Cooperative detection of areas of rapid change in spatial fields

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Abstract

This paper proposes a distributed coordination algorithm for robotic sensor networks to detect boundaries that separate areas of rapid change of planar spatial phenomena. We consider an aggregate objective function, termed wombliness, to measure the change of the spatial field along the closed polygonal curve defined by the location of the sensors. We encode the network task as the optimization of the wombliness and characterize the smoothness properties of the objective function. In general, the complexity of the spatial phenomena may make the gradient flow cause self-intersections in the polygonal curve described by the network. We design the **`hybrid wombling algorithm`** that allows for network splitting and merging and guarantees local convergence to the critical configurations of the wombliness, while monotonically optimizing it. The technical approach combines ideas from statistical estimation, dynamical systems, and hybrid modeling and design.

Key words: Robotic sensor networks; hybrid systems; distributed detection; spatial fields; wombling

1 Introduction

Consider a network of mobile sensors moving in a planar environment where a spatial process takes place. Our aim is to design a distributed algorithm that allows the group of sensors to determine boundaries separating areas with large differences in the spatial process. Such boundary phenomena are relevant in multiple applications, including upwelling boundaries in oceanographic studies, see e.g., Sousa et al. [2008] and cloud boundaries in weather forecasting, see e.g., [Nowak et al., 2008].

Literature review: Our work has connections with several domains. In statistical estimation [Fagan et al., 2003, Banerjee and Gelfand, 2006], wombling boundaries are curves that delimit areas of rapid change of some scientific phenomena of interest. Algorithms for detecting these boundaries are used in biology [Barbujani et al., 1989], computational ecology [Fagan et al., 2003], and medicine [Jacquez and Greiling, 2003]. Banerjee and Gelfand [2006] use Monte Carlo Markov Chain methods to detect wombling boundaries in spatial process models. In computer vision [Osher and Paragios, 2003, Kimmel, 2003, Paragios et al., 2005], image segmentation and edge detection problems are encoded as optimization problems for a variety of objective functionals. These problems are solved using PDE-based approaches that build on variational information of the functionals. Work in this area assumes the availability of full, complete information, and is not directly applicable to motion coordination of distributed sensors with partial information. Finally, we use modeling tools from hybrid systems [van der Schaft and Schumacher, 2000, Liberzon, 2003, Sanfelice et al., 2008] in the algorithm design.

Statement of contributions: The spatial phenomena is described by a time-independent twice continuously differentiable function on a planar compact set. We consider a network of agents with first-order dynamics capable of measuring the gradient and the Laplacian of the field along finite segments. The wombliness of a not self-intersecting, closed curve is a measure of the alignment of its normal direction with the gradient of the field. We use the wombliness associated to a closed polygonal curve to formulate the network objective as a distributed optimization problem. Our first contribution is the study of the smoothness properties of the wombliness, an explicit expression for its gradient and a characterization of its critical points. If the network were to follow a gradient ascent, situations may arise where the polygonal curve described by the sensors becomes self-intersecting and the ensuing flow ill-posed. To prevent this, our second contribution is the synthesis of the `hybrid wombling algorithm` for distributed wombliness optimization. This strategy allows for splitting and merging of curves and may require the inclusion of additional agents. Our third contribution is the characterization of its convergence properties.

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2 Preliminaries

Let us start with some notation. For $S \subset \mathbb{R}^d$, we denote by $\mathrm{int}(S)$, $\overline{S}$, and ∂S its interior, closure, and boundary, respectively. Let $|S|$ be the cardinality of a finite set S. Let unit : $\mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by $\mathrm{unit}(x) = x/\|x\|$ for $x \neq 0$ and $\mathrm{unit}(0) = 0$. A totally ordered S is a set paired with a total order, i.e., a transitive, skew-symmetric and total binary relation $\preceq$. Given $a \in S$, we let a_+ (resp. a_-) denote the smallest (with respect to $\preceq$) $b \in S$ such that $a \preceq b$ (resp. largest $b \in S$ such that $b \preceq a$). Given $a, b \in S$, let $\langle a, b\rangle = \{c \in S : a \preceq c \preceq b\}$ if $a \preceq b$ and $\langle a, b\rangle = \{c \in S : a \preceq c \text{ or } c \preceq b\}$ if $b \preceq a$.

2.1 Planar geometric notions

Given a vector $v = (v_1, v_2) \in \mathbb{R}^2$, let $v^\perp = (v_2, -v_1) \in \mathbb{R}^2$ denote the vector perpendicular to v to the right, i.e., the 90 degree clockwise rotation of v. Given $p \neq q \in \mathbb{R}^2$, let $]p, q[$ and $[p, q]$ be, respectively, the open and closed segments with end points p and q. We let $[p, q[= [p, q] \setminus \{q\}$. Let $u_{[p,q]} = \mathrm{unit}(q - p)$ be the unit vector in the direction from p to q and $n_{[p,q]} = u^\perp_{[p,q]}$ the unit normal vector to the right. In coordinates,

$$u_{[p,q]} = \frac{1}{\|q-p\|}(q_1 - p_1, q_2 - p_2),$$
$$n_{[p,q]} = \frac{1}{\|q-p\|}(q_2 - p_2, p_1 - q_1),$$

where $p = (p_1, p_2)$ and $q = (q_1, q_2)$. We denote by $H^{\mathrm{out}}_{[p,q]} = \{z \in \mathbb{R}^2 : (z - p)^T n_{[p,q]} \geq 0\}$ the halfplane of points in the positive direction of the normal vector with respect to the closed segment $[p, q]$. Likewise, we denote $H^{\mathrm{in}}_{[p,q]} = \{z \in \mathbb{R}^2 : (z - p)^T n_{[p,q]} \leq 0\}$.

Given $p \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$, let $\mathtt{ray}(p, v) = \{z \in \mathbb{R}^2 : z = p + tv,\ t \in \mathbb{R}_{\geq 0}\}$. The wedge $\mathtt{wedge}(p, (v_1, n_1), (v_2, n_2))$ is the open cone with vertex p and axes $\mathtt{ray}(p, v_1)$ and $\mathtt{ray}(p, v_2)$, i.e., the set of points towards which n_1 points along $\mathtt{ray}(p, v_1)$ and n_2 points along $\mathtt{ray}(p, v_2)$, see Fig. 1 for an illustration. For the wedge to be well-defined, the normal vectors n_1 and n_2 need to specify it uniquely. Given a simply connected set $\mathcal{D} \subset \mathbb{R}^2$ with nonempty

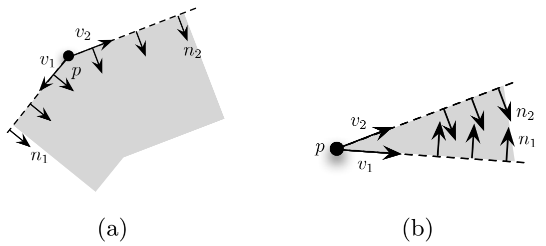


Fig. 1. Wedges determined by p, (v_1, n_1) and (v_2, n_2).

interior and $v \in \mathbb{R}^2$ with origin q in $\mathcal{D}$, let $\mathrm{pr}_\mathcal{D}(v)$ be the orthogonal projection of v onto the tangent space $T_q\mathcal{D}$ of $\mathcal{D}$ at q if $q \in \partial\mathcal{D}$ and v points outside $\mathcal{D}$, and v otherwise.

2.2 Curve parameterizations

A *curve* $\mathcal{C}$ in $\mathbb{R}^2$ is the image of a map $\gamma : [a, b] \to \mathbb{R}^2$. The map γ is called a *parametrization of* $\mathcal{C}$. We often identify a curve with its parametrization. We only deal with piecewise smooth and continuous curves. A curve $\mathcal{C}$ is *self-intersecting* if γ is not injective on (a, b). A curve $\mathcal{C}$ is *closed* if $\gamma(a) = \gamma(b)$. For a curve $\mathcal{C}$, $\dot\gamma$ denotes the tangent direction to $\mathcal{C}$ and $n_\mathcal{C} = \mathrm{unit}(\dot\gamma)^\perp$ the unit normal vector to $\mathcal{C}$. A closed, not self-intersecting curve $\mathcal{C}$ partitions $\mathbb{R}^2$ into two disjoint open and connected sets, $\mathrm{Inside}_\mathcal{C}$ and $\mathrm{Outside}_\mathcal{C}$, such that $n_\mathcal{C}$ along $\mathcal{C}$ points outside $\mathrm{Inside}_\mathcal{C}$ and inside $\mathrm{Outside}_\mathcal{C}$, respectively. The orientation of $\mathcal{C}$ affects the definition of $n_\mathcal{C}$ and $\mathrm{Inside}_\mathcal{C}$, $\mathrm{Outside}_\mathcal{C}$. Given a curve $\mathcal{C}$ parametrized by a piecewise smooth map $\gamma : [a, b] \to \mathcal{C}$, the line integral of a function $h : \mathcal{C} \subset \mathbb{R}^2 \to \mathbb{R}$ over $\mathcal{C}$ is

$$\int_\mathcal{C} h = \int_\mathcal{C} h(q)dq = \int_a^b h(\gamma(t))\,\|\dot\gamma(t)\|\,dt, \tag{1}$$

and is independent of the selected parametrization.

