

Global formation-shape stabilization of relative sensing networks

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Abstract—This paper proposes a simple, distributed algorithm that achieves global stabilization of formations for relative sensing networks in arbitrary dimensions. Assuming the network runs an initialization procedure to equally orient all the agent reference frames, convergence to the desired formation shape is guaranteed even in partially asynchronous settings. We also characterize the algorithm robustness to errors in the initialization procedure. The technical approach merges ideas from graph drawing, algebraic graph theory, multidimensional scaling, and distributed linear iterations.

I. INTRODUCTION

This paper proposes a distributed algorithm for relative sensing networks to achieve formation shape stabilization. A relative sensing network consists of a group of agents, each with its own reference frame, that can sense the relative position of their neighbors. The proposed algorithm guarantees that the network shape converges to the desired formation shape starting from any initial configuration.

Literature review: There is a large body of work on formation control in the multi-agent systems literature. A wide range of issues have been addressed, including pattern formation, stability, and merging, see e.g., [1], [2], [3] for a very small sample of works. Numerous continuous-time formation control strategies employ algebraic graph-theoretic tools, see e.g., [4], [5], [6], [7]. The works [8], [9], [10], [11] use graph rigidity ideas to achieve formation shape stabilization on the plane. However, in the rigidity approach, the desired formation shape is in general only locally stable (e.g., collinear network configurations are invariant and additional undesired locally stable equilibria exist). Another source of inspiration for this work is the literature on graph drawing [12], [13], multidimensional scaling and iterative majorization [14], where the design of global optimization algorithms that overcome the local stability properties of the desired configurations is a topic of vigorous research. Finally, groups of agents with only relative information about each other's state are considered in [15], [16], [17].

Statement of contributions: The main contribution of the paper is a simple, distributed coordination algorithm that stabilizes the shape of a relative sensing network to a desired formation. In contrast to previous work, the desired formation is not encoded using inter-agent distances and assuming that the interaction topology is rigid. Instead, in $d \in \mathbb{Z}_{>0}$ dimensions and assuming that the interaction topology has at least a globally reachable node, we encode the desired formation by assigning to each agent a vector in \mathbb{R}^d . The proposed strategy is executed in discrete time and is valid

for arbitrary dimensions. The formation control objective can be encoded by means of the global minimization of the stress function associated to the network. Our algorithmic design builds on additional contributions regarding the majorization of the stress function and its critical points when the interaction graph is directed. In particular, we show how the critical points can be characterized as the solutions to a sparse linear equation whose elements are computable in a distributed way over the interaction graph, in both the undirected and the directed cases. The coordination strategy then results from a Jacobi overrelaxation algorithm to solve the linear equation. We characterize the convergence properties of the algorithm as well as its descent properties with regards to the stress majorization function. In particular, we show that the algorithm guarantees that the network acquires the desired formation starting from any initial condition. We also analyze its performance in partially asynchronous executions and under errors in the initialization of the common agent frames.

Organization: Section II introduces notions from graph theory and distributed linear iterations. Section III states the formation control problem and the relative sensing network model. Section IV develops several results on the stress function from scaling theory. Section V presents the cooperative strategy and analyzes its convergence under asynchronism and errors in the agents' orientation. Section VI gathers our conclusions. For reasons of space, all proofs are omitted.

Notation: We do not distinguish between a vector $P = (p_1, \dots, p_n) \in (\mathbb{R}^d)^n$ and the matrix $P \in \mathbb{R}^{n \times d}$ whose i th row is p_i . We let $I_d \in \mathbb{R}^{d \times d}$ denote the identity matrix and $\mathbf{1}_d \in \mathbb{R}^d$ denote the vector whose entries are all 1. We let $\text{diag}(A_1, \dots, A_n) \in \mathbb{R}^{dn \times dn}$ denote the block-diagonal matrix that has $A_1, \dots, A_n \in \mathbb{R}^{d \times d}$ in the diagonal. Given $A \in \mathbb{R}^{d_1 \times d_2}$ and $B \in \mathbb{R}^{e_1 \times e_2}$, we let $A \otimes B \in \mathbb{R}^{d_1 e_1 \times d_2 e_2}$ denote its Kronecker product. For $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{e \times e}$, the eigenvalues of $A \otimes B$ are the product of the eigenvalues of A and B . The (Cartesian) product of $f_1 : X_1 \rightarrow Y_1$ and $f_2 : X_2 \rightarrow Y_2$ is $f_1 \times f_2 : X_1 \times X_2 \rightarrow Y_1 \times Y_2$, $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. The product of f_1, \dots, f_m is denoted $\prod_{k=1}^m f_k$. Finally, $\pi_\alpha : \mathbb{R}^d \rightarrow \mathbb{R}$, $\alpha \in \{1, \dots, d\}$, is the projection of a vector onto its α th-component.

