Global and robust formation-shape stabilization of relative sensing networks

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Abstract

This paper proposes a simple, distributed algorithm that achieves global stabilization of formations for relative sensing networks in arbitrary dimensions with fixed topology. Assuming the network runs an initialization procedure to equally orient all agent reference frames, convergence to the desired formation shape is guaranteed even in partially asynchronous settings. We characterize the algorithm robustness against several sources of errors: link failures, measurement errors, and frame initialization errors. The technical approach merges ideas from graph drawing, algebraic graph theory, multidimensional scaling, and distributed linear iterations.

Key words: Multi-agent systems; distributed algorithms; formation control; multidimensional scaling; linear iterations

1 Introduction

This paper proposes a distributed algorithm for relative sensing networks to achieve formation shape stabilization. A relative sensing network consists of a group of agents, each with its own reference frame, that can sense the relative position of their neighbors. The proposed algorithm assumes that the interaction topology remains fixed and guarantees that the network shape will converge to the desired formation shape starting from any initial configuration.

Literature review There is a large body of work on formation control in the multi-agent systems literature. A wide range of issues have been addressed, including pattern formation, stability, and merging, see e.g., [9, 15, 27, 29] for a very small sample of works. Numerous continuous-time formation control strategies employ algebraic graph-theoretic tools, see e.g., [10, 19, 26]. The work [13] proposes hybrid rendezvous-to-formation control strategies under state-dependent interaction topologies. The works [8, 17, 23, 30] use graph rigidity ideas to achieve formation shape stabilization on the plane. However, in the rigidity approach, the desired formation shape is in general only locally stable (e.g., collinear network configurations are invariant and additional undesired locally stable equilibria exist). Another source of inspiration for this work is the literature on graph drawing [16], multidimensional scaling [4], and iterative majorization [11], where the design of global optimization algorithms that overcome the local stability properties of the desired configurations is a topic of vigorous research. Finally, groups of agents with only relative information about each other's state are considered in [5, 21, 25].

Statement of contributions The main contribution of the paper is a simple, distributed coordination algorithm that stabilizes the shape of a relative sensing network to a desired formation. In contrast to previous work, the desired formation is not encoded using inter-agent distances and assuming that the interaction topology is rigid. Instead, in $d \in \mathbb{Z}_{>0}$ dimensions and assuming that the interaction topology has at least a globally reachable vertex, we encode the desired formation by assigning to each agent a vector in $\mathbb{R}^d$. The proposed strategy runs in discrete time and works in arbitrary dimensions. The formation control objective can be encoded by means of the global minimization of the stress function associated to the network. Our algorithmic design builds on additional contributions regarding the majorization of the stress function and its critical points when the interaction graph is directed. In particular, we show how the critical points can be characterized as the solutions to a sparse linear equation whose elements are computable in a distributed way over the interaction graph, in both the undirected and the directed cases. The coordination strategy then results from the design of a Jacobi overrelaxation algorithm to solve the linear equation. We characterize the global convergence properties of the algorithm as well as its descent

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properties with regards to the stress majorization function. We also establish the algorithm robustness against several sources of error: when agents fail to acquire the relative position of neighboring agents (a scenario that can be interpreted as a partially asynchronous network operation), when there are measurement errors in the relative position information of neighboring agents, and when there are errors in the initial computation of the common reference frame of the individual agents.

Organization The paper is organized as follows. Section 2 introduces basic notions from graph theory and distributed linear iterations. Section 3 states the formation control problem and the model for the relative sensing network. Section 4 develops several results on the stress function from scaling theory, with special emphasis on the directed graph case. Section 5 presents the coordination algorithm design and analyzes its convergence properties. Section 6 characterizes the algorithm robustness against several sources of error. Finally, Section 7 gathers our conclusions and ideas for future work.

Notation We let $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ denote the set of reals, positive reals, and nonnegative reals, respectively. We denote by $\|\cdot\|_\infty$ the ∞-norm in the Euclidean space, and we denote the ∞-distance between a point $p \in \mathbb{R}^d$ and a set $S \subset \mathbb{R}^d$ by $\operatorname{dist}_\infty(p, S)$. With a slight abuse of notation, we do not distinguish between a vector $P = (p_1, \dots, p_n) \in (\mathbb{R}^d)^n$ and the matrix $P \in \mathbb{R}^{n\times d}$ whose ith row is p_i. We let $I_d \in \mathbb{R}^{d\times d}$ denote the identity matrix and $\mathbf{1}_d \in \mathbb{R}^d$ denote the vector whose entries are all 1. For $A_1, \dots, A_n \in \mathbb{R}^{d\times d}$, we let $\operatorname{diag}(A_1, \dots, A_n) \in \mathbb{R}^{dn\times dn}$ denote the block-diagonal matrix that has $A_1, \dots, A_n$ in the diagonal. Given $A \in \mathbb{R}^{d_1\times d_2}$ and $B \in \mathbb{R}^{e_1\times e_2}$, we let $A \otimes B \in \mathbb{R}^{d_1 e_1 \times d_2 e_2}$ denote its Kronecker product. For square matrices $A \in \mathbb{R}^{d\times d}$, $B \in \mathbb{R}^{e\times e}$, the eigenvalues of $A \otimes B$ are the product of the eigenvalues of A and B. The Cartesian product of maps $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ is $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$, $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$. For brevity, the Cartesian product of $f_1, \dots, f_m$ is denoted $\Pi_{k=1}^m f_k$. Finally, we let $\pi_\alpha : \mathbb{R}^d \to \mathbb{R}$, $\alpha \in \{1, \dots, d\}$, denote the projection of a vector onto its αth-component.

2 Preliminary developments

Here we introduce basic notions on kinematic motions, algebraic graph theory, and linear iterations. For further details on these topics, we refer to [2, 5, 7].

2.1 Fixed and body reference frames

Let $\Sigma_{\text{fixed}} = (p_{\text{fixed}}, \{\boldsymbol{x}^1_{\text{fixed}}, \dots, \boldsymbol{x}^d_{\text{fixed}}\})$ be a fixed reference frame in $\mathbb{R}^d$ and let $\Sigma_{\text{b}} = (p_{\text{b}}, \{\boldsymbol{x}^1_{\text{b}}, \dots, \boldsymbol{x}^n_{\text{b}}\})$ be a reference frame fixed with a moving body. A point q and a vector v expressed with respect to the frames Σ_{fixed} and Σ_{b} are denoted by q^{fixed} and q^{b}, and v^{fixed} and v^{b}, respectively. The origin of Σ_{b} is the point p_{b}, denoted by $p^{\text{fixed}}_{\text{b}}$ when expressed with respect to Σ_{fixed}. The orientation of Σ_{b} is characterized by the rotation matrix $R^{\text{fixed}}_{\text{b}} \in \mathrm{SO}(d)$, whose columns are the frame vectors $\{\boldsymbol{x}^1_{\text{b}}, \dots, \boldsymbol{x}^d_{\text{b}}\}$ of Σ_{b} expressed with respect to Σ_{fixed}.

