# Distributed wombling by robotic sensor networks

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**Abstract.** This paper proposes a distributed coordination algorithm for robotic sensor networks to detect boundaries that separate areas of abrupt change of spatial phenomena. We consider an aggregate objective function, termed wombliness, that measures the change of the spatial field along the closed polygonal curve defined by the location of the sensors in the environment. We encode the network task as the optimization of the wombliness and characterize the smoothness properties of the objective function. In general, the complexity of the spatial phenomena makes the gradient flow cause self-intersections in the polygonal curve described by the network. Therefore, we design a distributed coordination algorithm that allows for network splitting and merging while guaranteeing the monotonic evolution of wombliness. The technical approach combines ideas from statistical estimation, dynamical systems, and hybrid modeling and design.

## 1 Introduction

Consider a network of mobile sensors moving in an environment with the objective of finding regions where large changes occur in a spatial phenomena of interest. Our aim is to design a distributed coordination algorithm that allows the group of sensors to determine boundaries that separate the areas with large differences in the spatial phenomena. The determination of such boundaries is relevant in multiple applications of robotic networks, including oceanographic surveys and weather forecasting. As an example, scientists are interested in determining regions of abrupt change in temperature fields over regions of the ocean, as they are related to upwelling and the food habits of fish.

The present work has connections with several scientific domains. In statistical estimation [1, 2], wombling boundaries are curves that delimit areas of rapid change of some scientific phenomena of interest. Algorithms for detecting these boundaries based on point-referenced data are widely used for various applications, including biology [3], computational ecology [1], and medicine [4]. In computer vision [5, 6], image segmentation and edge detection problems are encoded as optimization problems for a variety of objective functionals such as alignment, contrast, and geodesic active contour. These optimization problems are typically solved using PDE-based approaches that build on the variational information about the functionals. Finally, this work uses classical modeling and stabilization tools from hybrid systems theory [7–10] in the algorithm design.

The contributions of the paper are the following. We model the spatial phenomena as a deterministic spatial field. The wombliness of a non self-intersecting, closed curve is a measure of the alignment of the gradient of the spatial field along the normal direction to the curve. We use the notion of wombliness associated to a closed polygonal curve to formulate the network objective as a distributed optimization problem. We study the smoothness properties of the wombliness measure and provide an explicit expression for its gradient and a characterization of its critical points. If the network were to follow a gradient ascent law to optimize wombliness, then situations may arise where the polygonal curve described by the group of sensors becomes self-intersecting and the ensuing flow ill-posed. To prevent this from happening, we combine our analysis results with ideas from hybrid control design to synthesize a coordination algorithm for distributed wombliness optimization. The algorithm introduces the possibility of splitting and merging curves, and is guaranteed to monotonically optimize the wombliness measure associated to the network. Several simulations illustrate the results. For reasons of space, all proofs are omitted.

## 2 Preliminaries

Here, we gather some basic notions that will be frequently used along the paper. Let us start with some notation. We let unit : $\mathbb{R}^2 \to \mathbb{R}^2$ denote the map defined by $\mathrm{unit}(x) = x/\|x\|$ for $x \neq 0$ and $\mathrm{unit}(0) = 0$. Given $n \in \mathbb{Z}_{>0}$ and $i, j \leq n$, let $\langle i, \dots, j\rangle$ be the set defined by $\langle i, \dots, j\rangle = \{i, \dots, j\}$ if $i \leq j$ and $\langle i, \dots, j\rangle = \{i, \dots, n, 1, \dots, j\}$ if $i > j$. Next, we introduce some useful geometric concepts.

### 2.1 Planar geometric notions

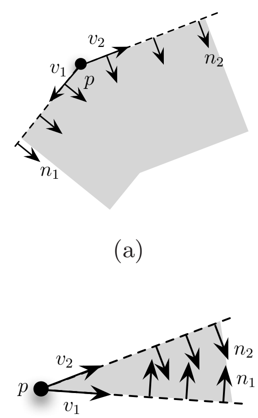
Given a vector $v = (v_1, v_2) \in \mathbb{R}^2$, we denote by $v^\perp = (v_2, -v_1) \in \mathbb{R}^2$ the vector perpendicular to $v$ to the right, i.e., the 90 degree clockwise rotation of $v$. Given $p \neq q \in \mathbb{R}^2$, let $]p, q[$ and $[p, q]$ denote, respectively, the open and closed segments with end points $p$ and $q$. We let $[p, q[$ denote the closed segment between $p$ and $q$ with the end point $q$ excluded. We let $u_{[p,q]} = \mathrm{unit}(q - p)$ denote the unit vector in the direction from $p$ to $q$ and $n_{[p,q]} = u_{[p,q]}^\perp$ the unit normal vector to the right. In coordinates, if $p = (p_1, p_2)$ and $q = (q_1, q_2)$, then

$$u_{[p,q]} = \frac{1}{\|q - p\|}(q_1 - p_1, q_2 - p_2), \quad n_{[p,q]} = \frac{1}{\|q - p\|}(q_2 - p_2, p_1 - q_1).$$

We denote by $H^{\mathrm{out}}_{[p,q]} = \{z \in \mathbb{R}^2 \mid (z - p)^T n_{[p,q]} \geq 0\}$ the halfplane of points in the positive direction of the normal vector with respect to the closed segment $[p, q]$. Likewise, we denote $H^{\mathrm{in}}_{[p,q]} = \{z \in \mathbb{R}^2 \mid (z - p)^T n_{[p,q]} \leq 0\}$.

Given $p \in \mathbb{R}^2$ and $v \in \mathbb{R}^2$, we use the notation $\mathtt{ray}(p, v) = \{z \in \mathbb{R}^2 \mid z = p + tv,\ t \in \mathbb{R}_{\geq 0}\}$. The wedge $\mathtt{wedge}(p, (v_1, n_1), (v_2, n_2))$ is the cone with vertex $p$ and axes $\mathtt{ray}(p, v_1)$ and $\mathtt{ray}(p, v_2)$. The interior of $\mathtt{wedge}(p, (v_1, n_1), (v_2, n_2))$ is the set of points towards which $n_1$ points along $\mathtt{ray}(p, v_1)$ and $n_2$ points along $\mathtt{ray}(p, v_2)$, see Figure 1 for an illustration. For the wedge to be well-defined, the normal vectors $n_1$ and $n_2$ need to specify the interior uniquely.

