Coverage optimization and spatial load balancing by robotic sensor networks

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Abstract

This paper studies robotic sensor networks performing static coverage optimization with area constraints. Given a density function describing the probability of events and a performance function measuring the cost to service a location, the objective is to position sensors in the environment so as to minimize the expected servicing cost. Moreover, because of load balancing considerations, the area of the region assigned to each robot is constrained to be a pre-specified amount. We characterize the optimal configurations as center generalized Voronoi configurations. The generalized Voronoi partition depends on a set of weights, one per robot, assigned to the network. We design a Jacobi iterative algorithm to find the weight assignment whose corresponding generalized Voronoi partition satisfies the area constraints. This algorithm is distributed over the generalized Delaunay graph. We also design the "move-to-center-and-compute-weight" strategy to steer the robotic network towards the set of center generalized Voronoi configurations while monotonically optimizing coverage.

I. INTRODUCTION

This paper studies a class of coverage optimization problems subject to area constraints. Our objective is to position a robotic sensor network in an environment Q so that any location is as close as possible to at least one agent in case an event that needs servicing happens. The probability of events is determined by a density function $\phi : Q \to \mathbb{R}$ and the cost of moving from one point p to another q to service the event is given by $f(\|p-q\|)$, where $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function. To allow for the possibility of agents with different capabilities, we further require that the areas of the regions assigned to the agents satisfy a desired set of constraints. This type of coverage problems finds applications in servicing, spatial estimation, and optimal resource allocation.

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Literature review: Facility location [1], [2] studies locational optimization problems and looks at optimal resource placement and optimal space partitioning. The notion of Voronoi partition, or generalized versions of it, plays an important role in facility location. The work [3] considers centroidal Voronoi partitions, [4] considers power diagrams, [5] considers additively-weighted Voronoi partitions, and [6] considers multiplicatively-weighted Voronoi partitions. From a computational geometric perspective, an important research issue is the design of efficient algorithms that, given a fixed set of locations, compute partitions of the space into regions of prescribed areas [4], [5], [6]. Among these, the equitable case is of special importance as it represents a balanced distribution of the overall load, see e.g. [7]. In the context of robotic sensor networks, this work builds on [8], where distributed algorithms based on centroidal Voronoi partitions are presented, and [9], where limited-range interactions are considered. Voronoi partitions are also employed in [10], [11], [12]. Other works on deployment coverage problems include [13], [14]. Finally, we note that the locational optimization problem considered here is a *static* coverage problem, in contrast to *dynamic* coverage problems, e.g., [15], [16], [17] and references therein, that seek to visit or continuously sense all points in the environment. In coverage path planning problems [15], [16], a robot equipped with a limited footprint sensor has to visit all points in the environment. In [17], [18], a group of mobile sensors seeks to dynamically survey a given search domain providing a certain preset level of coverage.

Statement of contributions: The contributions pertain both the analysis of a broad class of constrained locational optimization problems and the design of coordination algorithms for robotic sensor networks. Regarding analysis, we study the notion of generalized Voronoi partition associated with an increasing function. We study the map that, given fixed agent locations, maps a set of weights to the areas of the corresponding regions. We characterize the Jacobian of this map as the Laplacian matrix of a weighted version of the generalized Delaunay graph induced by the Voronoi partition. This characterization allows us to show that, given a network configuration and a performance function, there exist weights that make the regions of the generalized Voronoi partition have a prescribed set of areas. A second set of results deal with the analysis of the solutions of the area-constrained locational optimization problem. We show that the generalized Voronoi partition is optimal among all partitions satisfying the area constraints and characterize the critical points of the optimization problem as center generalized Voronoi configurations. Regarding design, we provide two distributed algorithms over the generalized Delaunay graph. We design the "move-to-center-and-compute-weight" algorithm to steer the robotic network towards the solutions of the optimization problem while monotonically optimizing coverage.

We also design a Jacobi algorithm to find the weight assignment that makes the generalized Voronoi partition satisfy the area constraints. The latter is an efficient solution to a classical computational geometric problem with applications to load balancing and space partitioning.

Notation: Let $\mathbf{1}_n = (1,\dots,1)^T \in \mathbb{R}^n$ and $\mathbf{0}_n = (0,\dots,0)^T \in \mathbb{R}^n$. We let $\mathrm{diag}(\mathbb{R}^n) = \{(a,\dots,a) \in \mathbb{R}^n \mid a \in \mathbb{R}\}$. We consider the orthogonal decomposition $\mathbb{R}^n = \mathrm{diag}(\mathbb{R}^n) \oplus \mathrm{diag}(\mathbb{R}^n)^\perp$, with projections $\pi_1 : \mathbb{R}^n \to \mathrm{diag}(\mathbb{R}^n)$ and $\pi_2 : \mathbb{R}^n \to \mathrm{diag}(\mathbb{R}^n)^\perp$. Note that

$$\pi_1(x) = \frac{\mathbf{1}_n^T x}{n}\mathbf{1}_n, \quad \pi_2(x) = x - \pi_1(x). \tag{1}$$

The diagonal set $\mathrm{diag}(\mathbb{R}^n)$ is 1-dimensional, and hence $\mathrm{diag}(\mathbb{R}^n)^{n-1}$ is $(n-1)$-dimensional.

II. Preliminaries

A. Notions from graph theory

Here we present some basic graph-theoretic notions [19], [20]. An (undirected) graph consists of a vertex set $\mathcal{V}$ and of a set $\mathcal{E}$ of unordered pairs of vertices. For $v_1, v_2 \in \mathcal{V}$ distinct, (v_1, v_2) denotes an undirected edge between v_1 and v_2. A path in a graph is an ordered sequence of vertices such that any two consecutive vertices in the sequence are an edge of the graph. A graph is connected if there exists a path between any two vertices.

A weighted graph is a triplet $G = (\mathcal{V}, \mathcal{E}, A)$ where $\mathcal{V}$ and $\mathcal{E}$ are a graph and where $A \in \mathbb{R}^{n\times n}_{\geq 0}$ is a weighted adjacency matrix such that, for $i, j \in \{1,\dots,n\}$, the entry $a_{ij} > 0$ if (v_i, v_j) is an edge of G, and $a_{ij} = 0$ otherwise. The weighted Laplacian is the matrix defined by $L = \mathrm{diag}(A\mathbf{1}_n) - A$. The Laplacian is symmetric, its eigenvalues are all nonnegative, and 0 is an eigenvalue of L with eigenvector $\mathbf{1}_n$. In addition, G is connected if and only if $\mathrm{rank}(L) = n-1$.

A proximity graph [21], [9] is a generalization of the notion of graph that captures the fact that, in some situations, the edges of the graph change as the vertices move. More formally, given a set $\mathcal{P} = \{p_1,\dots,p_n\} \subset \mathbb{R}^d$ of n distinct points, the proximity graph $\mathcal{G}$ at $\mathcal{P}$, denoted by $\mathcal{G}(\mathcal{P})$, is an undirected graph with vertex set $\mathcal{P}$ and with edge set $\mathcal{E}_\mathcal{G}(\mathcal{P})$. A graph G can be interpreted as a proximity graph whose edge set does not depend on the configuration $\mathcal{P}$.