II. PRELIMINARY DEVELOPMENTS

Here we introduce basic notions on kinematic motions, algebraic graph theory, and linear iterations. For further details on these topics, we refer to [18], [19], [16].

A. Fixed and body reference frames

Let $\Sigma_{\text{fixed}} = (p_{\text{fixed}}, \{\mathbf{x}_{\text{fixed}}^1, \dots, \mathbf{x}_{\text{fixed}}^d\})$ be a fixed reference frame in \mathbb{R}^d and let $\Sigma_{\text{b}} = (p_{\text{b}}, \{\mathbf{x}_{\text{b}}^1, \dots, \mathbf{x}_{\text{b}}^n\})$ be a reference

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frame fixed with a moving body. A point q and a vector v expressed with respect to the frames Σ_{fixed} and Σ_b are denoted by q^{fixed} and q^b , and v^{fixed} and v^b , respectively. The origin of Σ_b is the point p_b , denoted by p_b^{fixed} when expressed with respect to Σ_{fixed} . The orientation of Σ_b is characterized by the rotation matrix $R_b^{\text{fixed}} \in \text{SO}(d)$, whose columns are the frame vectors $\{x_b^1, \dots, x_b^d\}$ of Σ_b expressed with respect to Σ_{fixed} . With this notation, changes of frames read

$$q^{\text{fixed}} = R_b^{\text{fixed}} q^b + p_b^{\text{fixed}}, \quad (1a)$$

$$v^{\text{fixed}} = R_b^{\text{fixed}} v^b. \quad (1b)$$

B. Graph-theoretic notions

A *directed graph* (or *digraph*) $G = (\mathcal{V}, \mathcal{E})$ of order n consists of a *vertex set* \mathcal{V} with n elements, and an *edge set* $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. For simplicity, we take $\mathcal{V} = \{1, \dots, n\}$. A digraph is *undirected* if $(j, i) \in \mathcal{E}$ anytime $(i, j) \in \mathcal{E}$. In a digraph G with an edge $(i, j) \in \mathcal{E}$, i is called an *in-neighbor* of j , and j is called an *out-neighbor* of i . A *directed path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices in the sequence are an edge of the digraph. A vertex of a digraph is *globally reachable* if it can be reached from any other vertex by traversing a directed path. An undirected graph is *connected* if there exists a path between any two vertices. For an undirected graph, this is equivalent to the graph having a globally reachable vertex.

A *weighted digraph* is a triplet $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $(\mathcal{V}, \mathcal{E})$ is a digraph and where \mathcal{A} is an $n \times n$ *weighted adjacency matrix* with the following properties: for $i, j \in \{1, \dots, n\}$, the entry $a_{ij} > 0$ if (i, j) is an edge of G , and $a_{ij} = 0$ otherwise. A weighted digraph is *undirected* if $a_{ij} = a_{ji}$ for all $i, j \in \{1, \dots, n\}$. When convenient, we write $\mathcal{A}(G)$ to make clear the explicit dependence on the graph. Note that a digraph $G = (\mathcal{V}, \mathcal{E})$ can be naturally thought of as a weighted digraph by defining the weighted adjacency matrix \mathcal{A} with entries $a_{ij} = 1$ if (i, j) is an edge of G , and $a_{ij} = 0$ otherwise. Reciprocally, one can define the *unweighted version* of a weighted digraph $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ by simply considering the digraph $(\mathcal{V}, \mathcal{E})$.