A domain $\mathcal{D} \subset \mathbb{R}^2$ is an open and simply connected set. Given $q \in \mathcal{D}$, let $T_q\mathcal{D}$ denote the set of all vectors tangent to $\mathcal{D}$ with origin at $q$. For $q \in \text{int}(\mathcal{D})$, $T_q\mathcal{D}$ is 2-dimensional and can be identified with $\mathbb{R}^2$. However, for $q \in \partial\mathcal{D}$, $T_q\mathcal{D}$ is one-dimensional and can be identified with $\mathbb{R}$. Let $T\mathcal{D}$ denote the collection $\cup\{T_q\mathcal{D} \mid q \in \mathcal{D}\}$ of all tangent vectors to $\mathcal{D}$. We let $\text{pr}_{T\mathcal{D}} : T_{\mathcal{D}}\mathbb{R}^2 \to T\mathcal{D}$ assign to each vector in $\mathbb{R}^2$ with origin at $q \in \mathcal{D}$ the orthogonal projection onto $T_q\mathcal{D}$. Any vector $v \in \mathbb{R}^2$ with origin in $\mathcal{D}$ has $\text{pr}_{T\mathcal{D}}(v) = v$.

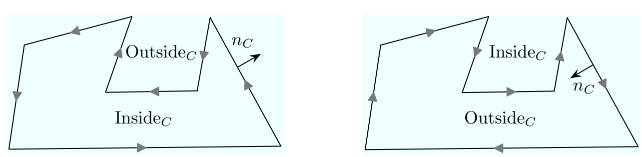


(b)

**Fig. 1.** Wedge determined by the point $p$ and the pairs of vectors $(v_1.n_1)$ and $(v_2, n_2)$.

### 2.2 Curve parameterizations

A *curve* $C$ in $\mathbb{R}^2$ is the image of a map $\gamma : [a, b] \to \mathbb{R}^2$. The map $\gamma$ is called a *parametrization of* $C$. We often identify a curve with its parametrization. A curve $C$ is *self-intersecting* if $\gamma$ is not injective on $(a, b)$. A curve $C$ is *closed* if $\gamma(a) = \gamma(b)$. For a closed curve $C$, we let $n_C = \text{unit}(\dot{\gamma})^\perp$ denote the unit normal vector to $C$. A closed, not self-intersecting curve $C$ partitions $\mathbb{R}^2$ into two disjoint open and connected sets, $\text{Inside}_C$ and $\text{Outside}_C$, such that $n_C$ along $C$ points outside $\text{Inside}_C$ and inside $\text{Outside}_C$, respectively. The orientation of $C$ affects the definition of $n_C$ and $\text{Inside}_C$, $\text{Outside}_C$, see Figure 2 for an illustration.



(a) (b)

**Fig. 2.** Closed curve oriented in a (a) counterclockwise and (b) clockwise fashion.

Given a curve $C$ parametrized by a piecewise smooth map $\gamma : [a, b] \to C$, the line integral of a function $f : C \subset \mathbb{R}^2 \to \mathbb{R}$ over $C$ is defined by

$$\int_C f = \int_C f(q)dq = \int_a^b f(\gamma(t)) \|\dot{\gamma}(t)\| \, dt, \tag{1}$$

and it is independent of the selected parametrization.

## 3 Problem statement

Let $Y : \mathbb{R}^2 \to \mathbb{R}$ be a twice continuously differentiable function modeling a planar spatial field. Consider a network of $n$ mobile agents moving in a compact domain

$\mathcal{D} \subset \mathbb{R}^2$ with positions $p_1, \dots, p_n$. Our objective is to find regions in $\mathcal{D}$ where large changes occur in the spatial field $Y$ by determining their boundaries.

Let us start by defining a measure of how fast the field changes along a given curve. Let $C$ be a non self-intersecting curve in $\mathbb{R}^2$ and define the *wombliness* or *alignment* of $C$ by

$$\mathcal{H}(C) = \int_C \langle \nabla Y, n_C \rangle, \tag{2}$$

see e.g., [11, 2]. The interpretation of the wombliness measure is as follows. At each point of the curve, we look at how much $Y$ is changing along the normal direction to $C$ (i.e., how much $Y$ is "flowing through $C$"). The integral sums this change throughout the curve. We are interested in using the robotic network to find curves whose corresponding value of $\mathcal{H}$ is large.

For a closed non-self-intersecting curve, the wombling measure $\mathcal{H}$ can be rewritten, using the Gauss Divergence Theorem [12], as

$$\mathcal{H}(C) = \int_C \langle \nabla Y, n_C \rangle = \int_D \operatorname{div} \nabla Y = \int_D \Delta Y, \tag{3}$$

where $D$ is the set in $\mathbb{R}^2$ whose boundary is $C$, and $\Delta Y = \frac{\partial^2 Y}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2}$ denotes the Laplacian of $Y$. It is interesting to observe that, in general, that the level curves of the spatial field are not optimizers of $\mathcal{H}$.

In general, the optimization of (2) is an infinite-dimensional problem. Our approach here is to order counterclockwise the agents according to their unique identifier, and consider the closed polygonal curve that result from joining the positions of consecutive robots. In general, such curves may be self-intersecting. Therefore, we restrict our attention to the subset $\mathcal{S}_c$ of $\mathcal{D}^n$ defined as follows. For $(p_1, \dots, p_n) \in \mathcal{D}^n$, let $\gamma_{\mathtt{cpc}}$ be the closed polygonal curve that results from the concatenation of the straight segments $[p_i, p_{i+1}]$, $i \in \{1, \dots, n-1\}$ and $[p_n, p_1]$. Then, we define the following open subset of $\mathcal{D}^n$,

$$\mathcal{S}_c = \{(p_1, \dots, p_n) \in \mathcal{D}^n \mid \gamma_{\mathtt{cpc}} \text{ is non-self-intersecting}\}.$$

Define the function $\mathcal{H}_c : \mathcal{S}_c \to \mathbb{R}$ by

$$\mathcal{H}_c(p_1, \dots, p_n) = \mathcal{H}(\gamma_{\mathtt{cpc}}) = \sum_{i=1}^{n} \int_{[p_i, p_{i+1}]} \langle \nabla Y, n_{[p_i, p_{i+1}]} \rangle. \tag{4}$$

The optimization of (4) is now a finite-dimensional problem. Note that $\mathcal{H}_c$ can be expressed in terms of the polygon determined by the concatenated straight segments. If $\mathcal{P}(p_1, \dots, p_n)$ denotes this polygon, then we have

$$\mathcal{H}_c(p_1, \dots, p_n) = \int_{\mathcal{P}(p_1, \dots, p_n)} \Delta Y. \tag{5}$$

For reasons that will become clear in the following sections, we assume that, at each network configuration, agent $i \in \{1, \dots, n\}$ can measure the gradient $\nabla Y$ and the Laplacian $\Delta Y$ along the segments $[p_{i-1}, p_i]$ and $[p_i, p_{i+1}]$.