With this notation, changes of frames read

$$q^{\text{fixed}} = R^{\text{fixed}}_{\text{b}} q^{\text{b}} + p^{\text{fixed}}_{\text{b}}, \tag{1a}$$

$$v^{\text{fixed}} = R^{\text{fixed}}_{\text{b}} v^{\text{b}}. \tag{1b}$$

2.2 Graph-theoretic notions

A *directed graph* (or *digraph*) $G = (\mathcal{V}, \mathcal{E})$ of order n consists of a *vertex set* $\mathcal{V}$ with n elements, and an *edge set* $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. For simplicity, we take $\mathcal{V} = \{1, \dots, n\}$. A digraph is *undirected* if $(j, i) \in \mathcal{E}$ anytime $(i, j) \in \mathcal{E}$. In a digraph G with an edge $(i, j) \in \mathcal{E}$, i is called an *in-neighbor* of j, and j is called an *out-neighbor* of i. A *directed path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices in the sequence are an edge of the digraph. A vertex of a digraph is *globally reachable* if it can be reached from any other vertex by traversing a directed path. An undirected graph is *connected* if there exists a path between any two vertices. For an undirected graph, this notion is equivalent to the graph having a globally reachable vertex.

A *weighted digraph* is a triplet $G = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $(\mathcal{V}, \mathcal{E})$ is a digraph and where $\mathcal{A}$ is an $n \times n$ *weighted adjacency matrix* with the following properties: for $i, j \in \{1, \dots, n\}$, the entry $a_{ij} > 0$ if (i, j) is an edge of G, and $a_{ij} = 0$ otherwise. A weighted digraph is *undirected* if $a_{ij} = a_{ji}$ for all $i, j \in \{1, \dots, n\}$. When convenient, we write $\mathcal{A}(G)$ to make clear the explicit graph dependence. A digraph $G = (\mathcal{V}, \mathcal{E})$ can be naturally thought of as a weighted digraph by defining entries $a_{ij} = 1$ if (i, j) is an edge of G, and $a_{ij} = 0$ otherwise. Reciprocally, one can define the *unweighted version* of a weighted digraph $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ by simply considering the digraph $(\mathcal{V}, \mathcal{E})$.

The *weighted out- and in-degree* matrices are the diagonal matrices $D_{\text{out}}(G) = \operatorname{diag}(\mathcal{A}\mathbf{1}_n)$ and $D_{\text{in}}(G) = \operatorname{diag}(\mathcal{A}^T\mathbf{1}_n)$. A weighted digraph is *regular* if there exists $a \in \mathbb{R}_{>0}$ such that $D_{\text{out}}(G) = D_{\text{in}}(G) = a\, I_n$. If G is undirected, we denote $D(G) = D_{\text{out}}(G) = D_{\text{in}}(G)$. The *graph Laplacian* of the weighted digraph G is

$$L(G) = D_{\text{out}}(G) - \mathcal{A}(G).$$

Note that $L(G)\mathbf{1}_n = 0$, and that G is undirected iff $L(G)$ is symmetric. For undirected graphs, the Laplacian is a symmetric, positive semidefinite matrix. The Laplacian also captures the connectivity properties of the graph: $L(G)$ has rank $n-1$ if and only if G has a globally reachable vertex. The following result states a useful property.

Lemma 2.1 *For any connected undirected weighted graph G, the matrix $2D(G)^{-1} - hD(G)^{-1}L(G)D(G)^{-1}$ is positive definite for $h \in (0, 1)$.*

PROOF. Since the graph is connected, every vertex has a positive degree, and the matrix $D(G)$ is positive definite. Now, note that

$$\begin{aligned}&2D(G)^{-1} - hD(G)^{-1}L(G)D(G)^{-1}\\ &= D(G)^{-1/2}(2I_n - hD(G)^{-1/2}L(G)D(G)^{-1/2})D(G)^{-1/2}.\end{aligned}$$

It suffices to show that the symmetric matrix $2I_n - hD(G)^{-1/2}L(G)D(G)^{-1/2}$ is positive definite. The matrix $D(G)^{-1/2}L(G)D(G)^{-1/2}$ is the *normalized Laplacian* considered in [6]. According to [6, Lemma 1.7], the eigenvalues of the normalized Laplacian all belong to the interval $[0, 2]$, and the result follows. □

Next, we define reverse and mirror digraphs. Let $\widetilde{\mathcal{E}}$ be the set of edges obtained by reversing the order of all pairs in $\mathcal{E}$. The *reverse digraph* $\widetilde{G}$ of G is $(\mathcal{V}, \widetilde{\mathcal{E}})$. Observe

$$\begin{aligned}\mathcal{A}(\widetilde{G}) &= \mathcal{A}(G)^T,\\ L(\widetilde{G}) &= D_{\text{out}}(\widetilde{G}) - \mathcal{A}(\widetilde{G}) = D_{\text{in}}(G) - \mathcal{A}(G)^T.\end{aligned}$$

In general, $L(\widetilde{G}) \neq L(G)^T$. The *mirror digraph* $\widehat{G}$ of G is $(\mathcal{V}, \mathcal{E} \cup \widetilde{\mathcal{E}})$ with

$$\begin{aligned}\mathcal{A}(\widehat{G}) &= \frac{1}{2}(\mathcal{A}(G) + \mathcal{A}(G)^T) = \text{Sym}(\mathcal{A}(G)),\\ L(\widehat{G}) &= \frac{1}{2}(L(G) + L(\widetilde{G})).\end{aligned}$$

The mirror graph is undirected.

Lemma 2.2 *Given a weighted digraph G and $x, y \in \mathbb{R}^n$,*

$$\frac{1}{2} \sum_{(i,j)\in\mathcal{E}} a_{ij}(x_i - x_j)(y_i - y_j) = y^T L(\widehat{G})x.$$

PROOF. Note that, for $x \in \mathbb{R}^n$, the ith coordinate of the vectors $L(G)x, L(\widetilde{G})x \in \mathbb{R}^n$ are

$$(L(G)x)_i = \sum_{j=1}^n a_{ij}(x_i - x_j), \quad (L(\widetilde{G})x)_i = \sum_{j=1}^n a_{ji}(x_i - x_j).$$

We use these expressions in the following manipulations,

$$\begin{aligned}\sum_{(i,j)\in\mathcal{E}} a_{ij}(x_i - x_j)(y_i - y_j) &= \sum_{i,j=1}^n a_{ij}(x_i - x_j)(y_i - y_j)\\ &= \sum_{i,j=1}^n a_{ij}(x_i - x_j)y_i - \sum_{i,j=1}^n a_{ij}(x_i - x_j)y_j\\ &= \sum_{i=1}^n y_i(L(G)x)_i + \sum_{i,j=1}^n a_{ji}(x_i - x_j)y_i\\ &= y^T(L(G) + L(\widetilde{G}))x,\end{aligned}$$

as claimed. □

2.3 Jacobi overrelaxation iteration

Given an invertible matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$, consider the linear system $Ax = b$. The Jacobi overrelaxation (JOR) algorithm is an iterative procedure to compute the unique solution $x = A^{-1}b \in \mathbb{R}^n$. It is formulated as the discrete-time dynamical system

$$x_i(\ell + 1) = (1 - h)x_i(\ell) - h\frac{1}{a_{ii}}\Big(\sum_{j\neq i} a_{ij}x_j(\ell) - b_i\Big),$$

with $\ell \in \mathbb{Z}_{\geq 0}$, $i \in \{1, \dots, n\}$, $x(0) \in \mathbb{R}^n$, and $h \in (0, 1)$. The convergence properties of the JOR algorithm can be fully characterized in terms of the eigenvalues of the matrix describing the linear iteration, see [2].

