Area-constrained coverage optimization by robotic sensor networks

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***Abstract*— This paper studies robotic sensor networks performing coverage optimization tasks with area constraints. The network coverage of the environment is a function of the robot locations and the partition of the space. The area of the region assigned to each robot is constrained to be a prespecified amount. We characterize the optimal configurations as center generalized Voronoi configurations. The generalized Voronoi partition depends on a set of weights, one per robot, assigned to the network. We design a Jacobi iterative algorithm to find the weight assignment whose corresponding generalized Voronoi partition satisfies the area constraints. This algorithm is distributed over the generalized Delaunay graph. We also design the "move-to-center-and-compute-weight" coordination algorithm that steers the robotic network towards the set of center generalized Voronoi configurations while monotonically optimizing coverage. Various simulations illustrate our results.**

I. Introduction

This paper studies a class of locational optimization problems subject to area constraints. Our objective is to design distributed coordination algorithms for robotic sensor networks that (i) guarantee optimal quality-of-service, i.e., optimize the agent location and the partitioning of the environment, and (ii) satisfy a desired set of constraints imposed on the areas of the regions assigned to the agents.

Our study is motivated by applications in milling, mine sweeping, and minimum servicing time problems. Consider the following sample scenario. Given an environment of interest, we represent the likelihood of a customer appearing at specific locations by a density function. Ideally, one would like to partition the environment into regions of the same area and, at the same time, minimize the expected time an agent has to travel to service a location. Initially, the location of the customers might be unknown, and this can be reflected in the density function. As agents move within the environment, the density function can be updated in a way that reflects both the location and the time required to service the newly-discovered customers. It is our belief that coordination algorithms that address these scenarios can be designed building on the results presented in this paper.

Literature review: The discipline of facility location [1], [2] studies locational optimization problems and looks at optimal resource placement and optimal space partitioning. The notion of Voronoi partition, or generalized versions of it, plays an important role in locational optimization. The work [3] considers centroidal Voronoi partitions, [4] considers power diagrams, [5] considers additively-weighted Voronoi partitions, and [6] considers multiplicatively-weighted Voronoi partitions. From a computational geometric perspective, an important research issue is the design of efficient algorithms that, given a fixed set of locations, compute partitions of the space into regions of prescribed areas [4], [5], [6]. Among these, the equitable case (i.e., all areas being equal) is of special importance as it represents a balanced distribution of the overall load. In the context of robotic sensor networks, this work builds on [7], where distributed algorithms based on centroidal Voronoi partitions are presented, and [8], where limited-range interactions are considered. Voronoi partitions are also employed in [9], [10], [11]. Other works on coverage problems include [12], [13].

Statement of contributions: The contributions of the paper pertain both the analysis of a broad class of constrained locational optimization problems and the design of coordination algorithms for robotic sensor networks. Regarding analysis, we study the notion of generalized Voronoi partition associated with a given performance function. We pay special attention to the properties of the map that, given a fixed set of agent locations, maps a set of weights to the areas of the corresponding regions. We characterize the Jacobian of this map as the Laplacian corresponding to a weighted version of the generalized Delaunay graph induced by the Voronoi partition. This characterization allows us to prove that, given any network configuration and any performance function, there exists a weight assignment that makes the regions of the generalized Voronoi partition have a prescribed set of areas. A second set of results deal with the analysis of the solutions of the area-constrained locational optimization problems. We show that the generalized Voronoi partition is optimal among all partitions satisfying the area constraints. We also characterize the critical points of the optimization problem as center generalized Voronoi configurations. Regarding design, we provide two distributed algorithms over the generalized Delaunay graph. We design the "move-to-center-and-compute-weight" coordination algorithm to steer the network agents towards the set of center Voronoi configurations. At the same time, the evolution of the network under this algorithm monotonically optimizes the coverage of the environment. We also design a Jacobi iterative algorithm to solve the problem of finding the weight assignment that makes the generalized Voronoi partition satisfy the area constraints. This algorithm is of interest by itself, as it constitutes an efficient approach from a dynamical systems perspective to a classical computational geometric problem. Because of space constraints, all proofs are omitted.