B. Generalized Voronoi partitions

Here we discuss the notion of (generalized) Voronoi partition [22], [2]. Let $Q \subset \mathbb{R}^d$ be convex. The *Voronoi partition* $\mathcal{V}(P) = \{V_1(P),\dots,V_n(P)\}$ of Q associated to $P = (p_1,\dots,p_n) \in Q^n$ is

$$V_i(P) = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|\}. \tag{2}$$

The collection $\mathcal{V}(P)$ partitions Q into convex sets whose interiors are pairwise disjoint. The Voronoi partition defines the *Delaunay graph* with vertices $\{p_1, \dots, p_n\}$. p_i and p_j are neighbors if and only if $V_i(P) \cap V_j(P) \neq \emptyset$. This graph is undirected and connected. For $f : \mathbb{R} \to \mathbb{R}$ strictly increasing, the *generalized Voronoi partition* $\mathcal{V}(P, \omega; f) = \{V_1(P, \omega; f), \dots, V_n(P, \omega; f)\}$ of Q associated to $P = (p_1, \dots, p_n) \in Q^n$, $\omega = (w_1, \dots, w_n) \in \mathbb{R}^n$ is

$$V_i(P, \omega; f) = \{q \in Q \mid f(\|q - p_i\|) - w_i \leq f(\|q - p_j\|) - w_j\}. \tag{3}$$

In general, the generalized Voronoi regions are neither convex nor star-shaped. The collection $\mathcal{V}(P, \omega; f)$ partitions Q into sets whose interiors are pairwise disjoint. One can interpret the weight w_i as a measure of the strength of the position p_i. In general, the larger w_i is with respect to the weights of its neighbors, the larger the Voronoi region of p_i is. In fact, depending on the selection of weights and agent locations, $V_i(P, \omega; f)$ might be empty for some i. Specifically, $V_i(P, \omega; f) = \emptyset$ if there exist $i, j \in \{1, \dots, n\}$ such that

$$w_j - w_i > f(\|p_i - p_j\|) - f(0). \tag{4}$$

Lemma II.1 (Properties of generalized Voronoi partition) *For $f : \mathbb{R} \to \mathbb{R}$ strictly increasing and locations $p_1, \dots, p_n \in Q$, the generalized Voronoi partition* (3) *of Q is*

(i) *equal to the Voronoi partition* (2) *if $w_1 = \dots = w_n$;*

(ii) *invariant under weight translations $\omega = (w_1, \dots, w_n) \mapsto \omega + \alpha \mathbf{1}_n = (w_1 + \alpha, \dots, w_n + \alpha)$, $\alpha \in \mathbb{R}$, i.e., $V_i(P, \omega + \alpha \mathbf{1}_n; f) = V_i(P, \omega; f)$, for $i \in \{1, \dots, n\}$;*

(iii) *monotonic in the set of weights, i.e., for any $i \in \{1, \dots, n\}$ and any $\omega = (w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_n)$, $\omega' = (w_1, \dots, w_{i-1}, w'_i, w_{i+1}, \dots, w_n) \in \mathbb{R}^n$ with $w'_i \geq w_i$,*

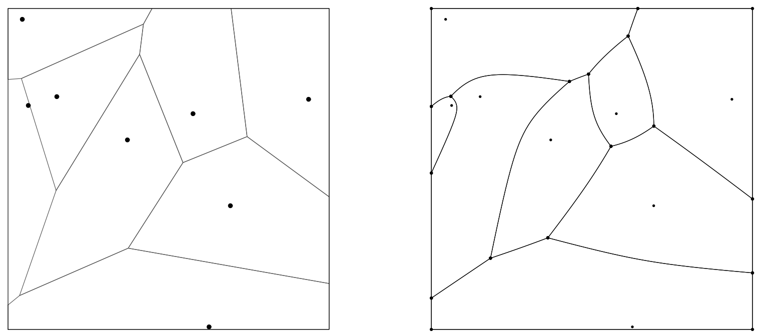
$$V_i(P, \omega; f) \subseteq V_i(P, \omega'; f), \qquad V_j(P, \omega'; f) \subseteq V_j(P, \omega; f), \quad j \neq i.$$

The generalized Voronoi partition takes different forms depending on the performance function:

a) Quadratic performance: for $f(x) = x^2$, the partition is the *power diagram*. The Voronoi regions are convex sets whose boundary is composed of straight segments, see Figure 1(a).

b) Linear performance: for $f(x) = x$, the partition is the *additively-weighted Voronoi partition*. The Voronoi regions are star-shaped sets whose boundary is composed of hyperbolic segments, see Figure 1(b).

c) Logarithmic performance: for $f(x) = \log x$, the partition is the *multiplicatively-weighted Voronoi partition*. The Voronoi regions are non-empty and might contain holes. In general, they are neither convex nor connected and their boundary is composed of circular segments.



(a) (b)

Fig. 1. Power diagram (a) and additively weighted Voronoi partition (b) defined by 8 agents with randomly assigned weights.

The generalized Voronoi partition defines the *generalized Delaunay graph* $\mathcal{G}_{\mathcal{V}}$ with vertices $\{(p_1, w_1), \ldots, (p_n, w_n)\}$. (p_i, w_i) and (p_j, w_j) are neighbors if and only if $\Delta_{ij}(P, \omega; f) = V_i(P, \omega; f) \cap V_j(P, \omega; f) \neq \emptyset$. $\mathcal{G}_{\mathcal{V}}$ is undirected and, if all Voronoi regions are non-empty, it is connected.

Remark II.2 (Computation of the Delaunay graph) In general, there exist configurations for which two neighbors in the generalized Delaunay graph can be arbitrarily far from each other. Given a specific configuration, distributed procedures exist [23] to compute the minimum radius r so that the Delaunay neighbors can be computed with information of other agents within distance r. Such procedures are not extensible to the generalized Delaunay graph unless additional assumptions are required on the difference among the weights. •

III. Problem statement

This section presents the area-constrained locational optimization problem. We start by briefly discussing the unconstrained optimization problem. Although the solution to this problem is known, it serves as a useful introduction to the problem of interest in this paper.

Let $Q \subset \mathbb{R}^d$ be convex. Consider n agents evolving in Q with positions $p_1, \ldots, p_n$ and define

$$\mathcal{H}(p_1, \ldots, p_n, W_1, \ldots, W_n) = \sum_{i=1}^{n} \int_{W_i} f(\|q - p_i\|)\phi(q)dq,$$

where $W_1, \ldots, W_n$ is a partition of Q, $f : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function, and $\phi : Q \to \mathbb{R}$ is a density function. $\phi(q)$ can be understood as a measure of the probability of an event happening at $q \in Q$, while $f(\|p - q\|)$ can be understood as the cost of moving from p to q to service the event. $\mathcal{H}$ represents an aggregate measure of the network performance when agent i is in charge of region W_i, $i \in \{1, \ldots, n\}$. Intuitively, the lower the value of $\mathcal{H}$ is, the better the network deployment and the space partitioning are.