The *weighted out-degree and in-degree matrices* are the diagonal matrices defined by

$$D_{\text{out}}(G) = \text{diag}(\mathcal{A}\mathbf{1}_n), \quad D_{\text{in}}(G) = \text{diag}(\mathcal{A}^T\mathbf{1}_n).$$

If G is undirected, we use the notation $D(G) = D_{\text{out}}(G) = D_{\text{in}}(G)$. The *graph Laplacian* of the weighted digraph G is

$$L(G) = D_{\text{out}}(G) - \mathcal{A}(G).$$

Note that $L(G)\mathbf{1}_n = 0$, and that G is undirected iff $L(G)$ is symmetric. For undirected graphs, the Laplacian is a symmetric, positive semidefinite matrix. The Laplacian also captures the connectivity properties of the graph: $L(G)$ has rank $n - 1$ iff G has a globally reachable vertex.

Next, we define reverse and mirror digraphs. Let $\tilde{\mathcal{E}}$ be the set of reverse edges of G obtained by reversing the order of all pairs in \mathcal{E} . The *reverse digraph* \tilde{G} of G is $(\mathcal{V}, \tilde{\mathcal{E}})$. Observe

$$\mathcal{A}(\tilde{G}) = \mathcal{A}(G)^T,$$

$$L(\tilde{G}) = D_{\text{out}}(\tilde{G}) - \mathcal{A}(\tilde{G}) = D_{\text{in}}(G) - \mathcal{A}(G)^T.$$

In general, $L(\tilde{G}) \neq L(G)^T$. The *mirror digraph* \hat{G} of G is the undirected graph $(\mathcal{V}, \mathcal{E} \cup \tilde{\mathcal{E}})$ with

$$\mathcal{A}(\hat{G}) = \frac{1}{2}(\mathcal{A}(G) + \mathcal{A}(G)^T) = \text{Sym}(\mathcal{A}(G)),$$

$$L(\hat{G}) = \frac{1}{2}(L(G) + L(\tilde{G})).$$

C. Jacobi overrelaxation iteration

Given an invertible matrix $A \in \mathbb{R}^{n \times n}$ and a vector $b \in \mathbb{R}^n$, consider the linear system $Ax = b$. The Jacobi overrelaxation (JOR) algorithm is an iterative procedure to compute the unique solution $x = A^{-1}b \in \mathbb{R}^n$. It is formulated as the discrete-time dynamical system

$$x_i(\ell + 1) = (1 - h)x_i(\ell) - h \frac{1}{a_{ii}} \left(\sum_{j \neq i} a_{ij} x_j(\ell) - b_i \right),$$

with $\ell \in \mathbb{Z}_{\geq 0}$, $i \in \{1, \dots, n\}$, $x(0) \in \mathbb{R}^n$, and $h \in (0, 1)$. The convergence properties of the JOR algorithm can be fully characterized in terms of the eigenvalues of the matrix describing the linear iteration, see [18]. Given a digraph G , as long as (i) agent i has access to b_i and a_{ii} , and (ii) if $a_{ij} \neq 0$, then $(i, j) \in \mathcal{E}$, the JOR algorithm is amenable to distributed implementation in the following sense: agent i can compute the i th component x_i of the solution $x = A^{-1}b$ with information gathered from its out-neighbors in G .

III. PROBLEM STATEMENT

Our objective is to synthesize a discrete-time distributed coordination algorithm that achieves global stabilization of the desired formation shape. Here we describe the capabilities of the robotic network and formally state the control objective.

A. Relative sensing network

Consider a group of n agents in \mathbb{R}^d . We assume that each agent has its own reference frame Σ_i . Expressed with respect to the fixed frame Σ_{fixed} , the i th frame Σ_i is described by a position $p_i^{\text{fixed}} \in \mathbb{R}^d$ and an orientation $R_i^{\text{fixed}} \in \text{SO}(d)$. The dynamical model of each agent is as follows. With its sensed information, agent $i \in \{1, \dots, n\}$ computes its own control input, expressed in its local frame Σ_i as u_i^i . In the local frame, each agent moves according to

$$\dot{p}_i^i(\ell + 1) = u_i^i.$$

According to (1), in the global frame Σ_{fixed} this reads as

$$\dot{p}_i^{\text{fixed}}(\ell + 1) = p_i^{\text{fixed}}(\ell) + R_i^{\text{fixed}} u_i^i.$$

The sensing interactions between agents are encoded by a digraph $G = (\{1, \dots, n\}, \mathcal{E})$. An edge (i, j) means that agent i can sense the relative position of agent j in its own local frame, p_j^j . There is no explicit communication among agents. We refer to this group of robots by S_G^{rs} . A coordination algorithm is a specification of an input u_i^i for each agent $i \in \{1, \dots, n\}$. The algorithm is *distributed over* S_G^{rs} if each agent can compute its control input with the information collected on the relative position of its neighbors in G .