Notation: We denote by $\mathrm{int}\,(U)$ the interior of a set $U \subset \mathbb{R}^n$. Unless otherwise noted, vectors are always understood as column vectors. Let $\mathbf{1}_n = (1,\dots,1)^T \in \mathbb{R}^n$ and $\mathbf{0}_n = (0,\dots,0)^T \in \mathbb{R}^n$. Let $\{e_1,\dots,e_n\}$ denote the canonical basis of $\mathbb{R}^n$. We let $\mathrm{diag}\,(\mathbb{R}^n) = \{(a,\dots,a) \in \mathbb{R}^n \mid a \in$

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$\mathbb{R}\}$. Of special interest to us is the orthogonal decomposition $\mathbb{R}^n = \operatorname{diag}(\mathbb{R}^n) \oplus \operatorname{diag}(\mathbb{R}^n)^\perp$, with associated projections $\pi_1 : \mathbb{R}^n \to \operatorname{diag}(\mathbb{R}^n)$ and $\pi_2 : \mathbb{R}^n \to \operatorname{diag}(\mathbb{R}^n)^\perp$. Note that

$$\pi_1(x) = \frac{\mathbf{1}_n^T x}{n}\mathbf{1}_n, \quad \pi_2(x) = x - \pi_1(x). \tag{1}$$

The diagonal set $\operatorname{diag}(\mathbb{R}^n)$ is 1-dimensional, and hence $\operatorname{diag}(\mathbb{R}^n)^{n-1}$ is $(n-1)$-dimensional.

II. Preliminaries

In this section we gather some preliminary notions on graph theory and computational geometry.

A. Notions from graph theory

Here we present some basic graph-theoretic notions [14], [15]. An (undirected) graph consists of a vertex set V and of a set E of unordered pairs of vertices. For $v_1, v_2 \in V$ distinct, (v_1, v_2) denotes an undirected edge between v_1 and v_2. A path in a graph is an ordered sequence of vertices such that any two consecutive vertices in the sequence are an edge of the graph. A graph is connected if there exists a path between any two vertices.

A weighted graph is a triplet $G = (V, E, A)$ where V and E are a graph and where $A \in \mathbb{R}^{n\times n}_{\geq 0}$ is a weighted adjacency matrix with the following properties: for $i, j \in \{1, \dots, n\}$, the entry $a_{ij} > 0$ if (v_i, v_j) is an edge of G, and $a_{ij} = 0$ otherwise. In other words, the scalars a_{ij}, for all $(v_i, v_j) \in E$, are a set of weights for the edges of G. The weighted Laplacian is the matrix defined by

$$L = \operatorname{diag}(A\mathbf{1}_n) - A.$$

The Laplacian matrix has several important properties: L is symmetric, all eigenvalues are nonnegative, and 0 is an eigenvalue of L with eigenvector $\mathbf{1}_n$. In addition, G is connected if and only if $\operatorname{rank}(L) = n - 1$.

A proximity graph [16], [8] is a generalization of the notion of graph that captures the fact that, in some situations, the edges of the graph change as the vertices move. More formally, given a set $\mathcal{P} = \{p_1, \dots, p_n\} \subset \mathbb{R}^d$ of n distinct points, the proximity graph $\mathcal{G}$ at $\mathcal{P}$, denoted by $\mathcal{G}(\mathcal{P})$, is an undirected graph with vertex set $\mathcal{P}$ and with edge set $\mathcal{E}_\mathcal{G}(\mathcal{P})$. A graph G can be interpreted as a proximity graph whose edge set does not depend on the specific configuration $\mathcal{P}$.

B. Generalized Voronoi partitions

Here we discuss the notion of Voronoi partition and some generalizations following [17], [2]. Let Q be a convex set in $\mathbb{R}^d$. The *Voronoi partition* $\mathcal{V}(P) = \{V_1(P), \dots, V_n(P)\}$ of Q associated to $P = (p_1, \dots, p_n) \in Q^n$ is defined by

$$V_i(P) = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|\}. \tag{2}$$

The collection $\mathcal{V}(P)$ partitions Q into sets whose interiors are pairwise disjoint. Note that each Voronoi region is convex.

Let $f : \mathbb{R} \to \mathbb{R}$ be a strictly increasing function. The *generalized Voronoi partition* $\mathcal{V}(P, \omega; f) = \{V_1(P, \omega; f), \dots, V_n(P, \omega; f)\}$ of Q associated to $P = (p_1, \dots, p_n) \in Q^n$ and $\omega = (w_1, \dots, w_n) \in \mathbb{R}^n$ is

$$\begin{aligned} &V_i(P, \omega; f) \\ &= \{q \in Q \mid f(\|q - p_i\|) - w_i \leq f(\|q - p_j\|) - w_j\}. \end{aligned} \tag{3}$$

