Coverage control by robotic networks with limited-range anisotropic sensory

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***Abstract*— This paper considers the deployment of a network of robotic agents with limited-range communication and anisotropic sensing capabilities. The natural locational optimization function to measure the network coverage of the environment has a gradient which is not amenable to distributed computation. We provide a constant-factor approximation of this measure via another objective function, whose gradient is spatially distributed over the limited-range Delaunay proximity graph. We characterize the smoothness properties of the latter function, and propose a distributed deployment algorithm to optimize it. Simulations illustrate the results.**

I. Introduction

Currently there is a large interest in the design of stable and decentralized control laws for distributed motion coordination. In this paper, we focus on the deployment of a robotic network where each agent is equipped with limited-range omnidirectional communication and anisotropic sensing capabilities (e.g., cameras). We model the restricted sensory range by defining a wedge-shaped region centered about each agent's orientation with an angular width less than or equal to π radians. Our objective is to design a distributed coordination algorithm that optimizes sensory coverage by the robotic network of a convex closed environment.

The literature on coordination tasks for robotic systems is becoming quite extensive. A sample of the research currently ongoing is presented in the recent special section [1] of the IEEE Control Systems Magazine. The deployment problem considered here falls within the field of facility location [2], [3], [4], where one seeks to optimize the position of a number of resources in order to provide better quality-of-service. In particular, this paper builds on [5], which provides an overview of coverage control for mobile networks, and [6], which models systems with limited-range interactions. Other works on coverage problems include [7], [8], [9], [10]. Our technical approach builds on concepts and notions from computational geometry and geometric optimization, such as Voronoi partitions [2], proximity graphs [11], and spatially distributed maps [6]. Lastly, we refer to [12], [13], [14] for an exposition of the nonsmooth stability analysis tools used in our discussion.

The contributions of the paper are threefold. First, we define a novel proximity graph, termed the (r, α)-limited wedge graph, which is relevant in our deployment problem. Second, we introduce a locational optimization function to measure the network coverage of the environment. Motivated by the fact that the gradient of this function is not amenable to distributed computation, we provide a constant-factor approximation via another aggregate objective function. We characterize the smoothness properties of the latter function and show that its gradient is spatially distributed over the r-limited Delaunay graph. Third, we propose a gradient ascent algorithm to optimize the network coverage of the environment and provide simulations to illustrate the algorithm execution.

The organization of this paper is as follows. Section II presents useful concepts on Voronoi partitions, proximity graphs, and spatially distributed maps. Section III introduces the locational optimization functions mentioned above, discusses a constant-factor approximation between them, and analyzes their distributed character. Based on these results, Section IV designs a deployment algorithm spatially distributed over the (r, α)-limited wedge graph. Section V presents simulations of our algorithm. Conclusions and plans for future research are discussed in Section VI. The appendices gather several auxiliary results. For reasons of space, the proofs of some results is omitted, and will be presented elsewhere.

II. Preliminary developments

In this section we present various notational conventions, and discuss useful notions from computational geometry. We begin by presenting some general notation. Let $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ be the set of real, positive real, and non-negative real numbers. Let $\mathbb{F}(\mathbb{R}^d)$ be the set of all finite pointsets in $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, let $x^{\mathbf{T}}$ denote the transpose of x. Given a set S in $\mathbb{R}^d$, let $\mathrm{co}(S)$ and $\mathrm{int}(S)$ be the convex hull and the interior of S, respectively. For $\phi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ integrable and $A \subset \mathbb{R}^n$, let $\mathrm{area}_\phi(A) = \int_A \phi(x) dx$. Let $\overline{B}(x, r)$ denote the closed ball centered at x with radius r, and $\mathrm{arc}(x, r)$ be an arc segment of $\partial \overline{B}(x, r)$. Throughout the paper $Q \subset \mathbb{R}^2$ denotes a simple convex polygon the *diameter* of Q is defined as $\mathrm{diam}(Q) = \max_{q,p \in Q} \|q - p\|$. Lastly, define the unit direction vector $u_\theta = [\cos\theta, \sin\theta]^{\mathbf{T}}$ and the matrices

$$\mathrm{Rot}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad \mathrm{Ref}_y = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

A. Voronoi partitions and boundary parameterizations

We begin by defining the notion of Voronoi partition generated by a set of distinct points, and then extend this notion to consider tuples of possibly coincident points.

1) Voronoi partitions for configurations with distinct points: Let $\mathcal{P} = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$ be a finite set. The *Voronoi partition* generated by $\mathcal{P}$ is the collection $\mathcal{V}(\mathcal{P}) = (V_1(\mathcal{P}), \dots, V_n(\mathcal{P}))$ where,

$$V_i(\mathcal{P}) = \{q \in \mathbb{R}^2 \mid \|q - p_i\| \leq \|q - p_j\|, \text{ for all } p_j \in \mathcal{P}\}.$$

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We will often use V_i instead of $V_i(\mathcal{P})$ to denote the Voronoi cell of agent i. Two agents i and j are Voronoi neighbors if $V_i \cap V_j \neq \emptyset$.

The section of the boundary of $V_i(\mathcal{P})$ that corresponds to the intersection with $V_j(\mathcal{P})$ is (counterclockwise) parameterized as

$$\gamma_{ij}(t) = \frac{p_i + p_j}{2} + t \operatorname{Rot}_{\frac{\pi}{2}}(p_j - p_i), \quad t \in [c_i, d_i], \tag{1}$$

for some $c_i, d_i \in \mathbb{R}$. The corresponding outward unit normal vector is $n_{ij} = \frac{p_j - p_i}{\|p_j - p_i\|}$, see Figure 1.

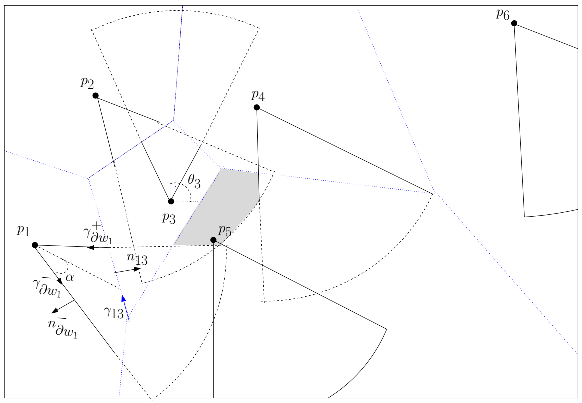

Fig. 1. Voronoi partition and sensing regions corresponding to a random deployment of a robotic network. Each agent has a wedge-shaped sensing region with $\alpha = \frac{\pi}{4}$ and $r = 0.5$. The various parameterizations of the Voronoi cell boundary and wedge sensing region are specified for agent 1.

2) Voronoi partitions for configurations with coincident points: We have introduced Voronoi partitions defined by sets of distinct points. We will also find it necessary to consider tuples of elements in $\mathbb{R}^2$ of the form $P = (p_1, \dots, p_n)$, i.e., ordered sets of possibly coincident points. For this reason, we introduce the natural immersion $i_{\mathbb{F}} : (\mathbb{R}^2)^n \to \mathbb{F}(\mathbb{R}^2)$ that maps P to the pointset $\mathcal{P}$ containing all distinct points in P. The cardinality of $\mathcal{P}$ is determined by whether P is an element of the set

$$\begin{aligned} S = \{(p_1, \dots, p_n) \in (\mathbb{R}^d)^n \\ \mid p_i = p_j \text{ for some } i, j \in \{1, \dots, n\},\ i \neq j\}. \end{aligned} \tag{2}$$

For $P \in S$, the cardinality of $\mathcal{P}$ is smaller than n, otherwise it is exactly n.

