t^*)) = \chi(P_0)$, i.e., χ is constant along $\mathbb{R}_{\geq 0} \ni t \mapsto P(t)$. •

The following result restricts the class of functions χ for which the consensus problem can be solved. The statement here is a generalization to functions with arbitrary domains of a result in [3].

Proposition 7 (χ is the identity on the diagonal) *Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Assume $\operatorname{Im}(i_{\mathbb{R}} \circ \chi) \subset \mathcal{V}$ and let $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a coordination algorithm asymptotically achieving χ-consensus. Then $\chi \circ i_{\mathbb{R}|\operatorname{Im}(\chi)} = \operatorname{Id}_{\mathbb{R}|\operatorname{Im}(\chi)}$.*

PROOF. We reason by contradiction. Assume there exists $p \in \operatorname{Im}(\chi)$ such that $\chi(p, \dots, p) \neq p$ (note that $(p, \dots, p) \in \mathcal{V}$ because $\operatorname{Im}(i_{\mathbb{R}} \circ \chi) \subset \mathcal{V}$). Let $\epsilon = |\chi(p, \dots, p) - p| > 0$. By continuity of χ, there exists $\delta > 0$ such that $\|P - (p, \dots, p)\| < \delta$ implies $|\chi(P) - \chi(p, \dots, p)| < \epsilon$. On the other hand, since $p \in \operatorname{Im}(\chi)$, there exists $P^* \in \mathcal{V} \setminus \{(p, \dots, p)\}$ such that $\chi(P^*) = p$. By hypothesis, u asymptotically achieves χ-consensus. In particular, this implies that the trajectory $\mathbb{R}_{\geq 0} \ni t \mapsto P(t)$ of (7) starting from $P(0) = P^*$ asymptotically converges to $(\chi(P^*), \dots, \chi(P^*)) = (p, \dots, p)$. For $\delta > 0$ above, there exists $T > 0$ such that $\|P(t) - (p, \dots, p)\| < \delta$ for $t \geq T$, which implies that $|\chi(P(t)) - \chi(p, \dots, p)| < \epsilon$. By Lemma 6, $\chi(P(t)) = \chi(P^*) = p$, and hence $|p - \chi(p, \dots, p)| < \epsilon$, contradicting $\epsilon = |\chi(p, \dots, p) - p|$. •

The following novel result characterizes when χ-consensus can be asymptotically achieved by showing that asymptotic convergence towards the set $\operatorname{diag}(\mathbb{R}^n)$ together with the necessary conditions in Lemma 6 and Proposition 7 are indeed sufficient for consensus.

Theorem 8 (Necessary and sufficient conditions for χ-consensus. I) *Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Assume that $i_{\mathbb{R}}^{-1}(\mathcal{V}) = \operatorname{Im}(\chi)$ and $\operatorname{Im}(\chi_{e|\operatorname{diag}(\mathbb{R}^n) \cap \partial \mathcal{V}}) \cap \operatorname{Im}(\chi) = \emptyset$. Let $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ be essentially locally bounded such that the trajectories of (7) are bounded and $\mathcal{V}$ is strongly invariant. Then, u guarantees that χ-consensus is asymptotically reached iff the following holds*

(i) the trajectories of (7) converge to $\operatorname{diag}(\mathbb{R}^n)$,
(ii) χ is constant along the trajectories of (7), and
(iii) $\chi \circ i_{\mathbb{R}|\operatorname{Im}(\chi)} = \operatorname{Id}_{\mathbb{R}|\operatorname{Im}(\chi)}$.

PROOF. If u guarantees that χ-consensus is asymptotically reached, then (i) holds by definition, (ii) holds by Lemma 6 and (iii) holds by Proposition 7. Now, assume that (i)-(iii) hold, and let us prove that u guarantees that χ-consensus is asymptotically reached. Let $P_0 \in \mathcal{V}$ and consider a trajectory $\mathbb{R}_{\geq 0} \ni t \mapsto P(t)$ of (7) starting at $P(0) = P_0$. Since the trajectory is bounded, its ω-limit set, denoted $\Omega(\{P(t)\}_{t \in \mathbb{R}_{\geq 0}}) \subset \mathbb{R}^n$, is non-empty, compact and invariant. By (i), $\Omega(\{P(t)\}_{t \in \mathbb{R}_{\geq 0}}) \subset \operatorname{diag}(\mathbb{R}^n)$. For each $(p, \dots, p) \in \Omega(\{P(t)\}_{t \in \mathbb{R}_{\geq 0}})$, there exists a convergent subsequence (that, for ease of notation, we also denote by $\{P(t)\}_{t \in \mathbb{R}_{\geq 0}}$) such that $P(t) \to (p, \dots, p)$. Note that $(p, \dots, p) \in \overline{\mathcal{V}}$. Extending χ by continuity if necessary, we have $\chi(p, \dots, p) = \lim_{t \to +\infty} \chi(P(t))$. Now (ii) implies $\chi(p, \dots, p) = \chi(P_0)$. This, together with $\operatorname{Im}(\chi_{e|\operatorname{diag}(\mathbb{R}^n) \cap \partial V}) \cap \operatorname{Im}(\chi) = \emptyset$, implies $p \in i_{\mathbb{R}}^{-1}(\mathcal{V}) = \operatorname{Im}(\chi)$. By (iii), we deduce $p = \chi(p, \dots, p) = \chi(P_0)$. Therefore, $\Omega(\{P(t)\}_{t \in \mathbb{R}_{\geq 0}}) = \{\chi(P_0)\mathbf{1}\}$, or, equivalently, the trajectory $\{P(t)\}_{t \in \mathbb{R}_{\geq 0}}$ converges to $\chi(P_0)\mathbf{1}$. •

The previous result takes a much simpler form when χ is defined over the whole space $\mathbb{R}^n$ and is surjective.