In general, the generalized Voronoi regions are neither convex nor star-shaped. The collection $\mathcal{V}(P, \omega; f)$ partitions Q into sets whose interiors are pairwise disjoint. Depending on the selection of weights and agent locations, $V_i(P, \omega; f)$ might be empty for some i. Indeed, $V_i(P, \omega; f) = \emptyset$ if there exist $i, j \in \{1, \dots, n\}$ such that

$$w_j - w_i > f(\|p_i - p_j\|) - f(0). \tag{4}$$

The generalized Voronoi partition induces the *generalized Delaunay proximity graph* $\mathcal{G}_\mathcal{V}$. The vertices of $\mathcal{G}_\mathcal{V}$ are $\{(p_1, w_1), \dots, (p_n, w_n)\}$ and its edges are determined as follows: (p_i, w_i) and (p_j, w_j) are neighbors if and only if their respective Voronoi regions intersect $V_i(P, \omega; f) \cap V_j(P, \omega; f) \neq \emptyset$. We use the shorthand notation

$$\Delta_{ij}(P, \omega; f) = V_i(P, \omega; f) \cap V_j(P, \omega; f),$$

for convenience. The graph $\mathcal{G}_\mathcal{V}$ is undirected and, if all Voronoi regions are non-empty, it is connected.

Lemma II.1 (Properties of generalized Voronoi partition) *For $f : \mathbb{R} \to \mathbb{R}$ strictly increasing and locations $p_1, \dots, p_n \in Q$, the generalized Voronoi partition* (3) *of Q is*

- (i) *equal to the Voronoi partition* (2) *if $w_1 = \dots = w_n$;*
- (ii) *invariant under weight translations $\omega = (w_1, \dots, w_n) \mapsto \omega + a\mathbf{1}_n = (w_1 + a, \dots, w_n + a)$, $a \in \mathbb{R}$, i.e., for $i \in \{1, \dots, n\}$,*

$$V_i(P, \omega + a\mathbf{1}_n; f) = V_i(P, \omega; f);$$

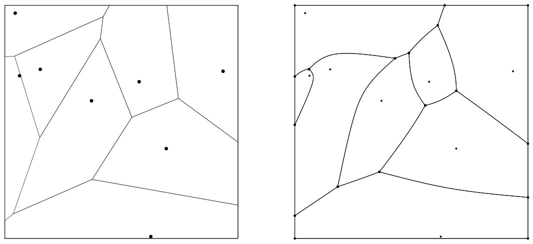
- (iii) *monotonic in the set of weights, i.e., for any $i \in \{1, \dots, n\}$ and any $\omega = (w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_n)$, $\omega' = (w_1, \dots, w_{i-1}, w_i', w_{i+1}, \dots, w_n) \in \mathbb{R}^n$ with $w_i' \geq w_i$,*

$$\begin{aligned} V_i(P, \omega; f) &\subseteq V_i(P, \omega'; f), \\ V_j(P, \omega'; f) &\subseteq V_j(P, \omega; f), \quad j \neq i. \end{aligned}$$

The generalized Voronoi partition takes different forms depending on the performance function. Examples include:

a) Quadratic performance: for $f(x) = x^2$, the generalized Voronoi partition is the *power diagram*. The boundary of $V_i(P, \omega; f)$ is composed of straight segments. For each Voronoi neighbor p_j, there is a segment that belongs to the bisector line between p_i and p_j, displaced towards either p_j or p_i depending on whether w_i is larger than w_j. The Voronoi regions are convex sets. Figure 1(a) shows an example.

b) Linear performance: for $f(x) = x$, the generalized Voronoi partition is the *additively-weighted Voronoi partition*. The boundary of $V_i(P, \omega; f)$ is composed of hyperbolic segments. For each Voronoi neighbor p_j, there is a hyperbolic segment of the hyperbola with foci p_i and p_j, and semimajor axis $|w_i - w_j|$. If $w_i > w_j$, then $\Delta_{ij}(P, \omega; f)$ belongs to the branch of the hyperbola closest to p_j, and if $w_i < w_j$, then



(a) (b)

Fig. 1. Power diagram (a) and additively weighted Voronoi partition (b) defined by 8 randomly deployed agents with randomly assigned weights.