We are now ready to consider the notion of Voronoi partition generated by $P = (p_1, \dots, p_n)$. Let $\mathcal{P} = i_{\mathbb{F}}(P) = \{r_1, \dots, r_m\}$, $m \leq n$. For each $j \in \{1, \dots, m\}$, if there is only one $i \in \{1, \dots, n\}$ such that $p_i = r_j$, set $V_i(P) = V_j(\mathcal{P})$. Consider the case when there exist $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$ such that $p_{i_1} = \dots = p_{i_k} = r_j$. Let Γ be a circle centered at r_j contained within V_{r_j}. Let $L_1, \dots, L_k$ be a collection of rays emanating from r_j and let x_i be the point $\Gamma \cap L_i$ for each $i \in \{1, \dots, k\}$. Define the angle ϕ_i as the length of the arc $[x_i, x_{i+1}]$, so that $\sum_{i=1}^k \phi_i = 2\pi$. Then the set of angles $\Phi = \{\phi_1, \dots, \phi_k\}$ define a partition of V_{r_j} into k sub-regions $\{R_1(\Phi), \dots, R_k(\Phi)\}$. We wish to put a constraint on Φ to ensure that these regions represent a Voronoi partition. This is done so by administering the criteria established in [15, Theorems 7 & 8]. Specifically, for n odd, Φ corresponds to a Voronoi partition of the unit circle if and only if

$$\sum_{j=0}^{n-1} (-1)^j \phi_{i+j} > 0, \quad \text{for all } i \in \{1, \dots, n\}. \tag{3a}$$

If n is even, Φ corresponds to a Voronoi partition of the unit circle if and only if

$$\sum_{i=r}^{s} (-1)^{i-r} \phi_i \begin{cases} = 0, & \text{if } r = 1 \text{ and } s = n, \\ > 0, & \text{if } s - r \text{ even, } 1 \leq r < s \leq n. \end{cases} \tag{3b}$$

We then assign $V_{p_{i_j}}(P) = R_j(\Phi)$ for each $j \in \{1, \dots, k\}$. The resulting partition of Q is referred to as $\mathcal{V}(P, \Phi)$. Since the partitioning of $V_{r_j}(\mathcal{P})$ varies with the choice of Φ, we define the set $\Delta = \{\Phi = \{\phi_1, \dots, \phi_n\} \mid \Phi \text{ satisfies (3)}\}$. We may now define the *collection of Voronoi partitions generated by* P as

$$\mathcal{V}(P) = \big\{\mathcal{V}(P, \Phi)\big\}_{\Phi \in \Delta}.$$

Note that if $P \notin S$, then $\mathcal{V}(P) = \{\mathcal{V}(\mathcal{P})\}$.

B. Limited-range anisotropic sensory

Let $P = (p_1, \dots, p_n) \in Q^n$ be a tuple of points in Q, where p_i denotes the position of robot i, and let $\Theta = (\theta_1, \dots, \theta_n) \in (\mathbb{S}^1)^n$ be a tuple of angles, where θ_i indicates the orientation of robot i. For $P = (p_1, \dots, p_n) \in (\mathbb{R}^2)^n$ and $\Theta = (\theta_1, \dots, \theta_n) \in (\mathbb{S}^1)^n$, we denote $((p_1, \theta_1), \dots, (p_n, \theta_n)) \in (\mathbb{R}^2 \times \mathbb{S}^1)^n$ by (P, Θ) with a slight abuse of notation. We define the sensory region of the ith agent as the sector of a circle of radius r, centered at p_i, with orientation θ_i and amplitude 2α, $\alpha \in (0, \frac{\pi}{2}]$. We call this region the wedge viewing region and denote it by $w_{r,\alpha}(p_i, \theta_i)$. For brevity, we occasionally refer to agent i's wedge by w_i. The *angular range indicator* function is defined as

$$1_\alpha(z) = \begin{cases} 1, & \text{if } \arccos\left(\frac{|(z - p_i) \cdot (\cos\theta_i, \sin\theta_i)|}{\|z - p_i\|}\right) \leq \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

We also define the *limited-range indicator* function as

$$1_{\overline{B}(p_i, r)}(z) = \begin{cases} 1, & \text{if } \|z - p_i\| \leq r, \\ 0, & \text{otherwise.} \end{cases}$$

Then the *wedge indicator* function is defined as

$$1_{w_{r,\alpha}(p_i, \theta_i)} = 1_\alpha \, 1_{\overline{B}(p_i, r)}.$$

It is convenient to decompose the boundary of the wedge into the union of two line segments and an arc. We denote the wedge boundary line segments by ∂w_i^+ and ∂w_i^-. We then introduce the following (counterclockwise) parameterization of $\partial w_{r,\alpha}(p_i, \theta_i) = \partial w_i^+ \cup \operatorname{arc}(p_i, r) \cup \partial w_i^-$ as

$$\gamma_{\partial w_i^-}(t) = p_i + t u_{\theta - \alpha}, \qquad t \in [0, r], \tag{4a}$$

$$\gamma_{\operatorname{arc}(p_i, r)}(t) = p_i + r u_{\theta + t}, \qquad t \in [-\alpha, \alpha], \tag{4b}$$

$$\gamma_{\partial w_i^+}(t) = p_i + (r - t) u_{\theta + \alpha}, \quad t \in [0, r]. \tag{4c}$$

The corresponding (outward) normal vectors are

$$n_{\partial w_i^+}(q) = \mathrm{Ref}_y\, u_{\theta+\alpha}, \qquad q \in \partial w_i^+, \tag{5a}$$

$$n_{\mathrm{arc}(p_i,r)}(q) = \frac{q - p_i}{\|q - p_i\|}, \qquad q \in \mathrm{arc}(p_i, r), \tag{5b}$$

$$n_{\partial w_i^-}(q) = -\,\mathrm{Ref}_y\, u_{\theta+\alpha}, \qquad q \in \partial w_i^-. \tag{5c}$$

These parameterizations are illustrated in Figure 1.

C. Proximity graphs and spatially distributed maps

A proximity graph functions assigns to a pointset a graph whose vertex set is the pointset, and whose edge set is determined by the relative location of its vertices. The notion of proximity graph is useful to model the changing interaction between the agents of a mobile network. The reader is referred to [6], [11] for a detailed treatment.

Let X be a d-dimensional space composed of either $\mathbb{R}^d$, $\mathbb{S}^d$ or the Cartesian product $\mathbb{R}^{d_1} \times \mathbb{S}^{d_2}$, where $d_1 + d_2 = d$. Let $\mathbb{G}(X)$ be the set of directed graphs whose vertex set is an element of $\mathbb{F}(X)$. A *proximity graph function* $\mathcal{G} : \mathbb{F}(X) \to \mathbb{G}(X)$ associates to $\mathcal{V} \in \mathbb{F}(X)$ a graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}_{\mathcal{G}}(\mathcal{V})$, where $\mathcal{E}_{\mathcal{G}} : \mathbb{F}(X) \to \mathbb{F}(X \times X)$ has the property that $\mathcal{E}_{\mathcal{G}}(\mathcal{V}) \subseteq \mathcal{V} \times \mathcal{V} \setminus \mathrm{diag}(\mathcal{V} \times \mathcal{V})$.