Corollary 9 (Necessary and sufficient conditions for χ-consensus. II) *Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be continuous and surjective. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be essentially locally bounded such that the trajectories of (7) are bounded. Then, u guarantees that χ-consensus is asymptotically reached iff the following holds*

(i) the trajectories of (7) converge to $\operatorname{diag}(\mathbb{R}^n)$,
(ii) χ is constant along the trajectories of (7), and
(iii) $\chi \circ i_{\mathbb{R}} = \operatorname{Id}_{\mathbb{R}}$.

Theorem 8 and Corollary 9 are important both from an analysis and a design viewpoint. From an analysis perspective, these results characterize when the network asymptotically achieves χ-consensus with a coordination algorithm of the form $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$. From a design perspective, these results establish a systematic methodology to synthesize solutions to the χ-consensus problem.

5 Distributed algorithms for χ-consensus

Here, we identify particular conditions on the function χ under which distributed coordination algorithms that asymptotically achieve consensus can be designed.

Proposition 10 (Design of distributed algorithms. I) *Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable such that $\chi \circ i_{\mathbb{R}|\operatorname{Im}(\chi)} = \operatorname{Id}_{\mathbb{R}|\operatorname{Im}(\chi)}$, $i_{\mathbb{R}}^{-1}(\mathcal{V}) = \operatorname{Im}(\chi)$ and $\operatorname{Im}(\chi_{e|\operatorname{diag}(\mathbb{R}^n)\cap\partial V}) \cap \operatorname{Im}(\chi) = \emptyset$. Let $\mathcal{G}$ be a weakly connected, weight-balanced digraph, and let $\operatorname{grad}\chi$ be out-distributed over $\mathcal{G}$, with all partial derivatives $\{\frac{\partial\chi}{\partial p_1}, \dots, \frac{\partial\chi}{\partial p_n}\}$ having the same constant sign on $\mathcal{V}$. Assume the coordination algorithm $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$,*

$$u_i = \frac{1}{\left|\frac{\partial\chi}{\partial p_i}\right|} \sum_{j=1}^n a_{ij}(p_j - p_i), \quad i \in \{1, \dots, n\} \qquad (8)$$

is essentially locally bounded and such that $\mathcal{V}$ is strongly invariant. Then, u is out-distributed over $\mathcal{G}$ and asymptotically achieves χ-consensus.

PROOF. Clearly, the map $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ is out-distributed over $\mathcal{G}$ since $a_{ij} \neq 0$ iff $j \in \mathcal{N}_{\text{out}}(i)$. We prove that u asymptotically achieves χ-consensus using Theorem 8. Let us first establish that each trajectory of (8) belongs to some bounded and invariant set. Consider the $\max : \mathbb{R}^n \to \mathbb{R}$ function, $\max(P) = \max_{i\in\{1,\dots,n\}}\{p_i\}$, and let us compute the set-valued Lie derivative $\widetilde{\mathcal{L}}_u(\max)$. If $a \in \widetilde{\mathcal{L}}_u(\max)$, then $a = u(P) \cdot \zeta$, for all $\zeta \in \partial\max$. The generalized gradient of max is

$$\partial \max(P) = \operatorname{co}\{e_i \mid i \in \{1, \dots, n\} \text{ with } p_i = \max(P)\}.$$

By hypothesis, $\mathcal{G}$ is weakly connected. From Proposition 2, $\mathcal{G}$ is strongly semiconnected. These two properties imply that $\mathcal{G}$ is actually strongly connected. If $P \in \operatorname{diag}(\mathbb{R}^n)$, then $u(P) = 0$, and therefore $a = 0$. If $P \notin \operatorname{diag}(\mathbb{R}^n)$, then using the fact that $\mathcal{G}$ is strongly connected, there exists $k \in \{1, \dots, n\}$ with $p_k = \max_{j\in\{1,\dots,n\}}\{p_j\}$ such that $\sum_{j=1}^n a_{kj}(p_j - p_k) < 0$. Consequently, $u_k(P) < 0$, and $a = u(P) \cdot e_k < 0$. Therefore, we conclude that either $\widetilde{\mathcal{L}}_u(\max) = \emptyset$ or $\max\widetilde{\mathcal{L}}_u(\max) \leq 0$. Theorem 4 implies that $p_i(t) \leq \max\{p_1(0), \dots, p_n(0)\}$. A similar argument with the min function shows that $\min\{p_1(0), \dots, p_n(0)\} \leq p_i(t)$. Hence, any trajectory of (8) belongs to a bounded and invariant set. Moreover, this argument guarantees that $Z_{u,\max} = \operatorname{diag}(\mathbb{R}^n)$. Given that any trajectory of (8) belongs to some bounded and invariant set, the application of Theorem 5 implies that all trajectories converge to $\operatorname{diag}(\mathbb{R}^n)$, i.e., (i) in Theorem 8 is satisfied. (ii) in Theorem 8 is easily verified since

