Analysis and design of distributed algorithms for χ-consensus

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***Abstract*— This paper presents analysis and design results for distributed consensus algorithms in multi-agent networks. We consider arbitrary consensus functions of the initial state of the network agents. Under mild smoothness assumptions, we obtain necessary and sufficient conditions characterizing any algorithm that asymptotically achieves consensus. This characterization is the building block to obtain various design results. We first identify a class of smooth functions for which one can synthesize in a systematic way distributed algorithms that achieve consensus. We apply this result to the family of weighted power mean functions, and characterize the exponential convergence properties of the resulting algorithms. We conclude with two distributed algorithms that achieve, respectively, max and min consensus in finite time.**

I. Introduction

Arguably, the ability to reach consensus, or agreement, upon some (a priori unknown) quantity is critical for any multi-agent system. Network coordination problems involving self-organization, formation pattern, distributed estimation or parallel processing, to name a few, require individual agents to agree on the identity of a leader, jointly synchronize their operation, decide which specific pattern to form, balance the computational load or fuse consistently the information gathered on some spatial process.

In this paper, we address the problem of designing (continuous-time) coordination algorithms that make a networked system asymptotically agree upon the value of a desired arbitrary function of the initial state of the individual agents. The motivation behind our approach is to make available broadly applicable tools and systematic design methodologies for coordination problems involving groups of robotic agents and mobile sensor networks.

Literature review: Distributed consensus algorithms have a long-standing tradition in computer science, e.g. [1]. Within the literature on cooperative control and multi-agent systems, recent years have witnessed the introduction of distributed strategies that achieve various forms of agreement. This interest is reflected in the recent surveys [2], [3]. A growing body of work focuses in designing and analyzing algorithms that make individual network agents agree upon the value of some function of their initial states. These include average consensus [4], [5], [6], average-max-min consensus [7], geometric-mean consensus [8] and power-mean consensus [9]. In these works, the state variables associated to the individual agents do not necessarily correspond to physical variables, such as spatial coordinates or velocities. Network coordination problems that focus, instead, on "spatial versions" of consensus include rendezvous [10], [11], [12], [13], flocking [14], [15], [16], [17] and cohesiveness [18], [19]. Applications of consensus algorithms to data fusion problems and distributed filtering include [20], [21], [22].

Statement of contributions: The contributions of this paper pertain both analysis and design of cooperative strategies for consensus. Regarding analysis, we identify a set of conditions that completely characterize when a coordination algorithm makes the network agents asymptotically agree upon the value of an arbitrary function of their individual states (cf. Theorem 4.3 and Corollary 4.4). This characterization holds under mild assumptions on the smoothness properties of both the consensus function and the coordination algorithm. We then particularize this result to the setting of real analytic consensus functions (cf. Proposition 4.6).

Regarding design, we identify a class of smooth consensus functions for which one can synthesize in a systematic way *distributed* coordination algorithms (cf. Proposition 5.1 and Corollary 5.2). The property common to these functions is that the computation of their gradients enjoys some special distributed features. Building on this result, we characterize the exponential rate of convergence of a class of distributed algorithms that achieve weighted power mean consensus originally introduced in [9] (cf. Proposition 5.4). The maximum and the minimum functions do not belong to the special class of functions mentioned above. The last contribution of the paper is the introduction of two distributed algorithms that achieve max and min consensus in finite time (cf. Proposition 5.5). The convergence proof relies on the characterization obtained in Corollary 4.4 and tools from nonsmooth stability analysis.

Organization: The paper is organized as follows. Section II presents some preliminary notions on undirected graphs, distributed maps and nonsmooth stability analysis. Section III formally introduces the consensus problem we are interested in solving. Section IV identifies necessary and sufficient conditions for any coordination algorithm that asymptotically achieves consensus. Section V investigates the design of distributed coordination algorithms for consensus, paying special attention to weighted power mean, max and min consensus. Finally, we present our conclusions and ideas for future research in Section VI.

Notation: Let $\mathbb{N}$, $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ denote, respectively, the set of natural numbers, the set of positive reals, and the set of non-negative reals. Let $i_{\mathbb{R}} : \mathbb{R} \to \mathrm{diag}(\mathbb{R}^n) \subset \mathbb{R}^n$ denote the natural inclusion, and $\mathbf{1}$ denote the vector $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^n$. Given $\chi : \mathcal{V} \subset \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$, $d_1, d_2 \in \mathbb{N}$, we denote $\mathrm{Im}(\chi) = \{\chi(P) \in \mathbb{R}^{d_2} \mid P \in \mathcal{V}\}$. Note

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that $\mathrm{Im}(i_{\mathbb{R}}) = \mathrm{diag}(\mathbb{R}^n)$. For a continuous function χ, its extension $\chi_e : \overline{\mathcal{V}} \subset \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is defined as $\chi_e(P) = \chi(P)$ for $P \in \mathcal{V}$, and $\chi_e(P) = \lim_{m\to+\infty} \chi(P_m)$ for $P \in \partial\mathcal{V}$ and $\mathcal{V} \ni P_m \to P$. Given a positive semidefinite matrix A, let $\ker(A) \subset \mathbb{R}^n$ denote the eigenspace corresponding to the eigenvalue 0 (if A is positive definite, then we set $\ker(A) = \{0\}$). Denote by $\pi_{\ker(A)} : \mathbb{R}^n \to \ker(A)$ the orthogonal projection onto $\ker(A)$. Let $\lambda_2(A)$ and $\lambda_n(A)$ be the smallest non-zero and greatest eigenvalue of A, respectively, i.e. $\lambda_2(A) = \min\{\lambda \mid \lambda > 0 \text{ and } \lambda \text{ eigenvalue of } A\}$ and $\lambda_n(A) = \max\{\lambda \mid \lambda \text{ eigenvalue of } A\}$. One can see that for $u \in \mathbb{R}^n$,

$$\lambda_2(A)\,\|u - \pi_{\ker(A)}(u)\|_2^2 \le u^T A\,u \le \lambda_n(A)\,\|u - \pi_{\ker(A)}(u)\|_2^2. \quad (1)$$

For a set X, we denote by $\mathfrak{P}(X)$ the collection of all subsets of X, and by $\mathbb{F}(X) \subset \mathfrak{P}(X)$ the collection of all finite subsets of X. Finally, let $\mathrm{sgn}_+, \mathrm{sgn}_-, \mathrm{sgn} : \mathbb{R} \to \mathbb{R}$ be

$$\mathrm{sgn}_+(x) = \begin{cases} 0, & x \le 0, \\ 1, & x > 0, \end{cases} \quad \mathrm{sgn}_-(x) = \begin{cases} 0, & x \ge 0, \\ -1, & x < 0, \end{cases}$$

and $\mathrm{sgn}(x) = \mathrm{sgn}_+(x) + \mathrm{sgn}_-(x)$.