B. The control objective

Our objective is to stabilize the group configuration to a desired formation. The desired formation is encoded as follows. Given $Z^* \in (\mathbb{R}^d)^n$, let $\text{Rgd}(Z^*)$ be the set of configurations in $(\mathbb{R}^d)^n$ which are related to Z^* by a translation and a rotation in \mathbb{R}^d . In other words, define

$$\text{Rgd}(Z^*) = \{W \in (\mathbb{R}^d)^n \mid \text{there exists } (q, R) \in \mathbb{R}^d \times SO(d) \text{ such that } w_i = Rz_i + q, i \in \{1, \dots, n\}\}.$$

Obviously, $Z^* \in \text{Rgd}(Z^*)$. Note that any two configurations in $\text{Rgd}(Z^*)$ have the same inter-agent distances, i.e., $\ell_{ij} = \|w_i - w_j\| \in \mathbb{R}_{>0}$, $i \neq j \in \{1, \dots, n\}$ are the same for any $W \in \text{Rgd}(Z^*)$. The control objective is then to stabilize the group of agents to a configuration that belongs to $\text{Rgd}(Z^*)$.

IV. SCALING THEORY AND STRESS MAJORIZATION

Here, we introduce the notion of stress function from multidimensional scaling theory [20], [14] and explain its relationship with the formation control problem. We also prove various results that will be instrumental in the algorithm design, paying particular attention to the directed graph case.

A. The stress function

The raw Stress function $\text{Stress}_G : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ is defined by

$$\text{Stress}_G(p_1, \dots, p_n) = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} (\|p_i - p_j\| - \ell_{ij})^2. \quad (2)$$

For an undirected graph G , this definition coincides with the classical one used in the multidimensional scaling. The desired formation configurations are global minimizers of Stress_G . Under additional assumptions on the rigidity of the graph, one can guarantee that they are the only ones. Alternatively, one may consider the S-Stress [21] function

$$\text{S-Stress}_G(p_1, \dots, p_n) = \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} (\|p_i - p_j\|^2 - \ell_{ij}^2)^2,$$

which has the same global minimizers. The S-Stress function is the Lyapunov function considered in [8], [9] in the context of formation control. Here, instead, we focus on the raw Stress, although the developments described later apply equally with the appropriate modifications. The partial derivative of Stress_G with respect to p_i , $i \in \{1, \dots, n\}$, is

$$\frac{\partial \text{Stress}_G}{\partial p_i} = 2 \sum_{j:(i,j) \in \mathcal{E}} (\|p_i - p_j\| - \ell_{ij}) \frac{p_i - p_j}{\|p_i - p_j\|}. \quad (3)$$

In an inter-agent distance approach, this partial derivative can be computed by agent i with local information, and hence one can design a gradient-descent algorithm to minimize Stress_G . Indeed, one can show [9] that the desired equilibria of the system are locally stable. However, the gradient system has other undesired equilibria (other local minimizers of Stress_G), which turn out to be also locally stable [9], [10]. In addition, it is not difficult to establish that the set of collinear network configurations is invariant under the gradient flow defined by (3). These observations are also valid for the gradient flow of S-Stress $_G$. Here, instead, we take an alternative approach that uses stress majorization.

B. Stress majorization

In general, the direct optimization of the stress function is prone to local minima. An alternative route involves the construction of majorization functions that are easier to optimize. This is what we discuss next.

Let us start with some notation. Given $Z = (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$ and an undirected graph G , let G^Z be the weighted graph with adjacency matrix $\mathcal{A}(G^Z)$ with entries

$$a_{ij} = \begin{cases} \ell_{ij} \text{inv}(\|z_i - z_j\|), & (i, j) \in \mathcal{E}, \\ 0, & (i, j) \notin \mathcal{E}, \end{cases}$$

where $\text{inv}(x) = 1/x$ if $x \neq 0$, and $\text{inv}(0) = 0$. Note that, for $Z \in \text{Rgd}(Z^*)$, the graphs G^Z and G are the same. The stress majorization function $\mathcal{F}_G^Z : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ is

$$\mathcal{F}_G^Z(P) = \text{tr}(P^T L(G)P) - 2 \text{tr}(P^T L(G^Z)Z) + \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \ell_{ij}^2.$$

The name of the function is justified by the following result.