## 4 Smoothness analysis of the wombliness measure

In this section, we analyze the smoothness properties of the wombliness measure, provide explicit expressions for the gradient, and characterize the critical points. We start by stating the expression of the partial derivative of $\mathcal{H}_c$.

**Proposition 1 (Gradient of $\mathcal{H}_c$).** *The function $\mathcal{H}_c : \mathcal{S}_c \to \mathbb{R}$ is continuously differentiable. For each $i \in \{1, \dots, n\}$, the partial derivative of $\mathcal{H}_c$ with respect to $p_i$ at $(p_1, \dots, p_n) \in \mathcal{S}_c$ is*

$$\frac{\partial \mathcal{H}_c}{\partial p_i} = \Big( \int_{[p_i, p_{i+1}]} \frac{\|p_{i+1} - q\|}{\|p_{i+1} - p_i\|} \Delta Y \Big) n_{[p_i, p_{i+1}]} + \Big( \int_{[p_{i-1}, p_i]} \frac{\|q - p_{i-1}\|}{\|p_i - p_{i-1}\|} \Delta Y \Big) n_{[p_{i-1}, p_i]}.$$

The proposition above implies in particular that the gradient of $\mathcal{H}_c$ is distributed over the ring graph: in other words, an agent $i$ only needs to know about the location of its neighbors in the ring graph (agents $i-1$ and $i+1$) in order to be able to compute $\partial \mathcal{H}_c p_i$.

Using Proposition 1, we can characterize the critical configurations of $\mathcal{H}_c$.

**Corollary 2 (Critical points of $\mathcal{H}_c$).** *With a slight abuse of notation, let $\mathcal{H}_c : \overline{\mathcal{S}_c} \to \mathbb{R}$ denote the extension by continuity of $\mathcal{H}_c$ to $\overline{\mathcal{S}_c}$. Let $(p_1, \dots, p_n) \in \mathcal{S}_c$ be a critical configuration of $\mathcal{H}_c$. Then, for $i \in \{1, \dots, n\}$,*

$$\mathrm{pr}_{T\mathcal{D}} \Big( \frac{\partial \mathcal{H}_c}{\partial p_i} \Big) = 0.$$

*Moreover, if $(p_1, \dots, p_n) \in \mathrm{int}\,(\mathcal{D}^n)$ and no three consecutive agents are aligned, this characterization can be alternatively described by, for $i \in \{1, \dots, n\}$,*

$$\int_{[p_i, p_{i+1}]} \|p_{i+1} - q\| \Delta Y = 0, \qquad \int_{[p_i, p_{i+1}]} \|q - p_i\| \Delta Y = 0. \tag{6}$$

*Remark 3 (Characterization of critical points of $\mathcal{H}_c$).* The characterization (6) of the critical configurations of $\mathcal{H}_c$ in the interior of $\mathcal{D}$ has the following interpretation. For each $i \in \{1, \dots, n\}$, define the map $G_i : [p_i, p_{i+1}] \to \mathbb{R}$ by

$$z \mapsto G_i(z) = \int_{[p_i, z]} \Delta Y.$$

Note that $G(p_i) = 0$ by definition. Moreover, after some manipulations, one can show that equations (6) are equivalent to

$$G_i(p_{i+1}) = 0, \qquad \int_{[p_i, p_{i+1}]} G_i(z) dz = 0. \tag{7}$$

Using the fact that $\Delta Y = \mathrm{div}(\nabla Y)$, we can interpret the first equation in (7) as follows: on a critical configuration, there is no net average change of the gradient $\nabla Y$ along the segment $[p_i, p_{i+1}]$. However, even if this condition holds true, $\nabla Y$ might exhibit a preferred orientation with respect to $[p_i, p_{i+1}]$. It is precisely the second equation in (7) that takes care of ensuring that there is no bias in the orientation of $\nabla Y$ with respect to $[p_i, p_{i+1}]$. •

## 5 Distributed hybrid design for wombliness optimization

Our approach to find boundaries that delimit areas where the spatial field changes abruptly consists of starting with an initial network configuration and optimizing the magnitude of the wombliness of the closed polygonal boundary defined by the network. To maximize $\mathcal{H}_c$, we implement the distributed gradient flow of this function, cf. Proposition 1, that is,

$$\dot{p}_i = \operatorname{sgn}(\mathcal{H}_c(P_0)) \operatorname{pr}_{T\mathcal{D}} \Big( \frac{\partial \mathcal{H}_c}{\partial p_i} \Big), \quad i \in \{1, \dots, n\}. \tag{8}$$

However, in general, the set $\overline{\mathcal{S}_c}$ is not invariant under (8). In other words, evolutions under (8) of the closed polygonal curve $\gamma_{\mathsf{cpc}}$ defined by the points $p_1, \dots, p_n$ become self-intersecting. To address this problem, we propose the following switching design, which is inspired on the interplay between the geometry of the polygonal curve $\gamma_{\mathsf{cpc}}$ and the value of the wombliness function $\mathcal{H}_c$.

### 5.1 Curve self-intersection

Let $\gamma_{\mathsf{cpc}}$ be the closed polygonal curve defined by the segments $\{[p_i, p_{i+1}] \mid i \in \{1, \dots, n-1\}\} \cup [p_n, p_1]$. Assume $(p_1, \dots, p_n) \in \overline{\mathcal{S}_c}$, i.e., the curve $\gamma_{\mathsf{cpc}}$ is self-intersecting. Note that when a self-intersection occurs, either $\text{Inside}_{\gamma_{\mathsf{cpc}}}$ becomes disconnected or $\text{Outside}_{\gamma_{\mathsf{cpc}}}$ becomes disconnected. We refer to these two cases as *inside* and *outside* self-intersections, respectively. Figure 3 presents an illustration. We further distinguish between whether the self-intersection occurs at an open segment or at a point's location.

**Self-intersection at an open segment.** For each $i \neq j \in \{1, \dots, n\}$ such that $p_i \in ]p_j, p_{j+1}[$, define $\lambda \in [0, 1)$ by $p_i = (1-\lambda)p_j + \lambda p_{j+1}$ and consider

$$v_i = (1-\lambda)u_j + \lambda u_{j+1}, \quad u_k = \operatorname{sgn}(\mathcal{H}_c(P)) \operatorname{pr}_{T\mathcal{D}} \Big( \frac{\partial \mathcal{H}_c}{\partial p_k} \Big),$$

where $k \in \{i, j, j+1\}$. The guards depend upon the type of self-intersection.