The following proximity graph functions are relevant to our discussion:

(i) the *Delaunay graph* $\mathcal{P} \mapsto \mathcal{G}_D(\mathcal{P}) = (\mathcal{P}, \mathcal{E}_{\mathcal{G}_D}(\mathcal{P}))$, with

$$\mathcal{E}_{\mathcal{G}_{\mathrm{D}}}(\mathcal{P}) = \{p_i, p_j \in \mathcal{P}, \times\mathcal{P} \mid V_i \cap V_j \neq \emptyset\};$$

(ii) the r-*disk graph* $\mathcal{P} \mapsto \mathcal{G}_{\mathrm{disk}}(\mathcal{P}, r) = (\mathcal{P}, \mathcal{E}_{\mathcal{G}_{\mathrm{disk}}}(\mathcal{P}, r))$, with

$$\begin{aligned}&\mathcal{E}_{\mathcal{G}_{\mathrm{disk}}}(\mathcal{P}, r)\\ &\quad = \{(p_i, p_j) \in \mathcal{P} \times \mathcal{P} \setminus \mathrm{diag}(\mathcal{P} \times \mathcal{P}) \mid \|p_i - p_j\| \leq r\}\,;\end{aligned}$$

(iii) the r-*limited Delaunay (or limited-range Delaunay) graph* $\mathcal{P} \mapsto \mathcal{G}_{\mathrm{LD}}(\mathcal{P}, r) = (\mathcal{P}, \mathcal{E}_{\mathcal{G}_{\mathrm{LD}}}(\mathcal{P}, r))$ consists of the edges $(p_i, p_j) \in \mathcal{P} \times \mathcal{P} \setminus \mathrm{diag}(\mathcal{P} \times \mathcal{P})$ with the property that

$$\big(V_i(\mathcal{P}) \cap \overline{B}(p_i, \tfrac{r}{2})\big) \cap \big(V_j(\mathcal{P}) \cap \overline{B}(p_j, \tfrac{r}{2})\big) \neq \emptyset\,;$$

(iv) the (r, α)-*limited wedge graph* $(\mathcal{P}, \Theta) \mapsto \mathcal{G}_{\mathrm{LW}}(\mathcal{P}, \Theta) = (\mathcal{P}, \mathcal{E}_{\mathcal{G}_{\mathrm{LW}}}(\mathcal{P}, \Theta))$ consists of the edges $((p_i, \theta_i), (p_j, \theta_j)) \in (\mathcal{P}, \Theta) \times (\mathcal{P}, \Theta)$ with the property that

$$\big(V_i(\mathcal{P}) \cap V_j(\mathcal{P})\big) \cap \big(w_{\frac{r}{2},\alpha}(p_i, \theta_i) \cup w_{\frac{r}{2},\alpha}(p_j, \theta_j)\big) \neq \emptyset.$$

Note that the orientation of the agents does not affect the computation of the r-limited Delaunay graph. The r-limited Delaunay graph is undirected, whereas the (r, α)-limited wedge graph is directed. Clearly it is possible for $V_i \cap V_j \cap w_{r,\alpha}(p_i, \theta_i) \neq \emptyset$ and $V_i \cap V_j \cap w_{r,\alpha}(p_j, \theta_j) = \emptyset$ simultaneously (see Figure 2).

For a directed proximity graph with vertex v, a vertex q is an *in-neighbor* of v if the ordered pair $(q, v) \in \mathcal{E}_{\mathcal{G}}(\mathcal{V})$. Likewise, vertex v is an *out-neighbor* of q. To a proximity graph $\mathcal{G}$ and a vertex v, one can associate the set of *in-neighbors* and *out-neighbors* maps $\mathcal{N}^{in}_{\mathcal{G},v}, \mathcal{N}^{out}_{\mathcal{G},v} : \mathbb{F}(X) \to \mathbb{F}(X)$, defined as

$$\begin{aligned}\mathcal{N}^{in}_{\mathcal{G},v}(\mathcal{V}) &= \{q \in \mathcal{V} \mid (q, v) \in \mathcal{E}_{\mathcal{G}}(\mathcal{V} \cup \{v\})\},\\ \mathcal{N}^{out}_{\mathcal{G},v}(\mathcal{V}) &= \{q \in \mathcal{V} \mid (v, q) \in \mathcal{E}_{\mathcal{G}}(\mathcal{V} \cup \{v\})\}.\end{aligned}$$

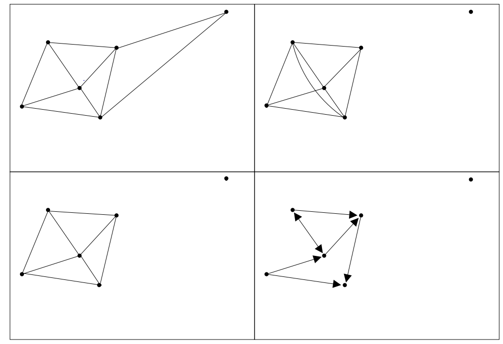

Fig. 2. From left to right, top to bottom: The Delaunay graph, the r-disk graph, the r-limited Delaunay graph, and the (r, α)-limited wedge graph which correspond to the configuration in Figure 1.

For the (r, α)-limited wedge graph, we define the neighbor maps

$$\begin{aligned}&\mathcal{N}^{in}_{\mathcal{G}_{\mathrm{LW}},(p_i,\theta_i)}(\mathcal{P}, \Theta)\\ &= \{(p_j, \theta_j) \in (\mathcal{P}, \Theta) \mid V_i(\mathcal{P}) \cap V_j(\mathcal{P}) \cap w_{\frac{r}{2},\alpha}(p_j, \theta_j) \neq \emptyset\},\\ &\mathcal{N}^{out}_{\mathcal{G}_{\mathrm{LW}},(p_i,\theta_i)}(\mathcal{P}, \Theta)\\ &= \{(p_j, \theta_j) \in (\mathcal{P}, \Theta) \mid V_i(\mathcal{P}) \cap V_j(\mathcal{P}) \cap w_{\frac{r}{2},\alpha}(p_i, \theta_i) \neq \emptyset\}.\end{aligned}$$

In other words, agent j is an in-neighbor of agent i if it is a nearby neighbor who "sees" agent i and is an out-neighbor of agent i if it is "seen" by agent i.

A proximity graph $\mathcal{G}_1$ is *spatially distributed* over an undirected proximity graph $\mathcal{G}_2$ if, for all $v \in \mathcal{V}$,

$$\begin{aligned}\mathcal{N}^{in}_{\mathcal{G}_1,v}(\mathcal{V}) &= \mathcal{N}^{in}_{\mathcal{G}_1,v}(\mathcal{N}_{\mathcal{G}_2,v}(\mathcal{V})),\\ \mathcal{N}^{out}_{\mathcal{G}_1,v}(\mathcal{V}) &= \mathcal{N}^{out}_{\mathcal{G}_1,v}(\mathcal{N}_{\mathcal{G}_2,v}(\mathcal{V})).\end{aligned}$$

Lemma II.1 *The (r, α)-limited wedge graph $\mathcal{G}_{\mathrm{LW}}$ is spatially distributed over the r-limited Delaunay graph $\mathcal{G}_{\mathrm{LD}}$.*

Functions which are spatially distributed over proximity graphs do not necessarily have to be defined as graph functions. More generally, given a set Y and an undirected proximity graph $\mathcal{G}$, the map $f : X^n \to Y^n$ is *spatially distributed* over $\mathcal{G}$ if there exists maps $\tilde{f}_i : X \times \mathbb{F}(X) \to Y^n$, for $i \in \{1, \ldots, n\}$ with the property that for all $(v_1, \ldots, v_n) \in X^n$,

$$f_i(v_1, \ldots, v_n) = \tilde{f}_i(v_i, \mathcal{N}_{\mathcal{G},i}(v_1, \ldots, v_n)),$$

where f_i denoted the ith component of f. Thus to compute the ith component of a spatially distributed function, one needs only to know the location of the vertex v_i and the location (and possibly orientation) of its neighbors on the graph $\mathcal{G}(\mathcal{V})$. This generalization allows for objects like vector fields and set-valued maps to seen as spatially distributed over proximity graphs when defined in the appropriate context.