2.3 Modeling of hybrid systems

We briefly review the framework for modeling hybrid systems in [Goebel et al., 2004].A *hybrid system* is defined by a tuple $\mathcal{H} = (C, F, D, G)$, where $C \subset \mathbb{R}^d$ is the flow set, $F : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ with $C \subset \mathrm{Dom}(F)$, is the flow (set-valued) map, $D \subset \mathbb{R}^d$ is the jump set, and $G : \mathbb{R}^d \rightrightarrows \mathbb{R}^d$ with $D \subset \mathrm{Dom}(G)$ is the jump (set-valued) map. The dynamics is given by

$$\begin{cases} \dot x \ \in F(x), & x \in C, \\ x^+ \in G(x), & x \in D. \end{cases} \tag{2}$$

The notation $\dot x$ indicates the time derivative of the state, whereas x^+ indicates the value of the state after an instantaneous change. In this paper, we use singleton-valued flow maps $F(x) = \{f(x)\}$, with $f : \mathbb{R}^d \to \mathbb{R}^d$. Equation (2) reflects the fact that solutions can evolve according to the differential inclusion $\dot x \in F(x)$ when they belong to C and according to the difference inclusion $x^+ \in G(x)$ when they belong to D. The concept of solution can be formalized using the notions of hybrid time domain and hybrid arc [Goebel et al., 2004]. The set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a compact hybrid time domain if $E = \cup_{j=0}^{J-1}([t_j, t_{j+1}], j)$, with $0 = t_0 \leq t_1 \leq \cdots \leq t_J$. The set $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for all $(t, j) \in E$, $E \cap ([0, t] \times \{0, 1, \ldots, j\})$ is a compact hybrid time domain. The length of E is $\mathrm{length}\,(E) = \sup_t E + \sup_j E$, i.e., the sum of the supremum of the t coordinates and the supremum of the j coordinates. A function $\phi : E \to \mathbb{R}^d$ is a hybrid arc if E is a hybrid time domain and, for each $j \in \mathbb{N}$, the function $t \mapsto \phi(t, j)$ is locally absolutely continuous on $I^j = \{t \in \mathbb{R}_{\geq 0} : (t, j) \in E\}$. Finally, a hybrid arc $\phi : E \to \mathbb{R}^d$ is a solution of (2) if $\phi(0, 0) \in \overline{C} \cup D$ and the following holds:

(i) for $j \in \mathbb{N}$ with $\text{int}(I^j) \neq \emptyset$, $\phi(t,j) \in C$ for $t \in \text{int}(I^j)$ and $\dot{\phi}(t,j) \in F(\phi(t,j))$ for almost all $t \in I^j$;
(ii) for $(t,j) \in E$ such that $(t,j+1) \in E$, $\phi(t,j) \in D$ and $\phi(t,j+1) \in G(\phi(t,j))$.

Of particular interest are complete (length $(E) = \infty$) and Zeno ($\sup_j E = \infty$ and $\sup_t E < \infty$) trajectories.

3 Problem statement

Let $\mathcal{D} \subset \mathbb{R}^2$ be a compact, simply connected set with nonempty interior, and let $Y : \mathcal{D} \to \mathbb{R}$ be a twice continuously differentiable function modeling a planar time-independent field. Consider a network Σ of agents with positions $p_1, \dots, p_n$ moving in $\mathcal{D}$. Our objective is to find regions in $\mathcal{D}$ where locally maximal changes occur in Y by determining their boundaries. We begin by defining a measure of how fast the field changes across a given curve. The *wombliness* or *alignment* of a curve $\mathcal{C}$ is

$$\mathcal{W}(\mathcal{C}) = \int_{\mathcal{C}} \langle \nabla Y, n_{\mathcal{C}} \rangle, \tag{3}$$

see e.g., [Kimmel, 2003, Banerjee and Gelfand, 2006]. The interpretation of $\mathcal{W}$ is as follows. At each point of the curve, we look at how much Y is changing along the normal direction to $\mathcal{C}$ (i.e., how much Y is "flowing through $\mathcal{C}$"). The integral sums this change throughout the curve. We are interested in using the robotic network to find curves whose value of $|\mathcal{W}|$ is large. For a closed not self-intersecting curve, $\mathcal{W}$ can be written,

$$\mathcal{W}(\mathcal{C}) = \int_{\mathcal{C}} \langle \nabla Y, n_{\mathcal{C}} \rangle = \int_D \text{div} \nabla Y = \int_D \Delta Y, \tag{4}$$

using the Gauss Divergence Theorem [Courant and John, 1999], where D is the set in $\mathbb{R}^2$ whose boundary is $\mathcal{C}$, and $\Delta Y = \frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2}$ is the Laplacian of Y. Note that $\mathcal{W}$ is a bounded function on the set of closed not self-intersecting curves in $\mathcal{D}$. Observe that, in general, the level curves of Y are not optimizers of $\mathcal{W}$. An example is given by $Y(x_1, x_2) = e^{-x_1^2 - 2x_2^2}$, whose level sets are the ellipses $x_1^2 + 2x_2^2 = c$, $c \geq 0$. One has $\Delta Y(x_1, x_2) = 2Y(x_1, x_2)(2x_1^2 + 8x_2^2 - 3)$, and hence, using (4), we deduce that the ellipse $2x_1^2 + 8x_2^2 - 3 = 0$ (which is not a level curve of Y) maximizes $|\mathcal{W}|$.

In general, the optimization of (3) is an infinite-dimensional problem. Our approach is to order counterclockwise the agents according to their unique identifier, and consider the closed polygonal curve that results from joining the positions of consecutive robots, i.e., for $(p_1, \dots, p_n) \in \mathcal{D}^n$, let γ_{cpc} be the closed polygonal curve that results from the concatenation of $[p_i, p_{i+1}]$, $i \in \{1, \dots, n-1\}$ and $[p_n, p_1]$. In general, such curves may be self-intersecting. Therefore, we restrict our attention to the following open subset of $\mathcal{D}^n$,

$$\mathcal{S}_c = \{(p_1, \dots, p_n) \in \mathcal{D}^n : \gamma_{\mathsf{cpc}} \text{ is not self-intersecting}\}.$$

Define $\mathcal{W}_c : \mathcal{S}_c \to \mathbb{R}$ by $\mathcal{W}_c(p_1, \dots, p_n) = \mathcal{W}(\gamma_{\mathsf{cpc}})$, i.e.,

$$\mathcal{W}_c(p_1, \dots, p_n) = \sum_{i=1}^{n} \int_{[p_i, p_{i+1}]} \langle \nabla Y, n_{[p_i, p_{i+1}]} \rangle. \tag{5}$$

The optimization of (5) is now a finite-dimensional problem. Our objective is then to have the network Σ locally optimize $\mathcal{W}_c$. Note that $\mathcal{W}_c$ can be expressed in terms of the polygon determined by the concatenated straight segments. If $\mathcal{P}(p_1, \dots, p_n)$ denotes this polygon, then

$$\mathcal{W}_c(p_1, \dots, p_n) = \int_{\mathcal{P}(p_1, \dots, p_n)} \Delta Y. \tag{6}$$

For reasons that will become clear later, we assume that, at each network configuration, agent $i \in \{1, \dots, n\}$ can measure ∇Y and ΔY along $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$.

4 Smoothness of the wombliness measure

In this section we analyze the smoothness properties of the wombliness measure, provide explicit expressions for the gradient, and characterize the critical points.