*Inside self-intersection.* If the self-intersection is of inside type, it is because the segment $[p_i, p_{i+1}]$ belongs to $H^{\text{in}}_{[p_j, p_{j+1}]}$ and there exists the possibility of $p_i$ crossing from $H^{\text{in}}_{[p_j, p_{j+1}]}$ to $H^{\text{out}}_{[p_j, p_{j+1}]}$, see Figure 3(a). The criterium to identify if a transition is needed in the network configuration is as follows. If

$$(u_i - v_i)^T n_{[p_j, p_{j+1}]} \le 0,$$

then $p_i$ does not cross, and the curve stays in $\overline{\mathcal{S}_c}$. If

$$(u_i - v_i)^T n_{[p_j, p_{j+1}]} > 0,$$

then the curve will move into $\mathcal{D}^n \setminus \overline{\mathcal{S}_c}$ unless the self-intersection is resolved.

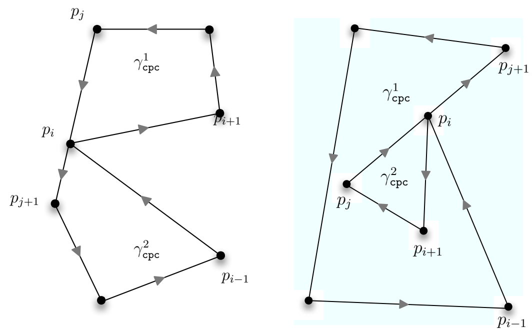
*Outside self-intersection.* If the self-intersection is of outside type, it is because the segment $[p_i, p_{i+1}]$ belongs to $H^{\text{out}}_{[p_j,p_{j+1}]}$ and there exists the possibility of $p_i$ crossing from $H^{\text{out}}_{[p_j,p_{j+1}]}$ to $H^{\text{in}}_{[p_j,p_{j+1}]}$, see Figure 3(b). The criterium to identify if a transition is needed in the network configuration is as follows. If

$$(u_i - v_i)^T n_{[p_j,p_{j+1}]} \geq 0,$$

then $p_i$ does not cross, and the curve stays in $\overline{\mathcal{S}_c}$. If

$$(u_i - v_i)^T n_{[p_j,p_{j+1}]} < 0,$$

the curve will move into $\mathcal{D}^n \setminus \overline{\mathcal{S}_c}$ unless the self-intersection is resolved.




(a) inside (b) outside

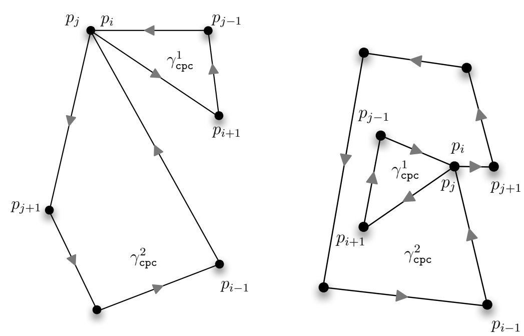
**Fig. 3.** The curve $\gamma_{\mathtt{cpc}}$ defined by $p_1, \ldots, p_n$ is self-intersecting at an open segment. (a) shows an inside self-intersection and (b) shows an outside self-intersection. In both cases, $\gamma_{\mathtt{cpc}}$ can be decomposed into two non-self-intersecting curves $\gamma^1_{\mathtt{cpc}}$ and $\gamma^2_{\mathtt{cpc}}$.

**Self-intersection at a point.** For each $i \neq j \in \{1, \ldots, n\}$ such that $p_i = p_j$, consider the vectors

$$u_i = \operatorname{sgn}(\mathcal{H}_c(P)) \frac{\partial \mathcal{H}_c}{\partial p_i},$$
$$u_j = \operatorname{sgn}(\mathcal{H}_c(P)) \operatorname{pr}_{T\mathcal{D}} \left( \frac{\partial \mathcal{H}_c}{\partial p_j} \right).$$

The guards depend upon the type of self-intersection.

*Inside self-intersection.* If the self-intersection is of inside type, see Figure 4(a), define the vectors




(a) inside (b) outside

**Fig. 4.** The curve $\gamma_{\mathtt{cpc}}$ defined by $p_1, \ldots, p_n$ is self-intersecting at a point's location. (a) shows an inside self-intersection and (b) shows an outside self-intersection. In both cases, $\gamma_{\mathtt{cpc}}$ can be decomposed into two non-self-intersecting curves $\gamma^1_{\mathtt{cpc}}$ and $\gamma^2_{\mathtt{cpc}}$.

$$v_1 = \begin{cases} u_{[p_{i-1},p_i]} & \text{if } [p_{j-1}, p_j] \subset H^{\text{in}}_{[p_{i-1},p_i]}, \\ u_{[p_j,p_{j-1}]} & \text{if } [p_{j-1}, p_j] \not\subset H^{\text{in}}_{[p_{i-1},p_i]}, \end{cases}$$
$$v_2 = \begin{cases} u_{[p_{i+1},p_i]} & \text{if } [p_j, p_{j+1}] \subset H^{\text{in}}_{[p_i,p_{i+1}]}, \\ u_{[p_j,p_{j+1}]} & \text{if } [p_j, p_{j+1}] \not\subset H^{\text{in}}_{[p_i,p_{i+1}]}. \end{cases}$$

The criterium to identify if a transition is needed in the network configuration is as follows. If

$$u_i - u_j \in \mathtt{wedge}(p_j, (v_1, v_1^{\perp}), (v_2, -v_2^{\perp})),$$

then the relative motion of $p_i$ and $p_j$ is such that the curve stays in $\overline{\mathcal{S}_c}$. If

$$u_i - u_j \notin \mathtt{wedge}(p_j, (v_1, v_1^{\perp}), (v_2, -v_2^{\perp})),$$

then the curve will move into $\mathcal{D}^n \setminus \overline{\mathcal{S}_c}$ unless the self-intersection is resolved.