$$\begin{aligned}\mathcal{L}_u\chi &= \sum_{i=1}^n \frac{\partial\chi}{\partial p_i} u_i = \sum_{i=1}^n \operatorname{sgn}\left(\frac{\partial\chi}{\partial p_i}\right) \cdot \sum_{j=1}^n a_{ij}(p_j - p_i) \\ &= \pm \sum_{i=1}^n\sum_{j=1}^n a_{ij}(p_j - p_i) = \mp\mathbf{1} \cdot L(\mathcal{G}) = 0,\end{aligned}$$

where $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$ and $\operatorname{sgn}(0) = 0$. The last equality follows from $\mathcal{G}$ being weight-balanced. Finally, (iii) in Theorem 8 is verified by hypothesis. •

The next result extends the applicability of Proposition 10 to functions χ whose gradient is not distributed over the interconnection topology, but admit a "distributing factor" that makes it distributed.

Corollary 11 (Design of distributed algorithms. II) *Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable such that $\chi \circ i_{\mathbb{R}|\operatorname{Im}(\chi)} = \operatorname{Id}_{\mathbb{R}|\operatorname{Im}(\chi)}$, $i_{\mathbb{R}}^{-1}(\mathcal{V}) = \operatorname{Im}(\chi)$ and $\operatorname{Im}(\chi_{e|\operatorname{diag}(\mathbb{R}^n)\cap\partial V}) \cap \operatorname{Im}(\chi) = \emptyset$. Let $\mathcal{G}$ be a weakly connected, weight-balanced digraph. Assume there exist $f : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ such that $f \cdot \operatorname{grad}\chi$ is out-distributed over $\mathcal{G}$, with all partial derivatives $\{\frac{\partial\chi}{\partial p_1}, \dots, \frac{\partial\chi}{\partial p_n}\}$ having the same constant sign on $\mathcal{V}$. Assume that the coordination algorithm $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$,*

$$u_i = \frac{1}{\left|f\frac{\partial\chi}{\partial p_i}\right|} \sum_{j=1}^n a_{ij}(p_j - p_i), \quad i \in \{1, \dots, n\} \qquad (9)$$

is essentially locally bounded and such that $\mathcal{V}$ is strongly invariant. Then, u is distributed over $\mathcal{G}$ and asymptotically achieves χ-consensus.

Note that Proposition 10 and Corollary 11 generalize [3, Theorem 2] by broadening the set of graphs and the set of functions for which the consensus problem can be solved.

5.1 *Networks with switching interconnection topologies*

The previous discussion can be extended to the scenario of networks with switching topologies. Note that, by Proposition 2, weakly connected, weight-balanced digraphs are strongly connected. Therefore, the unweighted versions of the (infinite) set of weakly connected, weight-balanced digraphs give rise to the (finite) set of strongly connected digraphs. Let $\Gamma = \{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ be a finite collection of weakly connected, weight-balanced digraphs of orden n. A *switching signal* σ is a map $\sigma : \mathbb{R}_{\geq 0} \to \{1, \dots, m\}$. For each time $t \in \mathbb{R}_{\geq 0}$, the switching signal σ establishes the network graph $\mathcal{G}_{\sigma(t)} \in \Gamma$. Now, consider a network of agents subject to the switching topology defined by σ and executing, at time t, the algorithm $u(\mathcal{G}_{\sigma(t)})$ in (8) corresponding to $\mathcal{G}_{\sigma(t)}$. In other words, consider the switching system

$$\dot{p}_i(t) = u_i(\mathcal{G}_{\sigma(t)}), \quad i \in \{1,\dots,n\}. \tag{10}$$

We call σ *feasible* if there exist solutions of (10) starting from any initial condition. For instance, piecewise constant signals are feasible. Noting that $\max : \mathbb{R}^n \to \mathbb{R}$ is a common Lyapunov function for (10), one can establish the next result.

Corollary 12 (Switching topologies) *Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable such that $\chi \circ i_{\mathbb{R}|\operatorname{Im}(\chi)} = \operatorname{Id}_{\mathbb{R}|\operatorname{Im}(\chi)}$, $i_{\mathbb{R}}^{-1}(\mathcal{V}) = \operatorname{Im}(\chi)$ and $\operatorname{Im}(\chi_{e|\operatorname{diag}(\mathbb{R}^n)\cap\partial V}) \cap \operatorname{Im}(\chi) = \emptyset$. Let $\Gamma = \{\mathcal{G}_1,\dots,\mathcal{G}_m\}$ be a finite collection of weakly connected, weight-balanced digraphs of orden n. Assume that*

(i) $\operatorname{grad}\chi$ *is out-distributed over $\mathcal{G}_k$, for $k \in \{1,\dots,m\}$;*
(ii) all partial derivatives $\{\frac{\partial\chi}{\partial p_1},\dots,\frac{\partial\chi}{\partial p_n}\}$ have the same constant sign on $\mathcal{V}$;
(iii) the coordination algorithm $u(\mathcal{G}_k) : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ defined in (8) *associated with $\mathcal{G}_k$ is essentially locally bounded and such that $\mathcal{V}$ is strongly invariant.*

Then, the switching system (10) *asymptotically achieves χ-consensus for any feasible signal $\sigma : \mathbb{R}_{\geq 0} \to \{1,\dots,m\}$.*

Similar convergence results can also be established for finite collection of weight-balanced digraphs whose union is weakly connected by restricting the set of allowable switching signals, as in [21]. We leave this for the reader.