Given a digraph G, as long as (i) agent i has access to b_i and a_{ii}, and (ii) if $a_{ij} \neq 0$, then $(i, j) \in \mathcal{E}$, the JOR algorithm is amenable to distributed implementation in the following sense: agent i can compute the ith component x_i of the solution $x = A^{-1}b$ with information gathered from its out-neighbors in G.

3 Problem statement

The objective of this paper is to synthesize a discrete-time distributed coordination algorithm that achieves global stabilization of the desired formation shape. Here we describe the capabilities of the robotic network and formally state the control objective.

3.1 Relative sensing network

Consider a group of n agents in $\mathbb{R}^d$. We assume that each agent has its own reference frame Σ_i. Expressed with respect to the fixed frame Σ_{fixed}, the ith frame Σ_i is described by a position $p_i^{\text{fixed}} \in \mathbb{R}^d$ and an orientation $R_i^{\text{fixed}} \in \text{SO}(d)$. The dynamical model of each agent is as follows. With its sensed information, agent $i \in \{1, \dots, n\}$ computes its own control input, expressed in its local frame Σ_i as u_i^i. In the local frame, each agent moves according to

$$p_i^i(\ell + 1) = u_i^i, \quad \ell \in \mathbb{Z}_{\geq 0}.$$

According to (1), in the global frame Σ_{fixed},

$$p_i^{\text{fixed}}(\ell + 1) = p_{\text{i}}^{\text{fixed}}(\ell) + R_i^{\text{fixed}} u_i^i, \quad \ell \in \mathbb{Z}_{\geq 0}.$$

The sensing interactions between agents are encoded by a digraph $G = (\{1, \dots, n\}, \mathcal{E})$. An edge (i, j) means that agent i can sense the relative position of agent j in its own local frame, p_j^i. There is no explicit communication among agents. We refer to this network by $\mathcal{S}_G^{\text{rs}}$.

A coordination algorithm is a specification of a control input u_i^i for each agent $i \in \{1, \dots, n\}$. The algorithm is *distributed over* $\mathcal{S}_G^{\text{rs}}$ if agent i can compute its control input with the sensing information collected on the relative position of its neighbors in the graph G.

3.2 The control objective

Our objective is to stabilize the group configuration to a desired formation. The desired formation is encoded as follows. Given $Z^* = (z_1^*, \dots, z_n^*) \in (\mathbb{R}^d)^n$, let $\texttt{Rgd}(Z^*)$

be the set of configurations in $(\mathbb{R}^d)^n$ which are related to Z^* by a translation and a rotation in $\mathbb{R}^d$, i.e.,

$$\texttt{Rgd}(Z^*) = \{(w_1,\dots,w_n) \in (\mathbb{R}^d)^n \mid \text{there is } (q,R) \in \mathbb{R}^d \times SO(d) \text{ such that } w_i = Rz_i^* + q,\ i \in \{1,\dots,n\}\},$$

Obviously, $Z^* \in \texttt{Rgd}(Z^*)$. Note that any two configurations in $\texttt{Rgd}(Z^*)$ have the same inter-agent distances, i.e., $\Bbbk_{ij} = \|w_i - w_j\| \in \mathbb{R}_{>0}$, $i \neq j \in \{1,\dots,n\}$ are the same for any $W \in \texttt{Rgd}(Z^*)$. The control objective is then to stabilize the group of agents to a configuration that belongs to $\texttt{Rgd}(Z^*)$.

4 Scaling theory and stress majorization

In this section, we introduce the notion of stress function from multidimensional scaling theory [4, 18] and explain its relationship with the formation control problem. We also prove various results that will be instrumental in the algorithm design, paying particular attention to the case of directed graphs.

4.1 The stress function

The *raw Stress* function $\mathrm{Stress}_G : (\mathbb{R}^d)^n \to \mathbb{R}$ is

$$\mathrm{Stress}_G\,(p_1,\dots,p_n) = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} (\|p_i - p_j\| - \Bbbk_{ij})^2. \quad (2)$$

For an undirected graph G, this definition coincides with the classical one used in the multidimensional scaling. Note that the desired formation configurations are global minimizers of Stress_G. Under additional assumptions on the rigidity of the graph, one can guarantee that they are the only ones. Alternatively, one may consider the *S-Stress* [28] function

$$\text{S-Stress}_G\,(p_1,\dots,p_n) = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} (\|p_i - p_j\|^2 - \Bbbk_{ij}^2)^2,$$

which has the same global minimizers. The S-Stress function is the Lyapunov function considered in [17, 23] in the context of formation control. Here instead we focus on the raw Stress, although the developments described below apply equally with the appropriate modifications.

The partial derivative of Stress_G with respect to p_i, $i \in \{1,\dots,n\}$, is given by

$$\frac{\partial \mathrm{Stress}_G}{\partial p_i} = 2\sum_{j:(i,j)\in\mathcal{E}} (\|p_i - p_j\| - \Bbbk_{ij})\frac{p_i - p_j}{\|p_i - p_j\|}. \quad (3)$$

This partial derivative can be computed by agent i with local information, and hence one can design a gradient-descent coordination algorithm to minimize Stress_G. Indeed, one can show [17] that the desired equilibria of the system are locally stable. However, the gradient system has other undesired equilibria (other local minimizers of Stress_G), which turn out to be also locally stable [8, 17]. In addition, it is not difficult to establish that the set of collinear network configurations is invariant under the gradient flow defined by (3). These observations are also valid for the gradient flow of S-Stress_G.

4.2 Stress majorization

In general, the direct optimization of the stress function is prone to local minima. An alternative route involves the construction of majorization functions that are easier to optimize. This is what we discuss next.

Let us start with some notation. Given $Z = (z_1,\dots,z_n) \in (\mathbb{R}^d)^n$ and an undirected graph G, let G^Z be the weighted graph with adjacency matrix $\mathcal{A}(G^Z)$ with entries

$$a_{ij} = \begin{cases} \Bbbk_{ij}\,\mathrm{inv}(\|z_i - z_j\|), & (i,j) \in \mathcal{E}, \\ 0, & (i,j) \notin \mathcal{E}, \end{cases}$$

where $\mathrm{inv}(x) = 1/x$ if $x \neq 0$, and $\mathrm{inv}(0) = 0$. Note that, for $Z \in \texttt{Rgd}(Z^*)$, the graphs G^Z and G are the same. The *stress majorization function* $\mathcal{F}_G^Z : (\mathbb{R}^d)^n \to \mathbb{R}$ is

$$\mathcal{F}_G^Z(P) = \mathrm{tr}\,(P^T L(G) P) - 2\,\mathrm{tr}\,(P^T L(G^Z) Z) + \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} \Bbbk_{ij}^2.$$

The name of the function is justified as follows.