*Outside self-intersection.* : If the self-intersection is of outside type, see Figure 4(b), define the vectors

$$v_1 = \begin{cases} u_{[p_j, p_{j-1}]} & \text{if } [p_{j-1}, p_j] \subset H^{\text{in}}_{[p_{i-1}, p_i]}, \\ u_{[p_{i-1}, p_i]} & \text{if } [p_{j-1}, p_j] \not\subset H^{\text{in}}_{[p_{i-1}, p_i]}, \end{cases}$$

$$v_2 = \begin{cases} u_{[p_j, p_{j+1}]} & \text{if } [p_j, p_{j+1}] \subset H^{\text{in}}_{[p_i, p_{i+1}]}, \\ u_{[p_{i+1}, p_i]} & \text{if } [p_j, p_{j+1}] \not\subset H^{\text{in}}_{[p_i, p_{i+1}]}. \end{cases}$$

The criterium to identify if a transition is needed in the network configuration is as follows. If

$$u_i - u_j \in \mathtt{wedge}(p_j, (v_1, -v_1^{\perp}), (v_2, v_2^{\perp})),$$

then the relative motion of $p_i$ and $p_j$ is such that the curve stays in $\overline{\mathcal{S}_c}$. If

$$u_i - u_j \notin \mathtt{wedge}(p_j, (v_1, -v_1^{\perp}), (v_2, v_2^{\perp})),$$

then the curve will move into $\mathcal{D}^n \setminus \overline{\mathcal{S}_c}$ unless the self-intersection is resolved.

**State transition.** We have encountered above the need to deal with self-intersections in $\gamma_{\mathsf{cpc}}$ to prevent it from stepping into $\mathcal{D}^n \setminus \overline{\mathcal{S}_c}$. Next, we deal with these situations. For simplicity, we begin by considering the case where there is only one agent causing the self-intersection. If this is the case, then $\gamma_{\mathsf{cpc}}$ can be decomposed into two polygonal curves $\gamma^1_{\mathsf{cpc}}$ and $\gamma^2_{\mathsf{cpc}}$, see Figures 3 and 4. The curve $\gamma^1_{\mathsf{cpc}}$ is defined by the concatenation of the segments $\{[p_k, p_{k+1}] \mid k \in \langle i, \dots, j-1\rangle\} \cup [p_j, p_i]$, if $p_i \in ]p_j, p_{j+1}[$, and $\{[p_k, p_{k+1}] \mid k \in \langle i+1, \dots, j-1\rangle\} \cup [p_j, p_{i+1}]$, if $p_i = p_j$. The curve $\gamma^2_{\mathsf{cpc}}$ is defined in an analogous way as the concatenation of the segments $\{[p_k, p_{k+1}] \mid k \in \langle j+1, \dots, i-1\rangle\} \cup [p_i, p_{j+1}]$, if $p_i \in ]p_j, p_{j+1}[$, and $\{[p_k, p_{k+1}] \mid k \in \langle j+1, \dots, i-1\rangle\} \cup [p_i, p_{j+1}]$, if $p_i = p_j$. Observe that $\gamma^2_{\mathsf{cpc}}$ might not be oriented in a counterclockwise fashion. Moreover, if we are dealing with a self-intersection at an open segment, i.e., $p_i$ belongs to $]p_j, p_{j+1}[$, note that $p_i$ appears in the definition of both $\gamma^1_{\mathsf{cpc}}$ and $\gamma^2_{\mathsf{cpc}}$. The wombliness of $\gamma_{\mathsf{cpc}}$ is split between $\gamma^1_{\mathsf{cpc}}$ and $\gamma^2_{\mathsf{cpc}}$ according to

$$\mathcal{H}(\gamma_{\mathsf{cpc}}) = \mathcal{H}(\gamma^1_{\mathsf{cpc}}) + \mathcal{H}(\gamma^2_{\mathsf{cpc}}).$$

We are now ready to detail the two possible outcomes if a self-intersection occurs:

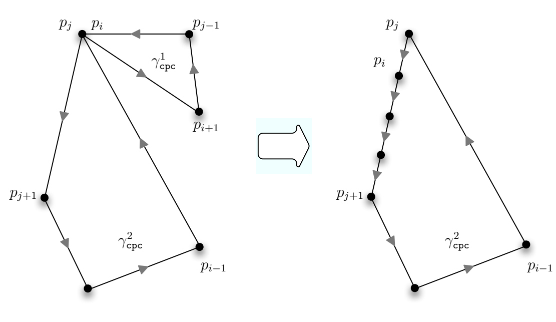



**Fig. 5.** Agent re-positioning. Agents in the curve $\gamma^1_{\mathsf{cpc}}$ get re-positioned onto the curve $\gamma^2_{\mathsf{cpc}}$.

*Agent re-positioning.* If $\mathcal{H}(\gamma^1_{\mathsf{cpc}})$ and $\mathcal{H}(\gamma^2_{\mathsf{cpc}})$ have different signs, we only keep the curve whose wombliness has the same sign as $\gamma_{\mathsf{cpc}}$. Without loss of generality, assume the curve we keep is $\gamma^2_{\mathsf{cpc}}$. Then, we re-position the agents in $\gamma^1_{\mathsf{cpc}}$ along the boundary of $\gamma^2_{\mathsf{cpc}}$. This process does not affect the value of the wombliness of $\gamma^2_{\mathsf{cpc}}$, and can be made in an arbitrary way. Note that the absolute value of the wombliness of the resulting non-self-intersecting curve is strictly larger than the value of the wombliness of the original self-intersecting curve $\gamma_{\mathsf{cpc}}$. This transition is illustrated in Figure 5.

*Curve splitting.* If $\mathcal{H}(\gamma^1_{\mathsf{cpc}})$ and $\mathcal{H}(\gamma^2_{\mathsf{cpc}})$ have the same sign as $\mathcal{H}(\gamma_{\mathsf{cpc}})$, then choosing only one curve would lead to a decrease in the value of the wombliness. Therefore, we consider both. If the self-intersection occurs on an open segment, we need to add one more agent to the network at the intersection location, according to the definition of $\gamma^1_{\mathsf{cpc}}$ and $\gamma^2_{\mathsf{cpc}}$ above. After the split, each curve evolves independently according to (8). This transition is illustrated in Figure 6.

If multiple self-intersections occur at different locations, then the state transitions corresponding to each one of them can be executed simultaneously. If multiple self-intersections occur at the same location, then the curve $\gamma_{\mathsf{cpc}}$ can be decomposed into 3 or more non self-intersecting curves, and the state transition as described above can be conveniently modified to jointly consider the wombliness of each individual curve.

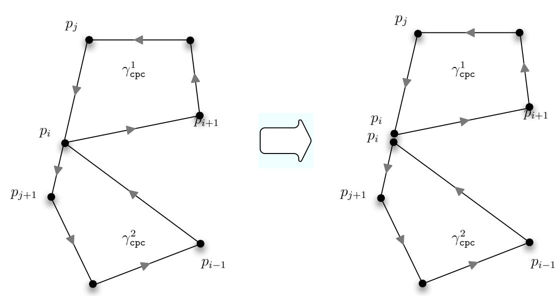



**Fig. 6.** Curve splitting. The curve $\gamma_{\mathsf{cpc}}$ is split into $\gamma^1_{\mathsf{cpc}}$ and $\gamma^2_{\mathsf{cpc}}$, and these curves evolve independently afterwards.