A. The 1-center problem

Consider the optimization of $\mathcal{H}$ when there is only one agent. The function takes the form

$$\mathcal{H}_1(p) = \int_Q f(\|q-p\|)\phi(q)dq.$$

If f is strictly convex, then $\mathcal{H}_1$ is strictly convex, and the next result follows.

Lemma III.1 (Minimizer of $\mathcal{H}_1$ is geometric center of Q) *For Q convex, there is a unique minimizer Cntr(Q) of $\mathcal{H}_1$.*

Observe that the dependence of the minimizer of $\mathcal{H}_1$ on the set Q is continuous, i.e., small changes in Q induce small changes in the optimal agent location Cntr(Q). If Q is not convex, then the center might not be unique. However, the continuous dependence of the minimizers of $\mathcal{H}_1$ on Q still holds. The minimizer depends on the performance function.

B. The unconstrained locational optimization problem

Consider now the multicenter optimization problem where we seek to minimize $\mathcal{H}$ among all agent locations and all partitions of Q,

$$\text{minimize } \mathcal{H}(p_1, \dots, p_n, W_1, \dots, W_n). \tag{5}$$

If we fix the partition $W_1, \dots, W_n$ of Q, then the problem of optimizing $\mathcal{H}$ consists of solving n 1-center optimization problems, one per individual agent. Therefore, Lemma III.1 implies that for fixed $W_1, \dots, W_n$, the optimal agent locations are Cntr$(W_1), \dots,$ Cntr(W_n), respectively.

Interestingly enough, for fixed agent locations $p_1, \dots, p_n \in Q$, the optimal partition of Q does not depend on the specific performance function [3], [8]. In general, the optimal partition is the Voronoi partition $\mathcal{V}(p_1, \dots, p_n)$ defined by (2). Therefore, we have the following result.

Lemma III.2 (Critical points of $\mathcal{H}$) *A solution $p_1^*, \dots, p_n^*$, $W_1^*, \dots, W_n^*$ of (5) is a center Voronoi configuration of Q, i.e., for all $i \in \{1, \dots, n\}$, $p_i^* =$ Cntr(W_i^*), $W_i^* = V_i(P^*)$.*

C. The area-constrained locational optimization problem

We consider an area-constrained multicenter optimization problem. We seek to minimize $\mathcal{H}$ among all agent locations and all partitions of Q, with the constraint that the area of each region must be a pre-specified amount. The constraints are motivated by the desire to balance the load across the network according to the agent capabilities. Formally, a *feasible collection of areas* $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$ satisfies $\sum_{i=1}^{n} a_i = \int_Q \phi(q)dq = \text{area}_\phi(Q)$. We set

$$\text{minimize } \mathcal{H}(p_1, \dots, p_n, W_1, \dots, W_n), \tag{6a}$$

$$\text{subject to } \int_{W_i} \phi(q)dq = a_i, \; i \in \{1, \dots, n\}. \tag{6b}$$

The *equitable partition* case, $a_i = \frac{1}{n}\int_Q \phi(q)dq$, for $i \in \{1, \dots, n\}$, is of particular interest.

Our next objective is to provide a similar result to Lemma III.2 for the constrained problem (6).

IV. Analysis of the weights-to-areas assignment of the Voronoi partition

In this section, we study the properties of the map that assigns to a set of weights the corresponding set of areas of the generalized Voronoi regions. This analysis will be key in the characterization of the optimal solution of (6).

Let $p_1, \dots, p_n \in Q$ be fixed agent locations. Consider the neighborhood of $\text{diag}(\mathbb{R}^n)$ defined by $U = \{\omega \in \mathbb{R}^n \mid |w_i - w_j| \le f(\|p_i - p_j\|) - f(0) \text{ for all } i, j \in \{1, \dots, n\}\}$. The *weights-to-areas map* $\mathcal{M} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is

$$\mathcal{M}(\omega) = \Big(\int_{V_1(P,\omega;f)} \phi(q)dq, \dots, \int_{V_n(P,\omega;f)} \phi(q)dq \Big),$$

where $P = (p_1, \dots, p_n)$. Note that, if $\omega \notin U$, then, according to (4), at least there is one empty generalized Voronoi region. We begin by establishing some important properties of $\mathcal{M}$.

Proposition IV.1 (Properties of the weights-to-areas map) *Let $p_1, \dots, p_n \in Q$. The map $\mathcal{M} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is invariant under translations and its range belongs to the $(n-1)$-dimensional space $\{m \in \mathbb{R}^n_{\ge 0} \mid \mathbf{1}^T m = 1\}$. Moreover, $\mathcal{M}$ is gradient, i.e., $\nabla F = -\mathcal{M}$, where $F : \mathbb{R}^n \to \mathbb{R}$,*

$$F(\omega) = \sum_{j=1}^{n} \int_{V_j(P,\omega;f)} \big(f(\|q - p_j\|) - w_j\big)\phi(q)dq.$$

Proof: The fact that $\mathcal{M}$ is invariant under translations follows from noting that the generalized Voronoi regions are also invariant under translations, cf. Lemma II.1(ii). On the other hand, since the Voronoi regions form a partition of Q, the sum of the areas is constant and equals $\text{area}_\phi(Q)$. Hence, the range of $\mathcal{M}$ belongs to the $(n-1)$-dimensional space $\{m \in \mathbb{R}^n_{\ge 0} \mid \mathbf{1}^T m = 1\}$. Furthermore, if ω belongs to the interior of U, then $\mathcal{M}(\omega) \in \mathbb{R}^n_{>0}$. Using the generalized conservation-of-mass law established in [9, Proposition A.1], we compute

$$\frac{\partial F}{\partial w_i} = -\int_{V_i(P,\omega;f)} \phi(q)dq + \sum_{j=1}^{n} \int_{\partial V_j(P,\omega;f)} \big(f(\|q - p_j\|) - w_j\big) n_j \frac{\partial q}{\partial w_i} \phi(q)dq.$$

We show that the second term vanishes. For $j = i$, we have

$$\int_{\partial V_i(P,\omega;f)} \big(f(\|q-p_i\|) - w_i\big) n_i \frac{\partial q}{\partial w_i} \phi(q) dq = \sum_{j=1}^{n} \int_{\Delta_{ij}(P,\omega;f)} \big(f(\|q-p_i\|) - w_i\big) n_i \frac{\partial q}{\partial w_i} \phi(q) dq,$$

while for $j \neq i$, we have

$$\int_{\partial V_j(P,\omega;f)} \big(f(\|q-p_j\|) - w_j\big) n_j \frac{\partial q}{\partial w_i} \phi(q) dq = \int_{\Delta_{ij}(P,\omega;f)} \big(f(\|q-p_j\|) - w_j\big) n_j \frac{\partial q}{\partial w_i} \phi(q) dq.$$

Since $f(\|q-p_i\|) - w_i = f(\|q-p_j\|) - w_j$ and $n_i = -n_j$ on $\Delta_{ij}(P,\omega;f)$, we deduce that

$$\sum_{j=1}^{n} \int_{\partial V_j(P,\omega;f)} \big(f(\|q-p_j\|) - w_j\big) n_j \frac{\partial q}{\partial w_i} \phi(q) dq = \sum_{j\neq i} \int_{\Delta_{ij}(P,\omega;f)} \big(f(\|q-p_i\|) - w_i\big)(n_i + n_j) \frac{\partial q}{\partial w_i} \phi(q) dq = 0,$$

and therefore $\frac{\partial F}{\partial w_i} = -\int_{V_i(P,\omega;f)} \phi(q) dq = -\mathcal{M}_i(\omega)$, as claimed. ■

Using Proposition IV.1, one can derive various interesting properties of the Jacobian of $\mathcal{M}$.