Proposition IV.1 ([14]) *Given an undirected graph G , for any $P = (p_1, \dots, p_n)$, $Z = (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$,*

$$\text{Stress}_G(p_1, \dots, p_n) \leq \mathcal{F}_G^Z(P).$$

Moreover, if $P = Z$, then $\mathcal{F}_G^P(P) = \text{Stress}_G(P)$.

The result can be extended to the digraph case.

Proposition IV.2 *Given a digraph G , for any $P = (p_1, \dots, p_n)$, $Z = (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$,*

$$\text{Stress}_G(p_1, \dots, p_n) \leq \mathcal{F}_G^Z(P).$$

Moreover, if $P = Z$, then $\mathcal{F}_G^P(P) = \text{Stress}_G(P)$.

Alternatively, the stress majorization function can be expressed using the Kronecker product as

$$\begin{aligned} \mathcal{F}_G^Z(P) &= P^T(L(G) \otimes I_d)P \\ &\quad - 2P^T(L(G^Z) \otimes I_d)Z + \frac{1}{2} \sum_{(i,j) \in \mathcal{E}} \ell_{ij}^2. \end{aligned} \quad (4)$$

This expression is useful in establishing the following key properties of the stress majorization function.

Proposition IV.3 *Given $Z = (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$ and an undirected graph G , the following holds:*

- (i) *The gradient and Hessian of \mathcal{F}_G^Z are, respectively,*

$$\begin{aligned} \nabla \mathcal{F}_G^Z &= 2(L(G) \otimes I_d)P - 2(L(G^Z) \otimes I_d)Z, \\ \nabla^2(\mathcal{F}_G^Z) &= 2L(G) \otimes I_d. \end{aligned}$$

In particular, both are distributed over the graph G ;

- (ii) *The function \mathcal{F}_G^Z is globally convex;*
- (iii) *$P \in (\mathbb{R}^d)^n$ is a global minimizer of \mathcal{F}_G^Z iff*

$$(L(G) \otimes I_d)P = (L(G^Z) \otimes I_d)Z. \quad (5)$$

In particular, any two minima of \mathcal{F}_G^Z are equal up to a translation in \mathbb{R}^d .

The following result will be important later for our distributed algorithmic design in the case of directed graphs.

Lemma IV.4 *Given $Z \in \text{Rgd}(Z^*)$ and a digraph G with a globally reachable vertex, P is a global minimizer of \mathcal{F}_G^Z iff*

$$(L(G) \otimes I_d)P = (L(G) \otimes I_d)Z. \quad (6)$$

The importance of Lemma IV.4 stems from the following observation: the critical points of \mathcal{F}_G^Z can be characterized by a linear equation (6) defined by the Laplacian matrix of G , which is distributed over the digraph G . Note that the original characterization (5) is defined by the Laplacian matrix of the mirror graph, which is not distributed over G .

V. COORDINATION ALGORITHM FOR GLOBAL STABILIZATION OF FORMATIONS

In this section, we propose a discrete-time distributed coordination algorithm that achieves global stabilization of the desired formation, i.e., it guarantees that the network acquires the desired formation starting from any initial condition. We begin by discussing the problem of finding a network-wide reference frame and then design the coordination algorithm.

A. Common orientation of local reference frames

The reference frames of the individual agents in $\mathcal{S}_G^{\text{rs}}$ might have different orientations with respect to the global reference frame. However, the network can execute some initialization algorithm to equally orient all agent reference frames. Here we describe one simple procedure based on a distributed implementation of the flooding algorithm [22] on the relative sensing network. Other solutions to the common reference frame problem are explored in [17].

Assume the digraph G has at least a globally reachable node. For simplicity, we describe the strategy first in \mathbb{R}^2 . At the first time step, a preselected globally reachable node moves a unit in the direction of its x -axis. All other agents that can sense the position of this agent measure the relative displacement in their local frames and figure out the x -axis direction of the agent. They rotate their frames to align them with the direction of the relative displacement. The process is repeated until all agents have rotated their frames to align them with the frame of the globally reachable node.