Proposition 4.1 (Gradient of $\mathcal{W}_c$) *The function $\mathcal{W}_c : \mathcal{S}_c \to \mathbb{R}$ is continuously differentiable. For each $i \in \{1, \dots, n\}$, at $(p_1, \dots, p_n) \in \mathcal{S}_c$, we have*

$$\begin{aligned} \frac{\partial \mathcal{W}_c}{\partial p_i} &= \Big(\int_{[p_i, p_{i+1}]} \frac{\|p_{i+1} - q\|}{\|p_{i+1} - p_i\|} \Delta Y(q) \Big) n_{[p_i, p_{i+1}]} dq \\ &\quad + \Big(\int_{[p_{i-1}, p_i]} \frac{\|q - p_{i-1}\|}{\|p_i - p_{i-1}\|} \Delta Y(q) \Big) n_{[p_{i-1}, p_i]} dq. \end{aligned} \tag{7}$$

PROOF. To compute the gradient of $\mathcal{W}_c$, we use (6). For a parameterization γ of the boundary of $\mathcal{P}(p_1, \dots, p_n)$, according to the generalized conservation-of-mass lemma, cf. [Bullo et al., 2009, Proposition 2.23],

$$\frac{\partial \mathcal{W}_c}{\partial p_i} = \int_{\partial \mathcal{P}} \Delta Y \cdot \frac{\partial \gamma}{\partial p_i} n_{\partial \mathcal{P}} \, d\gamma. \tag{8}$$

Next, we develop this expression. Let $\gamma_{ii+1} : [0,1] \to \mathbb{R}^2$ be a parametrization of $[p_i, p_{i+1}]$,

$$\gamma_{ii+1}(t) = p_i + t(p_{i+1} - p_i). \tag{9}$$

The collection of these parameterizations for $i \in \{1, \dots, n\}$ defines a parameterization γ of the boundary of $\mathcal{P}(p_1, \dots, p_n)$. Substituting into (8), we get

$$\begin{aligned} \frac{\partial \mathcal{W}_c}{\partial p_i} &= \int_{[p_i, p_{i+1}]} \Delta Y \cdot \frac{\partial \gamma_{ii+1}}{\partial p_i} n_{[p_i, p_{i+1}]} \, d\gamma_{ii+1} \\ &\quad + \int_{[p_{i-1}, p_i]} \Delta Y \cdot \frac{\partial \gamma_{i-1i}}{\partial p_i} n_{[p_{i-1}, p_i]} \, d\gamma_{i-1i}. \end{aligned}$$

According to (9), $\partial\gamma_{ii+1}/\partial p_i = (1-t)I_2$ and $\partial\gamma_{i-1i}/\partial p_i = tI_2$. Therefore,

$$\frac{\partial \mathcal{W}_c}{\partial p_i} = \int_0^1 (1-t)\Delta Y \cdot n_{[p_i,p_{i+1}]}\|p_{i+1}-p_i\|dt + \int_0^1 t\,\Delta Y \cdot n_{[p_{i-1},p_i]}\|p_i - p_{i-1}\|dt. \qquad (10)$$

Finally, from (9) and using $t \in [0,1]$, we have

$$(1-t)\|p_{i+1}-p_i\| = \|p_{i+1} - \gamma_{ii+1}(t)\|,$$
$$t\|p_i - p_{i-1}\| = \|\gamma_{i-1i}(t) - p_{i-1}\|.$$

The result follows from using these formula in (10). □

According to Proposition 4.1, agent i needs to know the location of its neighbors in the ring graph (agents $i-1$ and $i+1$) and the Laplacian of the field along the corresponding segments to compute $\frac{\partial \mathcal{W}_c}{\partial p_i}$. The following characterization of the critical configurations of $\mathcal{W}_c$ follows from observing that if three agents p_{i-1}, p_i, and p_{i+1} are not aligned, then $n_{[p_i,p_{i+1}]}$ and $n_{[p_{i-1},p_i]}$ are linearly independent. In the following result, with a slight abuse of notation, we let $\mathcal{W}_c : \overline{\mathcal{S}_c} \to \mathbb{R}$ denote the extension by continuity, see e.g., [Pedrick, 1994], of $\mathcal{W}_c$ to $\overline{\mathcal{S}_c}$.

Corollary 4.2 (Critical points of $\mathcal{W}_c$) *Let $(p_1,\dots,p_n) \in \overline{\mathcal{S}_c}$ be a critical configuration of $\mathcal{W}_c$. Then, for $i \in \{1,\dots,n\}$, $\mathrm{pr}_{\mathcal{D}}\left(\frac{\partial \mathcal{W}_c}{\partial p_i}\right) = 0$. Moreover, if $(p_1,\dots,p_n) \in \mathrm{int}(\mathcal{D}^n)$ and no three consecutive agents are aligned, this can be alternatively described by, for $i \in \{1,\dots,n\}$,*

$$\int_{[p_i,p_{i+1}]} \|p_{i+1} - q\|\Delta Y = 0, \qquad (11a)$$

$$\int_{[p_i,p_{i+1}]} \|q - p_i\|\Delta Y = 0. \qquad (11b)$$

5 The `hybrid wombling algorithm`

Following Section 3, we seek to optimize the function $\mathcal{W}_c$. The approach we propose is to implement a distributed gradient flow using the result of Proposition 4.1. However, the evolution under this flow of the closed polygonal curve γ_{cpc} defined by the network agents may become self-intersecting. To address this problem, we design here the **`hybrid wombling algorithm`** on Σ. This strategy prescribes state transitions that resolve the curve intersections. We will see that, in some cases, the state transition requires the inclusion of additional agents. In what follows, we describe the resulting hybrid system in detail within the framework of Section 2.3.

The flow set. Given $N \in \mathbb{N}$, let $\ell = \{\ell_1,\dots,\ell_{|\ell|}\}$ be a partition of $\{1,\dots,N\}$ into disjoint totally ordered subsets with $|\ell_1| \geq 1$ and $|\ell_\alpha| \geq 3$ for $\alpha \in \{2,\dots,|\ell|\}$. Let $\mathfrak{F}(\{1,\dots,N\})$ denote the (finite) class of all such collections. For $P \in \mathbb{R}^N$ and $\ell \in \mathfrak{F}(\{1,\dots,N\})$, let $\gamma^\alpha_{\mathsf{cpc}}$ be the closed polygonal curve defined by $\{p_k\}_{k\in\ell_\alpha}$, $\alpha \in \{2,\dots,|\ell|\}$. The agents in ℓ_1 will be used later in the definition of the jump map. We define the flow set by

$$C = \{(P,\ell) \in \mathbb{R}^N \times \mathfrak{F}(\{1,\dots,N\}) \mid \text{for } \mu = 2,\dots,|\ell|, \gamma^\mu_{\mathsf{cpc}} \in \overline{\mathcal{S}_c} \text{ w/ disjoint inside or disjoint outside sets}\}.$$

Note the slight abuse of notation in this expression when using $\overline{\mathcal{S}_c}$, since each curve $\gamma^\mu_{\mathsf{cpc}}$ might consist of a different number of points. Note also that the set C is closed.

The flow map. For each curve $\gamma^\alpha_{\mathsf{cpc}}$, let P^α be the vector whose components are $\{p_k\}_{k\in\ell_\alpha}$. With a slight abuse of notation, let $\mathcal{W}^\alpha_c(P^\alpha) = \mathcal{W}(\gamma^\alpha_{\mathsf{cpc}})$. The flow map $f : C \to \mathbb{R}^{N+1}$ is defined as follows. For $k \in \ell_1 \cup \{N+1\}$, $f_k(P,\ell) = 0$. For $\alpha \in \{2,\dots,|\ell|\}$ and $k \in \ell_\alpha$, its kth component is

$$f_k(P,\ell) = \begin{cases} \mathrm{pr}_{\mathcal{D}}\left(\frac{\partial \mathcal{W}^\alpha_c}{\partial p_k}\right) & \text{if } \mathcal{W}^\alpha_c(P^\alpha) \geq 0, \\ -\mathrm{pr}_{\mathcal{D}}\left(\frac{\partial \mathcal{W}^\alpha_c}{\partial p_k}\right) & \text{if } \mathcal{W}^\alpha_c(P^\alpha) < 0. \end{cases} \qquad (12)$$

Note that f_k is continuous except possibly at configurations for which $p_k \in \partial\mathcal{D}$ for some $k \in \ell_\alpha$ (due to the projection operation) and at configurations where $\mathcal{W}^\alpha_c(P^\alpha) = 0$ (due to the sign change).