$\Delta_{ij}(P,\omega;f)$ belongs to the branch of the hyperbola closest to p_i. If $w_i = w_j$, then the hyperbola is just the bisector line defined by p_i and p_j. The Voronoi regions are star-shaped sets. Figure 1(b) shows an example.

c) Logarithmic performance: for $f(x) = \log x$, the generalized Voronoi partition is the *multiplicatively-weighted Voronoi partition*. The boundary of $V_i(P,\omega;f)$ is composed of circular segments. For each Voronoi neighbor p_j, there is a circular segment of the circle with center $\frac{e^{2w_i}}{e^{2w_i}-e^{2w_j}}p_j + \frac{e^{2w_j}}{e^{2w_j}-e^{2w_i}}p_i$ and radius $\frac{e^{w_i+w_j}}{|e^{2w_i}-e^{2w_j}|}\|p_j - p_i\|$. If $w_i = w_j$, then the circle has infinite radius, i.e., is the bisector line defined by p_i and p_j. The Voronoi regions are non-empty and might contain holes. In general, they are neither convex nor connected.

III. Problem Statement

This section presents the area-constrained locational optimization problem. We start by briefly discussing the unconstrained optimization problem. Although the solution to this problem is known, it serves as a useful introduction to the problem of interest in this paper.

Let Q be a convex set in $\mathbb{R}^d$. Consider n agents evolving in Q with positions $p_1,\dots,p_n$. Consider the function

$$\mathcal{H}(p_1,\dots,p_n,W_1,\dots,W_n) = \sum_{i=1}^{n}\int_{W_i} f(\|q-p_i\|)\phi(q)dq,$$

where $W_1,\dots,W_n$ is a partition of the environment Q, $f : \mathbb{R}\to\mathbb{R}$ is a strictly increasing function modeling sensing performance, and $\phi : Q \to \mathbb{R}$ is a density function.

A. The 1-center problem

Consider the optimization of $\mathcal{H}$ when there is only one agent in the environment. The function takes the form

$$\mathcal{H}_1(p) = \int_Q f(\|q-p\|)\phi(q)dq.$$

It is not difficult to see that if f strictly convex, then $\mathcal{H}_1$ is strictly convex, and the next result follows.

Lemma III.1 (Minimizer of $\mathcal{H}_1$ is geometric center of Q) *For Q convex, there is a unique minimizer Cntr(Q) of $\mathcal{H}_1$.*

Observe that the dependence of the minimizer of $\mathcal{H}_1$ on the set Q is continuous, i.e., small changes in Q induce small changes in the optimal agent location Cntr(Q). If Q is not convex, then the center might not be unique. However, the continuous dependence of the minimizers of $\mathcal{H}_1$ on Q still holds. The minimizer depends on the performance function. The following are some relevant cases:

d) Quadratic performance: for $f(x) = x^2$, the gradient of $\mathcal{H}_1$ is

$$\frac{\partial\mathcal{H}_1}{\partial p} = 2\Big(p\int_Q \phi(q)dq - \int_Q q\phi(q)dq\Big).$$

The minimizer of $\mathcal{H}_1$ is the center of mass of Q,

$$\mathrm{CM}(Q) = \frac{\int_Q q\phi(q)dq}{\int_Q \phi(q)dq}.$$

e) Linear performance: for $f(x) = x$, the gradient of $\mathcal{H}_1$ is

$$\frac{\partial\mathcal{H}_1}{\partial p} = \int_Q \frac{p-q}{\|p-q\|}\phi(q)dq. \tag{5}$$

The minimizer of $\mathcal{H}_1$ is the unique point that makes (5) vanish. In general, the minimizer does not have an analytic expression. In the discrete version of this problem, the minimizer is called the Weber or Fermat-Torricelli point [18].

f) Logarithmic performance: for $f(x) = \log x$, the gradient of $\mathcal{H}_1$ is

$$\frac{\partial\mathcal{H}_1}{\partial p} = \int_Q \frac{p-q}{\|p-q\|^2}\phi(q)dq. \tag{6}$$

The minimizer of $\mathcal{H}_1$ is the unique point that makes (6) vanish. In general, it does not have an analytic expression.

B. The unconstrained locational optimization problem

Consider now the multicenter optimization problem where we seek to minimize the value of $\mathcal{H}$ among all possible agent locations and all possible partitions of Q,

$$\text{minimize } \mathcal{H}(p_1,\dots,p_n,W_1,\dots,W_n). \tag{7}$$

If we fix the partition $W_1,\dots,W_n$ of Q, then the problem of optimizing $\mathcal{H}$ consists of solving n 1-center optimization problems, one per individual agent. Therefore, Lemma III.1 implies that for fixed $W_1,\dots,W_n$, the optimal agent locations are Cntr$(W_1),\dots,$Cntr(W_n), respectively.

Interestingly enough, for fixed agent locations $p_1,\dots,p_n \in Q$, the optimal partition of Q does not depend on the specific performance function [3], [7]. In general, the optimal partition is the Voronoi partition $\mathcal{V}(p_1,\dots,p_n)$ defined by (2). Therefore, we have the following result.