### 5.2 Intersection between curves

As a result of the curve splitting transition described in Section 5.1, there might be more than one curve moving in $\mathcal{D}$. It is therefore conceivable that along the ensuing evolution two of these curves intersect each other. Let us consider this situation. For simplicity, we only treat the case where there are two curves evolving in $\mathcal{D}$. The case with more than two curves can be treated in an analogous way. Let $\gamma^{\alpha}_{\mathsf{cpc}}$ be a closed polygonal curve determined by $n_1$ agents at positions $P^{\alpha} = (p^{\alpha}_1, \dots, p^{\alpha}_{n_1})$ and wombliness $\mathcal{H}^{\alpha}_c(P^{\alpha}) = \mathcal{H}(\gamma^{\alpha}_{\mathsf{cpc}})$, and let $\gamma^{\beta}_{\mathsf{cpc}}$ be a closed polygonal curve determined by $n_2$ agents at positions $P^{\beta} = (p^{\beta}_1, \dots, p^{\beta}_{n_2})$ and wombliness $\mathcal{H}^{\beta}_c(P^{\beta}) = \mathcal{H}(\gamma^{\beta}_{\mathsf{cpc}})$. Note that the orientation of the curves is not necessarily counterclockwise. When a intersection occurs between the two curves, either $\text{Inside}_{\gamma^{\alpha}_{\mathsf{cpc}}} \cap \text{Inside}_{\gamma^{\beta}_{\mathsf{cpc}}}$ is connected or $\text{Outside}_{\gamma^{\alpha}_{\mathsf{cpc}}} \cap \text{Outside}_{\gamma^{\beta}_{\mathsf{cpc}}}$ is connected.

We refer to these two cases as *inside* and *outside* intersections, respectively. Figure 7 presents an illustration of these notions.

We further distinguish between whether the intersection occurs at an open segment or at a point's location.

**Intersection at an open segment.** For each $i \in \{1, \ldots, n_1\}$ such that $p_i^\alpha \in ]p_j^\beta, p_{j+1}^\beta[$ for some $j \in \{1, \ldots, n_2\}$, define $\lambda \in [0,1)$ by $p_i^\alpha = (1-\lambda)p_j^\beta + \lambda p_{j+1}^\beta$ and consider the vectors

$$v_i = (1-\lambda)u_j + \lambda u_{j+1},$$
$$u_i = \operatorname{sgn}(\mathcal{H}_c^\alpha(P^\alpha))\,\mathrm{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}_c^\alpha}{\partial p_i^\alpha}\right), \quad u_k = \operatorname{sgn}(\mathcal{H}_c^\beta(P^\beta))\,\mathrm{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}_c^\beta}{\partial p_k^\beta}\right),$$

where $k \in \{j, j+1\}$. The guards depend upon the type of self-intersection.

*Inside intersection.* If the intersection is of inside type, it is because the segment $[p_i^\alpha, p_{i+1}^\alpha]$ belongs to $H^{\text{in}}_{[p_j^\beta, p_{j+1}^\beta]}$ and there exists the possibility of $p_i^\alpha$ crossing from $H^{\text{in}}_{[p_j^\beta, p_{j+1}^\beta]}$ to $H^{\text{out}}_{[p_j^\beta, p_{j+1}^\beta]}$, see Figure 7(a). The criterium to identify if a transition is needed in the network configuration is as follows. If

$$(u_i - v_i)^T n_{[p_j^\beta, p_{j+1}^\beta]} \leq 0,$$

then $p_i^\alpha$ does not cross. If

$$(u_i - v_i)^T n_{[p_j^\beta, p_{j+1}^\beta]} > 0,$$

then $p_i^\alpha$ will cross unless the intersection is resolved.

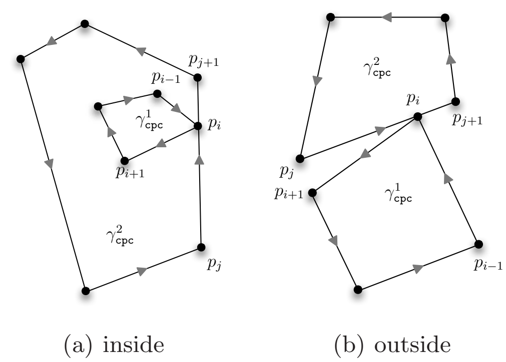



**Fig. 7.** The curves $\gamma^\alpha_{\text{cpc}}$ and $\gamma^\beta_{\text{cpc}}$ intersect at an open segment. (a) shows an inside intersection and (b) shows an outside intersection. In both cases, the curves $\gamma^\alpha_{\text{cpc}}$ and $\gamma^\beta_{\text{cpc}}$ can be merged into a new self-intersecting curve $\gamma_{\text{cpc}}$.

*Outside intersection.* If the intersection is of outside type, it is because the segment $[p_i^\alpha, p_{i+1}^\alpha]$ belongs to $H^{\text{out}}_{[p_j^\beta, p_{j+1}^\beta]}$ and there exists the possibility of $p_i^\alpha$ crossing from $H^{\text{out}}_{[p_j^\beta, p_{j+1}^\beta]}$ to $H^{\text{in}}_{[p_j^\beta, p_{j+1}^\beta]}$, see Figure 7(b). The criterium to identify if a transition is needed in the network configuration is as follows. If

$$(u_i - v_i)^T n_{[p_j^\beta, p_{j+1}^\beta]} \geq 0,$$

then $p_i^\alpha$ does not cross. If

$$(u_i - v_i)^T n_{[p_j^\beta, p_{j+1}^\beta]} < 0,$$

then $p_i^\alpha$ will cross unless the intersection is resolved.

**Intersection at a point.** For each $i \in \{1,\dots,n_1\}$ and $j \in \{1,\dots,n_2\}$ such that $p_i^\alpha = p_j^\beta$, consider the vectors

$$u_i = \operatorname{sgn}(\mathcal{H}_c(P^\alpha)) \operatorname{pr}_{T\mathcal{D}} \left(\frac{\partial \mathcal{H}_c^\alpha}{\partial p_i^\alpha}\right), \quad u_j = \operatorname{sgn}(\mathcal{H}_c(P^\beta)) \operatorname{pr}_{T\mathcal{D}} \left(\frac{\partial \mathcal{H}_c^\beta}{\partial p_j^\beta}\right).$$

The guards depend upon the type of intersection.