Proposition 4.1 ([4]) *Given an undirected graph G, for any $P = (p_1,\dots,p_n)$, $Z = (z_1,\dots,z_n) \in (\mathbb{R}^d)^n$,*

$$\mathit{Stress}_G\,(p_1,\dots,p_n) \leq \mathcal{F}_G^Z(P).$$

Moreover, if $P = Z$, then $\mathcal{F}_G^P(P) = \mathit{Stress}_G\,(P)$.

This result can be generalized to the case of directed graphs using Lemma 2.2.

Proposition 4.2 *Given a digraph G, for any $P = (p_1,\dots,p_n)$, $Z = (z_1,\dots,z_n) \in (\mathbb{R}^d)^n$,*

$$\mathit{Stress}_G\,(p_1,\dots,p_n) \leq \mathcal{F}_{\widehat{G}}^Z(P).$$

Moreover, if $P = Z$, then $\mathcal{F}_{\widehat{G}}^P(P) = \mathit{Stress}_G\,(P)$.

PROOF. We expand the expression of Stress_G as

$$\mathrm{Stress}_G\,(p_1,\dots,p_n) = \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} \|p_i - p_j\|^2 - \sum_{(i,j)\in\mathcal{E}} \|p_i - p_j\|\Bbbk_{ij} + \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} \Bbbk_{ij}^2.$$

Using Lemma 2.2, the first summand can be rewritten as $\mathrm{tr}\,(P^T L(\widehat{G}) P)$. Using the Laplacian of the graph G^Z, the second summand can be rewritten as

$$-\sum_{(i,j)\in\mathcal{E}} \|p_i - p_j\|\Bbbk_{ij} = -\sum_{(i,j)\in\mathcal{E}} \Bbbk_{ij}\,\mathrm{inv}(\|z_i - z_j\|)\|p_i - p_j\| \cdot \|z_i - z_j\|.$$

Using the Cauchy-Schwarz inequality and an argument similar to the one used in the proof of Lemma 2.2, we deduce that this summand is upper bounded $2\,\mathrm{tr}\,(P^T L(\widehat{G}^Z)Z)$, with equality if $P = Z$, and this concludes the proof. □

Alternatively, the stress majorization function can be expressed using the Kronecker product as

$$\mathcal{F}^Z(P) = P^T(L(G)\otimes I_d)P \\ - 2P^T(L(G^Z)\otimes I_d)Z + \frac{1}{2}\sum_{(i,j)\in\mathcal{E}} \Bbbk_{ij}^2. \quad (4)$$

This expression is useful in establishing the following key properties of the stress majorization function.

Proposition 4.3 *Given $Z = (z_1, \dots, z_n) \in (\mathbb{R}^d)^n$ and an undirected graph G, the following holds:*

(i) The gradient and Hessian of $\mathcal{F}_G^Z$ are, respectively,

$$\nabla\mathcal{F}_G^Z = 2(L(G)\otimes I_d)P - 2(L(G^Z)\otimes I_d)Z,$$
$$\nabla^2(\mathcal{F}_G^Z) = 2L(G)\otimes I_d.$$

In particular, both are distributed over the graph G;

(ii) The function $\mathcal{F}_G^Z$ is globally convex;

(iii) $P \in (\mathbb{R}^d)^n$ is a global minimizer of $\mathcal{F}_G^Z$ if and only if

$$(L(G)\otimes I_d)P = (L(G^Z)\otimes I_d)Z. \quad (5)$$

In particular, any two minima of $\mathcal{F}_G^Z$ are equal up to a translation in $\mathbb{R}^d$.

PROOF. Fact (i) readily follows from expression (4). Fact (ii) follows from noting that the eigenvalues of $\nabla^2(\mathcal{F}_G^Z)$ are the same as those of $L(G)$ with their multiplicity doubled. Since $L(G)$ is positive semidefinite, so is $\nabla^2(\mathcal{F}_G^Z)$. Finally, fact (iii) follows from fact (i). □

The next result will be important later for our distributed algorithm design in the case of directed graphs.

Lemma 4.4 *Given $Z \in \mathtt{Rgd}(Z^*)$ and a digraph G with a globally reachable vertex, P is a global minimizer of $\mathcal{F}_{\widehat{G}}^Z$ if and only if*

$$(L(G)\otimes I_d)P = (L(G)\otimes I_d)Z. \quad (6)$$

PROOF. Note that $G = G^Z$ because $Z \in \mathtt{Rgd}(Z^*)$. According to Proposition 4.3(iii), P is a global minimizer of $\mathcal{F}_{\widehat{G}}^Z$ if and only if

$$(L(\widehat{G})\otimes I_d)P = (L(\widehat{G})\otimes I_d)Z. \quad (7)$$

Since G has a globally reachable vertex, the mirror graph $\widehat{G}$ is connected, and therefore, (7) is equivalent to $P = Z + \mathbf{1}_n \otimes q$, for some $q \in \mathbb{R}^d$, which implies (6). The reverse implication follows by noting that, since G has a globally reachable vertex, its Laplacian has rank $n-1$, and therefore $(L(G)\otimes I_d)v = 0$ if and only if $v \in \mathbf{1}_n \otimes \mathbb{R}^d$. Consequently, (6) implies that $P = Z + \mathbf{1}_n \otimes q$, for some $q \in \mathbb{R}^d$, which is equivalent to (7). □

The importance of Lemma 4.4 stems from the following observation: the critical points of $\mathcal{F}_{\widehat{G}}^Z$ can be characterized by a linear equation (6) defined by the Laplacian matrix of G, which is distributed over the digraph G. Note that the original characterization (7) is defined by the Laplacian matrix of the mirror graph, which is not distributed over G.

5 Coordination algorithm for global stabilization of formations

In this section, we propose a discrete-time distributed coordination algorithm that achieves global stabilization of the desired formation. We begin by discussing the problem of finding a network-wide reference frame and then design the motion coordination algorithm.

5.1 Common orientation of local reference frames

The reference frames of the individual agents in $\mathcal{S}_G^{\mathrm{rs}}$ might have different orientations with respect to the global reference frame. However, the network can execute some suitable initialization algorithm to equally orient all agent reference frames. Here we describe one simple procedure based on a distributed implementation of the flooding algorithm [24] on the relative sensing network. Other solutions to the common reference frame problem are explored in [25].

Assume G has a globally reachable vertex. For simplicity, we describe the strategy first in $\mathbb{R}^2$. At the first time step, a preselected globally reachable vertex moves a unit in the direction of its x-axis. All other agents that can sense the position of this agent measure the relative displacement in their local frames and figure out the x-axis direction of the agent. They rotate their frames to align them with the direction of the relative displacement. The process is repeated until all agents have aligned their frames with the one of the globally reachable vertex.