Lemma III.2 (Critical points of $\mathcal{H}$ are center Voronoi configurations) *A solution $p_1^*,\dots,p_n^*,W_1^*,\dots,W_n^*$ of (7) is a center Voronoi configuration of Q, i.e., for $i\in\{1,\dots,n\}$,*

$$p_i^* = \mathit{Cntr}(W_i^*), \quad W_i^* = V_i(P^*).$$

C. The area-constrained locational optimization problem

Next, we consider an area-constrained multicenter optimization problem. We seek to minimize the value of $\mathcal{H}$ among all possible agent locations and all possible partitions of Q, but with the constraint that the (generalized) area of each region must be a pre-specified amount. Formally, a *feasible* *collection of areas* is a set $\{a_1, \ldots, a_n\} \subset \mathbb{R}_{>0}$ satisfying $\sum_{i=1}^n a_i = \int_Q \phi(q)dq = \text{area}_\phi(Q)$. We then set

$$\text{minimize } \mathcal{H}(p_1, \ldots, p_n, W_1, \ldots, W_n), \tag{8a}$$

$$\text{subject to } \int_{W_i} \phi(q)dq = a_i, \; i \in \{1, \ldots, n\}. \tag{8b}$$

A case of particular interest is the *equitable partition* case, when all areas are the same, i.e.,

$$a_i = \frac{1}{n} \int_Q \phi(q)dq, \quad i \in \{1, \ldots, n\}.$$

IV. Analysis of area-constrained locational optimization

In this section, we characterize the optimal solution of (8). For a fixed partition $W_1, \ldots, W_n$ of Q, the optimal agent locations depend on the performance function in the same way as for the unconstrained optimization problem, cf. Section III-B. The problem of optimizing $\mathcal{H}$ consists of solving n 1-center optimization problems. Therefore, Lemma III.1 implies that for fixed $W_1, \ldots, W_n$, the optimal agent locations are $\text{Cntr}(W_1), \ldots, \text{Cntr}(W_n)$, respectively.

Given fixed agent locations $p_1, \ldots, p_n \in Q$, our objective is to determine the optimal partition of Q with respect to $\mathcal{H}$. We show that, unlike for the problem (7), the optimal partition depends on the performance function. In order to do this, we will find it useful to characterize the properties of the areas of the generalized Voronoi regions. We discuss this next.

A. Weights-to-areas assignment

Here, we study the properties of the map that assigns to a set of weights the corresponding set of areas of the generalized Voronoi regions. Let $p_1, \ldots, p_n \in Q$ be fixed agent locations. Consider the neighborhood of $\text{diag}(\mathbb{R}^n)$ defined by

$$U = \{\omega \in \mathbb{R}^n \mid |w_i - w_j| \leq f(\|p_i - p_j\|) - f(0) \\ \text{for all } i, j \in \{1, \ldots, n\}\}.$$

The *weights-to-areas map* $\mathcal{M} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\mathcal{M}(\omega) = \Big(\int_{V_1(P,\omega;f)} \phi(q)dq, \ldots, \int_{V_n(P,\omega;f)} \phi(q)dq \Big),$$

where $P = (p_1, \ldots, p_n)$. Note that, if $\omega \notin U$, then, according to (4), at least there is one empty generalized Voronoi region. We begin by establishing some important properties of $\mathcal{M}$.

Proposition IV.1 (Properties of the weights-to-areas map) *Let $p_1, \ldots, p_n \in Q$. The map $\mathcal{M} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is invariant under translations and its range belongs to the $(n-1)$-dimensional space $\{m \in \mathbb{R}^n_{\geq 0} \mid \mathbf{1}^T m = 1\}$. Moreover, $\mathcal{M}$ is gradient, i.e., $\nabla F = -\mathcal{M}$, where $F : \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$F(\omega) = \sum_{j=1}^n \int_{V_j(P,\omega;f)} \big(f(\|q - p_j\|) - w_j\big)\phi(q)dq.$$

Using Proposition IV.1, one can derive various interesting properties of the Jacobian of $\mathcal{M}$. We state them next.