The jump set and the jump map. As mentioned above, the evolution under the gradient flow prescribed by f on C might lead to a curve self-intersection or, if there is more than one curve evolving in $\mathcal{D}$, to the intersection among several curves. A curve self-intersection corresponds to $D_1 = \{(P,\ell) \in C$: there exists $\alpha \in \{2,\dots,|\ell|\}$ and $k, z_1, z_2 \in \ell_\alpha$ with $p_k \in [p_{z_1}, p_{z_2}]\}$. An intersection of curves corresponds to $D_2 = \{(P,\ell) \in C$: there exist $\alpha \neq \beta \in \{2,\dots,|\ell|\}$ with $k \in \ell_\alpha$ and $p_k \in \gamma^\beta_{\mathsf{cpc}}\}$. Both sets are closed. Among these configurations, we need to identify the ones where f points outside C, and hence a state transition is necessary to have a well-defined network evolution. This is what we study next. Section 5.1 deals with curve self-intersection and Section 5.2 deals with intersection of curves. In both cases, we identify the configurations in the jump set $D \subset \partial C$ and define the action of the jump map G.

5.1 Curve self-intersection

Let $(P,\ell) \in D_1$. Here we consider the case of a single curve self-intersection. This discussion forms the basis for the treatment of the multiple self-intersection case in Section 5.1.4. Let $\gamma^\alpha_{\mathsf{cpc}}$ denote the self-intersecting curve. When a self-intersection occurs, either $\mathrm{Inside}_{\gamma_{\mathsf{cpc}}}$ or $\mathrm{Outside}_{\gamma_{\mathsf{cpc}}}$ become disconnected. We refer to these two cases as *inside* and *outside* self-intersections, respectively, see Fig. 2. Sections 5.1.1 and 5.1.2 describe the configurations in the jump set when the self-intersection occurs at an open segment or at a point's location, respectively. Section 5.1.3 describes the jump map.

5.1.1 *Self-intersection at an open segment*

For $i \neq j \in \ell_\alpha$ such that $p_i \in]p_j, p_{j_+}[$, define $\lambda \in [0,1)$ by $p_i = (1-\lambda)p_j + \lambda p_{j_+}$ and consider

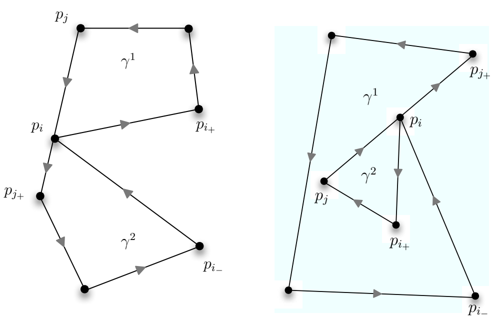
$$v_i = (1-\lambda)u_j + \lambda u_{j_+},\ u_k = \operatorname{sgn}(\mathcal{W}_c^\alpha(P^\alpha))\,\mathrm{pr}_{\mathcal{D}}\Big(\frac{\partial \mathcal{W}_c^\alpha}{\partial p_k}\Big),$$

where $k \in \{i, j, j_+\}$. The conditions characterizing when (P,ℓ) belongs to the jump set D depend on the type of self-intersection, and are detailed next.

Inside self-intersection. If the self-intersection is of inside type, the segment $[p_i, p_{i_+}]$ belongs to $H^{\text{in}}_{[p_j,p_{j_+}]}$ and there exists the possibility of p_i crossing from $H^{\text{in}}_{[p_j,p_{j_+}]}$ to $H^{\text{out}}_{[p_j,p_{j_+}]}$, see Fig. 2(a). Then, $(P,\ell) \in D$ iff

(i) $(P,\ell) \in D_1$ with inside self-intersection at an open segment and $(u_i - v_i)^T n_{[p_j,p_{j_+}]} \geq 0$.

Including $(u_i - v_i)^T n_{[p_j,p_{j_+}]} = 0$ (corresponding to p_i evolving along the segment $[p_j, p_{j_+}]$) in the definition of D simplifies the convergence analysis of Section 6.



(a) inside (b) outside

Fig. 2. $\gamma_{\texttt{cpc}}$ defined by $p_1, \ldots, p_n$ is self-intersecting at an open segment. (a) inside self-intersection and (b) outside self-intersection. $\gamma_{\texttt{cpc}}$ is the union of two not self-intersecting curves γ^1 and γ^2.

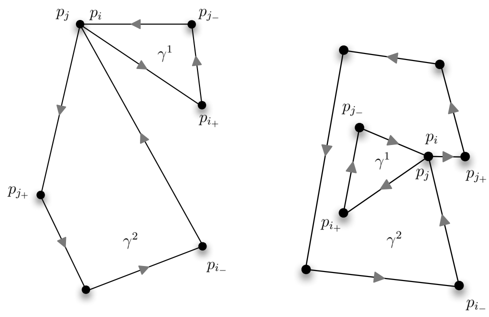
Outside self-intersection. If the self-intersection is of outside type, the segment $[p_i, p_{i_+}]$ belongs to $H^{\text{out}}_{[p_j,p_{j_+}]}$ and there exists the possibility of p_i crossing from $H^{\text{out}}_{[p_j,p_{j_+}]}$ to $H^{\text{in}}_{[p_j,p_{j_+}]}$, see Fig. 2(b). Then, $(P,\ell) \in D$ iff

(ii) $(P,\ell) \in D_1$ with outside self-intersection at an open segment and $(u_i - v_i)^T n_{[p_j,p_{j_+}]} \leq 0$.

5.1.2 *Self-intersection at a point*

For $i \neq j \in \ell_\alpha$ such that $p_i = p_j$, consider

$$u_i = \operatorname{sgn}(\mathcal{W}_c^\alpha(P^\alpha))\,\mathrm{pr}_{\mathcal{D}}\Big(\frac{\partial \mathcal{W}_c^\alpha}{\partial p_i}\Big),$$
$$u_j = \operatorname{sgn}(\mathcal{W}_c^\alpha(P^\alpha))\,\mathrm{pr}_{\mathcal{D}}\Big(\frac{\partial \mathcal{W}_c^\alpha}{\partial p_j}\Big).$$



(a) inside (b) outside

Fig. 3. $\gamma_{\texttt{cpc}}$ defined by $p_1, \ldots, p_n$ is self-intersecting at a point's location. (a) inside self-intersection and (b) outside self-intersection. $\gamma_{\texttt{cpc}}$ is the union of two not self-intersecting curves γ^1 and γ^2.

Inside self-intersection. If the self-intersection is of inside type, see Fig. 3(a), define the vectors

$$v_1 = \begin{cases} u_{[p_{i_-},p_i]} & \text{if } [p_{j_-}, p_j] \subset H^{\text{in}}_{[p_{i_-},p_i]}, \\ u_{[p_j,p_{j_-}]} & \text{if } [p_{j_-}, p_j] \not\subset H^{\text{in}}_{[p_{i_-},p_i]}, \end{cases}$$
$$v_2 = \begin{cases} u_{[p_{i_+},p_i]} & \text{if } [p_j, p_{j_+}] \subset H^{\text{in}}_{[p_i,p_{i_+}]}, \\ u_{[p_j,p_{j_+}]} & \text{if } [p_j, p_{j_+}] \not\subset H^{\text{in}}_{[p_i,p_{i_+}]}. \end{cases}$$

Then, $(P,\ell) \in D$ iff

(iii) $(P,\ell) \in D_1$ with inside self-intersection at a point and $u_i - u_j \notin \texttt{wedge}(p_j, (v_1, v_1^\perp), (v_2, -v_2^\perp))$.

Outside self-intersection. If the self-intersection is of outside type, see Fig. 3(b), define the vectors

$$v_1 = \begin{cases} u_{[p_j,p_{j_-}]} & \text{if } [p_{j_-}, p_j] \subset H^{\text{in}}_{[p_{i_-},p_i]}, \\ u_{[p_{i_-},p_i]} & \text{if } [p_{j_-}, p_j] \not\subset H^{\text{in}}_{[p_{i_-},p_i]}, \end{cases}$$
$$v_2 = \begin{cases} u_{[p_j,p_{j_+}]} & \text{if } [p_j, p_{j_+}] \subset H^{\text{in}}_{[p_i,p_{i_+}]}, \\ u_{[p_{i_+},p_i]} & \text{if } [p_j, p_{j_+}] \not\subset H^{\text{in}}_{[p_i,p_{i_+}]}. \end{cases}$$

Then, $(P,\ell) \in D$ iff

(iv) $(P,\ell) \in D_1$ with outside self-intersection at a point and $u_i - u_j \notin \texttt{wedge}(p_j, (v_1, -v_1^\perp), (v_2, v_2^\perp))$.