III. Locational optimization: network performance and smoothness analysis

We begin by introducing measures of the sensory coverage of the environment by our robotic network.

A. Locational optimization functions

Let $\phi : Q \to \mathbb{R}_{\geq 0}$ be an integrable function that we term *density function*. It can be thought of as a measure of the probability of some event taking place over Q. Due to noise and interference, the sensory performance of each agent p_i degrades at point q in proportion to the distance $\|q - p_i\|$. Thus, we introduce a continuously differentiable, strictly positive, non-increasing function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ to measure this degradation. In other words, $f(\|q - p_i\|)$ provides a quantitative assessment of sensory quality of the ith robot at point $q \in Q$. We refer to f as a *performance function*.

Consider the locational optimization function $\mathcal{H} : (Q \times \mathbb{S}^1)^n \to \mathbb{R}_{\geq 0}$ defined by

$$\mathcal{H}(P, \Theta) = \int_Q \max_{i \in \{1,\dots,n\}} f(\|q - p_i\|) 1_{w_{r,\alpha}(p_i,\theta_i)}(q)\phi(q)dq. \tag{6}$$

This function provides an expected value of the sensory performance by the robotic network weighted by the density function ϕ. Hence, it is of interest to find maximizers of $\mathcal{H}$. However, its gradient is in general not distributed over the r-limited Delaunay or (r, α)-limited wedge proximity graphs. Figure 1 illustrates this assertion. The sensing region of p_2 is the only one that contains the grey region depicted in the plot. Therefore, changes in p_5 will affect the gradient of $\mathcal{H}$ with respect to p_2. However, p_2 and p_5 are not neighbors in the r-limited Delaunay or (r, α)-limited wedge graphs.

Our approach is to provide an alternative locational optimization function that approximates $\mathcal{H}$. Specifically, we define

$$\mathcal{H}_{\mathcal{V}}(P, \Theta) = \sum_{i=1}^{n} \int_{V_i(P)} f(\|q - p_i\|) 1_{w_{r,\alpha}(p_i,\theta_i)}(q)\phi(q)dq. \tag{7}$$

Note that $\mathcal{H}_{\mathcal{V}}$ sums the individual sensory performance of the robots within the intersection of their respective sensing wedge and Voronoi cell. Next, we establish two important properties of this function. On one hand, we show that it provides a good approximation of $\mathcal{H}$. On the other hand, we characterize its smoothness properties, and compute its (generalized) gradient.

B. Additive and constant-factor approximations

We next provide an additive and constant factor approximation of the locational optimization function $\mathcal{H}$ by $\mathcal{H}_{\mathcal{V}}$. Given $\epsilon > 0$, let

$$\Sigma_\epsilon = \{(P, \Theta) \in (Q \times \mathbb{S}^1)^n \mid \sum_{i=1}^{n} \mathrm{area}_\phi(V_i \cap w_i) \geq \epsilon\}.$$

The constant-factor approximations are restricted to configurations in Σ_ϵ.

Proposition III.1 *Let f be a strictly positive performance function. Consider the objective functions defined in* (6) *and* (7)*. Then, for all $(P, \Theta) \in \Sigma_\epsilon$,*

$$\mathcal{H}_{\mathcal{V}}(P, \Theta) \leq \mathcal{H}(P, \Theta) \leq \mathcal{H}_{\mathcal{V}}(P, \Theta) + C,$$
$$\mathcal{H}_{\mathcal{V}}(P, \Theta) \leq \mathcal{H}(P, \Theta) \leq K\mathcal{H}_{\mathcal{V}}(P, \Theta),$$

where $C = \|f\|_\infty \, \mathrm{area}_\phi(Q)$ and $K = 1 + \frac{\|f\|_\infty \, \mathrm{area}_\phi(Q)}{\epsilon f(\mathrm{diam}(Q))}$.

Remark III.2 For any configuration $P \notin S$ with the property that $\|p_i - p_j\|$ is strictly greater than $2r$ for each $i, j \in \{1, \dots, n\}$, it is not hard to show that the values of $\mathcal{H}$ and $\mathcal{H}_{\mathcal{V}}$ coincide, i.e., $\mathcal{H}(P, \Theta) = \mathcal{H}_{\mathcal{V}}(P, \Theta)$, for any $\Theta \in (\mathbb{S}^1)^n$. •

C. Smoothness properties of $\mathcal{H}_{\mathcal{V}}$

We next explore the smoothness properties of the locational optimization function $\mathcal{H}_{\mathcal{V}}$ and compute its gradient.

Theorem III.3 *Given a density function ϕ and a performance function f, the locational optimization function $\mathcal{H}_{\mathcal{V}}$ is globally Lipschitz on $(Q \times \mathbb{S}^1)^n$.*

Proof: Let $P = (p_1, \dots, p_n) \in Q^n$ and for each $q \in Q$ define $p_{\mathrm{min}}(q) \in P$ so that $\|q - p_{\mathrm{min}}(q)\| = \min_{i \in \{1,\dots,n\}} \|q - p_i\|$. For the sake of simplicity, denote $p_{\mathrm{min}}(q)$ as p_{min}. Then we can write $\mathcal{H}_{\mathcal{V}}$ as

$$\mathcal{H}_{\mathcal{V}}(P, \Theta) = \int_Q f(\|q - p_{\mathrm{min}}\|) 1_{w_\alpha(p_{\mathrm{min}},\theta_{\mathrm{min}})}(q)\phi(q)dq,$$

We now introduce a useful partition of Q. For $W \subset Q$, recall $W^c = Q \backslash W$. Given $(P, \Theta) = \Big((p_1, \theta_1) \dots, (p_n, \theta_n)\Big)$, $(P', \Theta') = \Big((p'_1, \theta'_1) \dots, (p'_n, \theta'_n)\Big)$, define the following sets

$$W_1 = \Big(\cap_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_i, \theta_i)^c\Big) \cap \Big(\cup_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_{i'}, \theta_{i'})^c\Big),$$
$$W_2 = \Big(\cup_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_i, \theta_i)\Big) \cap \Big(\cup_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_{i'}, \theta_{i'})\Big),$$
$$W_3 = \Big(\cup_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_i, \theta_i)\Big) \cap \Big(\cap_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_{i'}, \theta_{i'})^c\Big),$$
$$W_4 = \Big(\cup_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_{i'}, \theta_{i'})\Big) \cap \Big(\cap_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_i, \theta_i)^c\Big).$$