II. Preliminaries

In this section, we gather some definitions and tools from algebraic graph theory, distributed maps and nonsmooth stability analysis.

A. Graph Laplacians, disagreement functions and distributed maps

The graph Laplacian matrix L associated with an undirected graph $G = (\{1, \dots, n\}, \mathcal{E})$ (see, for instance, [23]) is defined as $L = \Delta - A$, where Δ is the degree matrix and A is the adjacency matrix of the graph. The Laplacian matrix has the following relevant properties: it is symmetric, positive semidefinite and has $\lambda = 0$ as an eigenvalue with eigenvector $\mathbf{1}$. More importantly, the graph G is connected if and only if $\mathrm{rank}(L) = n - 1$, i.e., if the eigenvalue 0 has multiplicity one. This is the reason why the eigenvalue $\lambda_2(L) = \min\{\lambda \mid \lambda > 0 \text{ and } \lambda \text{ eigenvalue of } L\}$ is termed the *algebraic connectivity* of the graph G.

Let us associate a state $p_i \in \mathbb{R}$ to each vertex $i \in \{1, \dots, n\}$. Two nodes are said to *agree* if and only if $p_i = p_j$. A meaningful function that quantifies the group disagreement in a network is the so-called *disagreement function* or *Laplacian potential* $\Phi_G : \mathbb{R}^n \to \overline{\mathbb{R}}_+$ associated with G (see [4]), defined by

$$\Phi_G(p_1, \dots, p_n) = \frac{1}{2} P^T L P = \frac{1}{2} \sum_{\substack{i<j \\ (i,j)\in\mathcal{E}}} (p_j - p_i)^2,$$

with $P = (p_1, \dots, p_n) \in \mathbb{R}^n$. Clearly $\Phi_G(p_1, \dots, p_n) = 0$ if and only if all neighboring nodes in the graph G agree. If the graph G is connected, then all nodes in the graph agree and a consensus is reached. The Laplacian potential is smooth, and its gradient is given by

$$\frac{\partial \Phi_G}{\partial p_i} = \sum_{j\in\mathcal{N}_G(i)} (p_i - p_j), \quad i \in \{1, \dots, n\}. \quad (2)$$

Next, we introduce the notion of distributed map over an undirected graph G. Given two spaces X, Y, and a function $T : X^n \to Y^n$, we say that T *is (1-hop) distributed over* G if there exist functions $\tilde{T}_1, \dots, \tilde{T}_n : X \times \mathbb{F}(X) \to Y$ with

$$T_i(x_1, \dots, x_n) = \tilde{T}_i(x_i, \{x_j \mid j \in \mathcal{N}_G(i)\}),$$

for all $(x_1, \dots, x_n) \in X$ and all $i \in \{1, \dots, n\}$. Roughly speaking, the ith component of a distributed map over G can be computed only with information about the state of node i and its neighbors in the graph G. For example, from (2), we deduce that $\mathrm{grad}(\Phi_G) : \mathbb{R}^n \to \mathbb{R}^n$ is distributed over G. This notion was introduced in [13] for the more general class of proximity graphs.

B. Nonsmooth stability analysis

This section introduces differential equations with discontinuous right-hand sides and presents various nonsmooth tools to analyze their stability properties. The presentation follows the exposition in [24], [25].

For differential equations with discontinuous right-hand sides we understand the solutions in terms of differential inclusions following [26]. Let $F : \mathbb{R}^d \to 2^{\mathbb{R}^d}$, $d \in \mathbb{N}$, be a set-valued map. Consider the differential inclusion

$$\dot{x} \in F(x)\,. \quad (3)$$

A solution to this equation on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as an absolutely continuous function $x : [t_0, t_1] \to \mathbb{R}^d$ such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in [t_0, t_1]$. Now, consider the differential equation

$$\dot{x}(t) = X(x(t))\,, \quad (4)$$

where $X : \mathbb{R}^d \to \mathbb{R}^d$ is measurable and essentially locally bounded [26]. For each $x \in \mathbb{R}^d$, consider the set

$$K[X](x) = \bigcap_{\delta>0} \bigcap_{\mu(S)=0} \mathrm{co}\{X(B_d(x, \delta) \setminus S)\}\,, \quad (5)$$

where μ denotes the usual Lebesgue measure in $\mathbb{R}^d$. A Filippov solution of (4) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion

$$\dot{x} \in K[X](x)\,. \quad (6)$$

A set M is *weakly invariant* (respectively *strongly invariant*) for (4) if for each $x_0 \in M$, M contains a maximal solution (respectively all maximal solutions) of (4).

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. From Rademacher's Theorem [27], we know that locally Lipschitz functions are differentiable a.e. Let $\Omega_f \subset \mathbb{R}^d$ denote the set of points where f fails to be differentiable. The *generalized gradient of f at $x \in \mathbb{R}^d$* (cf. [27]) is defined by

$$\partial f(x) = \mathrm{co}\,\big\{ \lim_{i\to+\infty} df(x_i) \mid x_i \to x\,,\ x_i \notin S \cup \Omega_f \big\},$$

where S can be any set of zero measure. Note that if f is continuously differentiable, then $\partial f(x) = \{df(x)\}$.

Given a locally Lipschitz function f, the *set-valued Lie derivative of f with respect to X* at x (cf. [24], [25]) is

$$\widetilde{\mathcal{L}}_X f(x) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ such that } \zeta \cdot v = a\,,\ \forall \zeta \in \partial f(x)\}.$$

If f is continuously differentiable at x and X is continuous at x, then $\widetilde{\mathcal{L}}_X f(x)$ corresponds to the singleton $\{\mathcal{L}_X f(x)\}$, the usual Lie derivative of f in the direction of X at x. The next result states that the set-valued Lie derivative allows us to study the evolution of a function along the Filippov solutions.

Theorem 2.1: Let $x : [t_0, t_1] \to \mathbb{R}^d$ be a Filippov solution of (4). Let f be a locally Lipschitz and regular function. Then $t \mapsto f(x(t))$ is absolutely continuous, $\frac{d}{dt}\big(f(x(t))\big)$ exists a.e. and $\frac{d}{dt}\left(f(x(t))\right) \in \widetilde{\mathcal{L}}_X f(x(t))$ a.e.

The following result is a generalization of LaSalle principle for differential equations of the form (4) with nonsmooth Lyapunov functions.

Theorem 2.2: (LaSalle Invariance Principle): Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz and regular function. Let $x_0 \in S \subset \mathbb{R}^d$, with S compact and strongly invariant for (4). Assume that either $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ or $\widetilde{\mathcal{L}}_X f(x) = \emptyset$ for all $x \in S$. Let $Z_{X,f} = \{x \in \mathbb{R}^d \mid 0 \in \widetilde{\mathcal{L}}_X f(x)\}$. Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^d$ of (4) starting from x_0 converges to the largest weakly invariant set M contained in $\overline{Z}_{X,f} \cap S$. Moreover, if the set M is a finite collection of points, then the limit of all solutions starting at x_0 exists and equals one of them.