5.2 Weighted power mean consensus

In this section, we study distributed algorithms that asymptotically achieve weighted power mean consensus. This class of algorithms was originally presented in [3] for undirected graphs. Here, we introduce them for the more general setup of weighted digraphs as a particular application of Corollary 11 to the weighted power mean function. More importantly, we characterize their exponential rate of convergence.

For $w \in \mathbb{R}^n_{>0}$ with $\sum_{i=1}^n w_i = 1$ and $r \in \mathbb{R} \setminus \{0\}$, the *weighted power mean* $\chi_{w,r} : \mathbb{R}^n_{>0} \to \mathbb{R}$ is (cf. [5])

$$\chi_{w,r}(p_1,\dots,p_n) = \Big(\sum_{i=1}^n w_i p_i^r\Big)^{\frac{1}{r}}.$$

For $r \in \{0, \pm\infty\}$, the function $\chi_{w,r}$ is defined by

$$\chi_{w,r}(p_1,\dots,p_n) = \lim_{s\to r} \chi_{w,s}(p_1,\dots,p_n).$$

Alternatively, for $r = 0$, $\chi_{w,0}(p_1,\dots,p_n) = p_1^{w_1}\dots p_n^{w_n}$, for $r = +\infty$, $\chi_{w,+\infty}(p_1,\dots,p_n) = \max\{p_1,\dots,p_n\}$, and for $r = -\infty$, $\chi_{w,-\infty}(p_1,\dots,p_n) = \min\{p_1,\dots,p_n\}$. Table 1 summarizes some distinguished members of this class of functions.

$\chi_{w,-\infty}$	Minimum
$\chi_{w,-1}$	Harmonic Mean
$\chi_{w,0}$	Geometric Mean
$\chi_{w,1}$	Arithmetic Mean
$\chi_{w,2}$	Root-Mean-Square
$\chi_{w,\infty}$	Maximum

Table 1
Examples of weighted power means. This family of functions includes the sample raw statistical moments of any order.

Note that for specific values of the parameter $r \in \mathbb{R}^n \cup \{\pm\infty\}$, the domain of definition of $\chi_{w,r}$ can be larger than $\mathbb{R}^n_{>0}$. For instance, the function $\chi_{w,1}$ is well-defined on $\mathbb{R}^n$. The choice $w_i = \frac{1}{n}$, $i \in \{1,\dots,n\}$, yields the usual power mean function, that we simply denote χ_r.

Before presenting the main result of this section, we need to introduce a powerful pair of inequalities concerning differences of weighted power means. The following beautiful result is a particular case of [12, Corollary 2.2].

Proposition 13 (Differences of power means) *Let $r \in \mathbb{R}$ and $w \in \mathbb{R}^n_{>0}$ with $\sum_{i=1}^n w_i = 1$. For any $P = (p_1,\dots,p_n) \in \mathbb{R}^n_{>0}$, one has*

$$\frac{1}{2}\min\Big\{\frac{1}{\min\{p_i\}^{1-r}}, \frac{1}{\max\{p_i\}^{1-r}}\Big\}\chi_{w,2}^2\big(P - \chi_{w,1}(P)\mathbf{1}\big) \leq \frac{\chi_{w,r+1}^{r+1}(P) - \chi_{w,r}^{r+1}(P)}{r+1} \leq \frac{1}{2}\max\Big\{\frac{1}{\min\{p_i\}^{1-r}}, \frac{1}{\max\{p_i\}^{1-r}}\Big\}\chi_{w,2}^2\big(P - \chi_{w,1}(P)\mathbf{1}\big).$$

Here, when $r = -1$, the inequalities hold by setting $(\chi_{w,0}^0(P) - \chi_{w,-1}^0(P))/0 = \ln(\chi_{w,0}(P)/\chi_{w,-1}(P))$.

We are now ready to present a class of coordination algorithms that asymptotically achieve $\chi_{w,r}$-consensus, $r \in \mathbb{R}$, exponentially fast.