5.1.3 *Jump map*

Here, we specify the action of the jump map G on $(P,\ell) \in D \cap D_1$ with a single self-intersection. In this case, $\gamma^\alpha_{\texttt{cpc}}$ can be decomposed into two polygonal curves γ^1 and γ^2, see Figs. 2 and 3. The curve γ^1 is defined by the concatenation of the segments $\{[p_k, p_{k_+}] : k \in \langle i, j_- \rangle\} \cup [p_j, p_i]$, if $p_i \in]p_j, p_{j_+}[$, and $\{[p_k, p_{k_+}] : k \in \langle i_+, j_- \rangle\} \cup [p_j, p_{i_+}]$, if $p_i = p_j$. The curve γ^2 is defined by the concatenation of the segments $\{[p_k, p_{k_+}] : k \in \langle j_+, i_- \rangle\} \cup [p_i, p_{j_+}]$, if $p_i \in]p_j, p_{j_+}[$, and $\{[p_k, p_{k_+}] : k \in \langle j_+, i_- \rangle\} \cup [p_i, p_{j_+}]$, if

$p_i = p_j$. Let $\mathbb{k}_1$ and $\mathbb{k}_2$ denote the indices of the agents

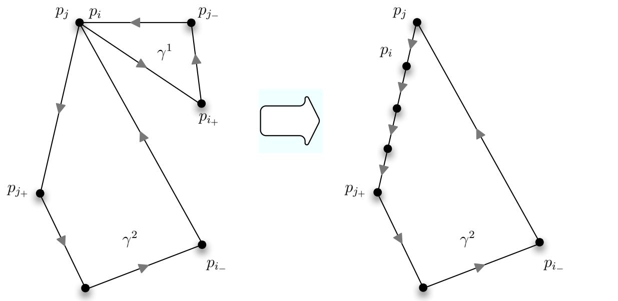


Fig. 4. Agent re-positioning. Agents in γ^1 get re-positioned onto γ^2.

defining γ^1 and γ^2, respectively. In the case of a self-intersection at an open segment, note that $i \in \mathbb{k}_1 \cap \mathbb{k}_2$. The wombliness of $\gamma^\alpha_{\mathsf{cpc}}$ is split between γ^1 and γ^2 as $\mathcal{W}(\gamma^\alpha_{\mathsf{cpc}}) = \mathcal{W}(\gamma^1) + \mathcal{W}(\gamma^2)$. Depending on the sign of $\mathcal{W}(\gamma^1)$ and $\mathcal{W}(\gamma^2)$, we define:

Agent re-positioning. If $\mathcal{W}(\gamma^1)$ and $\mathcal{W}(\gamma^2)$ have different signs, we only keep the curve whose wombliness has the same sign as $\gamma^\alpha_{\mathsf{cpc}}$ and reposition the agents of the other curve along it arbitrarily. Formally, if $\operatorname{sgn}(\mathcal{W}(\gamma^1))\operatorname{sgn}(\mathcal{W}(\gamma^2)) \leq 0$, then $G(P,\ell)$ is the set of configurations (P',ℓ) such that P' satisfies

$$\begin{aligned} p'_k &= p_k \quad \text{for } k \notin \mathbb{k}_{\bar\mu} \text{ or } k \in \mathbb{k}_\mu, \\ p'_k &\in \gamma^\mu \quad \text{otherwise}, \end{aligned}$$

where $\operatorname{sgn}(\mathcal{W}(\gamma^\mu)) = \operatorname{sgn}(\mathcal{W}(\gamma^\alpha_{\mathsf{cpc}}))$ and $\bar\mu \in \{1,2\} \setminus \{\mu\}$. Note that if both curves have nonzero wombliness, the absolute value of the wombliness of the resulting not self-intersecting curve is strictly larger than the value of the wombliness of the original self-intersecting curve $\gamma^\alpha_{\mathsf{cpc}}$. Fig. 4 illustrates this transition.

Curve splitting. If $\mathcal{W}(\gamma^1)$ and $\mathcal{W}(\gamma^2)$ have the same sign as $\mathcal{W}(\gamma^\alpha_{\mathsf{cpc}})$ (note that this implies $n \geq 5$), then the curves split. If the self-intersection occurs on an open segment, one additional agent must be added to the network at the intersection location. Formally, if $\operatorname{sgn}(\mathcal{W}(\gamma^1))\operatorname{sgn}(\mathcal{W}(\gamma^2)) > 0$ and the self-intersection is at a point, $G(P,\ell) = \{(P,\ell')\}$ with $|\ell'| = |\ell| + 1$ and

$$\begin{aligned} &\ell'_\beta = \ell_\beta, \text{ for } \beta \in \{1,\dots,|\ell|\} \setminus \{\alpha\}, \\ &\ell'_\alpha = \mathbb{k}_1, \qquad \ell'_{|\ell|+1} = \mathbb{k}_2. \end{aligned}$$

If $\operatorname{sgn}(\mathcal{W}(\gamma^1))\operatorname{sgn}(\mathcal{W}(\gamma^2)) > 0$ and the self-intersection is on an open segment, $G(P,\ell)$ is the set of configurations (P',ℓ') that satisfy $|\ell'| = |\ell| + 1$ and

$$\begin{aligned} &p'_k = p_k, \text{ for } k \neq m, \qquad p'_m = p_i, \\ &\ell'_\beta = \ell_\beta, \text{ for } \beta \in \{2,\dots,|\ell|\} \setminus \{\alpha\}, \\ &\ell'_1 = \ell_1 \setminus \{m\}, \qquad \ell'_\alpha = \mathbb{k}_1, \qquad \ell'_{|\ell|+1} = \mathbb{k}'_2, \end{aligned}$$

with $m \in \ell_1$. Here, $\mathbb{k}'_2$ is the result of substituting i by m in $\mathbb{k}_2$. This transition is illustrated in Fig. 5. G may not

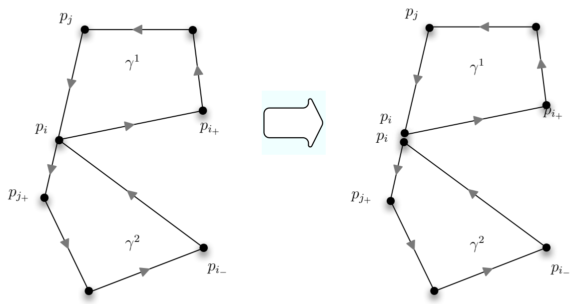


Fig. 5. Curve splitting. $\gamma^\alpha_{\mathsf{cpc}}$ is split into γ^1 and γ^2, and these curves evolve independently afterward.

be outer semicontinuous relative to D at configurations where the wombliness of one of the curves is zero.

5.1.4 Multiple self-intersections

Let $(P,\ell) \in D_1$. Here we discuss the case when $\gamma^\alpha_{\mathsf{cpc}}$ has multiple self-intersections. One proceeds by characterizing the self-intersections that need to be resolved because the flow map f points outside C, in an analogous way to cases (i)-(ii) in Section 5.1.1 and (iii)-(iv) in Section 5.1.2. If e is the number of self-intersections that need to be resolved, then $\gamma^\alpha_{\mathsf{cpc}}$ can be decomposed into $e+1$ not self-intersecting curves $\gamma^1,\dots,\gamma^{e+1}$. The wombliness of $\gamma^\alpha_{\mathsf{cpc}}$ is split as $\mathcal{W}(\gamma^\alpha_{\mathsf{cpc}}) = \sum_{\mu=1}^{e+1}\mathcal{W}(\gamma^\mu)$. Similarly to Section 5.1.3, the action of G is:

- For $\mu \in \{1,\dots,e+1\}$ with $\operatorname{sgn}(\mathcal{W}(\gamma^\mu)) = \operatorname{sgn}(\mathcal{W}(\gamma^\alpha_{\mathsf{cpc}}))$, γ^μ splits and remains after the jump as an independent curve. At least a curve always falls in this case;
- For $\mu \in \{1,\dots,e+1\}$ with $\operatorname{sgn}(\mathcal{W}(\gamma^\mu)) \neq \operatorname{sgn}(\mathcal{W}(\gamma^\alpha_{\mathsf{cpc}}))$, all agents in γ^μ are repositioned along curves in the previous case. The repositioning is arbitrary.