The following result establishes one condition under which convergence is attained in finite time.

Proposition 2.3: Under the same assumptions of Theorem 2.2, further assume that there exists a neighborhood U of $Z_{X,f} \cap S$ in S such that $\max \widetilde{\mathcal{L}}_X f < -\epsilon < 0$ a.e. on $U \setminus (Z_{X,f} \cap S)$. Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^d$ of (4) starting at $x_0 \in S$ reaches $Z_{X,f} \cap S$ in finite time.

III. PROBLEM STATEMENT

Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Consider a network of agents whose individual dynamics is given by

$$\dot{p}_i = u_i, \quad i \in \{1, \ldots, n\}. \tag{7}$$

We say that a coordination algorithm $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ *asymptotically achieves χ-consensus* if u is essentially locally bounded, and for any $(p_1(0), \ldots, p_n(0)) \in \mathcal{V}$, any solution of the dynamics (7) starting at $(p_1(0), \ldots, p_n(0))$ stays in $\mathcal{V}$ and verifies, for all $i \in \{1, \ldots, n\}$,

$$p_i(t) \longrightarrow \chi(p_1(0), \ldots, p_n(0)), \quad t \to +\infty.$$

Because the trajectories stay in $\mathcal{V}$, for consistency, $\mathrm{Im}(i_{\mathbb{R}} \circ \chi) \subset \mathcal{V}$ must hold. Note that we do not require u to be continuous. If u is discontinuous, then solutions are understood in the Filippov sense [26]. We usually refer to χ as the *consensus function*.

Now, assume the network interconnection topology is described by an undirected graph $G = (\{1, \ldots, n\}, \mathcal{E})$. Our objective is to design coordination algorithms that verify, at the same time,

(P.1): u is distributed over G, and
(P.2): u asymptotically achieves χ-consensus.

Property (P.1) guarantees that the control law u is implementable over the network (7), and property (P.2) guarantees that individual agents asymptotically agree on the value of χ.

IV. NECESSARY AND SUFFICIENT CONDITIONS FOR χ-CONSENSUS

In this section, we obtain necessary and sufficient conditions for any coordination algorithm that asymptotically achieve consensus (i.e., satisfy property (P.2) in Section III). We undertake this study as a necessary step previous to the synthesis of coordination algorithms with properties (P.1) and (P.2). The treatment of Section V builds on this discussion to design distributed algorithms for χ-consensus.

We start by showing that the function χ must be constant along the trajectories of a coordination algorithm that asymptotically achieves χ-consensus. The statement here is a generalization to continuous functions of a result in [9].

Lemma 4.1: Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Assume $\mathrm{Im}(i_{\mathbb{R}} \circ \chi) \subset \mathcal{V}$ and let $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a coordination algorithm asymptotically achieving χ-consensus. Then χ is constant along the trajectories of (7).

Proof: Let $P_0 \in \mathcal{V}$ and consider a trajectory $\overline{\mathbb{R}}_+ \ni t \mapsto P(t)$ of (7) that starts at P_0. Since u asymptotically achieves χ-consensus, then $P(t) \to (\chi(P_0), \ldots, \chi(P_0))$. Let $t^* \in \mathbb{R}_+$. The curve $\overline{\mathbb{R}}_+ \ni t \mapsto P(t+t^*)$ starts at $P(t^*) \in \mathcal{V}$ and it is a trajectory of (7). Since u asymptotically achieves χ-consensus, then $P(t + t^*) \to (\chi(P(t^*)), \ldots, \chi(P(t^*)))$. Since $P(t) \to (\chi(P_0), \ldots, \chi(P_0))$, we conclude $\chi(P(t^*)) = \chi(P_0)$, i.e., χ is constant along $\overline{\mathbb{R}}_+ \ni t \mapsto P(t)$. ■

The following result restricts the class of functions χ for which the consensus problem can be solved. The statement here is a generalization to functions with arbitrary domains of a result in [9].

Proposition 4.2: Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be a continuous function. Assume $\mathrm{Im}(i_{\mathbb{R}} \circ \chi) \subset \mathcal{V}$ and let $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ be a coordination algorithm asymptotically achieving χ-consensus. Then $\chi \circ i_{\mathbb{R}|\mathrm{Im}(\chi)} = \mathrm{Id}_{\mathbb{R}|\mathrm{Im}(\chi)}$.

Proof: We reason by contradiction. Assume there exists $p \in \mathrm{Im}(\chi)$ such that $\chi(p, \ldots, p) \neq p$ (note that $(p, \ldots, p) \in \mathcal{V}$ because $\mathrm{Im}(i_{\mathbb{R}} \circ \chi) \subset \mathcal{V}$). Let $\epsilon = |\chi(p, \ldots, p) - p| > 0$. By continuity of χ, there exists $\delta > 0$ such that $\|P - (p, \ldots, p)\| < \delta$ implies $|\chi(P) - \chi(p, \ldots, p)| < \epsilon$. On the other hand, since $p \in \mathrm{Im}(\chi)$, there exists $P^* \in \mathcal{V} \setminus \{(p, \ldots, p)\}$ such that $\chi(P^*) = p$. By hypothesis, u asymptotically achieves χ-consensus. In particular, this implies that the trajectory $\overline{\mathbb{R}}_+ \ni t \mapsto P(t)$ of (7) starting from $P(0) = P^*$ asymptotically converges to $(\chi(P^*), \ldots, \chi(P^*)) = (p, \ldots, p)$. For $\delta > 0$ above, there exists $T > 0$ such that $\|P(t) - (p, \ldots, p)\| < \delta$ for $t \geq T$, which implies that $|\chi(P(t)) - \chi(p, \ldots, p)| <$

ϵ. By Lemma 4.1, $\chi(P(t)) = \chi(P^*) = p$, and hence $|p-\chi(p,\dots,p)| < \epsilon$, contradicting $\epsilon = |\chi(p,\dots,p)-p|$. ■

The following result fully characterizes the situations where χ-consensus can be asymptotically achieved by a coordination algorithm.

Theorem 4.3: Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be continuous. Assume that $i_{\mathbb{R}}^{-1}(\mathcal{V}) = \mathrm{Im}(\chi)$ and $\mathrm{Im}(\chi_{e|\,\mathrm{diag}(\mathbb{R}^n)\cap\partial V}) \cap \mathrm{Im}(\chi) = \emptyset$. Let $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ be essentially locally bounded such that the trajectories of (7) are bounded and $\mathcal{V}$ is strongly invariant. Then, u guarantees that χ-consensus is asymptotically reached if and only if the following holds

(i) the trajectories of (7) converge to $\mathrm{diag}(\mathbb{R}^n)$,
(ii) χ is constant along the trajectories of (7), and
(iii) $\chi \circ i_{\mathbb{R}|\mathrm{Im}(\chi)} = \mathrm{Id}_{\mathbb{R}|\mathrm{Im}(\chi)}$.