Proposition IV.2 (The Jacobian of the weights-to-areas map is the Laplacian of the weighted generalized Delaunay graph) *Let $p_1, \ldots, p_n \in Q$ and let $J(\mathcal{M})$ denote the Jacobian matrix of $\mathcal{M} : U \subset \mathbb{R}^n \to \mathbb{R}^n$. Then,*

(i) *$J(\mathcal{M})$ is symmetric;*
(ii) *$\mathbf{1}_n$ is an eigenvector of $J(\mathcal{M})$ with eigenvalue 0;*
(iii) *The rank of $J(\mathcal{M})$ on $\text{int}(U)$ is $n-1$.*

Therefore, the Jacobian matrix of $\mathcal{M}$ is the Laplacian of the generalized Delaunay graph whose edges are weighted as

$$a_{ij} = \frac{\partial \mathcal{M}_i}{\partial w_j},$$

if i and j are Delaunay neighbors, and $a_{ij} = 0$ otherwise.

Since $\mathcal{M}$ is invariant under translations, cf. Proposition IV.1, we define the equivalence relation $\sim$ on $\mathbb{R}^n$:

$x \sim y$ if and only if there exists $(a, \ldots, a) \in \text{diag}(\mathbb{R}^n)$ such that $x = (a, \ldots, a) + y$.

Under this relation, any $\omega \in \mathbb{R}^n$ and its projection onto $\text{diag}(\mathbb{R}^n)^\perp$ are related, since there exists $(a, \ldots, a) = \pi_1(\omega) \in \text{diag}(\mathbb{R}^n)$ such that $\omega = \pi_1(\omega) + \pi_2(\omega)$. Therefore, we identify the quotient space $\mathbb{R}^n/\sim$ with $\text{diag}(\mathbb{R}^n)^\perp$ by means of the linear projection π_2.

$$\omega \mapsto \pi_2(\omega) = \omega - \pi_1(x). \tag{9}$$

The equivalence relation $\sim$ allows us to state a particularly useful property of $\mathcal{M}$ in an elegant way.

Corollary IV.3 *Let $p_1, \ldots, p_n \in Q$. The map $\mathcal{M} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ induces a local diffeomorphism $\widetilde{\mathcal{M}} : U/\sim \equiv \mathbb{R}^{n-1} \to \{m \in \mathbb{R}^n_{\geq 0} \mid \mathbf{1}^T m = 1\}$.*

We are now ready to establish that, given any network configuration and any feasible collection of areas, there exists a set of weights such that the associated generalized Voronoi partition satisfies the area constraints.

Proposition IV.4 (Existence of weight assignment that makes generalized Voronoi partition satisfy area constraints) *Let $p_1, \ldots, p_n \in Q$ and let $\{a_1, \ldots, a_n\} \subset \mathbb{R}_{>0}$ be a feasible collection of areas. Then there exists a set of weights $\omega = \{w_1, \ldots, w_n\} \subset \mathbb{R}$ such that*

$$\int_{V_i(P,\omega;f)} \phi(q)dq = a_i, \quad i \in \{1, \ldots, n\}.$$

Proposition IV.4, together with Corollary IV.3, states that, up to translations, the set of weights ω such that $\mathcal{M}(\omega) = (a_1, \ldots, a_n)$ is locally unique, that is, there exists a neighborhood of ω in $\mathbb{R}^n$ where no other set of weights (other than those equivalent to ω by translation) are mapped to $(a_1, \ldots, a_n)$ under $\mathcal{M}$.

B. Optimality of the generalized Voronoi partition

Next, we show that, for fixed agent locations, the optimal partition for the area-constrained locational optimization problem (8) is the generalized Voronoi partition.

Proposition IV.5 (Generalized Voronoi partition is $\mathcal{H}$-optimal among all partitions that satisfy area constraints)

Let $p_1, \dots, p_n \in Q$ be fixed agent locations and let $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$ be a feasible collection of areas. Let $\omega \in \mathbb{R}^n$ be such that $\mathcal{M}(\omega) = (a_1, \dots, a_n)$. Then, the generalized Voronoi partition $\mathcal{V}(P, \omega; f)$ optimizes $\mathcal{H}$ among all partitions satisfying the area constraints (8b).

We are now ready to state the analogue result to Lemma III.2 for the area-constrained problem.

Corollary IV.6 (Critical points of $\mathcal{H}$ with area constraints are center generalized Voronoi partitions) *A solution $p_1^*, \dots, p_n^*, W_1^*, \dots, W_n^*$ of* (8) *is a center generalized Voronoi configuration of Q, i.e., there exists a weight assignment $\omega^* \in \mathbb{R}^n$ such that, for $i \in \{1, \dots, n\}$,*

$$p_i^* = \mathit{Cntr}(W_i^*), \quad W_i^* = V_i(P^*, \omega^*; f).$$

V. Area-constrained locational optimization via distributed coordination

Here, we investigate distributed algorithmic solutions to the area-constrained locational optimization problem (8).