Note that $W_1, \dots, W_4$ form a pairwise disjoint partition of Q. We then write,

$$\begin{aligned}\mathcal{H}_{\mathcal{V}}(P, \Theta) &- \mathcal{H}_{\mathcal{V}}(P', \Theta') \\ &= \sum_{k=1}^{4} \int_{W_k} \Big(f(\|q - p_{\mathrm{min}}\|) 1_{w_\alpha(p_{\mathrm{min}},\theta_{\mathrm{min}})}(q) \\ &\quad - f(\|q - p'_{\mathrm{min}}\|) 1_{w_\alpha(p'_{\mathrm{min}},\theta'_{\mathrm{min}})(q)}(q)\Big)\phi(q)dq.\end{aligned}$$

Now we shall find an upper bound of this sum. For $q \in W_1$, both $1_{w_\alpha(p_{\mathrm{min}},\theta_{\mathrm{min}})}(q)$ and $1_{w_\alpha(p_{\mathrm{min}},\theta_{\mathrm{min}})}(q)$ are zero, so the integral over W_1 vanishes. For $q \in W_2$ we have,

$$\begin{aligned}
&\Big| \int_{W_2} \Big(f(\|q - p_{\min}\|) 1_{w_\alpha(p_{\min},\theta_{\min})}(q) \\
&\qquad - f(\|q - p'_{\min}\|) 1_{w_\alpha(p'_{\min},\theta'_{\min})}(q) \Big) \phi(q) dq \Big| \\
&\le \int_{W_2} |f(\|q - p_{\min}\|) - f(\|q - p'_{\min}\|)| \phi(q) dq \\
&\le \int_{W_2} \left\| \frac{df}{dx} \right\|_\infty \Big| \|q - p_{\min}\| - \|q - p'_{\min}\| \Big| \phi(q) dq \\
&\le \left\| \frac{df}{dx} \right\|_\infty \|P - P'\| \int_{W_2} \phi(q) dq \\
&= \left\| \frac{df}{dx} \right\|_\infty \text{area}_\phi(Q) \|P - P'\|.
\end{aligned}$$

Here, we have made use of the fact that for all $q \in Q$, the maps $\|q - p_{\min}\| \mapsto f(\|q - p_{\min}\|)$ and $P \mapsto \|q - p_{\min}\|$ are both continuous on a compact domain and therefore Lipschitz continuous with a Lipschitz constant equal to their respective derivatives. For $q \in W_3$ we have,

$$\begin{aligned}
&\Big| \int_{W_3} \Big(f(\|q - p_{\min}\|) 1_{w_\alpha(p_{\min},\theta_{\min})}(q) \\
&\qquad - f(\|q - p'_{\min}\|) 1_{w_\alpha(p'_{\min},\theta'_{\min})}(q) \Big) \phi(q) dq \Big| \\
&\le \int_{W_3} f(\|q - p_{\min}\|) \phi(q) dq \\
&\le \|f\|_\infty \|\phi\|_Q \sum_{i=1}^n \int_{w_{r,\alpha}(p_i,\theta_i) \cap_{i \in \{1,\dots,n\}} w_{r,\alpha}(p_{i'},\theta_{i'})^c} dq \\
&\le \|f\|_\infty \|\phi\|_Q \sum_{i=1}^n \int_{w_{r,\alpha}(p_i,\theta_i) \cap w_{r,\alpha}(p_{i'},\theta_{i'})^c} dq,
\end{aligned}$$

where $\|\phi\|_Q = \max_{q \in Q} \phi(q)$. We then observe

$$\begin{aligned}
&\int_{w_{r,\alpha}(p_i,\theta_i) \cap w_{r,\alpha}(p_{i'},\theta_{i'})^c} dq \\
&\qquad \le 2(1+\alpha) r \|p_i - p'_i\| + \frac{1}{2} r^2 |\theta_i - \theta'_i|
\end{aligned}$$

by Lemma C.1. Hence,

$$\begin{aligned}
&\Big| \int_{W_3} \Big(f(\|q - p_{\min}\|) 1_{w_\alpha(p_{\min},\theta_{\min})}(q) \\
&\qquad - f(\|q - p'_{\min}\|) 1_{w_\alpha(p'_{\min},\theta'_{\min})}(q) \Big) \phi(q) dq \Big| \\
&\le \|f\|_\infty \|\phi\|_Q \sum_{i=1}^n 2(1+\alpha) r \|p_i - p'_i\| + \frac{1}{2} r^2 |\theta_i - \theta'_i| \\
&= \|f\|_\infty \|\phi\|_Q (2(1+\alpha) r \|P - P'\| + \frac{1}{2} r^2 |\Theta - \Theta'|).
\end{aligned}$$

The integral over W_4 can be bounded in a similar fashion. We have now shown

$$\begin{aligned}
|\mathcal{H}_\mathcal{V}(P,\Theta) - \mathcal{H}_\mathcal{V}(P',\Theta')| &\le L\|P - P'\| + M\|\Theta - \Theta'\| \\
&\le \max\{M, L\}(\|P - P'\| + \|\Theta - \Theta'\|) \\
&= \max\{M, L\}\|(P,\Theta) - (P',\Theta')\|
\end{aligned}$$

where $L = \left\| \frac{df}{dx} \right\|_\infty \text{area}_\phi(Q) + 4r(1+\alpha)\|f\|_\infty \|\phi\|_Q$ and $M = \|f\|_\infty \|\phi\|_Q r^2$. Thus, $\mathcal{H}_\mathcal{V}$ is globally Lipschitz. ■

Theorem III.4 *Given a density function ϕ and a performance function f, the locational optimization function $\mathcal{H}_\mathcal{V}$ is continuously differentiable on $(\text{int}(Q) \times \mathbb{S}^1)^n \setminus S$ and for each $i \in \{1,\dots,n\}$ the gradient of $\mathcal{H}_\mathcal{V}$ is given as,*

$$\begin{aligned}
\frac{\partial \mathcal{H}_\mathcal{V}}{\partial p_i} &= \int_{V_i \cap w_{r,\alpha}(p_i,\theta_i)} \frac{\partial}{\partial p_i} f(\|q - p_i\|) \phi(q) dq \qquad (8a) \\
&+ \int_{(\partial w_i^+ \cup \partial w_i^-) \cap V_i} f(\|q - p_i\|) \phi(q) n_{\partial w_i^{(\cdot)}} dq \\
&+ \sum_{l=1}^{m_i} \int_{\text{arc}_l(p_i,r)} f(\|q - p_i\|) \phi(q) n_{\partial\,\text{arc}(p_i,r)} dq \\
&+ \sum_{\substack{j=1 \\ j \neq i}}^n \frac{1}{2\|p_j - p_i\|} \Big(\int_{V_i \cap V_j \cap w_i} f(\|q - p_i\|) \phi(q)(q - 2p_j) dq \\
&\qquad - \int_{V_i \cap V_j \cap w_j} f(\|q - p_j\|) \phi(q)(q - 2p_j) dq \Big),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \mathcal{H}_\mathcal{V}}{\partial \theta_i} &= \int_{\partial w_i^+ \cap V_i} \|q - p_i\| f(\|q - p_i\|) \phi(q) dq \qquad (8b) \\
&\quad - \int_{\partial w_i^- \cap V_i} \|q - p_i\| f(\|q - p_i\|) \phi(q) dq,
\end{aligned}$$

where $n_{(\cdot)} : \partial(V_k \cap w_{r,\alpha}(p_k,\theta_k)) \to \mathbb{R}^2$ is the normal vector parameterization of the specified boundaries as described in Section II-A and $\text{arc}_l(p_i,r)$, $l \in \{1,\dots,m_i\}$ are the arcs in the boundary $V_i \cap \text{arc}(p_i,r)$.