Proof: If u guarantees that χ-consensus is asymptotically reached, then (i) holds by definition, (ii) holds by Lemma 4.1 and (iii) holds by Proposition 4.2. Now, assume that (i)-(iii) hold, and let us prove that u guarantees that χ-consensus is asymptotically reached. Let $P_0 \in \mathcal{V}$ and consider a trajectory $\overline{\mathbb{R}}_+ \ni t \mapsto P(t)$ of (7) starting at $P(0) = P_0$. Since the trajectory is bounded, its ω-limit set, denoted $\Omega(\{P(t)\}_{t\in\overline{\mathbb{R}}_+}) \subset \mathbb{R}^n$, is non-empty, compact and invariant. By (i), $\Omega(\{P(t)\}_{t\in\overline{\mathbb{R}}_+}) \subset \mathrm{diag}(\mathbb{R}^n)$. For each $(p,\dots,p) \in \Omega(\{P(t)\}_{t\in\overline{\mathbb{R}}_+})$, there exists a convergent subsequence (that, for ease of notation, we also denote by $\{P(t)\}_{t\in\overline{\mathbb{R}}_+}$) such that $P(t) \to (p,\dots,p)$. Note that $(p,\dots,p) \in \overline{\mathcal{V}}$. Extending χ by continuity if necessary, we have $\chi(p,\dots,p) = \lim_{t\to+\infty}\chi(P(t))$. Now (ii) implies that actually $\chi(p,\dots,p) = \chi(P_0)$. This, together with $\mathrm{Im}(\chi_{e|\,\mathrm{diag}(\mathbb{R}^n)\cap\partial V}) \cap \mathrm{Im}(\chi) = \emptyset$, implies that $p \in i_{\mathbb{R}}^{-1}(\mathcal{V}) = \mathrm{Im}(\chi)$. By (iii), we deduce $p = \chi(p,\dots,p) = \chi(P_0)$. Therefore, we have established that $\Omega(\{P(t)\}_{t\in\overline{\mathbb{R}}_+}) = \{\chi(P_0)\mathbf{1}\}$, or, equivalently, that the trajectory $\{P(t)\}_{t\in\overline{\mathbb{R}}_+}$ converges to $\chi(P_0)\mathbf{1}$, as claimed. ■

The previous result takes a much simpler form when the function χ is defined over the whole space $\mathbb{R}^n$ and is surjective.

Corollary 4.4: Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be continuous and surjective. Let $u : \mathbb{R}^n \to \mathbb{R}^n$ be essentially locally bounded such that the trajectories of (7) are bounded. Then, u guarantees that χ-consensus is asymptotically reached if and only if the following holds

(i) the trajectories of (7) converge to $\mathrm{diag}(\mathbb{R}^n)$,
(ii) χ is constant along the trajectories of (7), and
(iii) $\chi \circ i_{\mathbb{R}} = \mathrm{Id}_{\mathbb{R}}$.

Theorem 4.3 and Corollary 4.4 are important both from an analysis and a design viewpoint. From an analysis perspective, these results characterize under what circumstances the network asymptotically achieves χ-consensus with a coordination algorithm of the form $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$. From a design perspective, these results establish a systematic methodology to synthesize solutions to the χ-consensus problem. Therefore, if faced with the task of analyzing the correctness properties of a given coordination algorithm, or the task of designing a new coordination algorithm to achieve χ-consensus, one can just check that the consensus function satisfies condition (iii) in Theorem 4.3, and that the coordination algorithm satisfies conditions (i) and (ii) in Theorem 4.3.

A. Real analytic consensus functions

In this section, we focus on our attention on real analytic functions $\chi : \mathbb{R}^n \to \mathbb{R}$. First, we show that condition (iii) in Theorem 4.3 determines, up to first-order, the consensus function χ.

Lemma 4.5: Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be real analytic. Assume $\chi \circ i_{\mathbb{R}} = \mathrm{Id}_{\mathbb{R}}$. Then, there exists $w = (w_1,\dots,w_n) \in \mathbb{R}^n$ with $\sum_{i=1}^n w_i = 1$ such that the first-order approximation of χ is the weighted average mean function $\sum_{i=1^n} w_i\, p_i$. We refer to w as the first-order weight vector associated to χ.

Proof: Let $(p^*,\dots,p^*) \in \mathrm{diag}(\mathbb{R}^n)$. By definition, there exists a neighborhood $\mathcal{U}$ of $(p^*,\dots,p^*)$ such that

$$\chi(p_1,\dots,p_n) = \sum_{k_1,\dots,k_n\geq 0} a_{k_1,\dots,k_n}(p_1-p^*)^{k_1}\dots(p_n-p^*)^{k_n},$$

for all $(p_1,\dots,p_n) \in \mathcal{U}$, where

$$a_{k_1,\dots,k_n} = \frac{1}{k_1!\dots k_n!}\frac{\partial^{k_1+\dots+k_n}\chi}{\partial p_1^{k_1}\dots\partial p_n^{k_n}}(p^*,\dots,p^*).$$

In particular, note that $a_{0,\dots,0} = \chi(p^*,\dots,p^*) = p^*$. Now, for any $p \in {i_{\mathbb{R}}}^{-1}(\mathcal{U})$, we have

$$p = \chi(p,\dots,p) = \sum_{k_1,\dots,k_n\geq 0} a_{k_1,\dots,k_n}(p-p^*)^{k_1}\dots(p-p^*)^{k_n}$$
$$= p^* + \sum_{\substack{k_1,\dots,k_n\geq 0\\ k_1+\dots+k_n\geq 1}} a_{k_1,\dots,k_n}(p-p^*)^{k_1+\dots+k_n}.$$

Since real analytic functions of one variable that are equal on an open set must be necessarily identical on the intersection of their domains of definition, see e.g. [28, Corollary 1.2.6], we deduce

$$\sum_{\substack{k_1,\dots,k_n\geq 0\\ k_1+\dots+k_n=\ell}} a_{k_1,\dots,k_n} = \begin{cases} 1, & \ell = 1,\\ 0, & \ell \geq 2.\end{cases}$$

Denoting for simplicity $w_1 = a_{1,0,\dots,0},\dots,w_n = a_{0,0,\dots,1}$, we get the following expression for χ on $\mathcal{U}$,

$$\chi(p_1,\dots,p_n) = \sum_{i=1}^n w_i\,p_i + \sum_{\substack{k_1,\dots,k_n\geq 0\\ k_1+\dots+k_n\geq 2}} a_{k_1,\dots,k_n}(p_1-p^*)^{k_1}\dots(p_n-p^*)^{k_n}.$$