A. The "move-to-center-and-compute-weight" algorithm

Our strategy to solve (8) is to make each agent go to the center of its own generalized Voronoi region while, at the same time, the individual agent weights are tuned to satisfy the area constraints. Let us formalize this approach.

For a feasible collection of areas $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$, Proposition IV.4 guarantees that there exists a map $\mathcal{A} : Q^n \to \mathbb{R}^n$, assigning agent locations to weights, that satisfies

$$\mathcal{M}(\mathcal{A}(p_1, \dots, p_n)) = (a_1, \dots, a_n).$$

Moreover, the weight assignment can be selected so that $\mathcal{A}$ is continuous. The "move-to-center-and-compute-weight" algorithm is the discrete-time map $\mathrm{T} : Q^n \to Q^n$ defined by

$$\begin{aligned} &\mathrm{T}(p_1, \dots, p_n) \\ &= (\mathrm{Cntr}(V_1(P, \mathcal{A}(P); f)), \dots, \mathrm{Cntr}(V_n(P, \mathcal{A}(P); f))). \end{aligned} \tag{10}$$

The map T is continuous because $\mathcal{A}$ is, the Voronoi partition (3) changes continuously with the agent locations, and the solution to the 1-center problem changes continuously with the set. Provided $\mathcal{A}$ is distributed over the generalized Delaunay graph, i.e., agent i only needs to interact with its neighbors in the graph to compute its weight, then T is also distributed over the generalized Delaunay graph.

Proposition V.1 (Asymptotic convergence of "move-to-center-and-compute-weight" algorithm) *The trajectories of the discrete-time coordination algorithm T converge asymptotically to the set of center generalized Voronoi configurations of Q, while monotonically decreasing $\mathcal{H}$.*

From Corollary IV.6, we know that the solutions of the area-constrained locational optimization problem are generalized center Voronoi configurations. Proposition V.1 guarantees that the "move-to-center-and-compute-weight" algorithm steers the network towards this desirable set.

B. Jacobi iterative algorithm for weight assignment

In general, an explicit expression of the weight-assignment map $\mathcal{A}$ is not available. Equivalently, it is not possible in general to obtain an explicit expression for an inverse of the map $\mathcal{M}$. Our approach to this problem is to synthesize a distributed Jacobi iterative algorithm that numerically finds an appropriate weight assignment.

Given $p_1, \dots, p_n \in Q$ and a feasible collection of areas $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$, define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(w_1, \dots, w_n) = \mathcal{M}(w_1, \dots, w_n) - (a_1, \dots, a_n).$$

From Proposition IV.1, we know that g is the gradient vector field corresponding to the function $G : \mathbb{R}^n \to \mathbb{R}$,

$$G(\omega) = -F(\omega) - \sum_{i=1}^{n} w_i a_i.$$

We look for $\omega \in \mathbb{R}^n$ such that

$$g(\omega) = \mathbf{0}_n. \tag{11}$$

Alternatively, we look for a weight assignment that optimizes the value of G. There are multiple methods that can be used to this end, see e.g., [19]. Here, we use the Jacobi algorithm

$$\begin{aligned} \omega_{k+1} &= \omega_k \\ &\quad - \gamma \operatorname{diag}\left(\frac{\partial g_1}{\partial w_1}(\omega_k), \dots, \frac{\partial g_n}{\partial w_n}(\omega_k)\right)^{-1} g(\omega_k), \end{aligned} \tag{12}$$

for $k \geq 0$, where $\operatorname{diag}(v) \in \mathbb{R}^{n \times n}$ is the diagonal matrix with the components of the vector $v \in \mathbb{R}^n$ in the diagonal. Here, $\gamma > 0$ is a parameter that can be chosen to guarantee convergence. Note that the Jacobian of g and $\mathcal{M}$ are the same, that is, $J(g) = J(\mathcal{M})$. Therefore, from Proposition IV.2, we can state that the Jacobi algorithm is distributed over the generalized Delaunay graph. In other words, agent i only needs to interact with its neighbors in the graph to compute the ith entry of ω_{k+1} as prescribed by (12). The following result states that the Jacobi algorithm converges to a weight assignment that satisfies (11) and is a consequence of [19, Section 3.2].