Proof: Let $P \in \text{int}(Q) \setminus S$. Then we can write equation (7) as

$$\begin{aligned}
\mathcal{H}_\mathcal{V} &= \sum_{i=1}^n \int_{V_i} f(\|q - p_i\|) 1_{w_{r,\alpha}(p_i,\theta_i)}(q) \phi(q) dq \\
&= \sum_{i=1}^n \int_{V_i \cap w_{r,\alpha}(p_i,\theta_i)} f(\|q - p_i\|) \phi(q) dq.
\end{aligned}$$

Note that $f(\|q - p_i\|)$ is continuously differentiable and for fixed $(P,\Theta) \in \text{int}(Q)^n$, the maps $q \mapsto f(\|q - p_i\|)$ and $q \mapsto \frac{\partial}{\partial P} f(\|q - p_i\|)$ are both measurable and integrable on $V_i \cap w_{r,\alpha}(p_i,\theta_i)$. Also note since both the Voronoi partition and the wedge are convex sets, their intersection is also convex [6]. By Proposition B.1, $\mathcal{H}_\mathcal{V}$ is continuously differentiable and for each $i \in \{1,\dots,n\}$,

$$\begin{aligned}
\frac{\partial \mathcal{H}_\mathcal{V}}{\partial p_i}(P,\Theta) &= \frac{\partial}{\partial p_i} \sum_{k=1}^n \int_{V_k \cap w_k} f(\|q - p_k\|) \phi(q) dq \qquad (9) \\
&= \int_{V_i \cap w_i} \frac{\partial}{\partial p_i} f(\|q - p_k\|) \phi(q) dq \\
&+ \sum_{k=1}^n \int_{\partial(V_k \cap w_k)} f(\|\gamma - p_k\|) \phi(\gamma) n^{\mathbf{T}}(\gamma) \frac{\partial \gamma}{\partial p_i} d\gamma
\end{aligned}$$

To simplify the second term in this equation, note that the boundary $\partial(V_k \cap w_{r,\alpha}(p_k,\theta_k))$ is composed of a finite number of line segments and arcs, all of which have been parameterized previously in Section II-A. We first integrate over the wedge boundary $V_k \cap \partial(w_{r,\alpha}(p_k,\theta_k)) = V_k \cap (\partial w_k^+ \cup$

$\partial w_k^-)\cup_{l=1}^m \mathrm{arc}_l(p_k,r)$ (see (4)). This integral is nonzero only when $k=i$. Note that when there is a displacement in the position of p_i, the motion of $w_{r,\alpha}(p_i,\theta_i)$ (when projected along the appropriate normal vector) is exactly the same as p_i i.e., $n_{(\cdot)}^{\mathbf{T}} \frac{\partial \gamma_{(\cdot)}}{\partial p_i} = n_{(\cdot)}$. Hence,

$$\begin{aligned}\int_{V_i\cap(\partial w_k^+\cup\partial w_k^-)\cup_{l=1}^m \mathrm{arc}(p_i,r))} & f(\|q-p_i\|)\phi(q) n_{(\cdot)}^{\mathbf{T}} \frac{\partial \gamma_{(\cdot)}}{\partial p_i} dq \\ &= \int_{V_i\cap(\partial w_i^+\cup\partial w_i^-)} f(\|q-p_i\|)\phi(q) n_{\partial w_i^{(\cdot)}} dq \\ &+ \sum_{l=1}^m \int_{\mathrm{arc}_l(p_i,r)} f(\|q-p_i\|)\phi(q) n_{\overline{B}(p_i,r)} dq.\end{aligned}$$

The remaining boundary segments that must be considered define the regions $V_k\cap V_j\cap w_{r,\alpha}(p_k,\theta_k)$ for $j\in\{1,\dots,n\}$. To parameterize these boundaries, consider the map given in (1). The derivative of this map with respect to p_i is non-zero only within the regions $V_i\cap V_j\cap w_{r,\alpha}(p_i,\theta_i)$ and $V_j\cap V_i\cap w_{r,\alpha}(p_j,\theta_j)$, i.e., when $p_j\in\mathcal{N}^{in}_{\mathcal{G}_{\mathrm{LW}},(p_i,\theta_i)}(P,\Theta)$ or when $p_j\in\mathcal{N}^{out}_{\mathcal{G}_{\mathrm{LW}},(p_i,\theta_i)}(P,\Theta)$. We separate these boundaries accordingly. Next observe,

$$\begin{aligned} n_{ij}^{\mathbf{T}}\frac{\partial\gamma_{ij}}{\partial p_i} &= \frac{1}{2}n_{ij}^{\mathbf{T}} + \frac{t}{\|p_j-p_i\|}\mathrm{Rot}_{\frac{\pi}{2}}(p_j-p_i) \\ &= \frac{1}{2}n_{ij}^{\mathbf{T}} + \frac{1}{\|p_j-p_i\|}(\gamma_{ij}-\frac{p_i+p_j}{2}) \\ &= \frac{2\gamma_{ij}-p_j}{\|p_j-p_i\|}.\end{aligned}$$

For $p_j\in\mathcal{N}^{in}_{\mathcal{G}_{\mathrm{LW}},(p_i,\theta_i)}(P,\Theta)$, the outward normal vector n_{ij} is used and for $p_j\in\mathcal{N}^{out}_{\mathcal{G}_{\mathrm{LW}},(p_i,\theta_i)}(P,\Theta)$, the inward normal vector $n_{ji}=-n_{ij}$ is used to preserve counter-clockwise orientation with respect to p_i. We place these formulations back into (9) to obtain the complete form of (8a).

To compute the partial derivative of $\mathcal{H}_{\mathcal{V}}$ with respect to θ_i, assume the same general form of (9) and same parameterization given in (4). Since the boundary $\partial(w_{r,\alpha}(p_i,\theta_i))\cap V_i$ contains the only parameterization with a dependency on θ_i, we have

$$\frac{\partial\mathcal{H}_{\mathcal{V}}}{\partial\theta_i}(P,\Theta) = \int_{\partial(w_{r,\alpha}(p_i,\theta_i))\cap V_i} f(\|\gamma-p_i\|)\phi(\gamma)n^{\mathbf{T}}(\gamma)\frac{\partial\gamma}{\partial\theta_i}d\gamma.$$

Notice however, that the normal vector $n_{\overline{B}(p_i,r)}$ is orthogonal to $\frac{\partial\gamma_{\mathrm{arc}(p_i,r)}}{\partial\theta_i}$. Hence, the only regions which need to be considered are the line segments $\partial w_i^+\cap V_i$ and $\partial w_i^-\cap V_i$. For $q\in\partial w_i^+$ we compute,

$$n_{\partial w_i^+}^{\mathbf{T}}\frac{\partial\gamma_{\partial w_i^+}}{\partial\theta_i} = m^+ - t = \|\gamma_{\partial w_i^+}-p_i\|.$$

Hence,

$$\begin{aligned}\int_{\partial w_i^+\cap V_i} & f(\|\gamma_{\partial w_i^+}-p_i\|)\phi(\gamma_{\partial w_i^+})n(\gamma_{\partial w_i^+})^{\mathbf{T}}\frac{\partial\gamma_{\partial w_i^+}}{\partial\theta_i}d\gamma_{\partial w_i^+} \\ &= \int_{\partial w_i^+\cap V_i}\|q-p_i\|f(\|q-p_i\|)\phi(q)dq.\end{aligned}$$

A similar calculation is made for the integral over $\partial w_i^-\cap V_i$. This completes the proof. ∎

Remark III.5 Using extension by continuity, we shall redefine the domain where $\mathcal{H}_{\mathcal{V}}$ is continuously differentiable to include the boundary of Q. •

The function $\mathcal{H}_{\mathcal{V}}$ fails to be differentiable on the set S defined in (2), so it will be necessary to characterize its generalized gradient (see Appendix A). Recall that for $P\in S$, $\mathcal{V}(P)$ is defined as an infinite family of Voronoi partitions (see Section II-A).

