SPATIAL DETECTION OF AREAS OF ABRUPT CHANGE BY ROBOTIC NETWORKS

Hanqiao Gao
Department of Mechanical and Aerospace Engineering
University of California, San Diego
California 92093, USA
Email: hagao@ucsd.edu

Jorge Cortés
Department of Mechanical and Aerospace Engineering
University of California, San Diego
California 92093, USA
Email: cortes@ucsd.edu

ABSTRACT

This paper studies robotic sensor networks performing spatial detection of areas of rapid change in physical phenomena. We encode the task by means of an objective function, called wombliness, which measures the change of the spatial field along the open polygonal curve defined by the positions of the robotic sensors. This curve can become self-intersecting when evolving along the gradient flow of the wombliness. Borrowing tools from discontinuous dynamics and hybrid systems, we design an algorithm that allows for network re-positioning, splitting, and merging, while guaranteeing the monotonic evolution of the wombliness. We analyze its convergence properties and illustrate our approach in simulations.

INTRODUCTION

Consider a physical phenomena in a spatial domain modeled by a deterministic field. Our aim is to design a distributed algorithm to allow robotic networks to detect areas of abrupt change in the field. The boundaries delimiting these areas can be closed (e.g., a highly localized bank of nutrients) or open (e.g., a moving front of cold water). The accurate location of such boundaries is relevant in various applications, including oceanographic surveys, animal monitoring, and weather forecasting.

In statistical estimation [1, 2], wombliness identifies the boundaries where abrupt change occurs. Algorithms based on point-referenced data to detect boundaries with large wombliness are used in various disciplines [3, 4]. In computer vision [5, 6], image segmentation and edge detection aim to optimize functionals such as alignment, contrast, and geodesic active contour by solving gradient-based PDEs. ODE-based approaches are proposed in [7, 8]. Our work builds on discontinuous dynamics [9], hybrid modeling [10, 11] and stability analysis [12, 13], and, In particular, on the body of work [14, 15] on extensions of LaSalle Invariance Principle to hybrid systems.

The contributions of the paper are the following. We introduce the wombliness objective function to measure the alignment of the gradient of the spatial field along the normal direction to a non self-intersecting, open polygonal curve. We study its smoothness properties and provide an explicit expression for its gradient. To optimize the wombliness of the open polygonal curve determined by the robotic sensor positions, we analyze the evolution of the gradient flow. We design a discontinuous wombling algorithm that is guaranteed to monotonically optimize the wombliness. The algorithm allows for network transitions (agent re-positioning, splitting, and merging) that prevent the polygonal curve from becoming self-intersecting. This paper encompasses our previous results in [16] for closed polygonal curves and extends them in several ways. First, the consideration of open polygonal curves leads us to study the smoothness properties of a different objective function. Second, we consider a richer set of possible network transitions. In particular, splitting and merging of open curves can give rise to closed curves, and this further complicates the convergence analysis. Third, the discontinuous control law proposed here guarantees that no agent additions are required to execute the transitions specified by the algorithm. Finally, we provide stronger convergence results.

PRELIMINARIES

This section collects useful geometric concepts. For $n \in \mathbb{Z}_{>0}$ and $i, j \leq n$, we use the notation $\langle i, \dots, j\rangle$ to denote $\langle i, \dots, j\rangle = \{i, \dots, j\}$ if $i \leq j$ and $\langle i, \dots, j\rangle = \{i, \dots, n, 1, \dots, j\}$ if $i > j$.

Planar Geometric Notions

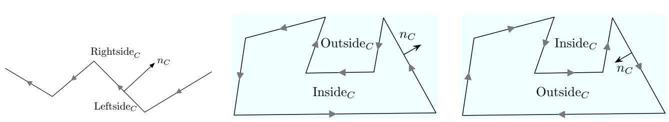
Given $v = (v_1, v_2) \in \mathbb{R}^2$, we denote by $v^\perp = (v_2, -v_1) \in \mathbb{R}^2$ the 90 degree clockwise rotation of v. Given $p \neq q \in \mathbb{R}^2$, $]p, q[$ and $[p, q]$ denote, respectively, the open and closed segments with end points p and q. We let $u_{[p,q]} = (q - p)/\|q - p\|$ and