In $\mathbb{R}^3$, it takes two time steps for each agent to figure out the orientation of the reference frame of the globally reachable vertex. This vertex first moves in the direction of its x-axis, and then moves in the direction of its y-axis. Then the process is repeated until all agents have equally-aligned reference frames.

Note that this algorithm does not provide the network with a common origin or reference point, and therefore agents cannot compute or move to positions in the environment specified in a global reference frame.

5.2 Motion coordination via Jacobi iteration

Here, we assume that all agent reference frames have the same orientation, i.e., $R_i^{\mathrm{fixed}} = R$, for $i \in \{1, \dots, n\}$, for some $R \in \mathrm{SO}(d)$ which can be unknown to the individual agents. Given the discussion in Section 4, our strategy to make the network achieve the desired formation shape is to globally optimize the stress majorization function $\mathcal{F}_G^{Z^*}$. From Proposition 4.3 and Lemma 4.4, this can be achieved by solving the sparse linear equation

$$(L(G) \otimes I_d)P = (L(G) \otimes I_d)Z^*.$$

To solve this equation, we propose to use a Jacobi over-relaxation iteration (JOR), cf. Section 2.3. Let

$$b = (b_1, \dots, b_n) = (L(G) \otimes I_d)Z^*, \tag{8}$$

with $b_i \in \mathbb{R}^d$, $i \in \{1, \dots, n\}$. In Cartesian coordinates, the JOR algorithm for each agent i is, for $\ell \in \mathbb{Z}_{\geq 0}$,

$$p_i(\ell+1) = (1-h)p_i(\ell) + h\frac{1}{d_i}\Big(\sum_{j \neq i} a_{ij}p_j(\ell) + b_i\Big),$$

where $(d_1, \dots, d_n)$ is the diagonal of $D_{\text{out}}(G)$. If $d_i = 0$, then we set $p_i(\ell+1) = p_i(\ell)$. In the local frame of agent i, this is written, for $\ell \in \mathbb{Z}_{\geq 0}$,

$$p_i^i(\ell+1) = h\frac{1}{d_i}\Big(\sum_{j \neq i} a_{ij}p_j^i(\ell) + (R_i^{\text{fixed}})^T b_i\Big), \tag{9}$$

if $d_i \neq 0$, and $p_i^i(\ell+1) = 0$ otherwise. The individual agent does not know the rotation matrix $R_i^{\text{fixed}} = R$. Therefore, instead of (9), agent i runs, for $\ell \in \mathbb{Z}_{\geq 0}$,

$$p_i^i(\ell+1) = h\frac{1}{d_i}\Big(\sum_{j \neq i} a_{ij}p_j^i(\ell) + b_i\Big), \tag{10}$$

if $d_i \neq 0$, and $p_i^i(\ell+1) = 0$ otherwise. In the global frame, using (1), the algorithm (10) reads, for $\ell \in \mathbb{Z}_{\geq 0}$,

$$p_i(\ell+1) = (1-h)p_i(\ell) + h\frac{1}{d_i}\Big(\sum_{j \neq i} a_{ij}p_j(\ell) + Rb_i\Big), \tag{11}$$

if $d_i \neq 0$, and $p_i(\ell+1) = p_i(\ell)$ otherwise. This corresponds to the JOR algorithm to solve the linear equation

$$(L(G) \otimes I_d)P = (I_n \otimes R)b. \tag{12}$$

Note that all the solutions of this equation correspond to translations of a rotated configuration of Z^*, and therefore, all belong to $\mathtt{Rgd}(Z^*)$, as desired.

Next, we characterize the distributed character of this algorithm as well as its convergence properties.

Proposition 5.1 *Consider the relative sensing network $\mathcal{S}_G^{\text{rs}}$, where G has a globally reachable vertex. Let $h \in (0,1)$ and assume all agent reference frames are equally oriented. Then, the following holds*

(i) the coordination algorithm (10) *is distributed over $\mathcal{S}_G^{\text{rs}}$. Moreover, as initial information, each agent only needs to store a vector in $\mathbb{R}^d$;*

(ii) the coordination algorithm (10) *converges to a configuration W in $\mathtt{Rgd}(Z^*)$;*

(iii) if G is undirected, the stress majorization function $\mathcal{F}_G^W$ is monotonically decreasing along (10).

PROOF. Fact (i) follows from directly from (10). The only initial information that agent $i \in \{1, \dots, n\}$ requires is the vector $b_i \in \mathbb{R}^d$. To show fact (ii), we reason on the expression (11) of the algorithm in the global frame. Some basic manipulations yield

$$\begin{aligned}(I_n \otimes R)b &= (I_n \otimes R)(L(G) \otimes I_d)Z^* \\ &= (L(G) \otimes I_d)(I_n \otimes R)Z^*.\end{aligned}$$

Consider the change of coordinates $Y = (y_1, \dots, y_n) = P - (I_n \otimes R)Z^*$. Then, the linear equation (12) reads as

$$(L(G) \otimes I_d)Y = 0. \tag{13}$$

For $\alpha \in \{1, \dots, d\}$, let

$$\widetilde{y}_\alpha = \Pi_{i=1}^n \pi_\alpha(Y) = (\pi_\alpha(y_1), \dots, \pi_\alpha(y_n)) \in \mathbb{R}^n.$$

Then the JOR algorithm (11) reads, for $\ell \in \mathbb{Z}_{\geq 0}$,

$$(\widetilde{y}_\alpha)_i(\ell+1) = (\widetilde{y}_\alpha)_i(\ell) + h\frac{1}{d_i}\sum_{j \neq i} a_{ij}((\widetilde{y}_\alpha)_j(\ell) - (\widetilde{y}_\alpha)_i(\ell)),$$

if $d_i \neq 0$, and $(\widetilde{y}_\alpha)_i(\ell+1) = (\widetilde{y}_\alpha)_i(\ell)$ otherwise, for $\alpha \in \{1, \dots, d\}$. Therefore, we have d copies of the same linear dynamical system, and we just need to analyze the convergence properties of

$$z(\ell+1) = (I_n - hD_{\text{out}}^{-1}(G)L(G))z(\ell), \tag{14}$$

with $z(0) \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}_{\geq 0}$, where for convenience set $D_{\text{out}}^{-1}(G)_{ii} = 0$ if $d_i = 0$. It is easy to see that, for $h \in (0,1)$, the matrix $F = I_n - hD_{\text{out}}^{-1}(G)L(G)$ is stochastic. Using [5, Proposition 1.65(ii)], we deduce that the trajectory of (14) starting at $z(0)$ converges to $(v^T z(0)/v^T \mathbf{1}_n)\mathbf{1}_n$, where v denotes a left eigenvector of F with eigenvalue 1. Therefore, for each $\alpha \in \{1, \dots, d\}$, there exists $s_\alpha \in \mathbb{R}$ such that $\widetilde{y}_\alpha(\ell) \to s_\alpha \mathbf{1}_n$ as $\ell \to +\infty$. Denoting $s = (s_1, \dots, s_d) \in \mathbb{R}^d$, we conclude that $y_i(\ell) \to s$ as $\ell \to +\infty$, for $i \in \{1, \dots, n\}$, or equivalently, that the coordination algorithm (11) converges to the configuration $W = (RZ_1^* + s, \dots, RZ_n^* + s) = (I_n \otimes R)Z^* + \mathbf{1}_n \otimes s \in \mathtt{Rgd}(Z^*)$.