To conclude, let us establish that the weights $w_1,\dots,w_n$ are independent of the selected point in $\mathrm{diag}(\mathbb{R}^n)$ where the series expansion of χ is derived. We reason by contradiction. Assume there exist $P_1^*, P_2^* \in \mathrm{diag}(\mathbb{R}^n)$, with corresponding neighborhoods $\mathcal{U}_1$ and $\mathcal{U}_2$, and different weights in the series expansion. Consider the compact segment with extreme points P_1^* and P_2^*. For each point in this segment, there exists a neighborhood where χ admits a convergent series expansion. Since the segment is compact, there exist a finite number of pairwise-intersecting neighborhoods whose union contains the segment. Without loss of generality, we can assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ belong to this finite family. Using again [28, Corollary 1.2.6], it is not difficult to see that any two points whose corresponding neighborhoods intersect must have the same weights in the series expansion. Therefore, the weights obtained throughout the segment are constant, which contradicts the fact that P_1^* and P_2^* have different weights in their series expansion. ■

Next, given a connected undirected graph G, we show that under some additional conditions, there always exist a (generally non distributed) coordination algorithm that asymptotically achieve consensus. In the forthcoming statement, we denote by $v/w \in \mathbb{R}^n$ with $v, w \in \mathbb{R}^n$, the vector whose ith component is v_i/w_i, $i \in \{1, \ldots, n\}$.

Proposition 4.6: Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be real analytic. Assume $\chi \circ i_{\mathbb{R}} = \mathrm{Id}_{\mathbb{R}}$. Let $w \in \mathbb{R}^n_+$ be the first-order weight vector associated to χ, and assume $(\operatorname{grad} \chi(P) - w) \cdot \mathbf{1} = 0$, for all $P \in \mathbb{R}^n$. Let G be a connected undirected graph. Then, the coordination algorithm $u : \mathbb{R}^n \to \mathbb{R}^n$ with ith component, $i \in \{1, \ldots, n\}$, given by

$$u_i(P) = \frac{1}{w_i} \sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) + \big(\frac{1}{w} \operatorname{grad} \chi(P)\big)^T LP, \quad (8)$$

asymptotically achieves χ-consensus.

Proof: Our proof strategy is to check the conditions of Corollary 4.4. Clearly, χ is continuous and surjective. The map u is differentiable. Condition (iii) is readily verified by hypothesis. Condition (ii) is a consequence of following simple computation

$$\begin{aligned}\mathcal{L}_u \chi &= \sum_{i=1}^n \frac{\partial \chi}{\partial p_i} u_i = \sum_{i=1}^n w_i u_i + \sum_{i=1}^n \big(\frac{\partial \chi}{\partial p_i} - w_i\big) u_i \\ &= \big(\frac{1}{w} \operatorname{grad} \chi(P)\big)^T LP - \sum_{i=1}^n \big(\frac{\partial \chi}{\partial p_i} - w_i\big) \frac{1}{w_i} (LP)_i = 0.\end{aligned}$$

Finally, condition (i) follows from

$$\mathcal{L}_u \Phi_G = \sum_{i=1}^n \frac{\partial \Phi_G}{\partial p_i} u_i = -\sum_{i=1}^n \frac{1}{w_i} \Big(\sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) \Big)^2 \leq 0.$$

Using the connectedness of G, it is not difficult to establish that $\mathcal{L}_u \Phi_G(P) = 0$ if and only if $P \in \operatorname{diag}(\mathbb{R}^n)$. Therefore, the trajectories of (8) converge to $\operatorname{diag}(\mathbb{R}^n)$. Using this property and the fact that χ is analytic, one can also deduce that the trajectories are bounded. ■

Remark 4.7: Note that the coordination algorithm (8) is, in general, not distributed over the graph G, since each agent needs to compute the term $\big(\frac{1}{w} \operatorname{grad} \chi(P)\big)^T LP$. In the next section, we focus our attention on a special class of functions that admit distributed coordination algorithms. •

V. Distributed coordination algorithms for χ-consensus

In this section, we identify particular conditions on the consensus function χ under which distributed coordination algorithms that asymptotically achieve consensus can be designed.

Proposition 5.1: Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable such that $\chi \circ i_{\mathbb{R}|\mathrm{Im}(\chi)} = \mathrm{Id}_{\mathbb{R}|\mathrm{Im}(\chi)}$, $i_{\mathbb{R}}^{-1}(\mathcal{V}) = \mathrm{Im}(\chi)$ and $\mathrm{Im}(\chi_{e|\operatorname{diag}(\mathbb{R}^n) \cap \partial V}) \cap \mathrm{Im}(\chi) = \emptyset$. Let G be a connected undirected graph, and let $\operatorname{grad} \chi$ be distributed over G, with all partial derivatives $\{\frac{\partial \chi}{\partial p_1}, \ldots, \frac{\partial \chi}{\partial p_n}\}$ having the same constant sign on $\mathcal{V}$. Assume that the coordination algorithm $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$,

$$u_i = \frac{1}{\big|\frac{\partial \chi}{\partial p_i}\big|} \sum_{j \in \mathcal{N}_G(i)} (p_j - p_i), \quad i \in \{1, \ldots, n\} \qquad (9)$$

is essentially locally bounded and such that $\mathcal{V}$ is strongly invariant. Then, u is distributed over G and asymptotically achieves χ-consensus.

Proof: Clearly, the map $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$ is distributed over G. We prove that u asymptotically achieves χ-consensus by using Theorem 4.3. Let us first establish that each trajectory of (9) belongs to some bounded and invariant set. Consider the $\max : \mathbb{R}^n \to \mathbb{R}$ function, $\max(P) = \max_{i \in \{1,\ldots,n\}}\{p_i\}$, and let us compute the set-valued Lie derivative $\widetilde{\mathcal{L}}_u \max$. If $a \in \widetilde{\mathcal{L}}_u \max$, then $a = u(P) \cdot \zeta$, for all $\zeta \in \partial \max$. The generalized gradient of $\max$ is

$$\partial \max(P) = \mathrm{co}\{e_i \mid i \in \{1, \ldots, n\} \text{ with } p_i = \max(P)\}.$$

If $P \in \operatorname{diag}(\mathbb{R}^n)$, then $u(P) = 0$, and therefore $a = 0$. If $P \notin \operatorname{diag}(\mathbb{R}^n)$, then using the fact that G is connected, there exists $k \in \{1, \ldots, n\}$ with $p_k = \max_{j \in \{1,\ldots,n\}}\{p_j\}$ such that $\sum_{j \in \mathcal{N}_{G,k}} (p_j - p_k) < 0$. Consequently, $u_k(P) < 0$, and $a = u(P) \cdot e_k < 0$. Therefore, we conclude that either $\widetilde{\mathcal{L}}_u \max = \emptyset$ or $\max \widetilde{\mathcal{L}}_u \max \leq 0$. Theorem 2.1 implies that $p_i(t) \leq \max\{p_1(0), \ldots, p_n(0)\}$. A similar argument with the $\min$ function shows that $\min\{p_1(0), \ldots, p_n(0)\} \leq p_i(t)$. Hence, any trajectory of (9) belongs to a bounded and invariant set.