Proposition 14 (Weighted power mean consensus) *Let $r \in \mathbb{R}$ and $w \in \mathbb{R}^n_{>0}$ with $\sum_{i=1}^n w_i = 1$. For any weakly connected, weight-balanced digraph $\mathcal{G}$, consider the coordination algorithm $u_{w,r} : \mathbb{R}^n_{>0} \to \mathbb{R}^n$ with ith component*

$$(u_{w,r})_i(p_1,\dots,p_n) = \frac{1}{w_i} p_i^{1-r} \sum_{j=1}^n a_{ij}(p_j - p_i). \tag{11}$$

Then, $\mathbb{R}^n_{>0}$ is strongly invariant for (11)*, and $u_{w,r}$ is out-distributed over $\mathcal{G}$ and asymptotically achieves weighted power mean-consensus with exponential rate of convergence greater than or equal to*

$$c\,\frac{\lambda_2(\operatorname{Sym}(L(\mathcal{G})))}{n\max\{w_1,\dots,w_n\}},$$

with

$$c = \begin{cases} \max\{p_1(0),\dots,p_n(0)\}^{1-r}, & \text{if } r > 1, \\ 1, & \text{if } r = 1, \\ \min\{p_1(0),\dots,p_n(0)\}^{1-r}, & \text{if } r < 1. \end{cases}$$

PROOF. Our proof strategy to assess the correctness of $u_{w,r}$ is to verify that the conditions in Corollary 11 hold. Clearly, $\chi_{w,r}$ is continuously differentiable in $\mathcal{V} = \mathbb{R}^n_+$, $\chi_{w,r} \circ i_{\mathbb{R}} = \mathrm{Id}_{\mathbb{R}}$ on $\mathbb{R}_{>0}$, and $\mathrm{Im}(\chi_{w,r}) = \mathbb{R}_{>0} = i_{\mathbb{R}}{}^{-1}(\mathbb{R}^n_{>0})$. Since $\mathrm{diag}(\mathbb{R}^n) \cap \partial\mathbb{R}^n_+ = \{0\}$, we also have $\mathrm{Im}((\chi_{w,r})_{e|\,\mathrm{diag}(\mathbb{R}^n)\cap\partial V}) \cap \mathrm{Im}(\chi_{w,r}) = \emptyset$. The partial derivative of $\chi_{w,r}$ with respect to p_i, $i \in \{1,\dots,n\}$, is

$$\frac{\partial \chi_{w,r}}{\partial p_i} = w_i\Big(\frac{p_i}{\chi_{w,r}(p_1,\dots,p_n)}\Big)^{r-1}.$$

Clearly, on $\mathbb{R}^n_+$, all partial derivatives are strictly positive. Selecting $f(p_1,\dots,p_n) = \chi_{w,r}(p_1,\dots,p_n)^{r-1}$ in Corollary 11, we see that $f \cdot \mathrm{grad}\,\chi_{w,r}$ is out-distributed over $\mathcal{G}$. The coordination algorithm defined by (9) corresponds precisely to $u_{w,r}$. Moreover, from the fact that $\chi_{w,r}$ is constant along the trajectories, we deduce that $\mathbb{R}^n_+$ is strongly invariant. The application of Corollary 11 yields the convergence result.

We conclude the proof by assessing the rate of convergence of the trajectories of the system. Let $\mathbb{R}_{\geq 0} \ni t \mapsto P(t) = (p_1(t),\dots,p_n(t)) \in \mathbb{R}^n_{>0}$ be a trajectory starting from $P(0) = P_0 \in \mathbb{R}^n_{>0}$. Consider the function $V : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$V(t) = \frac{\chi^{r+1}_{w,r+1}(P(t)) - \chi^{r+1}_{w,r}(P(t))}{r+1}.$$

For $r = -1$, we define V instead by

$$V(t) = \ln\Big(\frac{\chi_{w,0}(P(t))}{\chi_{w,-1}(P(t))}\Big).$$

Two things are worth noticing concerning this function: V takes non-negative values, and V vanishes iff $P(t) \in \mathrm{diag}(\mathbb{R}^n)$. Both facts are a consequence of the power means inequality [5], that asserts that $\chi_{w,r}(P) \geq \chi_{w,s}(P)$ for $r > s$ and any $P \in \mathbb{R}^n_{>0}$, with the equality holding iff $P \in \mathrm{diag}(\mathbb{R}^n)$.

Let us examine the evolution of V. Note that the coordination algorithm (11) preserves the function $\chi_{w,r}$ by design. Therefore, we have

$$\begin{aligned}\frac{dV}{dt}(t) &= -\sum_{i=1}^n w_i p_i^r(t) \frac{1}{w_i} p_i^{1-r}(t)(L(\mathcal{G})P(t))_i \\ &= -P(t)^T L(\mathcal{G}) P(t) \\ &\leq -\lambda_2(\mathrm{Sym}(L(\mathcal{G})))\,\|P(t) - \chi_1(P(t))\mathbf{1}\|_2^2 \\ &\leq -\frac{\lambda_2(\mathrm{Sym}(L(\mathcal{G})))}{n\max\{w_i\}}\chi^2_{w,2}\big(P(t) - \chi_{w,1}(P(t))\mathbf{1}\big).\end{aligned}$$

In the first inequality, we have used (1) for $\mathrm{Sym}(L(\mathcal{G}))$ (since $\mathcal{G}$ is weakly connected and weight-balanced, then $\ker(\mathrm{Sym}(L(\mathcal{G}))) = \mathrm{span}\{\mathbf{1}\}$). In the second inequality, we have used the fact that, for any $P \in \mathbb{R}^n_{>0}$, $n\|P - \chi_1(P)\|_2^2 \geq \|P - \chi_{w,1}(P)\|_2^2$ and

$$\begin{aligned}\min\{w_i\}\|P - \chi_{w,1}(P)\mathbf{1}\|_2^2 &\leq \chi^2_{w,2}\big(P - \chi_{w,1}(P)\mathbf{1}\big) \\ &\leq \max\{w_i\}\|P - \chi_{w,1}(P)\mathbf{1}\|_2^2. \quad (12)\end{aligned}$$