To show fact (iii), we examine the differences $\mathcal{F}_G^W(P(\ell+1)) - \mathcal{F}_G^W(P(\ell))$, where $\{P(\ell) \in (\mathbb{R}^d)^n \mid \ell \in \mathbb{Z}_{\geq 0}\}$ is the sequence of network configurations generated by the JOR algorithm (11). After some manipulations, this algorithm can be written in closed form as

$$P(\ell+1) = P(\ell) - h(D(G)^{-1}L(G) \otimes I_d)(P(\ell) - W),$$

for $\ell \in \mathbb{Z}_{\geq 0}$. Next, we compute the evolution of the stress majorization function as

$$\mathcal{F}_G^W(P(\ell+1)) - \mathcal{F}_G^W(P(\ell)) =$$
$$= P(\ell+1)^T(L(G)\otimes I_d)P(\ell+1) - P(\ell)^T(L(G)\otimes I_d)P(\ell)$$
$$- 2(P(\ell+1) - P(\ell))^T(L(G)\otimes I_d)W.$$

Using the identity $(A_1 \otimes B_1)(A_2 \otimes B_2) = (A_1A_2) \otimes (B_1B_2)$, we can rewrite the evolution of $\mathcal{F}_G^W$ as

$$\mathcal{F}_G^W(P(\ell+1)) - \mathcal{F}_G^W(P(\ell)) =$$
$$= -2hz^T(D(G)^{-1}\otimes I_d)(L(G)\otimes I_d)P(\ell)$$
$$+ h^2z^T(D(G)^{-1}L(G)D(G)^{-1}\otimes I_d)z$$
$$+ 2hz^T(D(G)^{-1}\otimes I_d)(L(G)\otimes I_d)W,$$

where, for brevity, we use the shorthand notation $z = (L(G)\otimes I_d)(P(\ell) - W)$. A final simplification yields

$$\mathcal{F}_G^W(P(\ell+1)) - \mathcal{F}_G^W(P(\ell)) =$$
$$= -hz^T((2D(G)^{-1} - hD(G)^{-1}L(G)D(G)^{-1})\otimes I_d)z.$$

Using Lemma 2.1, we can guarantee that for $h \in (0,1)$, the matrix $(2D(G)^{-1} - hD(G)^{-1}L(G)D(G)^{-1})\otimes I_d$ is positive definite, and therefore

$$\mathcal{F}_G^W(P(\ell+1)) - \mathcal{F}_G^W(P(\ell)) \leq 0,$$

with equality holding if and only if $z = (L(G)\otimes I_d)(P(\ell) - W) = 0$, i.e., $P(\ell) \in \texttt{Rgd}(Z^*)$. □

Remark 5.2 Proposition 5.1(iii) does not hold in general if G is directed. A counter example is given by the digraph plotted in Figure 1. For this digraph, the matrix

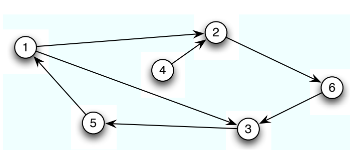
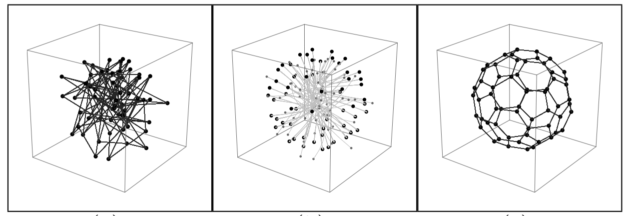


Fig. 1. Counter example to Proposition 5.1(iii) in the directed case. For this digraph, there exist initial network configurations for which $\mathcal{F}_G^W$ is not monotonically decreasing along the execution of (10).

$\mathrm{Sym}(L(\widetilde{G})D(G)^{-1}L(G))$ has a negative eigenvalue, and therefore there exist initial network configurations for which $\mathcal{F}_G^W$ is not monotonically decreasing along (10).•

Figure 2 shows an execution of (10) over a relative sensing network in $\mathbb{R}^3$ composed of 60 agents, with interaction topology given by the Buckminster Fuller geodesic dome [1]. The desired formation, shaped as a soccer ball, is encoded via (8), and agent $i \in \{1,\dots,n\}$ is provided with $b_i \in \mathbb{R}^3$. Proposition 5.1 guarantees that convergence to the desired formation shape is achieved.



(a) (b) (c)

Fig. 2. Execution of the algorithm (10) with $h = .25$ over a relative sensing network in $\mathbb{R}^3$ composed of 60 agents. (a) initial configuration, (b) evolution, and (c) final formation.

6 Robustness against link failures, measurement errors, and frame orientation errors

In this section, we examine the robustness properties of the coordination algorithm (10) against several sources of error. First, we consider the situation where one or more of agents fail to sense the relative position of other neighboring agents. We refer to such occurrences as *link failures*. Second, we consider measurement errors in the acquisition of the relative position of neighboring agents. Finally, we study the algorithm robustness to errors in the initialization of the orientation of the agents' frames.

6.1 Robustness against link failures and convergence of asynchronous executions

Consider the scenario where an agent fails to acquire the relative position of a neighbor. In such case, it seems logical for that agent to use the last recorded information about the relative position of the neighbor. Interestingly, this situation can be interpreted as an asynchronous execution of (10), where, at each time step, the information that an agent has about other neighboring agents is outdated to some degree. In our case, the outdatedness is due to a sensor failure, but it could also correspond to other reasons, such as delays in processing information.

This connection allows us to study the convergence of the resulting algorithm using well-established results from distributed algorithms. The asynchronous model for the network operation here corresponds to the *partially asynchronous model* from [2, Chapter 7]. Roughly speaking, when agent i, at time ℓ, uses the value p_j^i from another agent, that value is not necessarily the most recent one, $p_j^i(\ell)$, but rather an outdated one,

$$p_j^i(\tau_j^i(\ell)), \quad 0 \leq \tau_j^i(\ell) \leq \ell. \tag{15}$$

The quantity $\ell - \tau_j^i(\ell)$ represents the delay. Therefore, the JOR algorithm (10) then reads, for $\ell \in \mathbb{Z}_{\geq 0}$,

$$p_i^i(\ell+1) = h\frac{1}{d_i}\Big(\sum_{j\neq i} a_{ij}p_j^i(\tau_j^i(\ell)) + b_i\Big). \tag{16}$$

The following result states that the convergence of (16) is still guaranteed if the frequency of the failures is bounded (alternatively, the delays are bounded).