Let us study the evolution of the disagreement function Φ_G along the trajectories of the system

$$\mathcal{L}_u \Phi_G = \sum_{i=1}^n \frac{\partial \Phi_G}{\partial p_i} u_i = -\sum_{i=1}^n \frac{1}{\big|\frac{\partial \chi}{\partial p_i}\big|} \Big(\sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) \Big)^2 \leq 0.$$

Using the connectedness of G, it is not difficult to establish that $Z_{u,\Phi_G} = \operatorname{diag}(\mathbb{R}^n)$. Given that any trajectory of (9) belongs to some bounded and invariant set, the LaSalle Invariance Principle guarantees that all trajectories converge to $\operatorname{diag}(\mathbb{R}^n)$, i.e., condition (i) is satisfied. Condition (ii) is easily verified since

$$\begin{aligned}\mathcal{L}_u \chi &= \sum_{i=1}^n \frac{\partial \chi}{\partial p_i} u_i = \sum_{i=1}^n \operatorname{sgn}\big(\frac{\partial \chi}{\partial p_i}\big) \cdot \sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) \\ &= \pm \sum_{i=1}^n \sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) = 0.\end{aligned}$$

Condition (iii) is verified by hypothesis, and this concludes the result. ■

The next result extends the applicability of Proposition 5.1 to functions χ whose gradient is not distributed over the interconnection topology, but that admit a "distributing factor" that makes it distributed. We formalize this idea as follows.

Corollary 5.2: Let $\chi : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable such that $\chi \circ i_{\mathbb{R}|\mathrm{Im}(\chi)} = \mathrm{Id}_{\mathbb{R}|\mathrm{Im}(\chi)}$, $i_{\mathbb{R}}^{-1}(\mathcal{V}) = \mathrm{Im}(\chi)$ and $\mathrm{Im}(\chi_{e|\,\mathrm{diag}(\mathbb{R}^n)\cap\partial V}) \cap \mathrm{Im}(\chi) = \emptyset$. Let G be a connected undirected graph. Assume there exist $f : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}$ such that $f \cdot \mathrm{grad}\,\chi$ is distributed over G, with all partial derivatives $\{\frac{\partial\chi}{\partial p_1}, \dots, \frac{\partial\chi}{\partial p_n}\}$ having the same constant sign on $\mathcal{V}$. Assume that the coordination algorithm $u : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^n$,

$$u_i = \frac{1}{\left|f\frac{\partial\chi}{\partial p_i}\right|} \sum_{j\in\mathcal{N}_G(i)} (p_j - p_i), \quad i \in \{1, \dots, n\} \tag{10}$$

is essentially locally bounded and such that $\mathcal{V}$ is strongly invariant. Then, u is distributed over G and asymptotically achieves χ-consensus.

Remark 5.3: Note that Proposition 5.1 and Corollary 5.2 generalize the main result in [9, Theorem 2] by broadening the set of functions for which the consensus problem can be solved. The result in [9] presents a class of distributed coordination algorithms for functions that admit an expression of the form

$$\chi(p_1, \dots, p_n) = f\big(\sum_{i=1}^n g(p_i)\big), \tag{11}$$

for some functions $f, g : \mathbb{R} \to \mathbb{R}$ with $g'(x) \neq 0$ for all $x \in \mathbb{R}$. However, the set of functions to which Corollary 5.2 can be applied strictly contains this class of functions. As an example, consider the function $\chi^* : \mathbb{R}_+^3 \to \mathbb{R}$ defined by

$$\chi(p_1, p_2, p_3) = \frac{1}{2}(\sqrt{p_1 p_2} + \sqrt{p_2 p_3}).$$

This function does not fall into the category (11). This can be see by contradiction. Assuming $\chi^*(p_1, p_2, p_3) = f\big(g(p_1) + g(p_2) + g(p_3)\big)$, for some appropriate $f, g : \mathbb{R} \to \mathbb{R}$. Then, $\frac{\partial\chi^*}{\partial p_2}\big/\frac{\partial\chi^*}{\partial p_1} = g'(p_2)/g'(p_1)$, i.e., the quotient only depends on p_1 and p_2. However,

$$\frac{\frac{\partial\chi^*}{\partial p_2}}{\frac{\partial\chi^*}{\partial p_1}} = \frac{p_3\sqrt{p_1} + p_1\sqrt{p_3}}{p_2\sqrt{p_3}},$$

which depends on p_3, and therefore, χ^* is not of the form (11). On the other hand, the function $\chi^* : \mathbb{R}_+^3 \to \mathbb{R}$ verifies the hypotheses of Proposition 5.1, and is distributed over the connected undirected graph $G = (\{1, 2, 3\}, \mathcal{E})$, with $\mathcal{E} = \{(1, 2), (2, 3\}$. •

A. Distributed coordination algorithms for weighted power mean consensus

In this section, we study distributed algorithms that asymptotically achieve weighted power mean consensus. This class of algorithms was originally presented in [9]. Here, we introduce them as a particular application of Corollary 5.2 the weighted power mean function. More importantly, we characterize their exponential rate of convergence.

For $w \in \mathbb{R}_+^n$ with $\sum_{i=1}^n w_i = 1$ and $r \in \mathbb{R} \setminus \{0\}$, the *weighted power mean* $\chi_{w,r} : \mathbb{R}_+^n \to \mathbb{R}$ is defined by

$$\chi_{w,r}(p_1, \dots, p_n) = \big(\sum_{i=1}^n w_i p_i^r\big)^{\frac{1}{r}}.$$

For $r \in \{0, \pm\infty\}$, the function $\chi_{w,r}$ is defined by

$$\chi_{w,r}(p_1, \dots, p_n) = \lim_{s\to r} \chi_{w,s}(p_1, \dots, p_n).$$

Alternatively, one has

$$\chi_{w,0}(p_1, \dots, p_n) = p_1^{w_1} \dots p_n^{w_n},$$
$$\chi_{w,+\infty}(p_1, \dots, p_n) = \max\{p_1, \dots, p_n\},$$
$$\chi_{w,-\infty}(p_1, \dots, p_n) = \min\{p_1, \dots, p_n\}.$$

Note that for specific values of the parameter $r \in \mathbb{R}^n \cup \{\pm\infty\}$, the domain of definition of $\chi_{w,r}$ can be larger than $\mathbb{R}_+^n$. For instance, the function $\chi_{w,1}$ is well-defined on $\mathbb{R}^n$. The choice $w_i = \frac{1}{n}$, $i \in \{1, \dots, n\}$, yields the usual power mean function, that we simply denote by χ_r. Table I summarizes some distinguished members of this class of functions. The next result presents a class of coordination

$\chi_{-\infty}$	Minimum
χ_{-1}	Harmonic Mean
χ_0	Geometric Mean
χ_1	Arithmetic Mean or Average
χ_2	Root-Mean-Square
χ_∞	Maximum

TABLE I
SOME EXAMPLES OF POWER MEANS.

algorithms that asymptotically achieve $\chi_{w,r}$-consensus, with $r \in \mathbb{R}$, exponentially fast.