Proposition IV.2 (The Jacobian of the weights-to-areas map is the Laplacian of the weighted generalized Delaunay graph) *Let $p_1, \dots, p_n \in Q$ and let $J(\mathcal{M})$ denote the Jacobian matrix of $\mathcal{M} : U \subset \mathbb{R}^n \to \mathbb{R}^n$. Then,*

(i) *$J(\mathcal{M})$ is symmetric;*
(ii) *$\mathbf{1}_n$ is an eigenvector of $J(\mathcal{M})$ with eigenvalue 0;*
(iii) *The rank of $J(\mathcal{M})$ on the interior of U is $n-1$.*

Therefore, the Jacobian matrix of $\mathcal{M}$ is the Laplacian of the generalized Delaunay graph whose edges are weighted as $a_{ij} = \frac{\partial \mathcal{M}_i}{\partial w_j}$, if i and j are Delaunay neighbors, and $a_{ij} = 0$ otherwise.

Proof: Fact (i) follows from $\mathcal{M}$ being gradient, cf. Proposition IV.1. Fact (ii) follows from noting that the range of $\mathcal{M}$ is in $\{m \in \mathbb{R}^n_{\geq 0} \mid \mathbf{1}^T m = 1\}$, and hence, $\mathbf{1}_n^T J(\mathcal{M}) = \mathbf{0}_n$. Since $J(\mathcal{M})$ is symmetric, this implies that $\mathbf{1}_n$ is also a right eigenvector with eigenvalue 0. To show (iii), note that the monotonic properties of the Voronoi partition, cf. Lemma II.1(iii), imply, for $i \in \{1, \dots, n\}$ and $(w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_n)$, $(w_1, \dots, w_{i-1}, w_i', w_{i+1}, \dots, w_n)$ with $w_i' \geq w_i$,

$$\mathcal{M}_i(w_1, \dots, w_{i-1}, w_i', w_{i+1}, \dots, w_n) \geq \mathcal{M}_i(w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_n),$$

$$\mathcal{M}_j(w_1, \dots, w_{i-1}, w_i', w_{i+1}, \dots, w_n) \leq \mathcal{M}_j(w_1, \dots, w_{i-1}, w_i, w_{i+1}, \dots, w_n), \quad j \neq i.$$

The definition of partial derivative implies that the entries in $J(\mathcal{M})$ satisfy

$$\frac{\partial \mathcal{M}_i}{\partial w_i} \geq 0, \qquad \frac{\partial \mathcal{M}_i}{\partial w_j} \leq 0, \quad j \neq i,$$

and the inequalities are strict on the interior of U. This observation, together with (i) and (ii), imply that, when evaluated at ω in the interior of U, the Jacobian is the Laplacian matrix of a weighted generalized Delaunay graph. Since the graph is connected, rank $J(\mathcal{M}) = n-1$. ■

Since $\mathcal{M}$ is invariant under translations, cf. Proposition IV.1, define the equivalence relation $\sim$ on $\mathbb{R}^n$: $x \sim y$ if and only if there exists $(a,\dots,a) \in \operatorname{diag}(\mathbb{R}^n)$ such that $x = (a,\dots,a)+y$. Any $\omega \in \mathbb{R}^n$ and its projection onto $\operatorname{diag}(\mathbb{R}^n)^\perp$ are related under $\sim$, since there exists $(a,\dots,a) = \pi_1(\omega) \in \operatorname{diag}(\mathbb{R}^n)$ such that $\omega = \pi_1(\omega) + \pi_2(\omega)$. Therefore, we identify the quotient space $\mathbb{R}^n/\sim$ with $\operatorname{diag}(\mathbb{R}^n)^\perp$ by means of the linear projection π_2.

$$\omega \mapsto \pi_2(\omega) = \omega - \pi_1(x). \tag{7}$$

The relation $\sim$ allows us to state a particularly useful property of $\mathcal{M}$ in an elegant way.

Corollary IV.3 *Let $p_1,\dots,p_n \in Q$. The map $\mathcal{M} : U \subset \mathbb{R}^n \to \mathbb{R}^n$ induces a local diffeomorphism $\widetilde{\mathcal{M}} : U/\sim \equiv \mathbb{R}^{n-1} \to \{m \in \mathbb{R}^n_{\geq 0} \mid \mathbf{1}^T m = 1\}$.*

We are now ready to establish that, given a network configuration and a feasible collection of areas, there exist weights such that the generalized Voronoi partition satisfies the constraints.

Proposition IV.4 (Existence of weight assignment that makes generalized Voronoi partition satisfy area constraints) *Let $p_1,\dots,p_n \in Q$ and let $\{a_1,\dots,a_n\} \subset \mathbb{R}_{>0}$ be a feasible collection of areas. Then there exists a set of weights $\omega = \{w_1,\dots,w_n\} \subset \mathbb{R}$ such that*

$$\int_{V_i(P,\omega;f)} \phi(q)dq = a_i, \quad i \in \{1,\dots,n\}.$$

Proof: Consider the function from U to $\mathbb{R}$ defined by

$$\omega \mapsto \frac{1}{2}\|\mathcal{M}(\omega) - (a_1,\dots,a_n)\|^2. \tag{8}$$

This function is continuous and invariant under translations, and therefore, induces a continuous function from $U/\sim$ to $\mathbb{R}$. Next, we show that $U/\sim$ is compact. Note that $U/\sim$ is closed because U is. To show that $U/\sim$ is bounded, take $K > 0$ large enough so that $f(\|p_i - p_j\|) - f(0) \leq K$, for all $i,j \in \{1,\dots,n\}$. Under (7), any element of $U/\sim$ is of the form $\omega - \pi_1(\omega)$, with $\omega \in U$. Now, we bound the ith component of $\omega - \pi_1(\omega)$ by

$$|(\omega - \pi_1(\omega))_i| = |\frac{1}{n}\sum_{j=1}^n (w_i - w_j)| \leq \frac{1}{n}\sum_{j=1}^n |w_i - w_j| \leq K,$$

using the fact that $\omega \in U$. Therefore, we deduce that $U/\sim$ is contained in the closed ball $\overline{B}(0, K\sqrt{n})$, and hence is bounded.