Now, we are ready to use the second inequality in Proposition 13 to deduce that

$$\frac{dV}{dt}(t) \leq -\frac{\lambda_2(\mathrm{Sym}(L(\mathcal{G})))}{n\max\{w_i\}} \cdot 2\min\big\{\min\{p_i(t)\}^{1-r}, \max\{p_i(t)\}^{1-r}\big\}V(t).$$

Using the fact that $\min\{p_i(0)\} \leq \min\{p_i(t)\} \leq \max\{p_i(t)\} \leq \max\{p_i(0)\}$ for all $t \in \mathbb{R}_{\geq 0}$, we conclude

$$\frac{dV}{dt}(t) \leq -2\,c\,\frac{\lambda_2(\mathrm{Sym}(L(\mathcal{G})))}{n\max\{w_i\}}V(t),$$

where c is as defined in the statement of the proposition. Therefore, we have

$$V(t) \leq V(0)\exp\Big(-2\,c\,\frac{\lambda_2(\mathrm{Sym}(L(\mathcal{G})))}{n\max\{w_i\}}t\Big).$$

Using (12) and the first inequality in Proposition 13, we deduce

$$\begin{aligned}&\|P(t) - \chi_{w,1}(P(t))\mathbf{1}\|_2^2 \leq \\ &\frac{2}{\min\{w_i\}}\max\big\{\min\{p_i(t)\}^{1-r}, \max\{p_i(t)\}^{1-r}\big\}V(t) \leq \\ &\frac{2d}{\min\{w_i\}}V(0)\exp\Big(-2\,c\,\frac{\lambda_2(\mathrm{Sym}(L(\mathcal{G})))}{n\max\{w_i\}}t\Big) \leq \frac{\max\{w_i\}}{\min\{w_i\}} \cdot \\ &\quad\frac{d}{c}\|P(0) - \chi_{w,1}(P(0))\mathbf{1}\|_2^2\exp\Big(-2\,c\,\frac{\lambda_2(\mathrm{Sym}(L(\mathcal{G})))}{n\max\{w_i\}}t\Big),\end{aligned}$$

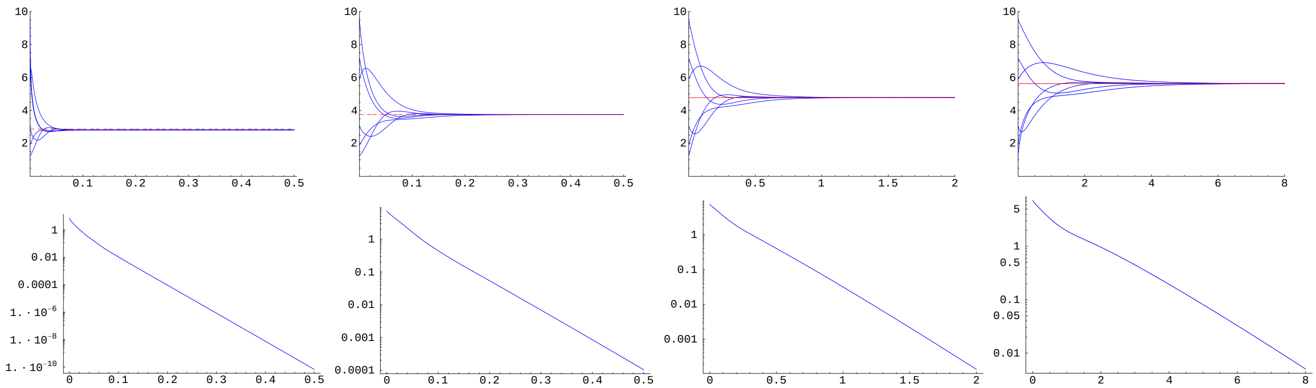
where in the second inequality, we have introduced

$$d = \begin{cases} \min\{p_1(0),\dots,p_n(0)\}^{1-r}, & \text{if } r > 1, \\ 1, & \text{if } r = 1, \\ \max\{p_1(0),\dots,p_n(0)\}^{1-r}, & \text{if } r < 1. \end{cases}$$

Finally,

$$\|P(t) - \chi_{w,1}(P(t))\mathbf{1}\|_2 \leq \sqrt{\frac{d}{c}\frac{\max\{w_i\}}{\min\{w_i\}}} \cdot \|P(0) - \chi_{w,1}(P(0))\mathbf{1}\|_2\exp\Big(-c\,\frac{\lambda_2(\mathrm{Sym}(L(\mathcal{G})))}{n\max\{w_i\}}t\Big),$$

as claimed. •



(a) Harmonic mean (b) Geometric mean (c) Arithmetic mean (d) Two-norm mean

Fig. 2. A network executing the distributed algorithm (11) asymptotically achieves power mean-consensus (top) with exponential rate of convergence (bottom). In these executions, the network topology is as in Figure 1(c), and the agents' initial state is chosen randomly.