Proposition 5.4: Let $r \in \mathbb{R}$ and $w \in \mathbb{R}_+^n$ with $\sum_{i=1}^n w_i = 1$. For any connected undirected graph $G = (\{1, \dots, n\}, \mathcal{E})$, the coordination algorithm $u_{w,r} : \mathbb{R}_+^n \to \mathbb{R}^n$ whose ith component is given by

$$(u_{w,r})_i(p_1, \dots, p_n) = \frac{1}{w_i} p_i^{1-r} \sum_{j\in\mathcal{N}_G(i)} (p_j - p_i), \tag{12}$$

is distributed over G and asymptotically achieves weighted power mean-consensus with exponential rate of convergence greater than or equal to $c\,\lambda_2^2(L)/\lambda_n(L)$, with

$$c = \begin{cases} \max\{p_1(0), \dots, p_n(0)\}^{1-r}, & r > 1, \\ 1, & r = 1, \\ \min\{p_1(0), \dots, p_n(0)\}^{1-r}, & r < 1. \end{cases}$$

Proof: Our proof strategy to assess the correctness of $u_{w,r}$ is to verify that the conditions in Corollary 5.2 hold. Clearly, $\chi_{w,r}$ is continuously differentiable in $\mathcal{V} = \mathbb{R}_+^n$, $\chi_{w,r} \circ i_{\mathbb{R}} = \mathrm{Id}_{\mathbb{R}}$ on $\mathbb{R}_+$, and $\mathrm{Im}(\chi_{w,r}) = \mathbb{R}_+ = {i_{\mathbb{R}}}^{-1}(\mathbb{R}_+^n)$. Since $\mathrm{diag}(\mathbb{R}^n) \cap \partial\mathbb{R}_+^n = \{0\}$, we also have $\mathrm{Im}\big((\chi_{w,r})_{e|\,\mathrm{diag}(\mathbb{R}^n)\cap\partial V}\big) \cap \mathrm{Im}(\chi_{w,r}) = \emptyset$. The partial derivative of $\chi_{w,r}$ with respect to p_i, $i \in \{1, \dots, n\}$, is

$$\frac{\partial\chi_{w,r}}{\partial p_i} = w_i\Big(\frac{p_i}{\chi_{w,r}(p_1, \dots, p_n)}\Big)^{r-1}.$$

Clearly, on $\mathbb{R}_+^n$, all partial derivatives are strictly positive. Selecting $f(p_1, \dots, p_n) = \chi_{w,r}(p_1, \dots, p_n)^{r-1}$ in Corollary 5.2, we see that $f \cdot \mathrm{grad}\,\chi_{w,r}$ is distributed over G. The coordination algorithm defined by (10) corresponds precisely to $u_{w,r}$. Moreover, from the fact that $\chi_{w,r}$ is constant along the trajectories, we deduce that $\mathbb{R}^n_+$ is strongly invariant. The application of Corollary 5.2 yields the convergence result.

We conclude the proof by assessing the rate of convergence of the trajectories of the system. Let $[0,+\infty) \ni t \mapsto P(t) = (p_1(t), \dots, p_n(t)) \in \mathbb{R}^n_+$ be a trajectory starting from $P(0) = P_0 \in \mathbb{R}^n_+$. To this trajectory, we associate a curve $[0,+\infty) \ni t \mapsto \delta(t) \in \mathbb{R}^n$ defined by

$$(p_1(t), \dots, p_n(t)) = \chi_1(p_1(t), \dots, p_n(t))\mathbf{1} + \delta(t).$$

Note that $\mathbf{1}^T \cdot \delta(t) = 0$. Let us study the evolution of $\delta(t)$. For each $i \in \{1, \dots, n\}$,

$$\begin{aligned}\dot{\delta}_i(t) &= \dot{p}_i(t) - \frac{d}{dt}\big(\chi_1(P(t))\big) \\ &= \frac{1}{w_i} p_i^{1-r}(t) \sum_{j \in \mathcal{N}_G(i)} (\delta_j(t) - \delta_i(t)) - \frac{d}{dt}\big(\chi_1(P(t))\big).\end{aligned}$$

Consider now the function $t \to V(t) = \frac{1}{2}\delta(t)^T L \delta(t)$. Since G is connected, the eigenspace of L corresponding to the eigenvalue 0 is $\ker(L) = \operatorname{span}\{\mathbf{1}\}$. From (1) and using the fact that $\delta(t)$ is orthogonal to $\mathbf{1}$, we deduce that $\lambda_2(L)\|\delta(t)\|_2^2 \leq V(t) \leq \lambda_n(L)\|\delta(t)\|_2^2$. The evolution of this function is governed by

$$\begin{aligned}\dot{V}(t) &= (L\delta(t))^T \cdot \dot{\delta}(t) \\ &= -\sum_{i=1}^n \frac{1}{w_i} p_i^{1-r}(t) \Big(\sum_{j \in \mathcal{N}_G(i)} (\delta_i(t) - \delta_j(t)) \Big)^2.\end{aligned}$$

Now, taking into account that, for $i \in \{1, \dots, n\}$, one has $w_i \leq 1$ and $p_i(t)^{1-r} \geq c$ for all $r \in \mathbb{R}$, we deduce

$$\begin{aligned}\dot{V}(t) &\leq -c \sum_{i=1}^n \Big(\sum_{j \in \mathcal{N}_G(i)} (\delta_i(t) - \delta_j(t)) \Big)^2 = -c \, \|L\,\delta(t)\|_2^2 \\ &\leq -c\, \lambda_2^2(L)\, \|\delta(t)\|_2^2 \leq -2c \frac{\lambda_2^2(L)}{\lambda_n(L)} V(t).\end{aligned}$$

Therefore, we conclude

$$\|\delta(t)\|_2^2 \leq \frac{2}{\lambda_2(L)} V(t) \leq \frac{2}{\lambda_2(L)} V(0) \exp\big(-2c \frac{\lambda_2^2(L)}{\lambda_n(L)} t\big),$$

which implies the result. ■

B. Distributed coordination algorithms for max and min consensus

In this section, we describe two distributed coordination algorithms for max and min consensus. Since neither the maximum nor the minimum are differentiable functions, we cannot rely on Proposition 5.1 or Corollary 5.2. Instead, we will build on the characterization obtained in Section IV.