Using that $U/\sim$ is compact and the function (8) is continuous and invariant under $\sim$, we deduce that there exists a minimizer ω_* of the function. We want to show that the value of the minimum is 0. When evaluated at ω_*, we have

$$0 = \frac{\partial}{\partial w_i}_{|\omega=\omega_*}\big(\frac{1}{2}\|\mathcal{M}(w_1,\dots,w_n)-(a_1,\dots,a_n)\|^2\big) = \sum_{k=1}^{n}\big(\mathcal{M}_k(\omega_*)-a_k\big)\frac{\partial \mathcal{M}_k}{\partial w_i},$$

for $i\in\{1,\dots,n\}$. Equivalently, we can express this set of equalities as $(\mathcal{M}(\omega_*)-(a_1,\dots,a_n))J(\mathcal{M}) = \mathbf{0}_n$. Because $\mathbf{1}_n^T J(\mathcal{M}) = \mathbf{0}_n$ and $\operatorname{rank} J(\mathcal{M}) = n-1$, we deduce $\mathcal{M}(\omega_*)-(a_1,\dots,a_n) = \alpha\mathbf{1}_n$ for some $\alpha\in\mathbb{R}$. Finally, $0 = \mathbf{1}_n^T(\mathcal{M}(\omega_*)-(a_1,\dots,a_n)) = \alpha n$, and therefore we conclude $\alpha = 0$, i.e., $\mathcal{M}(\omega_*) = (a_1,\dots,a_n)$. ■

Proposition IV.4, together with Corollary IV.3, states that, up to translations, the set of weights ω such that $\mathcal{M}(\omega) = (a_1,\dots,a_n)$ is locally unique, that is, there exists a neighborhood of ω in $\mathbb{R}^n$ where no other set of weights (other than those equivalent to ω by translation) are mapped to $(a_1,\dots,a_n)$ under $\mathcal{M}$.

A. Jacobi iterative algorithm for weight assignment

The existence result in Proposition IV.4 leads naturally to the question of how to determine the set of weights that make the generalized Voronoi partition satisfy the constraints. Given a network configuration, in general it is not possible to obtain an analytic expression for the map that assigns to a feasible collection of areas the corresponding distribution of weights (i.e., an inverse of $\mathcal{M}$). Our approach then is to synthesize a distributed Jacobi algorithm that numerically finds an appropriate weight assignment. Given $p_1,\dots,p_n\in Q$ and a feasible collection of areas $\{a_1,\dots,a_n\}\subset\mathbb{R}_{>0}$, define $g:\mathbb{R}^n\to\mathbb{R}^n$ by $g(w_1,\dots,w_n) = \mathcal{M}(w_1,\dots,w_n)-(a_1,\dots,a_n)$. From Proposition IV.1, we know that g is the gradient vector field corresponding to the function $\mathcal{F}:\mathbb{R}^n\to\mathbb{R}$, $\mathcal{F}(\omega) = -F(\omega)-\sum_{i=1}^n w_i a_i$. We look for $\omega\in U$ such that

$$\nabla\mathcal{F}(\omega) = g(\omega) = \mathbf{0}_n. \tag{9}$$

Alternatively, we look for a weight assignment that optimizes the value of $\mathcal{F}$. There are multiple methods that can be used to this end, see e.g., [26]. Here, we use the Jacobi algorithm

$$\omega_{k+1} = \omega_k - \gamma\,\mathrm{diag}\left(\frac{\partial g_1}{\partial w_1}(\omega_k),\dots,\frac{\partial g_n}{\partial w_n}(\omega_k)\right)^{-1} g(\omega_k), \tag{10}$$

for $k\geq 0$, where $\mathrm{diag}\,(v)\in\mathbb{R}^{n\times n}$ is the diagonal matrix with the components of the vector $v\in\mathbb{R}^n$ in the diagonal. Here, $\gamma>0$ is a parameter that can be chosen to guarantee convergence. Note that the Jacobian of g and $\mathcal{M}$ are the same, that is, $J(g) = J(\mathcal{M})$. Therefore, given

Proposition IV.2, the Jacobi algorithm is distributed over the generalized Delaunay graph. In other words, agent i only needs to interact with its neighbors in the graph to compute the ith entry of ω_{k+1} as prescribed by (10). The following result is a consequence of [26, Section 3.2].

Proposition IV.5 (Convergence of Jacobi algorithm to desired weight assignment) *Given $p_1, \dots, p_n \in Q$ and a feasible collection of areas $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$, let ω_0 in the interior of U. Consider the level set $\mathcal{L} = \{\omega \in U \mid \mathcal{F}(\omega) \le \mathcal{F}(\omega_0)\}$ of $\mathcal{F}$, and define*

$$A = \min_{i\in\{1,\dots,n\}} \min_{\omega\in\mathcal{L}} \frac{\partial g_i}{\partial w_i}(\omega) > 0, \quad B = \max_{i\in\{1,\dots,n\}} \max_{\omega\in\mathcal{L}} \frac{\partial g_i}{\partial w_i} > 0.$$

Then, for $0 < \gamma < A/B$, the sequence $\{\omega_k \in \mathbb{R}^n \mid k \in \mathbb{Z}_{\ge 0}\}$ generated by the Jacobi algorithm (10) *satisfies* $\lim_{k\to+\infty} g(\omega_k) = 0$.

Proof: Using that $\mathcal{F}$ is invariant under translations, it is not difficult to show that A and B are well-defined and positive. Additionally, the algorithm (10) is symmetric under translations, i.e., the sequence generated starting from $\omega_0 + \alpha\mathbf{1}_n$ is equal to $\{\omega_k + \alpha\mathbf{1}_n \in \mathbb{R}^n \mid k \in \mathbb{Z}_{\ge 0}\}$. This observation allows us to perform the convergence analysis invoking the compact set $U/\sim$. For convenience, we denote by $\widetilde{\mathcal{F}}$ the function induced by $\mathcal{F}$ on $\mathbb{R}^n/\sim$. The convergence of the Jacobi algorithm can be established as a consequence of the following facts. Since $\widetilde{\mathcal{F}}$ is continuous, it is bounded from below on the compact set $U \sim$, and so is $\mathcal{F}$. The level sets of $\mathcal{F}$ are mapped under the equivalence relation $\sim$ onto the bounded level sets of $\widetilde{\mathcal{F}}$. The gradient $\nabla\mathcal{F} = g$ is globally Lipschitz on each level set with a Lipschitz constant K that can be upper bounded as follows. Using the mean value theorem for vector-valued functions [27], we get

$$\|g(\omega_1) - g(\omega_2)\|_2 \le \sup_{t\in[0,1]} \|J(g)(t\omega_1 + (1-t)\omega_2)\|_2 \, \|\omega_1 - \omega_2\|_2 \le \sup_{\omega\in U} \|J(g)(\omega)\|_2 \, \|\omega_1 - \omega_2\|_2.$$