Proposition V.2 (Convergence of Jacobi algorithm to desired weight assignment) *For any initial condition $\omega_0 \in \mathbb{R}^n$, there exists γ_* such that if $0 < \gamma < \gamma_*$, then the sequence $\{\omega_k \in \mathbb{R}^n \mid k \in \mathbb{Z}_{\geq 0}\}$ generated by the Jacobi algorithm* (12) *satisfies $\lim_{k \to +\infty} g(\omega_k) = 0$.*

C. Simulations

We present simulations of the Jacobi iterative algorithm (12) in Figure 2 and of the "move-to-center-and-compute-weight" algorithm (10) in Figure 3 for the linear performance case, $f(x) = x$. Each algorithm has been implemented in Mathematica® as a main, centralized program that makes use of a library of routines for the computation of generalized Voronoi cells, line and area integrals, and geometric centers. As previously noted, both algorithms are distributed over the generalized Delaunay graph, i.e., each agent only needs to interact with its neighbors in the graph to execute the algorithms. On average, if all weights are similar, this means that each agent interacts with six neighbors [2].

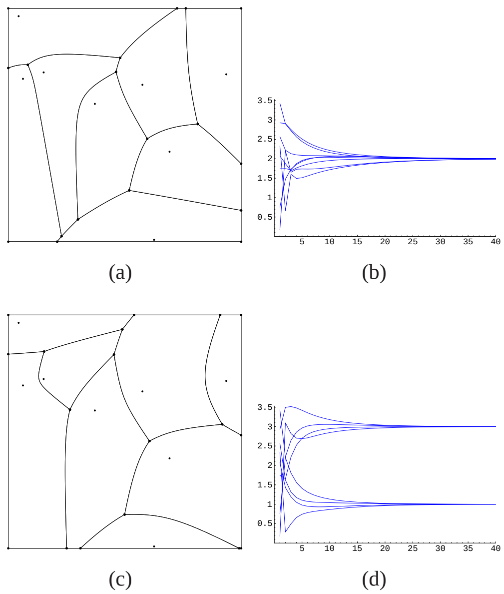


Fig. 2. Two executions of the Jacobi iterative algorithm (12). The location and initial weight assignment of the 8 agents in the square $[0, 4] \times [0, 4]$ is as in Figure 1(b), and the density is constant and equal to 1. The Jacobi algorithm is run with $\gamma = .3$. In the upper case, the target areas are $a_i = 2$, $i \in \{1, \ldots, 8\}$. In the lower case, the target areas are $a_i = 1$ if i is even, and $a_i = 3$ if i is odd. (a) and (c) show the final additively weighted Voronoi partitions obtained by the Jacobi algorithm in each case, whereas (b) and (d) show the corresponding evolution of the areas during the execution. In both cases, after 40 iterations, the executions are very close to the solution.

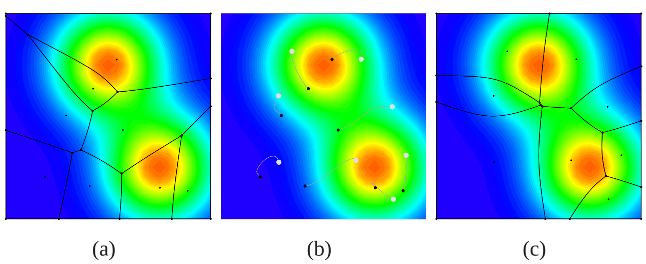


Fig. 3. Execution of the "move-to-center-and-compute-weight" algorithm (10). (a) shows the initial network configuration, (b) shows the evolution of the algorithm, and (c) shows the final center generalized Voronoi configuration attained after 80 iterations of T. All figures show the contour plot of $\phi = 1 + 2\,e^{-(x-3)^2-(y-1)^2} + 2\,e^{-(x-2)^2-(y-3)^2}$. The feasible collection of areas that constrain the partition are $a_i = \text{area}_\phi(Q)/16$ for i even, and $a_i = 3\,\text{area}_\phi(Q)/16$ for i odd.

VI. Conclusions

We have studied the area-constrained locational problem, where a group of robots seeks to optimize an appropriate notion of environmental coverage by partitioning the space into regions that have a pre-specified area. We have characterized the critical points of this optimization problem as center generalized Voronoi configurations. We have also designed a distributed coordination algorithm that steers the network towards this desirable set while at the same time monotonically optimizing the aggregate objective function. We have also obtained a distributed algorithm that, given a network configuration and a feasible collection of areas, computes a weight assignment whose associated generalized Voronoi configuration satisfies the constraints.

Future work will explore the area-constrained locational problem under limited-range interactions and time-dependent density functions. Limited-range interactions occur naturally in wireless sensor networks. Time-dependent density functions can model changing conditions in the environment. We are particularly interested in servicing problems where agents need to spend a fixed amount of time taking care of locations distributed throughout the environment.

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