Proposition 6.1 *Consider the relative sensing network $\mathcal{S}_G^{\mathrm{rs}}$ operating asynchronously, where G has a globally reachable vertex. Let $h \in (0,1)$ and assume all agent reference frames are equally oriented. Assume there exists $B > 0$ such that $\ell - B + 1 \le \tau_j^i(\ell) \le \ell$, for all $\ell \in \mathbb{Z}_{\ge 0}$, and all $(i,j) \in \mathcal{E}$. Then, the algorithm* (16) *converges to a configuration W in $\mathtt{Rgd}(Z^*)$.*

Proposition 6.1 is established by rewriting the coordination algorithm in a manner analogous to the expressions obtained in the proof of Proposition 5.1(ii), and then applying the results in [3, 22]. We omit it for space reasons.

Figure 3 shows an execution of (16) over a relative sensing network composed of 20 agents, with interaction topology given by a directed version of the Desargues graph [12]. The maximum delay is $B = 10$ steps, i.e., no agent has relative position information on its neighbors that is more than 10 steps outdated. As forecasted by Proposition 6.1, convergence to the desired formation shape is achieved.

6.2 Robustness against measurement errors

Here we study the convergence properties of the coordination algorithm (10) in the presence of measurement errors. Assume that, for each edge (i,j) of G, agent i senses the position of agent j in its own local frame with an error bounded by $\sigma \in \mathbb{R}_{\ge 0}$. In other words, agent i uses the erroneous relative position

$$p_j^i + e_{ij}, \qquad (17)$$

of agent j, with $e_{ij} \in \mathbb{R}^d$, $\|e_{ij}\|_\infty \le \sigma$, instead of the correct relative position p_j^i. Under such measurement errors, the JOR algorithm (10) then reads, for $\ell \in \mathbb{Z}_{\ge 0}$,

$$p_i^i(\ell+1) = h\frac{1}{d_i}\Big(\sum_{j\neq i} a_{ij}\big(p_j^i(\ell) + e_{ij}(\ell)\big) + b_i\Big), \qquad (18)$$

with $\|e_{ij}(\ell)\|_\infty \le \sigma$ for each edge (i,j) of G. The following result characterizes the convergence of (18).

Proposition 6.2 *Consider the relative sensing network $\mathcal{S}_G^{\mathrm{rs}}$, where G has a globally reachable vertex. Let $h \in (0,1)$ and assume all agent reference frames are equally oriented. Assume the network agents acquire erroneous relative position information according to the model* (17) *with mismatch bounded by $\sigma \in \mathbb{R}_{\ge 0}$. Then, there exists $K \in \mathbb{R}_{>0}$ such that the algorithm* (18) *converges to*

$$\{Z \in (\mathbb{R}^d)^n \mid \text{there exists } W \in \mathtt{Rgd}(Z^*) \text{ such that } \|Z - W\|_\infty \le Kh\sigma\}.$$

PROOF. As in the proof of Proposition 5.1, define the change of coordinates $Y = (y_1,\dots,y_n) = P - (I_n \otimes R)Z^*$, and, for each $\alpha \in \{1,\dots,d\}$, let

$$\widetilde{y}_\alpha = \Pi_{i=1}^n \pi_\alpha(Y) = (\pi_\alpha(y_1),\dots,\pi_\alpha(y_n)) \in \mathbb{R}^n.$$

Then, the algorithm (18) can be rewritten, for $\ell \in \mathbb{Z}_{\ge 0}$,

$$(\widetilde{y}_\alpha)_i(\ell+1) = (\widetilde{y}_\alpha)_i(\ell) + h\frac{1}{d_i}\sum_{j\neq i} a_{ij}((\widetilde{y}_\alpha)_j(\ell) - (\widetilde{y}_\alpha)_i(\ell) + \pi_\alpha(e_{ij}(\ell))),$$

if $d_i \neq 0$, and $(\widetilde{y}_\alpha)_i(\ell+1) = (\widetilde{y}_\alpha)_i(\ell)$ otherwise, for $\alpha \in \{1,\dots,d\}$. Note that

$$|\sum_{j\neq i} a_{ij}\pi_\alpha(e_{ij}(\ell))| \le \sum_{j\neq i} a_{ij}|\pi_\alpha(e_{ij}(\ell))| \le \sum_{j\neq i} a_{ij}\|e_{ij}(\ell)\|_\infty \le d_i\sigma \qquad (19)$$

Therefore, we have d instances of a linear dynamical system with bounded disturbances, and we just need to analyze the convergence properties of

$$z(\ell+1) = (I_n - hD_{\mathrm{out}}^{-1}(G)L(G))z(\ell) + hv(\ell), \qquad (20)$$

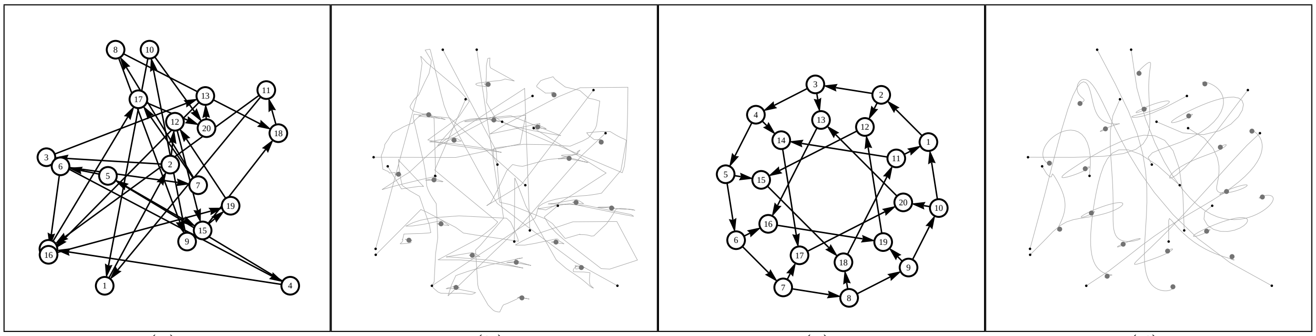
with $z(0) \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}_{\ge 0}$. From (19), we deduce that $\|v(\ell)\|_\infty \le \sigma$. Note that $F = I_n - hD_{\mathrm{out}}^{-1}(G)L(G)$ is stochastic. Moreover, since G is globally reachable, F has 1 as the unique eigenvalue of maximum magnitude and with multiplicity 1. Therefore, the linear system (20) is input-to-state stable [14] with respect to the agreement configurations $\{s\mathbf{1}_n \mid s \in \mathbb{R}\}$. The convergence result then follows from this observation. The constant K is obtained by observing that $F - \frac{1}{v^T\mathbf{1}_n}\mathbf{1}_n v^T$, where v denotes a left eigenvector of F with eigenvalue 1, has all its eigenvalues in the unit complex disk centered at 0, i.e., the matrix is *Schur stable*. Therefore, from [20, Chapter 5], there exist $c > 1$ and $b \in (0,1)$ such that $\|(F - \frac{1}{v^T\mathbf{1}_n}\mathbf{1}_n v^T)^k\|_\infty \le c\,b^k$. After upper bounding the evolution in (20), we obtain $K = \frac{c}{1-b}$. □

Note that Proposition 6.2 only guarantees convergence to a set. According to the statement, the configurations in this set correspond to translations and rotations of the desired formation slightly deformed by the effect of the erroneous relative position readings of the sensors.