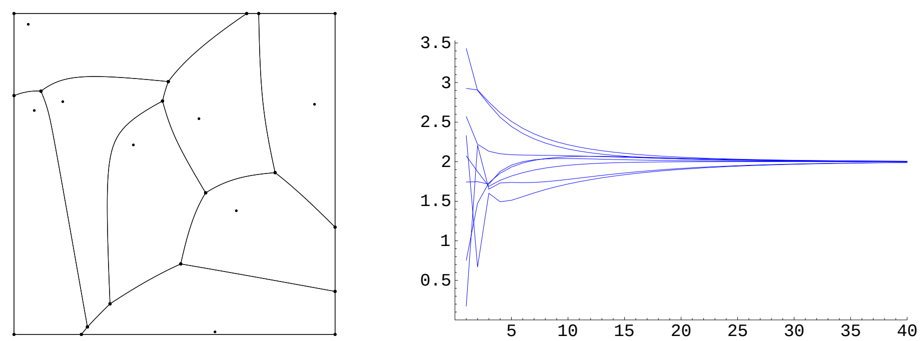
Additionally, since $J(g) = J(\mathcal{M})$ can be interpreted as the Laplacian of a weighted version of the generalized Delaunay graph, we can use [28] to upper bound the 2-norm of $J(g)$ as follows

$$\|J(g)(\omega)\|_2 \le \max_{i,j \text{ Delaunay neighbors}} \left\{ \frac{\partial g_i}{\partial w_i}(\omega) + \frac{\partial g_j}{\partial w_j}(\omega) \right\} \le 2 \max_{i\in\{1,\dots,n\}} \frac{\partial g_i}{\partial w_i}(\omega).$$

Therefore, on $\mathcal{L}$, we take $K = 2B$. According to [26, Section 3.2], these properties guarantee that, if $0 < \gamma < 2A/K$, the evolution of (10) from ω_0 remains in $\mathcal{L}$ and is convergent. ■

Figure 2 presents a simulation of the Jacobi algorithm (10) for the linear performance case.

Remark IV.6 (Explicit expressions of the Jacobian of the weights-to-areas map) For each performance function, we can get explicit expressions for $J(g)$. We omit the details for space rea-



(a) (b)

Fig. 2. Execution of the Jacobi algorithm (10). The location and initial weight assignment of the 8 agents in the square $[0, 4] \times [0, 4]$ is as in Figure 1(b), and the density is constant and equal to 1. The algorithm is run with $\gamma = .3$. The target areas are $a_i = 2$, $i \in \{1, \ldots, 8\}$. (a) shows the final additively weighted Voronoi partition obtained by the algorithm and (b) shows the corresponding evolution of the areas during the execution. After 40 iterations, the executions are very close to the solution.

sons, but note that they can be obtained with the conservation-of-mass law [9, Proposition A.1] as

$$\frac{\partial g_i}{\partial w_j} = \int_{\partial V_i} \phi \, \mathrm{n}_i \frac{\partial q}{\partial w_j} dq = \int_{\Delta_{ij}} \phi \, \mathrm{n}_i \frac{\partial q}{\partial w_j} dq,$$

where $\mathrm{n}_i(q)$ is the unit outward normal to V_i at $q \in \partial V_i$, and substituting an appropriate parametrization of the boundary of the generalized Voronoi region. •

Remark IV.7 (Dynamic density functions) Dynamic density functions may arise because of external (e.g., changes in environmental conditions) or internal (e.g., evolving conditions in the power reserves of individual agents) network factors. The dynamic evolution of ϕ causes changes in the area constraints and makes (9) a time-dependent equation. A natural question is to identify conditions that guarantee that the Jacobi algorithm (10) can cope with the dynamic evolution of ϕ to find a suitable weight assignment. Since the Jacobi algorithm has a linear root-convergence factor [29], one can deduce that convergence is guaranteed if changes in ϕ occur between increasingly longer time instants. •

V. Analysis and design for area-constrained locational optimization

Here, we build on the results of the previous sections to characterize the optimal configurations of (6) and design a distributed coordination algorithm that steers the network toward them.

A. The optimal configurations are the center generalized Voronoi configurations

For a fixed partition $W_1, \ldots, W_n$ of Q, the problem of optimizing $\mathcal{H}$ consists of solving n 1-center optimization problems (as for the unconstrained optimization problem, cf. Section III-B).

Therefore, Lemma III.1 implies that for fixed $W_1, \dots, W_n$, the optimal agent locations are $\text{Cntr}(W_1), \dots, \text{Cntr}(W_n)$, respectively.

Given fixed agent locations $p_1, \dots, p_n \in Q$, we show next that the optimal partition of Q for the problem (6) is the generalized Voronoi partition. Observe that, unlike for the problem (5), the optimal partition depends on the performance function.

Proposition V.1 (Generalized Voronoi partition is $\mathcal{H}$-optimal among all partitions that satisfy area constraints) *Let $p_1, \dots, p_n \in Q$ be fixed agent locations and let $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$ be a feasible collection of areas. Let $\omega \in \mathbb{R}^n$ be such that $\mathcal{M}(\omega) = (a_1, \dots, a_n)$. Then, the Voronoi partition $\mathcal{V}(P, \omega; f)$ optimizes $\mathcal{H}$ among all partitions satisfying the area constraints* (6b).

Proof: Given any partition $W_1, \dots, W_n$ of Q satisfying (6b), consider the sum

$$\sum_{i=1}^{n} \int_{W_i} \big(f(\|q - p_i\|) - w_i\big)\phi(q)dq.$$

The definition (3) guarantees that the expression is minimized when the partition $W_1, \dots, W_n$ is the generalized Voronoi partition. On the other hand, this sum can be decomposed as

$$\begin{aligned}\sum_{i=1}^{n} \int_{W_i} \big(f(\|q - p_i\|) - w_i\big)\phi(q)dq &= \sum_{i=1}^{n} \int_{W_i} f(\|q - p_i\|)\phi(q)dq - \sum_{i=1}^{n} w_i \int_{W_i} \phi(q)dq \\ &= \mathcal{H}(p_1, \dots, p_n, W_1, \dots, W_n) - \sum_{i=1}^{n} w_i a_i.\end{aligned}$$

The result follows by noting that the last term is constant for any partition satisfying (6b). ■

We are now ready to state the analogue result to Lemma III.2 for the area-constrained problem.

Corollary V.2 (Critical points of $\mathcal{H}$ with area constraints) *A solution $p_1^*, \dots, p_n^*, W_1^*, \dots, W_n^*$ of* (6) *is a center generalized Voronoi configuration of Q, i.e., there exists a weight assignment $\omega^* \in \mathbb{R}^n$ such that, for all $i \in \{1, \dots, n\}$, $p_i^* = \text{Cntr}(W_i^*)$, $W_i^* = V_i(P^*, \omega^*; f)$.*

B. Distributed design: the "move-to-center-and-compute-weight" algorithm

Here, we investigate algorithmic solutions to the constrained locational optimization problem (6). Our strategy is to make each agent go to the center of its generalized Voronoi region while, at the same time, have the individual weights tuned to satisfy the constraints.