In \mathbb{R}^3 , it takes two time steps for each agent to figure out the orientation of the frame of the globally reachable node. This node first moves in the direction of its x -axis, and then moves in the direction of its y -axis. The process is repeated until all agents have frames with the same orientation.

B. Motion coordination via Jacobi iteration

Here, we assume that all agent reference frames have the same orientation, i.e., $R_i^{\text{fixed}} = R$, for $i \in \{1, \dots, n\}$, for some $R \in \text{SO}(d)$ which may be unknown to the agents. Given the discussion in Section IV, our strategy to make the network achieve the desired formation shape is to globally

optimize $\mathcal{F}_G^{Z^*}$. From Proposition IV.3 and Lemma IV.4, this can be achieved by solving the sparse linear equation

$$(L(G) \otimes I_d)P = (L(G) \otimes I_d)Z^*.$$

To solve this equation, we propose to use a Jacobi overrelaxation iteration (JOR), as described in Section II-C. Let

$$b = (b_1, \dots, b_n) = (L(G) \otimes I_d)Z^*,$$

with $b_i \in \mathbb{R}^d$, $i \in \{1, \dots, n\}$. In Cartesian coordinates, the JOR algorithm for each agent i is

$$p_i(\ell + 1) = (1 - h)p_i(\ell) + h \frac{1}{d_i} \left(\sum_{j \neq i} a_{ij} p_j(\ell) + b_i \right),$$

where (d_1, \dots, d_n) is the diagonal of $D_{\text{out}}(G)$. If $d_i = 0$, then we set $p_i(\ell + 1) = p_i(\ell)$. In the local frame of agent i , this is written as

$$p_i^i(\ell + 1) = h \frac{1}{d_i} \left(\sum_{j \neq i} a_{ij} p_j^i(\ell) + (R_i^{\text{fixed}})^T b_i \right), \quad (7)$$

if $d_i \neq 0$, and $p_i^i(\ell + 1) = 0$ otherwise. The individual agent does not know the rotation matrix $R_i^{\text{fixed}} = R$. Therefore, instead of (7), agent i implements

$$p_i^i(\ell + 1) = h \frac{1}{d_i} \left(\sum_{j \neq i} a_{ij} p_j^i(\ell) + b_i \right), \quad (8)$$

if $d_i \neq 0$, and $p_i^i(\ell + 1) = 0$ otherwise. In the global frame, using (1), the algorithm (8) can be written as

$$p_i(\ell + 1) = (1 - h)p_i(\ell) + h \frac{1}{d_i} \left(\sum_{j \neq i} a_{ij} p_j(\ell) + R b_i \right), \quad (9)$$

if $d_i \neq 0$, and $p_i(\ell + 1) = p_i(\ell)$ otherwise. This corresponds to the JOR algorithm to solve the linear equation

$$(L(G) \otimes I_d)P = (I_n \otimes R)b. \quad (10)$$

Note that all the solutions of this equation correspond to translations of a rotated configuration of Z^* , and therefore, all belong to $\text{Rgd}(Z^*)$, as desired.

The following result characterizes the distributed character of this algorithm as well as its convergence properties.

Proposition V.1 *Consider the relative sensing network $\mathcal{S}_G^{\text{rs}}$, where G has a globally reachable vertex. Let $h \in (0, 1)$ and assume all agent frames are equally oriented. Then,*

- (i) *the coordination algorithm (8) is distributed over $\mathcal{S}_G^{\text{rs}}$. Moreover, as initial information, each agent only needs to store a vector in \mathbb{R}^d ;*
- (ii) *the coordination algorithm (8) converges to a configuration W in $\text{Rgd}(Z^*)$;*
- (iii) *if G is undirected, the stress majorization function \mathcal{F}_G^W is monotonically decreasing along (8).*

Remark V.2 Proposition V.1(iii) does not hold in general if G is directed. A counter example is given by the digraph plotted in Figure 1. For this digraph, the matrix $\text{Sym}(L(\tilde{G})D(G)^{-1}L(G))$ has a negative eigenvalue, and therefore there exist initial network configurations for which \mathcal{F}_G^W is not monotonically decreasing along (8). •

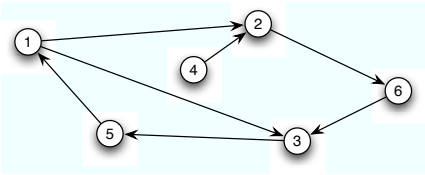


Fig. 1. Counter example to Proposition V.1(iii) in the directed case. For this digraph, there exist initial network configurations for which \mathcal{F}_G^W is not monotonically decreasing along the execution of (8).