Figure 4 shows an execution of (18) over the relative sensing network of Figure 3 under measurement errors in the acquisition of the relative position of neighboring agents. The norm of the measurement errors is upper bounded by $\sigma = .2$. As forecasted by Proposition 6.2, the network shape converges to a formation close to the desired one, while the group of agents moves in the plane.

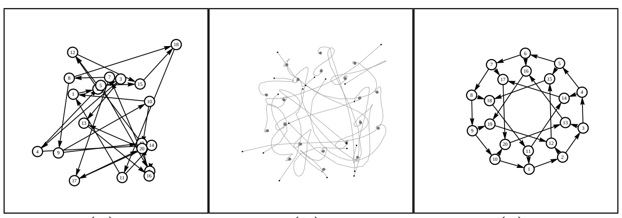
6.3 Robustness against frame orientation errors

Here we explore the robustness properties of the coordination algorithm (10) against errors in the computation of the common orientation of the reference frame, cf. Section 5.1. Suppose that the algorithm selected by the network to orient all agent frames equally is not executed perfectly and consequently the final orientation



(a) (b) (c) (d)

Fig. 3. Execution of the algorithm (16) with $h = .2$ over a relative sensing network in $\mathbb{R}^2$ composed of 20 agents. (a) initial configuration, (b) evolution, (c) final formation, and (d) evolution of (10) from the same initial condition for comparison.



(a) (b) (c)

Fig. 4. Execution of the algorithm (18) with $h = .25$ and $\sigma = .2$ over a relative sensing network in $\mathbb{R}^2$ composed of 20 agents. (a) initial configuration, (b) evolution, and (c) formation after 80 iterations.

of the frame of agent $i \in \{1, \dots, n\}$ is of the form

$$R_i^{\text{fixed}} = R + \Xi_i, \tag{21}$$

where $R \in \mathrm{SO}(d)$ and the error matrix Ξ_i satisfies $\|\Xi_i\|_\infty \leq \varepsilon$. Given the difference in the orientation of the frames, (10) reads now in the global frame as

$$p_i(\ell + 1) = (1 - h)p_i(\ell) + h\frac{1}{d_i}\Big(\sum_{j \neq i} a_{ij}p_j(\ell) + R_i^{\text{fixed}} b_i\Big), \tag{22}$$

for $\ell \in \mathbb{Z}_{\geq 0}$, which corresponds to the JOR algorithm to solve the linear algebraic equation

$$(L(G) \otimes I_d)P = \operatorname{diag}\left(R_1^{\text{fixed}}, \dots, R_n^{\text{fixed}}\right) b. \tag{23}$$

Observe that the mismatch in the orientation of the frames makes this linear equation ill-posed. In other words, the vector $\operatorname{diag}\left(R_1^{\text{fixed}}, \dots, R_n^{\text{fixed}}\right) b$ does not belong to the range of the matrix $L(G) \otimes I_d$, and therefore, there does not exist a solution P of (23). Intuitively, this observation is consistent with the fact that the algorithm design assumes all frames are equally oriented.

Even though (23) has no solution, the question about the convergence properties of (22) still remains. The following result provides an answer to it.

Proposition 6.3 *Consider the relative sensing network $\mathcal{S}_G^{\text{rs}}$, where G has a globally reachable vertex. Let $h \in (0, 1)$. Let the orientation of the frame of agent $i \in \{1, \dots, n\}$ be given by (21). Then, there exists $\widetilde{K} \in \mathbb{R}_{>0}$ such that the algorithm (10) converges to*

$$\{Z \in (\mathbb{R}^d)^n \mid \text{there exists } W \in \mathtt{Rgd}(Z^*) \text{ such that } \|Z - W\|_\infty \leq \widetilde{K} h \varepsilon\}.$$

The proof of this result is analogous to the proof of Proposition 6.2, and we omit it in the interest of space. After some manipulations, one can show that

$$\widetilde{K} = K \frac{\max\{\|b_i\|_\infty \mid i \in \{1, \dots, n\}\}}{\min\{d_i \mid d_i \neq 0,\ i \in \{1, \dots, n\}\}}.$$

Note that Proposition 6.3 only guarantees convergence to a set. According to the statement, the configurations in this set correspond to translations and rotations of the desired formation slightly deformed by the effect of the mismatch in the orientation of the agent frames.

Figure 5 shows an execution of (10) over the relative sensing network of Figure 3 under errors in the computation of the common orientation of the agent frames. Each agent orients its own frame with an angle of 90 degrees with an error whose absolute value is bounded by 9 degrees. The desired formation is the one given by Figure 3(c). As forecasted by Proposition 6.3, the network shape converges to a formation close to the desired one, while the group of agents moves in the plane.

Under additional conditions on the interaction topology, it is possible to state a slightly stronger result.

Corollary 6.4 *Under the same assumptions of Proposition 6.3, further assume that G is regular. Then, for each initial condition, there exists $s \in \mathbb{R}^d$ and $W \in (\mathbb{R}^d)^n$ with $\operatorname{dist}_\infty(W, \mathtt{Rgd}(Z^*)) \leq \widehat{K}\varepsilon$ (where $\widehat{K}$ only depends on the graph and the desired formation) such that the evolution of the coordination algorithm (10) converges to the line $\{Z + a(\mathbf{1}_n \otimes s) \mid a \in \mathbb{R}\}$.*

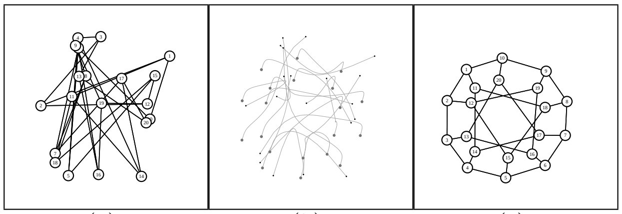


Fig. 5. Execution of the algorithm (10) with $h = .25$ over a relative sensing network in $\mathbb{R}^2$ composed of 20 agents under errors in the orientation of the agent frames. (a) initial configuration, (b) evolution, and (c) final formation shape.

7 Conclusions and future work

We have proposed a distributed formation control strategy for relative sensing networks. The algorithm design combines ideas on stress majorization from scaling theory with Jacobi overrelaxation algorithms from the theory of distributed linear iterations. We have analyzed the convergence properties of the proposed algorithm, showing that, for any interaction topology with a globally reachable vertex, it globally stabilizes the desired formation shape. We have established the algorithm robustness against complete failures, measurement errors in the acquisition of the relative position of neighboring agents, and errors in the initial computation of the common reference frame. Future work will include the design of error-correcting algorithms that completely eliminate any mismatch in the orientation of the agent frames and the extension of the results to consider switching interaction topologies and individual agent dynamics.

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