Proposition III.6 *Let $(P,\Theta)\in(Q\times\mathbb{S}^1)^n$. Then the generalized gradient of $\mathcal{H}_{\mathcal{V}}$ is given by*

$$\partial\mathcal{H}_{\mathcal{V}}(P,\Theta) = \begin{cases}(\frac{\partial\mathcal{H}_{\mathcal{V}}}{\partial P},\frac{\partial\mathcal{H}_{\mathcal{V}}}{\partial\Theta}), & \text{if } P\notin S,\\ \mathrm{co}\{(\frac{\partial\mathcal{H}_{\mathcal{V}}}{\partial P},\frac{\partial\mathcal{H}_{\mathcal{V}}}{\partial\Theta})|_{V=W} : W\in\mathcal{V}(P)\}, & \text{if } P\in S.\end{cases}$$

Proof: By $(\frac{\partial\mathcal{H}_{\mathcal{V}}}{\partial P},\frac{\partial\mathcal{H}_{\mathcal{V}}}{\partial\Theta})|_{V=W}$, we mean the gradient direction resulting from taking W as the Voronoi partition in the evaluation of the expression (8). The result follows from the definition of the generalized gradient (cf. (11)). Note that the configurations in S represent the set of points over which $\mathcal{H}_{\mathcal{V}}$ fails to be differentiable. ∎

IV. A DISTRIBUTED ALGORITHM FOR LOCATIONAL OPTIMIZATION

Here we present a distributed algorithm which maximizes the locational optimization function $\mathcal{H}_{\mathcal{V}}$. We implement our control law in continuous time and analyze its convergence properties. Assume the evolution of the robotic agents obeys a first-order dynamical system described by

$$\begin{bmatrix}\dot{p}_i\\ \dot{\theta}_i\end{bmatrix} = \begin{bmatrix}u_i\\ v_i\end{bmatrix}, \quad i\in\{1,\dots,n\}.$$

We implement a (generalized) gradient ascent of the locational optimization function $\mathcal{H}_{\mathcal{V}}$ using the result in Proposition III.6. In other words, we set

$$\begin{bmatrix}u_i\\ v_i\end{bmatrix} = \mathrm{Ln}(\partial\mathcal{H}_{\mathcal{V}})(P,\Theta). \tag{10}$$

We assume that the partition $\mathcal{V}(P)$ is updated continuously. Note that this vector field is discontinuous and therefore the solutions of this system must be understood in the Filippov sense. The following result can be stated.

Theorem IV.1 *Given a density function ϕ and a performance function f, the control law on $(Q\times\mathbb{S}^1)^n$ defined by* (10) *has the following properties:*

(i) *the law is spatially distributed over the r-limited Delaunay graph $\mathcal{G}_{LD}(\mathcal{P},r)$, and;*
(ii) *the agents' location with initial configuration $(P_0,\Theta_0)\in(Q\times\mathbb{S}^1)^n$ converges asymptotically to the set of critical points of $\mathcal{H}_{\mathcal{V}}$.*

Proof: Statement (i) follows from the fact that, according to (8), the gradient of $\mathcal{H}_{\mathcal{V}}$ depends only on the position and orientation of p_i as well as those of its in and out neighbors in the (r,α)-limited wedge graph $\mathcal{G}_{\mathrm{LW}}$, and, according to Lemma II.1, this graph is spatially distributed over $\mathcal{G}_{LD}$. Statement (ii) follows from considering the dynamical system defined by (10) on the compact and strongly invariant domain $(Q \times \mathbb{S}^1)^n$. Theorem III.3 guarantees that the function $\mathcal{H}_{\mathcal{V}}$ is globally Lipschitz, and Proposition A.1 ensures the gradient ascent system converges to the set of critical points of $\mathcal{H}_{\mathcal{V}}$. ∎

V. Simulations

To illustrate the performance of the network under the coordination algorithm (10), we present some numerical simulations. The algorithm is implemented in Mathematica® as a main program running the simulation that makes use of a library of routines. The structure of this simulation is loosely described by the following procedure: first, the intersection of the bounded Voronoi cell V_i and the wedge $w_{r,\alpha}(p_i, \theta_i)$, for $i \in \{1, \ldots, n\}$, is computed. Next, the r-limited Delaunay and (r, α)-limited wedge proximity graphs are constructed. Then, for each agent, information of its in/out neighbors is collected and used to construct the various parameterizations necessary for the gradient computation. Finally, the various surface and boundary integrals involved in the gradient of the locational optimization function $\mathcal{H}_{\mathcal{V}}$ are computed using the Mathematica® numerical integration routine `NIntegrate`. The position and orientation of each agent are then updated according to these results. Figure V illustrates an execution.

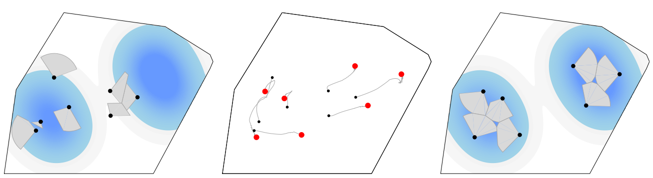

Fig. 3. Execution of the gradient ascent strategy (10) by a network of 7 agents. The plot on the left (resp. right) illustrates the initial (resp. final) configuration after 75 milliseconds. The central figure illustrates the gradient ascent flow of the system, with the smaller dots representing the initial configuration and the larger dots representing the final one. For this example, the performance measure is given by $f(x) = 2 - x^2$ and the distribution density function ϕ (represented by means of its contour plot) is the sum of three Gaussian functions of the form $50e^{-10((x-x_{\text{cntr}})^2+(y-y_{\text{cntr}})^2)}$.

VI. Conclusion

We have introduced two aggregate objective functions to measure the coverage of the environment provided by a group of robotic agents with limited-range anisotropic sensory. Based on considerations about the distributed computation of the gradient information, we have selected the objective function whose definition is based on a partitioning of the environment. We have characterized the smoothness properties of this objective function, computed its gradient, and characterized its spatially-distributed character. Combining these results with tools from nonsmooth analysis, we have designed a gradient ascent strategy that is guaranteed to achieve optimal network deployment. Further research will include the design and analysis of coordination algorithms implemented in discrete time, the synthesis of cooperative strategies to attain global optima of the aggregate objective function, and the study of similar deployment problems in nonconvex environments.

Acknowledgments

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Appendix A
Nonsmooth analysis

Here we review some basic concepts from nonsmooth analysis. The reader is referred to [12], [13], [14] for a more detailed treatment.