For a feasible collection of areas $\{a_1, \dots, a_n\} \subset \mathbb{R}_{>0}$, Proposition IV.4 guarantees that there exists a map $\mathcal{A} : Q^n \to \mathbb{R}^n$, assigning agent locations to weights, that satisfies

$$\mathcal{M}(\mathcal{A}(p_1, \dots, p_n)) = (a_1, \dots, a_n).$$

Moreover, the weight assignment can be selected so that $\mathcal{A}$ is continuous. The "move-to-center-and-compute-weight" algorithm is the discrete-time map $\mathrm{T} : Q^n \to Q^n$ defined by

$$\mathrm{T}(p_1, \dots, p_n) = (\mathrm{Cntr}(V_1(P, \mathcal{A}(P); f)), \dots, \mathrm{Cntr}(V_n(P, \mathcal{A}(P); f))). \quad (11)$$

The map T is continuous because $\mathcal{A}$ is, the Voronoi partition (3) changes continuously with the agent locations, and the solution to the 1-center problem changes continuously with the set. Provided $\mathcal{A}$ is distributed over the generalized Delaunay graph, i.e., agent i only needs to interact with its neighbors in the graph to compute its weight, then T is distributed too.

Proposition V.3 (Asymptotic convergence of "move-to-center-and-compute-weight" algorithm) *The trajectories of the discrete-time coordination algorithm* T *converge asymptotically to the set of center generalized Voronoi configurations of* Q*, while monotonically decreasing* $\mathcal{H}$*.*

Proof: We prove the result using the discrete-time LaSalle Invariance Principle [30]. The set Q^n is compact and invariant for the discrete-time dynamics defined by T. Consider the function

$$\mathcal{H}_V(p_1, \dots, p_n) = \mathcal{H}(p_1, \dots, p_n, V_1(f, \mathcal{A}(P)), \dots, V_n(f, \mathcal{A}(P))).$$

Let us show that the trajectories of T monotonically decrease the value of $\mathcal{H}_V$. First, note that

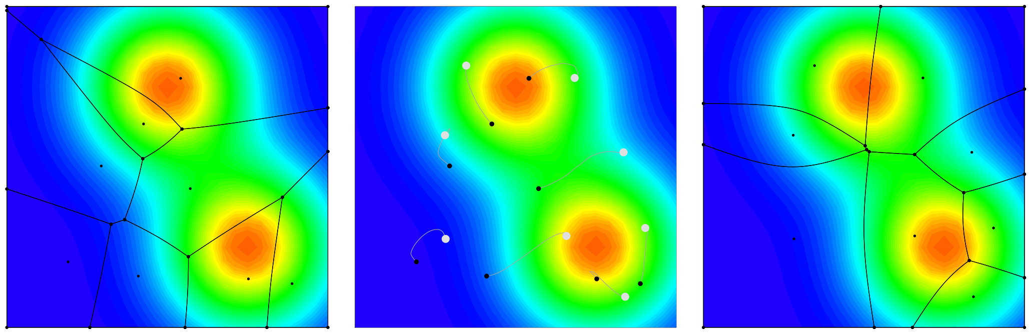
$$\begin{aligned}&\mathcal{H}(p_1, \dots, p_n, V_1(f, \mathcal{A}(P)), \dots, V_n(f, \mathcal{A}(P)))\\ &\geq \mathcal{H}(\mathrm{Cntr}(V_1(f, \mathcal{A}(P))), \dots, \mathrm{Cntr}(V_n(f, \mathcal{A}(P))), V_1(f, \mathcal{A}(P)), \dots, V_n(f, \mathcal{A}(P))),\end{aligned}$$

because for a fixed partition of Q, the center positions of the individual regions optimize the value of $\mathcal{H}$, according to Lemma III.1. Second, note that

$$\begin{aligned}&\mathcal{H}(\mathrm{Cntr}(V_1(f, \mathcal{A}(P))), \dots, \mathrm{Cntr}(V_n(f, \mathcal{A}(P))), V_1(f, \mathcal{A}(P)), \dots, V_n(f, \mathcal{A}(P)))\\ &\geq \mathcal{H}(\mathrm{Cntr}(V_1(f, \mathcal{A}(P))), \dots, \mathrm{Cntr}(V_n(f, \mathcal{A}(P))), V_1(f, \mathcal{A}(\mathrm{T}(P))), \dots, V_n(f, \mathcal{A}(\mathrm{T}(P)))) = \mathcal{H}_V(\mathrm{T}(P)),\end{aligned}$$

because for fixed agent locations, the Voronoi partition (3) is optimal for $\mathcal{H}$ among all partitions that verify the area constraints (6b), according to Proposition V.1. Therefore, the trajectories of T monotonically decrease the value of $\mathcal{H}_V$. Moreover, $\mathcal{H}_V(P) = \mathcal{H}_V(\mathrm{T}(P))$ if and only if $\mathrm{T}(P) = P$, i.e., each agent sits at the center of its own Voronoi region. The application of the discrete-time LaSalle Invariance Principle now guarantees that the trajectories of T converge to the largest invariant set contained in $\mathcal{Z} = \{P \in Q^n \mid \mathcal{H}_V(P) = \mathcal{H}_V(\mathrm{T}(P))\}$. The above discussion implies that $\mathcal{Z}$ is the set of center generalized Voronoi configurations. ■

Figure 3 presents a simulation of the algorithm (11) for the linear performance case.



(a) (b) (c)

Fig. 3. Execution of the "move-to-center-and-compute-weight" algorithm (11). (a) shows the initial network configuration, (b) shows the evolution of the algorithm, and (c) shows the final center generalized Voronoi configuration attained after 80 iterations of T. All figures show the contour plot of $\phi = 1 + 2\,e^{-(x-3)^2-(y-1)^2} + 2\,e^{-(x-2)^2-(y-3)^2}$. The feasible collection of areas that constrain the partition are $a_i = \mathrm{area}_\phi(Q)/16$ for i even, and $a_i = 3\,\mathrm{area}_\phi(Q)/16$ for i odd.