C. Convergence of asynchronous executions

In this section, we analyze the asymptotic convergence properties of (8) when agents operate asynchronously. The idea is that, at each time step, the information that an agent has about other neighboring agents is outdated to some degree.

The asynchronous model for the network operation here corresponds to the *partially asynchronous model* from [18, Chapter 7]. Roughly speaking, when agent i , at time ℓ , uses the value p_j^i from another agent, that value is not necessarily the most recent one, $p_j^i(\ell)$, but rather an outdated one, $p_j^i(\tau_j^i(\ell))$, where $0 \leq \tau_j^i(\ell) \leq \ell$. The quantity $\ell - \tau_j^i(\ell)$ represents the delay. Therefore, the JOR algorithm (8) reads

$$p_i^i(\ell + 1) = h \frac{1}{d_i} \left(\sum_{j \neq i} a_{ij} p_j^i(\tau_j^i(\ell)) + b_i \right). \quad (11)$$

Next, we characterize the convergence of (11).

Proposition V.3 Consider the relative sensing network $\mathcal{S}_G^{\text{rs}}$ operating asynchronously, where G has a globally reachable vertex. Let $h \in (0, 1)$ and assume all agent reference frames are equally oriented. Assume there exists $B > 0$ such that $\ell - B + 1 \leq \tau_j^i(\ell) \leq \ell$, for all ℓ and all $(i, j) \in \mathcal{E}$. Then, (11) converges to a configuration W in $\text{Rgd}(Z^*)$.

Proposition V.3 is established by rewriting the coordination algorithm in a manner analogous to the expressions obtained in the proof of Proposition V.1(ii), and then applying the results in [23], [24]. We omit it for reasons of space.

D. Robustness to errors in orientation of reference frames

Here we explore the robustness properties of the coordination algorithm (8) against errors in the computation of the common orientation of the reference frame, cf. Section V-A. Suppose that the algorithm selected by the network to orient all agent frames equally is not executed perfectly and consequently the final orientation of the frame of agent $i \in \{1, \dots, n\}$ is of the form

$$R_i^{\text{fixed}} = R + \Xi_i, \quad (12)$$

where $R \in \text{SO}(d)$ and the error matrix Ξ_i satisfies $\|\Xi_i\|_\infty \leq \varepsilon$. Given the difference in the orientation of the agent frames, the algorithm (8) reads now in the global frame as

$$p_i(\ell + 1) = (1 - h)p_i(\ell) + h \frac{1}{d_i} \left(\sum_{j \neq i} a_{ij} p_j(\ell) + R_i^{\text{fixed}} b_i \right). \quad (13)$$

which corresponds to the JOR algorithm to solve the linear algebraic equation

$$(L(G) \otimes I_d)P = \text{diag}(R_1^{\text{fixed}}, \dots, R_n^{\text{fixed}})b. \quad (14)$$

Observe that the mismatch in the orientation of the frames makes this linear equation ill-posed. In other words, the vector $\text{diag}(R_1^{\text{fixed}}, \dots, R_n^{\text{fixed}})b$ does not belong to the range of $L(G) \otimes I_d$, and therefore, there does not exist a solution P of (14). Intuitively, this observation is consistent with the fact that the algorithm design assumes all frames are equally oriented. Even though (14) has no solution, the question about the convergence properties of (13) still remains. Next, we analyze the convergence of (8) under errors in the computation of the common orientation of the frames.

Proposition V.4 Consider the relative sensing network $\mathcal{S}_G^{\text{rs}}$, where G has a globally reachable vertex. Let $h \in (0, 1)$. Let the orientation of the frame of agent $i \in \{1, \dots, n\}$ be given by (12). Then, there exists $K \in \mathbb{R}_{>0}$ such that the algorithm (8) converges to $\{Z \in (\mathbb{R}^d)^n \mid \text{there exists } W \in \text{Rgd}(Z^*) \text{ such that } \|Z - W\|_\infty \leq Kh\varepsilon\}$.