For space reasons, we leave to the reader the formal description of the action of the jump map. Finally, if (P,ℓ) is a configuration where multiple curves self-intersect, the jump map acts on each one as described above.

5.2 Intersection between curves

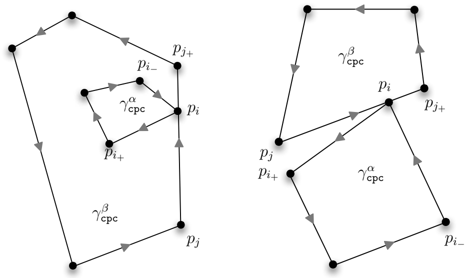
Let $(P,\ell) \in D_2$. Here we consider the case of a single intersection of two curves. This discussion is the basis for the treatment of the multiple intersection case in Section 5.2.4. Let $\gamma^\alpha_{\mathsf{cpc}}$ and $\gamma^\beta_{\mathsf{cpc}}$ denote the two intersecting curves. When an intersection occurs, either $\mathrm{Inside}_{\gamma^\alpha_{\mathsf{cpc}}} \cap \mathrm{Inside}_{\gamma^\beta_{\mathsf{cpc}}}$ (*inside* intersection) or $\mathrm{Outside}_{\gamma^\alpha_{\mathsf{cpc}}} \cap \mathrm{Outside}_{\gamma^\beta_{\mathsf{cpc}}}$ (*outside* intersection) is nonempty, see Fig. 6. Sections 5.2.1 and 5.2.2 describe the configurations in the jump set when the intersection occurs at an open segment or at a point's location, respectively. Section 5.2.3 describes the jump map.

5.2.1 *Intersection at an open segment*

For $i \in \ell_\alpha$ and $j \in \ell_\beta$ such that $p_i \in]p_j, p_{j_+}[$, define $\lambda \in [0,1)$ by $p_i = (1-\lambda)p_j + \lambda p_{j_+}$. Consider

$$\begin{aligned} v_i &= (1-\lambda)u_j + \lambda u_{j_+}, \\ u_i &= \operatorname{sgn}(\mathcal{W}_c^\alpha(P^\alpha)) \operatorname{pr}_{\mathcal{D}} \Big(\frac{\partial \mathcal{W}_c^\alpha}{\partial p_i}\Big), \\ u_k &= \operatorname{sgn}(\mathcal{W}_c^\beta(P^\beta)) \operatorname{pr}_{\mathcal{D}} \Big(\frac{\partial \mathcal{W}_c^\beta}{\partial p_k}\Big), \end{aligned}$$

where $k \in \{j, j_+\}$. The conditions characterizing when (P, ℓ) belongs to the jump set D depend on the type of self-intersection, and are detailed next.



(a) inside (b) outside

Fig. 6. $\gamma_{\mathsf{cpc}}^\alpha$ and $\gamma_{\mathsf{cpc}}^\beta$ intersect at an open segment. (a) inside intersection and (b) outside intersection. In both cases, $\gamma_{\mathsf{cpc}}^\alpha$ and $\gamma_{\mathsf{cpc}}^\beta$ can be merged into a new self-intersecting curve γ.

Inside intersection. If the intersection is of inside type, the segment $[p_i, p_{i_+}]$ of $\gamma_{\mathsf{cpc}}^\alpha$ belongs to $H^{\text{in}}_{[p_j, p_{j_+}]}$ and there exists the possibility of p_i crossing from $H^{\text{in}}_{[p_j, p_{j_+}]}$ to $H^{\text{out}}_{[p_j, p_{j_+}]}$, see Fig. 6(a). Then, $(P, \ell) \in D$ iff

(i) $(P, \ell) \in D_2$ with inside intersection at an open segment and $(u_i - v_i)^T n_{[p_j, p_{j_+}]} \geq 0$.

Outside intersection. If the intersection is of outside type, the segment $[p_i, p_{i_+}]$ of $\gamma_{\mathsf{cpc}}^\alpha$ belongs to $H^{\text{out}}_{[p_j, p_{j_+}]}$ and there exists the possibility of p_i crossing from $H^{\text{out}}_{[p_j, p_{j_+}]}$ to $H^{\text{in}}_{[p_j, p_{j_+}]}$, see Fig. 6(b). Then, $(P, \ell) \in D$ iff

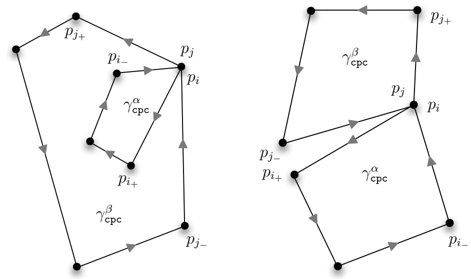
(ii) $(P, \ell) \in D_2$ with outside intersection at an open segment and $(u_i - v_i)^T n_{[p_j, p_{j_+}]} \leq 0$.

5.2.2 *Intersection at a point*

For $i \in \ell_\alpha$ and $j \in \ell_\beta$ such that $p_i = p_j$, consider

$$\begin{aligned} u_i &= \operatorname{sgn}(\mathcal{W}_c(P^\alpha)) \operatorname{pr}_{\mathcal{D}} \Big(\frac{\partial \mathcal{W}_c^\alpha}{\partial p_i}\Big), \\ u_j &= \operatorname{sgn}(\mathcal{W}_c(P^\beta)) \operatorname{pr}_{\mathcal{D}} \Big(\frac{\partial \mathcal{W}_c^\beta}{\partial p_j}\Big). \end{aligned}$$

Inside intersection. If the intersection is of inside type, see Fig. 7(a), define



(a) inside (b) outside

Fig. 7. $\gamma_{\mathsf{cpc}}^\alpha$ and $\gamma_{\mathsf{cpc}}^\beta$ intersect at a point's location. (a) inside intersection and (b) outside intersection. In both cases, $\gamma_{\mathsf{cpc}}^\alpha$ and $\gamma_{\mathsf{cpc}}^\beta$ can be merged into a new self-intersecting curve γ.

$$v_1 = \begin{cases} u_{[p_{i_-}, p_i]} & \text{if } [p_{j_-}, p_j] \subset H^{\text{in}}_{[p_{i_-}, p_i]}, \\ u_{[p_j, p_{j_-}]} & \text{if } [p_{j_-}, p_j] \not\subset H^{\text{in}}_{[p_{i_-}, p_i]}, \end{cases}$$

$$v_2 = \begin{cases} u_{[p_{i_+}, p_i]} & \text{if } [p_j, p_{j_+}] \subset H^{\text{in}}_{[p_i, p_{i_+}]}, \\ u_{[p_j, p_{j_+}]} & \text{if } [p_j, p_{j_+}] \not\subset H^{\text{in}}_{[p_i, p_{i_+}]}. \end{cases}$$

Then, $(P, \ell) \in D$ iff

(iii) $(P, \ell) \in D_2$ with inside intersection at a point and $u_i - u_j \notin \mathtt{wedge}(p_j, (v_1, v_1^\perp), (v_2, -v_2^\perp))$.

Outside intersection. If the intersection is of outside type, see Fig. 7(b), define

$$v_1 = \begin{cases} u_{[p_j, p_{j_-}]} & \text{if } [p_{j_-}, p_j] \subset H^{\text{in}}_{[p_{i_-}, p_i]}, \\ u_{[p_{i_-}, p_i]} & \text{if } [p_{j_-}, p_j] \not\subset H^{\text{in}}_{[p_{i_-}, p_i]}, \end{cases}$$

$$v_2 = \begin{cases} u_{[p_j, p_{j_+}]} & \text{if } [p_j, p_{j_+}] \subset H^{\text{in}}_{[p_i, p_{i_+}]}, \\ u_{[p_{i_+}, p_i]} & \text{if } [p_j, p_{j_+}] \not\subset H^{\text{in}}_{[p_i, p_{i_+}]}. \end{cases}$$

Then, $(P, \ell) \in D$ iff

(iv) $(P, \ell) \in D_1$ with outside intersection at a point and $u_i - u_j \notin \mathtt{wedge}(p_j, (v_1, -v_1^\perp), (v_2, v_2^\perp))$.