Figure 2 illustrates the evolution of the distributed coordination algorithm (11) for various power mean functions.

Remark 15 (Input-to-state stability) *Note that Proposition 14 implies that the coordination algorithm* (11) *enjoys some strong robustness properties: for instance, in scenarios where the network is subject to external disturbances affecting the execution of* (11)*, the exponential rate of convergence implies that the system is input-to-state stable, see e.g., [16, Lemma 4.6].* •

Remark 16 (Networks with switching interconnection topologies revisited) *Let* Γ *be a finite collection of weakly connected, weight-balanced digraphs. Then, building on the proof of Proposition 14, one can show that the switching system consisting of coordination algorithms* (11) *associated with digraphs in* Γ *asymptotically achieves weighted power mean consensus with exponential rate of convergence greater than or equal to*

$$c\,\frac{\min_{\mathcal{G}\in\Gamma}\{\mathrm{Sym}(\lambda_2(L(\mathcal{G})))\}}{n\max\{w_1,\dots,w_n\}}. \quad \bullet$$

5.3 Max and min consensus

Here, we describe two distributed coordination algorithms for max and min consensus. Since neither the maximum nor the minimum are differentiable functions, we cannot rely on Proposition 10 or Corollary 11. Instead, we use the characterization obtained in Section 4. Consider the dynamical systems

$$\dot{p}_i = \mathrm{sgn}_+\Big(\sum_{j=1}^{n} a_{ij}(p_j - p_i)\Big), \tag{13a}$$

$$\dot{p}_i = \mathrm{sgn}_-\Big(\sum_{j=1}^{n} a_{ij}(p_j - p_i)\Big), \tag{13b}$$

where $\mathrm{sgn}_+, \mathrm{sgn}_- : \mathbb{R} \to \mathbb{R}$ are defined by $\mathrm{sgn}_+(x) = 0$ if $x \leq 0$ and $\mathrm{sgn}_+(x) = 1$ if $x > 0$; and $\mathrm{sgn}_-(x) = 0$ if $x \geq 0$ and $\mathrm{sgn}_-(x) = -1$ if $x < 0$.

For ease of notation, we will refer to these flows by X_{sgn_+} and X_{sgn_-}, respectively. Note that both right-hand sides are discontinuous. We understand their solution in the Filippov sense [10]. The following result characterizes the asymptotic convergence properties of these systems.

Proposition 17 (Max and min consensus) *Let* $\mathcal{G}$ *be a strongly connected weighted digraph. Then, the coordination algorithm* (13a) *(respectively, the coordination algorithm* (13b)*) is out-distributed over* $\mathcal{G}$ *and asymptotically achieves max consensus (respectively, min consensus) in finite time.*

PROOF. Our proof strategy is to verify that the conditions in Corollary 9 hold. We prove it for the max function and the flow (13a), and leave to the reader the analogous proof for the min function and the flow (13b). Clearly, $\max : \mathbb{R}^n \to \mathbb{R}$, $\max(P) = \max_{i\in\{1,\dots,n\}}\{p_i\}$, is continuous and surjective. Moreover, $\max(p,\dots,p) = p$, so condition (iii) in Corollary 9 is satisfied.

Next, we show that max is preserved by the flow (13a). The set-valued map associated to (13a) is

$$K[X_{\mathrm{sgn}_+}](P) = \{v \in \mathbb{R}^n \mid v_i \in [0,1] \text{ if} \sum_{j=1}^{n} a_{ij}(p_j - p_i) = 0,$$
$$v_i = \mathrm{sgn}_+\Big(\sum_{j=1}^{n} a_{ij}(p_j - p_i)\Big), \text{ otherwise}\}.$$

Let $a \in \tilde{\mathcal{L}}_{X_{\mathrm{sgn}_+}} \max(P)$. By definition, there exists $v \in K[X_{\mathrm{sgn}_+}](P)$ with $a = v \cdot \zeta$, for all $\zeta \in \partial\max(P)$. If $P \in \mathrm{diag}(\mathbb{R}^n)$, then $\partial\max(P) = \mathbb{R}^n$, and, necessarily $v = (0,\dots,0)$. Therefore, $a = 0$. If $P \notin \mathrm{diag}(\mathbb{R}^n)$, then using the fact that $\mathcal{G}$ is strongly connected, there exists

$k \in \{1, \dots, n\}$ with $p_k = \max_{i \in \{1,\dots,n\}}\{p_i\}$ such that

$$\sum_{i=1}^{n} a_{ki}(p_i - p_k) < 0.$$

Therefore, $v_k = 0$, and $a = v \cdot e_k = 0$. Note that 0 always belongs to $\tilde{\mathcal{L}}_{X_{\mathrm{sgn}_+}} \max(P)$. We conclude $\tilde{\mathcal{L}}_{X_{\mathrm{sgn}_+}} \max(P) = \{0\}$, and therefore, by Theorem 4, max is constant along the trajectories of (13a), i.e., condition (ii) in Corollary 9 is satisfied.