*Inside intersection.* If the intersection is of inside type, see Figure 8(a), define

$$v_1 = \begin{cases} u_{[p_{i-1}^\alpha, p_i^\alpha]} & \text{if } [p_{j-1}^\beta, p_j^\beta] \subset H^{\text{in}}_{[p_{i-1}^\alpha, p_i^\alpha]}, \\ u_{[p_j^\beta, p_{j-1}^\beta]} & \text{if } [p_{j-1}^\beta, p_j^\beta] \not\subset H^{\text{in}}_{[p_{i-1}^\alpha, p_i^\alpha]}, \end{cases}$$

$$v_2 = \begin{cases} u_{[p_{i+1}^\alpha, p_i^\alpha]} & \text{if } [p_j^\beta, p_{j+1}^\beta] \subset H^{\text{in}}_{[p_i^\alpha, p_{i+1}^\alpha]}, \\ u_{[p_j^\beta, p_{j+1}^\beta]} & \text{if } [p_j^\beta, p_{j+1}^\beta] \not\subset H^{\text{in}}_{[p_i^\alpha, p_{i+1}^\alpha]}. \end{cases}$$

The criterium to identify if a transition is needed is as follows. If

$$u_i - u_j \in \texttt{wedge}(p_j^\beta, (v_1, v_1^\perp), (v_2, -v_2^\perp)),$$

then the relative motion of $p_i^\alpha$ and $p_j^\beta$ is such that the curves $\gamma_{\texttt{cpc}}^\alpha$ and $\gamma_{\texttt{cpc}}^\beta$ evolve without "crossing each other." If

$$u_i - u_j \notin \texttt{wedge}(p_j^\beta, (v_1, v_1^\perp), (v_2, -v_2^\perp)),$$

then the intersection needs to be resolved.

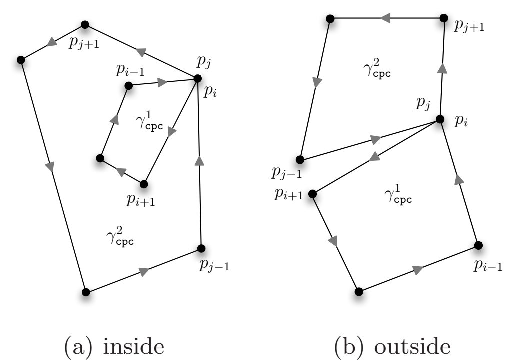



**Fig. 8.** The curves $\gamma_{\texttt{cpc}}^\alpha$ and $\gamma_{\texttt{cpc}}^\beta$ intersect at a point's location. (a) shows an inside intersection and (b) shows an outside intersection. In both cases, the curves $\gamma_{\texttt{cpc}}^\alpha$ and $\gamma_{\texttt{cpc}}^\beta$ can be merged into a new self-intersecting curve $\gamma_{\texttt{cpc}}$.

*Outside intersection.* If the intersection is of outside type, see Figure 8(b), define

$$v_1 = \begin{cases} u_{[p_j^\beta, p_{j-1}^\beta]} & \text{if } [p_{j-1}^\beta, p_j^\beta] \subset H^{\text{in}}_{[p_{i-1}^\alpha, p_i^\alpha]}, \\ u_{[p_{i-1}^\alpha, p_i^\alpha]} & \text{if } [p_{j-1}^\beta, p_j^\beta] \not\subset H^{\text{in}}_{[p_{i-1}^\alpha, p_i^\alpha]}, \end{cases}$$

$$v_2 = \begin{cases} u_{[p_j^\beta, p_{j+1}^\beta]} & \text{if } [p_j^\beta, p_{j+1}^\beta] \subset H^{\text{in}}_{[p_i^\alpha, p_{i+1}^\alpha]}, \\ u_{[p_{i+1}^\alpha, p_i^\alpha]} & \text{if } [p_j^\beta, p_{j+1}^\beta] \not\subset H^{\text{in}}_{[p_i^\alpha, p_{i+1}^\alpha]}. \end{cases}$$

The criterium to identify if a transition is needed is as follows. If

$$u_i - u_j \in \texttt{wedge}(p_j^\beta, (v_1, -v_1^\perp), (v_2, v_2^\perp)),$$

then the relative motion of $p_i^\alpha$ and $p_j^\beta$ is such that the curves $\gamma_{\texttt{cpc}}^\alpha$ and $\gamma_{\texttt{cpc}}^\beta$ evolve without "crossing each other." If

$$u_i - u_j \notin \texttt{wedge}(p_j^\beta, (v_1, -v_1^\perp), (v_2, v_2^\perp)),$$

then the intersection needs to be resolved.

**State transition.** We have encountered above the necessity to deal with intersections between the curves $\gamma_{\texttt{cpc}}^1$ and $\gamma_{\texttt{cpc}}^2$. For simplicity, we begin by considering the case where there is only one agent causing the intersection. If this is the case, then the two curves can be merged into a single one, see Figures 7 and 8. The closed polygonal curve $\gamma_{\texttt{cpc}}$ is defined by the concatenation of the segments

$$\{[p_k^\alpha, p_{k+1}^\alpha] \mid k \in \langle i, \dots, i-1\rangle\} \cup [p_i^\alpha, p_{j+1}^\beta] \\ \cup \{[p_k^\beta, p_{k+1}^\beta] \mid k \in \langle j+1, \dots, j-1\rangle\} \cup [p_j^\beta, p_i^\alpha],$$

if $p_i^\alpha \in ]p_j^\beta, p_{j+1}^\beta[$, and $\{[p_k^\alpha, p_{k+1}^\alpha] \mid k \in \langle i, \dots, i-1\rangle\} \cup \{[p_k^\beta, p_{k+1}^\beta] \mid k \in \langle j, \dots, j-1\rangle\}$, if $p_i^\alpha = p_j^\beta$. Observe that if we are dealing with a curve intersection at an open segment, i.e., $p_i^\alpha$ belongs to $]p_j^\beta, p_{j+1}^\beta[$, then $p_i^\alpha$ appears twice in the definition of $\gamma_{\texttt{cpc}}$. The wombliness of $\gamma_{\texttt{cpc}}^\alpha$ and $\gamma_{\texttt{cpc}}^\beta$ is summed up according to

$$\mathcal{H}(\gamma_{\texttt{cpc}}) = \mathcal{H}(\gamma_{\texttt{cpc}}^\alpha) + \mathcal{H}(\gamma_{\texttt{cpc}}^\beta).$$

We are now ready to detail the two possible outcomes of a curve intersection:

*Agent re-positioning.* If $\mathcal{H}(\gamma_{\texttt{cpc}}^\alpha)$ and $\mathcal{H}(\gamma_{\texttt{cpc}}^\beta)$ have different signs, we only keep the curve whose wombliness is larger in absolute value. Without loss of generality, assume the curve we keep is $\gamma_{\texttt{cpc}}^2$. Then, we re-position the agents in $\gamma_{\texttt{cpc}}^1$ along the boundary of $\gamma_{\texttt{cpc}}^2$. This process does not affect the value of the wombliness of $\gamma_{\texttt{cpc}}^2$, and can be made in an arbitrary way. Note that the absolute value of the wombliness of the resulting non-self-intersecting curve is strictly larger than the value of the wombliness of $\gamma_{\texttt{cpc}}$.