$n_{[p,q]} = u_{[p,q]}^{\perp}$. We denote by $H_{[p,q]}^{rs} = \{z \in \mathbb{R}^2 \mid (z-p)^T n_{[p,q]} \geq 0\}$ the half plane of points in the positive direction of $n_{[p,q]}$. Likewise, we denote $H_{[p,q]}^{ls} = \{z \in \mathbb{R}^2 \mid (z-p)^T n_{[p,q]} \leq 0\}$. For $p,v \in \mathbb{R}^2$, let $\mathtt{ray}(p,v) = \{z \in \mathbb{R}^2 \mid z = p + tv,\ t \in \mathbb{R}_{\geq 0}\}$. Given $v_1, v_2, n_1, n_2 \in \mathbb{R}^2$, with v_i orthogonal to n_i, $i \in \{1,2\}$, $\mathtt{wedge}(p,(v_1,n_1),(v_2,n_2))$ is the cone with vertex p, axes $\mathtt{ray}(p,v_1)$ and $\mathtt{ray}(p,v_2)$, and as interior, the set towards which n_1 points along $\mathtt{ray}(p,v_1)$ and n_2 points along $\mathtt{ray}(p,v_2)$. A domain $\mathcal{D} \subset \mathbb{R}^2$ is an open and simply connected set. Given $q \in \mathcal{D}$, $T_q\mathcal{D}$ denotes the set of vectors tangent to $\mathcal{D}$ with origin at q. For $q \in \mathrm{int}(\mathcal{D})$, $T_q\mathcal{D}$ is 2-dimensional. For $q \in \partial\mathcal{D}$, $T_q\mathcal{D}$ is the half plane divided by the tangent line to $\partial\mathcal{D}$ at q and containing $\mathcal{D}$. We let $\mathrm{pr}_{T\mathcal{D}} : T_{\mathcal{D}}\mathbb{R}^2 \to T\mathcal{D} = \cup\{T_q\mathcal{D} \mid q \in \mathcal{D}\}$ map a vector in $\mathbb{R}^2$ with origin at $q \in \mathcal{D}$ to its orthogonal projection onto $T_q\mathcal{D}$.

Curve Parameterizations

A *curve* C in $\mathbb{R}^2$ is the image of a map $\gamma : [a,b] \to \mathbb{R}^2$. The map γ is called a *parameterization of* C. A curve C is *self-intersecting* if γ is not injective on (a,b). A curve C is *open* if $\gamma(a) \neq \gamma(b)$. For an open curve C, we let $n_C = \dot{\gamma}/\|\dot{\gamma}\|^{\perp}$ denote the unit normal vector to C. Given a curve C parameterized by a piecewise smooth map $\gamma : [a,b] \to C$, the line integral of $f : C \subset \mathbb{R}^2 \to \mathbb{R}$ over C is $\int_C f = \int_C f(q)dq = \int_a^b f(\gamma(t))\,\|\dot{\gamma}(t)\|\,dt$, and it is independent of the selected parameterization.

We deal with polygonal curves. An open, not self-intersecting, polygonal curve C partitions $\mathbb{R}^2$ into two closed and connected sets, $\mathrm{Leftside}_C = \cup_{i=2}^{i=n-1}\mathtt{wedge}(p_i,(w_{i-1},n_{[p_i,p_{i-1}]}),(w_{i+1},n_{[p_{i+1},p_i]}))$ and $\mathrm{Rightside}_C = \cup_{i=2}^{i=n-1}\mathtt{wedge}(p_i,(w_{i-1},n_{[p_{i-1},p_i]}),(w_{i+1},n_{[p_i,p_{i+1}]}))$, such that n_C along C points outside $\mathrm{Leftside}_C$ and inside $\mathrm{Rightside}_C$, see Fig. 1(a). Here, $w_{i-1} = p_{i-1} - p_i$, $w_{i+1} = p_{i+1} - p_i$. A closed, not self-intersecting, polygonal curve C partitions $\mathbb{R}^2$ into two disjoint open and connected sets, Inside_C and $\mathrm{Outside}_C$, such that n_C along C points outside Inside_C and inside $\mathrm{Outside}_C$, respectively, see Fig. 1(b,c).



(a) (b) (c)

Figure 1. (a) OPEN, (b) COUNTERCLOCKWISE CLOSED AND (c) CLOCKWISE CLOSED CURVES.

ANALYSIS OF THE WOMBLINESS OF OPEN CURVES

Let $Y : \mathbb{R}^2 \to \mathbb{R}$ be a function of class C^2 modeling the spatial field. Our objective is to find boundaries of the spatial field Y where abrupt change occurs. Let C be a non self-intersecting curve in $\mathbb{R}^2$, and define its wombliness by

$$\mathcal{H}(C) = \int_C \langle \nabla Y, n_C \rangle. \tag{1}$$

This function measures how much Y changes along the normal direction of C. We are interested in finding the curves which optimize $\mathcal{H}$. Consider a group of n agents with locations $p_1, \ldots, p_n$ moving in a compact domain $\mathcal{D} \subset \mathbb{R}^2$. Here, we order the network agents in counterclockwise order, then join the positions of consecutive agents. Let γ_{opc} be the open polygonal curve concatenating the segments $[p_i, p_{i+1}]$, $i \in \{1, \ldots, n-1\}$. Let $\mathcal{S} = \{(p_1, \ldots, p_n) \in \mathcal{D}^n \mid \gamma_{\mathsf{opc}}$ is non-self-intersecting$\} \subset \mathcal{D}^n$. The wombliness $\mathcal{H}_o : \mathcal{S} \to \mathbb{R}$ of the group of robots is $\mathcal{H}_o(p_1, \ldots, p_n) = \mathcal{H}(\gamma_{\mathsf{opc}}) = \sum_{i=1}^{n-1}\int_{[p_i,p_{i+1}]}\langle \nabla Y, n_{[p_i,p_{i+1}]}\rangle$.