5.2.3 *Jump map*

Here we specify the action of G on $(P, \ell) \in D \cap D_2$ with a single intersection of curves. In this case, the two curves can be merged into a single one, γ, see Figs. 6 and 7, defined by the concatenation of the segments $\{[p_k, p_{k_+}] : k \in \ell_\alpha\} \cup [p_i, p_{j_+}] \cup \{[p_k, p_{k_+}] : k \in \ell_\beta \setminus \{j\}\} \cup [p_j, p_i]$, if $p_i \in]p_j, p_{j_+}[$, and $\{[p_k, p_{k_+}] : k \in \ell_\alpha\} \cup \{[p_k, p_{k_+}] : k \in \ell_\beta\}$, if $p_i = p_j$. If the curve intersection is at an open segment, p_i appears twice in γ. The wombliness of $\gamma_{\mathsf{cpc}}^\alpha$ and $\gamma_{\mathsf{cpc}}^\beta$ is summed up as $\mathcal{W}(\gamma) = \mathcal{W}(\gamma_{\mathsf{cpc}}^\alpha) + \mathcal{W}(\gamma_{\mathsf{cpc}}^\beta)$. Depending on the sign of $\mathcal{W}(\gamma_{\mathsf{cpc}}^\alpha)$ and $\mathcal{W}(\gamma_{\mathsf{cpc}}^\beta)$, we have:

Agent re-positioning. If $\mathcal{W}(\gamma^{\alpha}_{\mathsf{cpc}})$ and $\mathcal{W}(\gamma^{\beta}_{\mathsf{cpc}})$ have different signs, we only keep the curve whose wombliness has the same sign as γ and reposition the agents of the other curve along it arbitrarily. Formally, if $\operatorname{sgn}(\mathcal{W}(\gamma^{\alpha}_{\mathsf{cpc}}))\operatorname{sgn}(\mathcal{W}(\gamma^{\beta}_{\mathsf{cpc}})) \leq 0$, then $G(P,\ell)$ is the set of configurations (P',ℓ') with $|\ell'| = |\ell| - 1$ and

$$\ell'_\eta = \ell_\eta, \text{ for } \eta \in \{1,\dots,|\ell|\} \setminus \{\alpha,\beta\}, \qquad \ell'_\mu = \ell_\alpha \cup \ell_\beta,$$
$$p'_k = p_k, \text{ for } k \notin \ell_{\bar{\mu}}, \qquad p'_k \in \gamma^{\mu}_{\mathsf{cpc}}, \text{ for } k \in \ell_{\bar{\mu}},$$

where $\operatorname{sgn}(\mathcal{W}(\gamma^\mu)) = \operatorname{sgn}(\mathcal{W}(\gamma^{\alpha}_{\mathsf{cpc}}))$ and $\bar{\mu} \in \{\alpha,\beta\} \setminus \{\mu\}$. If both curves have nonzero wombliness, the absolute value of the wombliness of the resulting not self-intersecting curve is strictly larger than the value of the wombliness of γ.

Curve merging. If $\mathcal{W}(\gamma^{\alpha}_{\mathsf{cpc}})$ and $\mathcal{W}(\gamma^{\beta}_{\mathsf{cpc}})$ have the same sign, then we merge the respective curves into γ. If the intersection occurs on an open segment, one additional agent must be added to the network at the intersection location. Formally, if $\operatorname{sgn}(\mathcal{W}(\gamma^{\alpha}_{\mathsf{cpc}}))\operatorname{sgn}(\mathcal{W}(\gamma^{\beta}_{\mathsf{cpc}})) > 0$ and the self-intersection is at a point, $G(P,\ell) = \{(P,\ell')\}$ with $|\ell'| = |\ell| - 1$ and

$$\ell'_\eta = \ell_\eta, \text{ for } \eta \in \{1,\dots,|\ell|\} \setminus \{\alpha,\beta\}, \qquad \ell'_\alpha = \ell_\alpha \cup \ell_\beta.$$

If $\operatorname{sgn}(\mathcal{W}(\gamma^1))\operatorname{sgn}(\mathcal{W}(\gamma^2)) > 0$ and the self-intersection is on an open segment, $G(P,\ell)$ is the set of configurations (P',ℓ') with $|\ell'| = |\ell| - 1$, $m \in \ell_1$, and

$$p'_k = p_k, \text{ for } k \neq m, \qquad p'_m = p_i,$$
$$\ell'_\eta = \ell_\eta, \text{ for } \eta \in \{2,\dots,|\ell|\} \setminus \{\alpha,\beta\},$$
$$\ell'_\alpha = \ell_\alpha \cup \ell_\beta \cup \{m\}, \qquad \ell'_1 = \ell_1 \setminus \{m\}.$$

G may not be outer semicontinuous relative to D at configurations where the wombliness of one of the curves is zero. Such configurations are never attained by the continuous flow of `hybrid wombling algorithm`.

5.2.4 Multiple intersections

Let $(P,\ell) \in D_2$. Having described above the case of a single intersection of curves, we discuss next the case when two or more curves intersect at multiple points. One proceeds by characterizing the intersections that need to be resolved because the flow map f points outside C, in an analogous way to cases (i)-(ii) in Section 5.2.1 and (iii)-(iv) in Section 5.2.2. The action of G can be described along the same lines of Section 5.2.3. The sum of the wombliness of the curves involved in the intersections plays a key role. Agents in curves whose wombliness has the opposite sign as the total sum get re-positioned (this can be done in an arbitrary way). Curves whose wombliness has the same sign as the total sum and are still intersecting after the agent re-positioning are merged together. If the curve resulting from this operation is self-intersecting, then it is handled according to Section 5.1. For reasons of space, we leave to the reader the formal description of the action of the jump map.

Remark 5.1 (Algorithm implementation and distributed character) The implementation of the `hybrid wombling algorithm` by the network requires agents to be able to detect intersections that need resolving (an event of local nature) and to be capable of aggregating the wombliness of the curves to which they belong or are involved in an intersection. This aggregation can be done in a distributed way with a number of algorithms that rely on network connectivity and interaction among neighbors, see e.g., [Lynch, 1997, Bullo et al., 2009]. This capability allows individual agents to be able to determine the type of transition. Executing a given transition is another event of local nature, since the agents involved simply have to change their neighbors to the left and to the right. Finally, the network waits for agents which are repositioning to travel to their new positions before continuing with its evolution. •

6 Convergence analysis

Here, we characterize the convergence properties of a robotic network evolving under the `hybrid wombling algorithm`. In our analysis, we only consider trajectories which are not Zeno. In particular, this excludes the possibility of blocking transitions, i.e., a curve splitting jump resulting in a new configuration that satisfies the conditions for a curve merging; or a curve merging jump resulting in a new configuration that satisfies the conditions for a curve splitting. The wombliness $\mathcal{W}_\Sigma : C \to \mathbb{R}$ associated to a network Σ with state (P,ℓ) is given by

$$\mathcal{W}_\Sigma(P,\ell) = |\mathcal{W}(\gamma^2_{\mathsf{cpc}})| + \cdots + |\mathcal{W}(\gamma^{|\ell|}_{\mathsf{cpc}})|.$$

Since the wombliness is bounded on the set of closed not self-intersecting curves in $\mathcal{D}$, $\mathcal{W}_\Sigma$ is bounded.

Theorem 6.1 *A not Zeno network trajectory that evolves under the `hybrid wombling algorithm` and undergoes a finite number of transitions resulting in agent additions converges to the set of critical configurations of $\mathcal{W}_\Sigma$. Moreover, $\mathcal{W}_\Sigma$ is monotonically increasing along the network trajectory.*

PROOF. Since the network trajectory undergoes a finite number of transitions that result in agent additions, there exists $N \in \mathbb{N}$ with $N \geq n$ such that the network evolution is contained in C. Consider the hybrid system (C, f, D, G). Note that C and D are closed by construction. However, as noted above, the map f is not continuous and the set-valued map G is not outer semicontinuous. Hence, we use their Krasovskii regularizations $\widehat{f}$ and $\widehat{G}$, defined respectively by, see [Sanfelice et al., 2008],

$$\widehat{f}(x) = \cap_{\delta>0}\overline{\operatorname{con}} f((x + \delta B(0,1)) \cap C), \ x \in C, \quad (13a)$$
$$\widehat{G}(x) = \cap_{\delta>0}\overline{G((x + \delta B(0,1)) \cap D)}, \ x \in D. \quad (13b)$$