Note that while Propositions V.1 and V.3 guarantee convergence to a point, Proposition V.4 only guarantees convergence to a set. According to the statement, the configurations in this set correspond to translations and rotations of the desired formation slightly deformed by the effect of the mismatch in the orientation of the agent frames.

E. Simulations

In this section we show various executions of the discrete-time distributed coordination algorithm (8) to illustrate its convergence properties, paying attention to asynchronism and robustness against errors in the common orientation of the agent frames. Figure 2 shows an execution of (8) over a relative sensing network in \mathbb{R}^3 composed of 60 agents, with interaction topology given by the Buckminster Fuller geodesic dome [25]. Proposition V.1 guarantees that convergence to the desired formation shape is achieved.

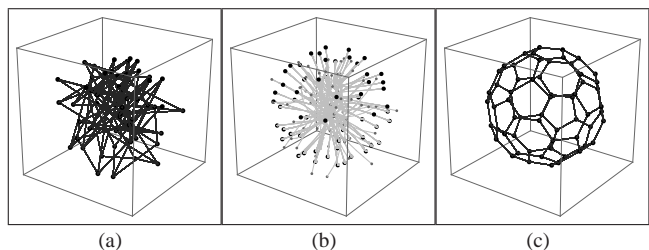


Fig. 2. Execution of the coordination algorithm (8) with $h = .25$ over a relative sensing network in \mathbb{R}^3 composed of 60 agents. (a) shows the initial configuration, (b) shows the evolution, and (c) shows the final formation.

Figure 3 shows an execution of (11) over a relative sensing network composed of 20 agents, with interaction topology given by a directed version of the Desargues graph [26]. The maximum delay is $B = 25$ steps, i.e., no agent has relative position information on its neighbors that is more

than 25 steps outdated. As forecasted by Proposition V.3, convergence to the desired formation shape is achieved.

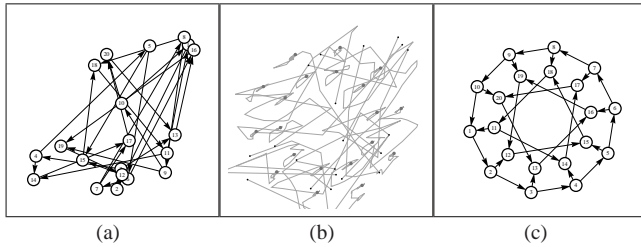


Fig. 3. Execution of the coordination algorithm (11) with $h = .25$ over a relative sensing network in \mathbb{R}^2 composed of 20 agents. (a) shows the initial configuration, (b) shows the evolution, and (c) shows the final formation.

Figure 4 shows an execution of (8) over the relative sensing network of Figure 1 under errors in the computation of the common orientation of the agent frames. Each agent orients its own frame with an angle of 90 degrees with an error whose absolute value is bounded by 9 degrees. The desired formation is a regular hexagon. As forecasted by Proposition V.4, the network shape converges to a formation close to the desired one, while the group of agents moves along a straight line in the plane.

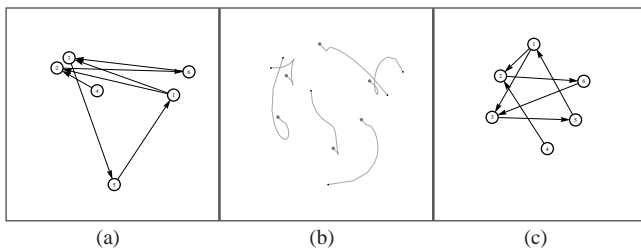


Fig. 4. Execution of the coordination algorithm (8) with $h = .25$ over the relative sensing network of Figure 1 in \mathbb{R}^2 under errors in the orientation of the agent frames. (a) shows the initial configuration, (b) shows the evolution, and (c) shows the final shape of the formation.

VI. CONCLUSIONS AND FUTURE WORK

We have proposed a distributed formation control strategy for relative sensing networks. The algorithm design combines ideas on stress majorization from scaling theory with Jacobi overrelaxation algorithms from distributed linear iterations. We have analyzed the convergence properties of the proposed algorithm in partially asynchronous settings and under errors in the initial computation of a common reference frame. Future work will include the study of robustness for general digraphs, the design of error-correcting algorithms that completely eliminate any mismatch in the orientation of the agent frames, and the extension of the results to switching interaction topologies and individual agent dynamics.

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