From Rademacher's Theorem [12], locally Lipschitz functions are differentiable a.e. If Ω_f denotes the set of points in $\mathbb{R}^N$ where f fails to be differentiable and S is any set of measure zero, the *generalized gradient* of f is

$$\partial f(x) = \mathrm{co}\{\lim_{i \to +\infty} df(x_i) \mid x_i \to x\,,\ x_i \notin S \cup \Omega_f\}\,. \quad (11)$$

A point $x \in \mathbb{R}^N$ with $0 \in \partial f(x)$ is a *critical point of f*.

Let $F : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ be a set-valued map. A solution to the differential inclusion $\dot{x} \in F(x)$ on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as an absolutely continuous function $x : [t_0, t_1] \to$

$\mathbb{R}^N$ such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in [t_0, t_1]$. Now, consider the equation

$$\dot{x}(t) = X(x(t)), \tag{12}$$

where $X : \mathbb{R}^N \to \mathbb{R}^N$ is measurable and essentially locally bounded. Let

$$K[X](x) = \bigcap_{\delta > 0} \bigcap_{\mu(S)=0} \operatorname{co}\{X(B_N(x,\delta) \setminus S)\}, \quad x \in \mathbb{R}^N.$$

A Filippov solution of (12) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion $\dot{x} \in K[X](x)$. A set M is *strongly invariant* for (12) if for each $x_0 \in M$, contains all maximal solutions of (12).

Let $\operatorname{Ln} : 2^{\mathbb{R}^N} \to \mathbb{R}$ be the map that associates to each convex set $S \subset \mathbb{R}^N$ its least-norm element, $\operatorname{Ln}(S) = \operatorname{proj} S(0)$. Given a locally Lipschitz and regular function f, consider

$$\dot{x}(t) = -\operatorname{Ln}(\partial f)(x(t)). \tag{13}$$

In general, the vector field $\operatorname{Ln}(\partial f)$ is discontinuous, and therefore the solution of (13) must be understood in the Filippov sense. Since f is locally Lipschitz, $\operatorname{Ln}(\partial f) = df$ almost everywhere. The following result guarantees the convergence to the set of critical points of f.

Proposition A.1 *Let $\mathcal{S} \subset \mathbb{R}^N$ be compact and strongly invariant for* (13)*. Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^N$ of* (13) *starting from a point in $\mathcal{S}$ converges asymptotically to the set of critical points of f contained in $\mathcal{S}$.*

APPENDIX B
A GENERALIZED CONSERVATION OF MASS LAW

The following result, originally stated and proved in [6], is an extension of the integral form of the Conservation-of-Mass law in fluid mechanics.

Proposition B.1 *Let $\{\Omega(x) \subset \mathbb{R}^2 \mid x \in (a,b)\}$ be a piece-wise smooth family of curves such that $\Omega(x)$ is strictly star-shaped for all $x \in (a,b)$. Let the function $\varphi : \mathbb{R}^2 \times (a,b) \to \mathbb{R}$ be continuous on $\mathbb{R}^2 \times (a,b)$, continuously differentiable with respect to its second argument for all $x \in (a,b)$ and almost all $q \in \Omega(x)$, and such that for each $x \in (a,b)$, the maps $q \mapsto \varphi(q,x)$ and $q \mapsto \frac{\partial \varphi}{\partial x}(q,x)$ are measurable, and integrable on $\Omega(x)$. Then, the function*

$$(a,b) \ni x \mapsto \int_{\Omega(x)} \varphi(q,x)dq$$

is continuously differentiable and

$$\frac{d}{dx}\int_{\Omega(x)} \varphi(q,x)dq = \int_{\Omega(x)} \frac{\partial \varphi}{\partial x}(q,x)dq + \int_{\partial\Omega(x)} \varphi(\gamma,x)n^t(\gamma)\frac{\partial \gamma}{\partial x}\, d\gamma, \tag{14}$$

where $n : \partial\Omega(x) \to \mathbb{R}^2$, $q \mapsto n(q)$, denotes the unit outward normal to $\partial\Omega(x)$ at $q \in \partial\Omega(x)$, and $\gamma : \mathbb{S}^1 \times (a,b) \to \mathbb{R}^2$ is a parameterization for the family $\{\Omega(x) \subset \mathbb{R}^2 \mid x \in (a,b)\}$.

APPENDIX C
AN UPPER BOUND ON THE AREA OF INTERSECTION BETWEEN TWO CIRCULAR SECTORS

Lemma C.1 *Denote a circular sector centered at point $p \in \mathbb{R}^2$ with orientation $\theta \in \mathbb{S}^1$, radius $r \in \mathbb{R}$, and angular width $\alpha \in (0, \frac{\pi}{2}]$ by $w_{r,\alpha}(p,\theta)$. Then the area A of $w_{r,\alpha}(p,\theta) \cap w_{r,\alpha}(p',\theta')^c$ satisfies*

$$A \leq 2(1+\alpha)r\|p-p'\| + \frac{1}{2}r^2|\theta - \theta'|.$$

Proof: The area A equals area$(w_{r,\alpha}(p,\theta) \cup w_{r,\alpha}(p',\theta')) - \frac{1}{2}r^2\alpha$. To find an upper bound to area$(w_{r,\alpha}(p,\theta) \cup w_{r,\alpha}(p',\theta'))$, consider the change in area caused by a rotation and translation of $w_{r,\alpha}(p',\theta')$ through the angle $\|\theta - \theta'\|$ and along the vector $p - p'$. In particular, let β_1, β_2 be the projection angles of $p - p'$ onto the segments l_1, l_2 respectively. Construct the regions B_1, B_2 as parallelograms formed by $p - p'$ and the segments l_1 and l_2 respectively (see Fig 4). Note the area of both B_1 and B_2 can

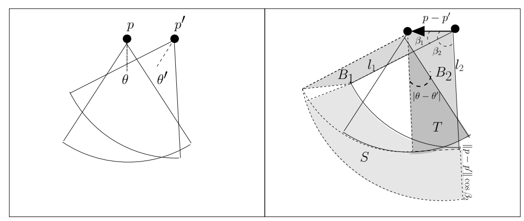

Fig. 4. On the left is an arbitrary configuration of two intersecting circular sectors. On the right is an illustration of the various regions which bound the union of the sectors.

be bounded by $r\|p-p'\|\cos\beta_{1,2} \leq r\|p-p'\|$. Next, construct the region S as an annular sector with inner radius r, outer radius $r + \|p-p'\|\cos\beta_2$, and angular width α. The area of S can be bounded by $\frac{1}{2}(r + \|p-p'\|\cos\beta_2)^2\alpha - \frac{1}{2}r^2\alpha \leq 2r\alpha\|p-p'\|$. Lastly, construct the region T as a circular sector with radius r and angular width $\|\theta-\theta'\|$. The area of the sector T can be computed as $\frac{1}{2}r^2\|\theta-\theta'\|$. Since any configuration of $w_{r,\alpha}(p,\theta) \cup w_{r,\alpha}(p',\theta')$ can be contained within the union of these regions with $w_{r,\alpha}(p',\theta')$, we are able to define the upper bound to A by

$$\begin{aligned} A &= \text{area}(w_{r,\alpha}(p,\theta) \cup w_{r,\alpha}(p',\theta')) - \frac{1}{2}r^2\alpha \\ &\leq \text{area}(B_1 + B_2 + S + T) \\ &\leq 2(1+\alpha)r\|p-p'\| + \frac{1}{2}r^2|\theta-\theta'|, \end{aligned}$$

as claimed. ■