Consider the dynamical systems

$$\dot{p}_i = \operatorname{sgn}_+\big(\sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) \big), \tag{13a}$$

$$\dot{p}_i = \operatorname{sgn}_-\big(\sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) \big). \tag{13b}$$

For ease of notation, we will refer to these flows by X_{sgn_+} and X_{sgn_-}, respectively. Note that both right-hand sides are discontinuous. We understand their solution in the Filippov sense [26]. The following result characterizes the asymptotic convergence properties of these systems.

Proposition 5.5: Let $G = (\{1, \dots, n\}, \mathcal{E})$ be a connected undirected graph. Then, the coordination algorithm (13a) (respectively, the coordination algorithm (13b)) is distributed over G and asymptotically achieves max consensus (respectively, min consensus) in finite time.

Proof: Our proof strategy is to verify that the conditions in Corollary 4.4 hold. We prove it for the $\max$ function and the flow (13a), and leave to the reader the analogous proof for the $\min$ function and the flow (13b). Clearly, $\max : \mathbb{R}^n \to \mathbb{R}$, $\max(P) = \max_{i \in \{1,\dots,n\}}\{p_i\}$, is continuous and surjective. Moreover, $\max(p, \dots, p) = p$, so condition (iii) in Corollary 4.4 is satisfied.

Let us show that $\max$ is preserved by the flow (13a) using Theorem 2.1. We start by noting that the set-valued map associated to (13a) is

$$\begin{aligned}K[X_{\operatorname{sgn}_+}](P) = \{v \in \mathbb{R}^n \mid v_i \in [0,1] \text{ if} \sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) = 0, \\ v_i = \operatorname{sgn}_+\big(\sum_{j \in \mathcal{N}_G(i)} (p_j - p_i) \big) \text{ otherwise}\}.\end{aligned}$$

Let $a \in \tilde{\mathcal{L}}_{X_{\operatorname{sgn}_+}} \max(P)$. By definition, there exists $v \in K[X_{\operatorname{sgn}_+}](P)$ with $a = v \cdot \zeta$, for all $\zeta \in \partial \max(P)$. If $P \in \operatorname{diag}(\mathbb{R}^n)$, then $\partial \max(P) = \mathbb{R}^n$, and, necessarily $v = (0, \dots, 0)$. Therefore, $a = 0$. If $P \notin \operatorname{diag}(\mathbb{R}^n)$, then using the fact that G is connected, there exists $k \in \{1, \dots, n\}$ with $p_k = \max_{i \in \{1,\dots,n\}}\{p_i\}$ such that

$$\sum_{i \in \mathcal{N}_{G,k}} (p_i - p_k) < 0.$$

Therefore, $v_k = 0$. We deduce then $a = v \cdot e_k = 0$. Note that 0 always belongs to $\tilde{\mathcal{L}}_{X_{\operatorname{sgn}_+}} \max(P)$. Finally, we conclude $\tilde{\mathcal{L}}_{X_{\operatorname{sgn}_+}} \max(P) = \{0\}$, and therefore, by Theorem 2.1, $\max$ is constant along the trajectories of (13a), i.e., condition (ii) in Corollary 4.4 is satisfied.

Let us see that the trajectories of (13a) converge to $\operatorname{diag}(\mathbb{R}^n)$. To do this, we rely on the nonsmooth LaSalle Invariance Principle. Consider as candidate Lyapunov function $V = -\min$. Reasoning in a similar way as before, one can show that the set-valued Lie derivative is

$$\widetilde{\mathcal{L}}_{X_{\operatorname{sgn}_+}}(-\min)(P) = \begin{cases} \{0\}, & P \in \operatorname{diag}(\mathbb{R}^n), \\ \{-1\}, & P \notin \operatorname{diag}(\mathbb{R}^n). \end{cases}$$

Invoking Theorem 2.1, we deduce that $\min P(0) \leq p_i(t)$ for all $i \in \{1, \dots, n\}$. Since the $\max$ function is conserved along the trajectories, we deduce

$$\min P(0) \leq p_i(t) \leq \max P(0)\,, \quad i \in \{1, \dots, n\},$$

and therefore, the trajectories of (13a) are bounded. Note that $Z_{X_{\operatorname{sgn}_+}, -\min} = \operatorname{diag}(\mathbb{R}^n)$. The application of Theorem 2.2 yields, in particular, that all trajectories of the system converge to $\operatorname{diag}(\mathbb{R}^n)$, which establishes condition (i) in Corollary 4.4. The application of Proposition 2.3 with $\epsilon = 1$ implies that convergence is attained in finite time (actually, in exactly $\max(P_0) - \min(P_0)$ units). ■

Remark 5.6: The class of coordination algorithms $u : \mathbb{R}^n \to \mathbb{R}^n$ proposed in [9] to asymptotically achieve min consensus are of the following form

$$u_i(P) = h(p_i, \min_{j \in \mathcal{N}_G(i)} p_j), \quad i \in \{1, \dots, n\}, \tag{14}$$

with $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ verifying $h(x, x) = 0$ and $h(x, y) < 0$ if $x < y$. However, it is easy to see that this class of algorithms violates condition (i) in Theorem 4.3, and therefore, cannot asymptotically achieve min consensus. Consider for instance the function $h(x, y) = y - x$. Any trajectory starting from a configuration $(p_1, \dots, p_n) \in \mathbb{R}^n$ with some $i \in \{1, \dots, n\}$ such that $p_i = \min_{j=1,\dots,n} p_j$, and $p_i < p_j$ for all $j \neq i$ has $u_i(P) > 0$, and therefore, the value of $\min$ is not preserved along it. A similar objection can be raised for the max consensus case. •

VI. Conclusions

We have presented necessary and sufficient conditions for any coordination algorithm that asymptotically achieves consensus upon the value of an arbitrary function. Building on this characterization, we have (i) explored the setting of real analytic consensus functions; (ii) identified particular conditions on the consensus function under which distributed coordination algorithms can be automatically designed, (iii) characterized the exponential convergence properties of a class of distributed coordination algorithms that achieve weighted power mean consensus, and (iv) introduced distributed coordination algorithms that achieve max and min consensus in finite time.

Future work will proceed along three lines of research: (i) the investigation of results similar to the ones obtained here in the setting of networks with dynamically changing interconnection topologies; (ii) the further development of systematic methodologies to design distributed coordination algorithms for general consensus functions; and (iii) the application of the results to the synthesis of cooperative strategies for distributed estimation, data processing and fusion problems.

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