Here, $B(0,1)$ is the closed ball centered 0 with radius 1, and $\overline{\text{con}}$ is the closed convex hull. Since f and G are locally bounded, so are their regularizations. Moreover, $\widehat{f}$ is outer semicontinuous relative to C and $\widehat{G}$ is outer semicontinuous relative to D, cf. [Sanfelice et al., 2008]. Since $f(x) \in \widehat{f}(x)$ and $G(x) \subset \widehat{G}(x)$, solutions of (C, f, D, G) are solutions of $(C, \widehat{f}, D, \widehat{G})$ too. The effect of the regularization (13) is to extend the set of allowed network evolutions. Specifically, in $(C, \widehat{f}, D, \widehat{G})$, a curve with zero wombliness can flow in either the positive or negative direction of the gradient; if a curve self-intersects and one of the curves has zero wombliness and non-empty interior, then both agent re-positioning and curve splitting are allowed; and similarly, if a curve of zero wombliness intersects another curve, then both agent re-positioning and curve merging are allowed. The bounded function $V = -\mathcal{W}_\Sigma$ is a Lyapunov function for $(C, \widehat{f}, D, \widehat{G})$, i.e., for each fixed $\ell \in \mathfrak{F}(\{1,\dots,N\})$, V is monotonically non-increasing along the continuous evolution of the network, and V is monotonically non-increasing when network transitions take place. Moreover, the network trajectory has the property that all but (at most) a finite of the elapsed times between jumps are bounded below by a positive constant. Formally, this means that there exists $\tau > 0$ such that, for all but a finite number of $j \in \mathbb{N}$,

$$\sup\{|t - t'| : (t,j), (t',j) \in E\} \geq \tau.$$

This fact can be shown by contradiction. Suppose the above statement does not hold, i.e, for $\tau_m = 1/m$, there exists $j_m \in \mathbb{N}$ such that $\sup\{|t - t'| : (t, j_m), (t', j_m) \in E\} < 1/m$. This implies that $\lim_{m\to\infty} \sup\{|t - t'| : (t, j_m), (t', j_m) \in E\} = 0$. Note that a jump corresponds to either (i) an agent re-positioning, (ii) a curve splitting, or (iii) a curve merging. Note that, since the trajectory is not Zeno, the new network configuration after (ii) takes place does not satisfy the requirements to execute (iii), and vice versa. In case (i), once the jump takes place and using the fact that f is upper bounded, one can deduce that there exists $M > 0$ such that the length of the ensuing continuous time interval is lower bounded by $M \min \|p_i - \gamma_{\texttt{cpc}} \setminus (]p_{i-}, p_i[\cup]p_i, p_{i+}[)\|$. In case (ii), once all agent additions have taken place (which, by hypothesis, occurs after a finite number of jumps), the only ensuing jumps that can occur correspond to network configurations with (self-)intersection at a point. In case (ii), this implies that, after splitting, if the curves slide along each other (hence corresponding to a configuration where two curves intersect), this cannot result in a jump (since it cannot correspond to either an agent repositioning because of the signs of the wombliness of each curve or a curve merging because the intersection is at an open segment), and hence a similar bound to case (i) can be found. This bound is also valid if the curves instead move away from each other, given the definition of f. An analogous argument can be made for case (iii). Therefore, in order for $\lim_{m\to\infty} \sup\{|t-t'| : (t, j_m), (t', j_m) \in E\} = 0$ to hold, the inter-agent distances in at least one of the curves should decrease all the way to zero. This contradicts the fact that such configurations can never be reached under the `hybrid wombling algorithm`. The convergence result then follows from the Hybrid Invariance Principle [Sanfelice et al., 2007, Theorem 4.7 and Corollary 5.9], which implies that there exists $r \in \mathbb{R}$ such that the trajectory converges to the largest invariant set contained in $V^{-1}(r) \cap (\mathcal{D}^n \times \mathfrak{F}(\{1,\dots,N\})) \cap \mathcal{S}_1$, with $\mathcal{S}_1 = \{(P,\ell) \in C : \mathrm{pr}_{\mathcal{D}}(\nabla \mathcal{W}_\Sigma(P,\ell)) = 0\}$. □

The statement of the result does not preclude the possibility of the trajectory experiencing an infinite number of transitions. Also, convergence to global optima is not guaranteed. Figs. 8 and 9 present illustrations of the execution of the `hybrid wombling algorithm`.

Remark 6.2 (Resolving situations that lead to multiple instantaneous jumps) Theorem 6.1 can be extended to trajectories with a finite number of blocking transitions. This is because such transitions can be resolved at the cost of adding more agents to the network. We describe why next. First, note that it is sufficient to discuss a blocking transition that takes place at a point, say $q_* = p_i = p_j$. Now, if $\Delta Y(q_*) > 0$, then two new neighbors must be added to i in the ring graph at $q_1 \in]p_{i-}, p_i[$ and $q_2 \in]p_i, p_{i+}[$ such that

$$\begin{aligned} &\Big(\int_{[p_i,q_2]} \frac{\|q_2 - q\|}{\|q_2 - p_i\|} \Delta Y(q) \Big) n_{[p_i,q_2]} dq \\ &= \Big(\int_{[q_1,p_i]} \frac{\|q - q_1\|}{\|p_i - q_1\|} \Delta Y(q) \Big) n_{[q_1,p_i]} dq > 0, \end{aligned}$$

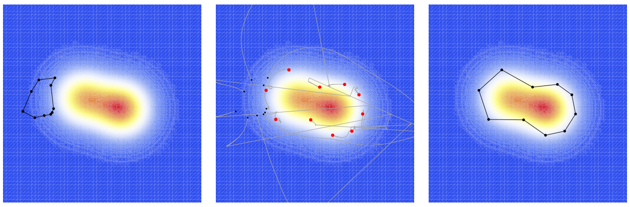
and two new neighbors must be added to j in the ring graph at $q_3 \in]p_{j-}, p_j[$ and $q_4 \in]p_j, p_{j+}[$ such that

$$\begin{aligned} &\Big(\int_{[p_j,q_4]} \frac{\|q_4 - q\|}{\|q_4 - p_j\|} \Delta Y(q) \Big) n_{[p_j,q_4]} dq \\ &= \Big(\int_{[q_3,p_j]} \frac{\|q - q_3\|}{\|p_j - q_3\|} \Delta Y(q) \Big) n_{[q_3,p_j]} dq > 0. \end{aligned}$$

Because of the geometry of the intersecting curves, it is not difficult to show that the resulting network only satisfies the requirements for a curve merging and not for a curve splitting, and hence the transition becomes non-blocking. A similar procedure can be done if $\Delta Y(q_*) < 0$ to yield a network that only satisfies the conditions for a curve splitting and not for a curve merging. •

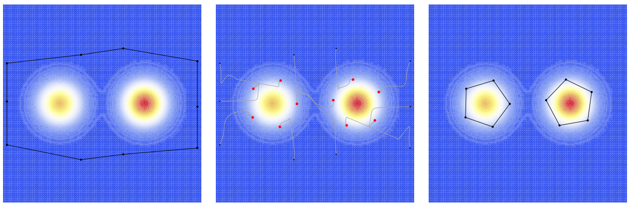
7 Conclusions

We have proposed the `hybrid wombling algorithm` for robotic sensor networks that seek to detect areas of rapid change of a spatial phenomena. Our algorithm design combines notions from statistical estimation and computer vision with tools from hybrid systems. The strategy allows for network splitting and re-grouping, and is guaranteed to monotonically increase the wombliness of the overall ensemble. In future work we plan to study



(a) (b) (c)

Fig. 8. Execution of the **`hybrid wombling algorithm`** with 10 agents in $\mathcal{D} = [-3, 3] \times [-3, 3]$. (a) initial configuration, (b) robot trajectories, and (c) final configuration. The field is $Y(x_1, x_2) = 1.25e^{-(x_1+.75)^2-(x_2-.2)^2} + 1.75e^{-(x_1-.75)^2-(x_2+.2)^2}$. Along the evolution, 1 outside self-intersection and then 2 inside self-intersections are triggered, with all transitions resulting in agent re-positionings.



(a) (b) (c)

Fig. 9. Execution of the **`hybrid wombling algorithm`** with 10 agents in $\mathcal{D} = [-4, 4] \times [-4, 4]$. (a) initial configuration, (b) robot trajectories, and (c) final configuration. The field is $Y(x_1, x_2) = e^{-(x_1+2)^2-x_2^2} + 1.25e^{-(x_1-2)^2-x_2^2}$. Along the evolution, an inside self-intersection is triggered that results in a curve splitting. After this, each new curve undergoes an inside self-intersection resulting in agent re-positionings.

extensions to three dimensions and consider stochastic scenarios, where agents take point measurements of the field and build estimates of its gradient and Laplacian.

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