Remark V.4 (Distributed properties of T) As noted in Remark II.2, two neighbors in the generalized Delaunay graph might be arbitrarily far form each other. However, given the continuous dependence of the generalized Voronoi partition on the agents' locations and weights and assuming no agent losses or arrivals, one can deduce that changes in the neighboring relationships along the execution of T occur because 2-hop neighbors become 1-hop neighbors or vice versa. This observation highlights the fact that, in general, agents do not need to know the location of every other agent in the network in order to execute T. •

Remark V.5 (Alternative specification of the feasible collection of areas) A feasible collection of areas can always be specified as follows: given $\{\alpha_1, \dots, \alpha_n\} \subset [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$, one defines $a_i = \alpha_i \,\mathrm{area}_\phi(Q)$, for $i \in \{1, \dots, n\}$. The number α_i represents then the percentage of the total area tasked to agent i. This formulation eliminates the need to pre-specify a feasible collection of areas at the expense of assuming that each agent knows the total weighted area of the environment. Since this area is the sum of the weighted area of the region of each agent, it can be computed using several distributed techniques, e.g., consensus algorithms [24]. If the environment or the density function (and hence the total area) are changing with time, one can instead use dynamic consensus algorithms, e.g., [25] and references therein, to track its evolution during the exection of T. •

VI. Conclusions

We have studied a class of area-constrained locational problem where a group of robots seeks to optimize a notion of environmental coverage by partitioning the space into regions that have a pre-specified area. We have characterized the critical points of this optimization problem as center generalized Voronoi configurations. We have designed two distributed coordination algorithms. Given a network configuration and a feasible collection of areas, the first algorithm computes a weight assignment whose associated generalized Voronoi configuration satisfies the constraints. The second algorithm steers the network towards the set of center Voronoi configurations.

Future work will explore the area-constrained locational problem under limited-range agent interactions and time-dependent density functions. We are interested in balancing the load in servicing problems, where agents discover previously unknown customer locations as they move through the environment. The location and required servicing time can be modeled as changes in a density function that is being learned during the algorithm execution. We also plan to incorporate annealing techniques in the design for finding global optima and investigate the trade-offs between the optimality of the partition in terms of the objective function $\mathcal{H}$ and the optimality of the shape of individual regions for the motion and sensing capabilities of the agents.

REFERENCES

[1] Z. Drezner and H. W. Hamacher, Eds., *Facility Location: Applications and Theory*. Springer, 2001.

[2] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*, 2nd ed., ser. Wiley Series in Probability and Statistics. John Wiley, 2000.

[3] Q. Du, V. Faber, and M. Gunzburger, "Centroidal Voronoi tessellations: Applications and algorithms," *SIAM Review*, vol. 41, no. 4, pp. 637–676, 1999.

[4] F. Aurenhammer, F. Hoffmann, and B. Aronov, "Minkowski-type theorems and least-squares clustering," *Algorithmica*, vol. 20, pp. 61–76, 1998.

[5] B. Aronov, P. Carmi, and M. J. Katz, "Minimum-cost load-balancing partitions," in *22nd ACM Symposium on Computational Geometry*, Sedona, AZ, 2006, pp. 301–308.

[6] R. Reitsma, S. Trubin, and E. Mortensen, "Weight-proportional space partitioning using adaptive Voronoi diagrams," *Geoinformatica*, vol. 11, pp. 383–405, 2007.

[7] J. G. Carlsson, B. Armbruster, and Y. Ye, "Finding equitable convex partitions of points in a polygon efficiently," *ACM Transactions on Algorithms*, 2008, to appear. Electronically available at `http://www.stanford.edu/~yyye/`.

[8] J. Cortés, S. Martínez, T. Karatas, and F. Bullo, "Coverage control for mobile sensing networks," *IEEE Transactions on Robotics and Automation*, vol. 20, no. 2, pp. 243–255, 2004.

[9] J. Cortés, S. Martínez, and F. Bullo, "Spatially-distributed coverage optimization and control with limited-range interactions," *ESAIM. Control, Optimisation & Calculus of Variations*, vol. 11, no. 4, pp. 691–719, 2005.

[10] A. Arsie and E. Frazzoli, "Efficient routing of multiple vehicles with no communications," *International Journal on Robust and Nonlinear Control*, vol. 18, no. 2, pp. 154–164, 2007.

[11] A. Kwok and S. Martínez, "Energy-balancing cooperative strategies for sensor deployment," in *IEEE Conf. on Decision and Control*, New Orleans, LA, 2007, pp. 6136–6141.

[12] M. Pavone, E. Frazzoli, and F. Bullo, "Decentralized algorithms for stochastic and dynamic vehicle routing with general target distribution," in *IEEE Conf. on Decision and Control*, New Orleans, LA, Dec. 2007, pp. 4869–4874.

[13] A. Howard, M. J. Matarić, and G. S. Sukhatme, "Mobile sensor network deployment using potential fields: A distributed scalable solution to the area coverage problem," in *International Conference on Distributed Autonomous Robotic Systems*, Fukuoka, Japan, June 2002, pp. 299–308.

[14] M. Schwager, D. Rus, and J. J. Slotine, "Decentralized, adaptive coverage control for networked robots," *International Journal of Robotics Research*, 2008, submitted.

[15] H. Choset, "Coverage for robotics – A survey of recent results," *Annals of Mathematics and Artificial Intelligence*, vol. 31, no. 1-4, pp. 113–126, 2001.

[16] E. U. Acar and H. Choset, "Sensor-based coverage of unknown environments: incremental construction of Morse decompositions," *International Journal of Robotics Research*, vol. 21, no. 4, pp. 345–366, 2002.

[17] I. I. Hussein and D. M. Stipanoviċ, "Effective coverage control for mobile sensor networks with guaranteed collision avoidance," *IEEE Transactions on Control Systems Technology*, vol. 15, no. 4, pp. 642–657, 2007.

[18] Y. Wang and I. I. Hussein, "Awareness coverage control over large scale domains with intermittent communications," in *American Control Conference*, Seattle, WA, June 2008, pp. 4370–4375.

[19] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein, *Introduction to Algorithms*, 2nd ed. MIT Press, 2001.

[20] C. D. Godsil and G. F. Royle, *Algebraic Graph Theory*, ser. Graduate Texts in Mathematics. Springer, 2001, vol. 207.

[21] J. W. Jaromczyk and G. T. Toussaint, "Relative neighborhood graphs and their relatives," *Proceedings of the IEEE*, vol. 80, no. 9, pp. 1502–1517, 1992.

[22] P. F. Ash and E. D. Bolker, "Generalized Dirichlet tesselations," *Geometriae Dedicata*, vol. 20, pp. 209–243, 1986.

[23] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*, ser. Applied Mathematics Series. Princeton University Press, 2009, electronically available at http://coordinationbook.info.

[24] W. Ren and R. W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control*, ser. Communications and Control Engineering. Springer, 2008.

[25] R. A. Freeman, P. Yang, and K. M. Lynch, "Stability and convergence properties of dynamic average consensus estimators," in *IEEE Conf. on Decision and Control*, San Diego, CA, Dec. 2006, pp. 398–403.

[26] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, 1997.

[27] W. S. Hall and M. L. Newell, "The mean value theorem for vector valued functions: A simple proof," *Mathematics Magazine*, vol. 52, no. 3, pp. 157–158, 1979.

[28] K. C. Das and R. B. Bapat, "A sharp upper bound on the largest Laplacian eigenvalue of weighted graphs," *Linear Algebra and its Applications*, vol. 409, pp. 153–165, 2005.

[29] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables*, ser. Classics in Applied Mathematics. Philadelphia, PA: SIAM, 2000, vol. 30.

[30] J. P. LaSalle, *The Stability and Control of Discrete Processes*, ser. Applied Mathematical Sciences. Springer, 1986, vol. 62.