*Curve merging.* If $\mathcal{H}(\gamma_{\texttt{cpc}}^\alpha)$ and $\mathcal{H}(\gamma_{\texttt{cpc}}^\beta)$ have the same sign, then choosing only one curve would lead to a decrease in the value of the wombliness. Therefore, we consider their merge into the curve $\gamma_{\texttt{cpc}}$. If the intersection occurs on an open segment, we need to add one more agent to the network at the intersection location, according to the definition of $\gamma_{\texttt{cpc}}$ above. After the merge, the curve $\gamma_{\texttt{cpc}}$ evolves according to (8).

The case when multiple intersections occur at the same time can be dealt with in a similar fashion to the discussion in Section 5.1.

## 5.3 Convergence analysis

We refer to the distributed hybrid control design described in Sections 5.1 and 5.2 as the *wombling coordination algorithm*. The next result follows from a simple application of LaSalle's Invariance principle [13].

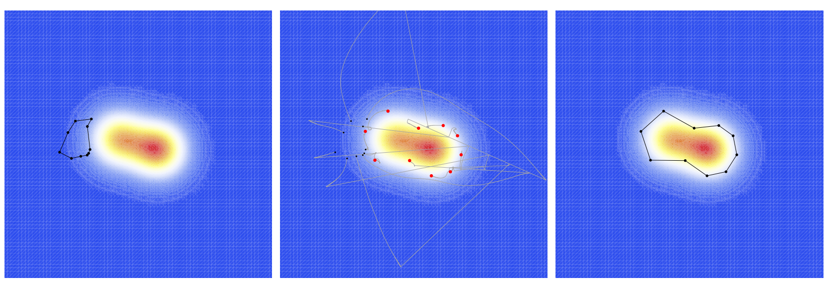
**Proposition 4.** *Any network trajectory evolving under the wombling coordination algorithm that does not undergo curve-splitting or curve-merging transitions converges to a critical configuration of $\mathcal{H}_c$ while monotonically optimizing the total wombliness.*

From Proposition 4, we can deduce the following result for network trajectories that undergo curve-splitting and curve-merging transitions.

**Corollary 5.** *A network trajectory that undergoes a finite number of curve-splitting and curve-merging transitions monotonically optimizes the total wombliness. Moreover, the subnetworks that result after these transitions have taken place each converge to a critical configuration of $\mathcal{H}_c$.*

*Remark 6.* Note that no conditions are imposed in Corollary 5 on the number of agent re-positioning transitions. A similar result could be established for network trajectories that undergo an infinite number of curve-splitting and curve-merging transitions but are non-Zeno executions of the hybrid system [7, 14]. •

Figures 9 and 10 present illustrations of the execution of the wombling coordination algorithm. The domain in all plots is $\mathcal{D} = [-4, 4] \times [-4, 4]$.

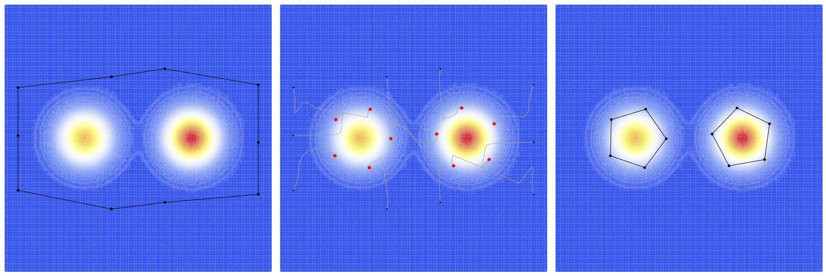


(a) (b) (c)

**Fig. 9.** Robotic network of 10 agents evolving under the wombling coordination algorithm. (a) shows the initial configuration, (b) shows the robot trajectories, and (c) shows the final configuration. The spatial field is $Y(x_1, x_2) = 1.25e^{-(x_1+.75)^2-(x_2-.2)^2} + 1.75e^{-(x_1-.75)^2-(x_2+.2)^2}$. The gradient flow (8) first triggers 1 outside self-intersection and then 2 inside self-intersections. All transitions result in agent re-positionings.

## 6 Conclusions

We have proposed a distributed coordination algorithm for robotic sensor networks that seek to detect areas of abrupt change of a spatial phenomena of interest. Our algorithm design has combined notions borrowed from statistical



(a) (b) (c)

**Fig. 10.** Robotic network of 10 agents evolving under the wombling coordination algorithm. (a) shows the initial configuration, (b) shows the robot trajectories, and (c) shows the final configuration. The spatial field is $Y(x_1, x_2) = e^{-(x_1+2)^2-x_2^2} + 1.25e^{-(x_1-2)^2-x_2^2}$. The gradient flow (8) first triggers an inside self-intersection that results in a curve splitting. After this, each of the new curves undergoes an inside self-intersection that result in agent re-positionings.

estimation and computer vision with tools from hybrid systems theory. The proposed algorithm allows for network splitting and re-grouping, and is guaranteed to monotonically increase the wombliness of the overall ensemble.

In order to make the proposed hybrid control design more amenable to implementation in practical scenarios, future work will address two limitations of the present approach. We need to move beyond the assumption that individual agents have gradient and Laplacian information on the spatial field along their immediate counterclockwise and clockwise boundary. When a curve merging or splitting occurs, the addition of an agent to the network can be done in a number of ways - e.g., individual agents might carry several smaller, lighter agents that can be deployed if needed. However, we need to better understand the number of switchings that can occur along the evolution, and provide conditions for their finiteness. We also plan to extend the present approach to open polygonal curves to detect "fronts" of abrupt change in the spatial phenomena.

## Acknowledgments

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