Let us see that the trajectories of (13a) converge to $\mathrm{diag}(\mathbb{R}^n)$. Consider as candidate Lyapunov function $V = -\min$. Reasoning similarly as above, one can show

$$\widetilde{\mathcal{L}}_{X_{\mathrm{sgn}_+}}(-\min)(P) = \begin{cases} \{0\}, & P \in \mathrm{diag}(\mathbb{R}^n), \\ \{-1\}, & P \notin \mathrm{diag}(\mathbb{R}^n). \end{cases}$$

Invoking Theorem 4, we deduce that $\min P(0) \le p_i(t)$ for all $i \in \{1, \dots, n\}$. Since the max function is conserved along the trajectories, we deduce

$$\min P(0) \le p_i(t) \le \max P(0)\,, \quad i \in \{1, \dots, n\},$$

and therefore, the trajectories of (13a) are bounded. Note that $Z_{X_{\mathrm{sgn}_+}, -\min} = \mathrm{diag}(\mathbb{R}^n)$. Theorem 5 now yields that all system trajectories converge to $\mathrm{diag}(\mathbb{R}^n)$, which establishes condition (i) in Corollary 9. The application of [8, Proposition 4] with $\epsilon = 1$ implies that convergence is attained in $\max(P_0) - \min(P_0)$ time units.•

6 Conclusions

We have presented necessary and sufficient conditions for any coordination algorithm that asymptotically achieves consensus upon the value of a general continuous function. Building on this characterization, and considering coordination algorithms over weighted digraphs, we have identified particular conditions on the consensus function under which distributed algorithms can be automatically designed, characterized the exponential convergence properties of a class of distributed coordination algorithms that achieve weighted power mean consensus, and introduced distributed coordination algorithms that achieve max and min consensus in finite time. We have also established the validity of the results for networks with dynamically changing interconnection topologies

Future work will investigate similar results for discrete-time coordination algorithms, characterize the class of functions whose gradient is out-distributed, examine the connection of the results with network games, analyze the robustness of the proposed algorithms against noise and time delays, and explore the application of the results of the paper to the synthesis of cooperative strategies for distributed estimation and fusion problems.

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7 Appendix: proof of Proposition 2

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a digraph of order n. The implication from right to left is proven in [13, Theorem 2]. If $\mathcal{G}$ is strongly semiconnected, then there exists an integer-valued $\mathcal{A} \in \mathbb{Z}_{\geq 0}^{n\times n}$ such that $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ is weight-balanced.

Let us prove the implication from left to right. Assume there exists a weight-balanced digraph $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ and let us show that $\mathcal{G}$ must be strongly semiconnected. By definition, $\mathcal{A} \in \mathbb{R}_{\geq 0}^{n\times n}$. Let m be the number of edges of $\mathcal{G}$, and consider the incidence matrix $\mathcal{B} = (b_{ij}) \in \mathbb{R}^{n\times m}$ with entries $b_{ij} = +1$, if $e_j = (i, *)$; $b_{ij} = -1$ if $e_j = (*, i)$; and $b_{ij} = 0$, otherwise.

The fact that $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ is weight-balanced can alternatively be expressed by saying that the linear system

$$\mathcal{B}w = 0 \qquad (14)$$

admits a solution of the form $w^* \in \mathbb{R}_{>0}^m$. Our proof strategy is to show that if this is true, then the linear system (14) must also admit a solution of the form $w \in \mathbb{Z}_{>0}^m$. By [13, Theorem 2], this would imply that $\mathcal{G}$ is strongly semiconnected. After performing Gaussian elimination (see e.g., [17]) in (14), let ℓ and $m-\ell$ be the number of leading and free variables, respectively. Without loss of generality, assume that $w_1, \dots, w_\ell$ are the leading variables, and $w_{\ell+1}, \dots, w_m$ are the free variables. Then,

$$w_k = \sum_{s=\ell+1}^{m} c_{ks} w_s, \quad k \in \{1, \dots, \ell\},$$

with $c_{ks} \in \mathbb{Z}$. Take any constant $\alpha \in \mathbb{R}_{>0}$ verifying that

$$\alpha > \frac{\max_{k\in\{1,\dots,\ell\}} \sum_{s=\ell+1}^{m} |c_{ks}|}{\min_{k\in\{1,\dots,\ell\}} w_k^*} > 0,$$

and consider the solution $w' \in \mathbb{Z}^m$ determined by the free variables $\lfloor \alpha w_{\ell+1}^* \rfloor, \dots, \lfloor \alpha w_m^* \rfloor \in \mathbb{Z}_{>0}$, i.e.,

$$\begin{aligned} w'_k &= \sum_{s=\ell+1}^{m} c_{ks} \lfloor \alpha w_s^* \rfloor, \quad && k \in \{1, \dots, \ell\}, \\ w'_s &= \lfloor \alpha w_s^* \rfloor, && s \in \{\ell+1, \dots, m\}. \end{aligned}$$

Note that, for all $k \in \{1, \dots, \ell\}$, one has

$$w'_k = \alpha w_k^* - \sum_{s=\ell+1}^{m} c_{ks}\{\alpha w_s^*\} > \\ \alpha \min_{k\in\{1,\dots,\ell\}} w_k^* - \max_{k\in\{1,\dots,\ell\}} \sum_{s=\ell+1}^{m} |c_{ks}| > 0,$$

and therefore, there exists $w' \in \mathbb{Z}_{>0}^m$ solving (14), which concludes the proof. •