The following result shows that the gradient of $\mathcal{H}_o$ is distributed over the ring graph. We omit the proof for space reasons.

Proposition 0.1 (Gradient of $\mathcal{H}_o$) *The function $\mathcal{H}_o : \mathcal{S} \to \mathbb{R}$ is continuously differentiable. For $i \in \{2, \ldots, n-1\}$,*

$$\begin{aligned}
\frac{\partial \mathcal{H}_o}{\partial p_i} &= \Big(\int_{[p_i,p_{i+1}]} \frac{\|p_{i+1}-q\|}{\|p_{i+1}-p_i\|}\Delta Y dq\Big) n_{[p_i,p_{i+1}]} \\
&\quad + \Big(\int_{[p_{i-1},p_i]} \frac{\|q-p_{i-1}\|}{\|p_i-p_{i-1}\|}\Delta Y dq\Big) n_{[p_{i-1},p_i]}, \\
\frac{\partial \mathcal{H}_o}{\partial p_1} &= \Big(\int_{[p_1,p_2]} \frac{\|p_2-q\|}{\|p_2-p_1\|}\Delta Y dq\Big) n_{[p_1,p_2]} + \nabla Y^{\perp}(p_1), \\
\frac{\partial \mathcal{H}_o}{\partial p_n} &= \Big(\int_{[p_{n-1},p_n]} \frac{\|q-p_{n-1}\|}{\|p_n-p_{n-1}\|}\Delta Y dq\Big) n_{[p_{n-1},p_n]} - \nabla Y^{\perp}(p_n),
\end{aligned}$$

where ΔY is the Laplacian of Y.

HYBRID DESIGN FOR WOMBLINESS OPTIMIZATION

Our approach to find boundaries where the spatial field change abruptly starts with an initial network configuration and optimizes the wombliness of the open polygonal curve defined by the network. To maximize $\mathcal{H}_o$, we implement the distributed gradient flow of this function, cf. Proposition 0.1,

$$\dot{p}_i = \mathrm{sgn}(\mathcal{H}_o(P_0))\,\mathrm{pr}_{T\mathcal{D}}\Big(\frac{\partial \mathcal{H}_o}{\partial p_i}\Big), \quad i \in \{1, \ldots, n\}, \tag{2}$$

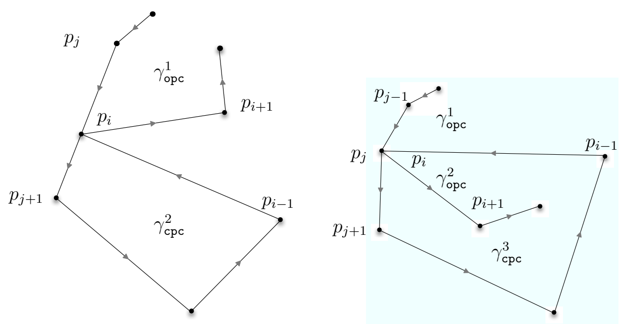
where $P(0) = P_0$ is the initial configuration. Evolutions under (2) of the open curve defined by $p_1, \ldots, p_n$ may become self-intersecting. To address this, we propose a switching design.

Curve Self-intersection

Assume the curve γ_{opc} is self-intersecting as in Fig. 2(a,b). Denote these cases as *outside* and *inside* self-intersection, respectively. For the outside self-intersection, none of the open curves is inside of the closed curve, while for the inside self-intersection, there is one open curve inside of a closed curve. We first discuss these self-intersections and then consider the transitions the network may experience. We further distinguish between whether the self-intersection occurs on an open segment or at a point.

Self-intersection On An Open Segment. For each $i \neq j \in \{1,\ldots,n\}$ such that $p_i \in]p_j, p_{j+1}[$, define $\lambda \in [0,1)$ by $p_i = (1-\lambda)p_j + \lambda p_{j+1}$ and consider $v_i = (1-\lambda)u_j + \lambda u_{j+1}$ and $u_k = \operatorname{sgn}(\mathcal{H}_o(P))\operatorname{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}_o}{\partial p_k}\right)$, where $k \in \{i, j, j+1\}$. The self-intersection happens either in the *left-side* of the segment or in the *right-side* of the segment. The criterium to identify if a transition is needed in the network configuration is as follows.

Left-side self-intersection If the self-intersection is of left-side type, there exists the possibility of p_i crossing from $H^{ls}_{[p_j,p_{j+1}]}$ to $H^{rs}_{[p_j,p_{j+1}]}$, see Fig. 2(a). If $(u_i - v_i)^T n_{[p_j,p_{j+1}]} \geq 0$, then the intersection will happen unless it is resolved.



(a) OUTSIDE ON A SEGMENT (b) INSIDE AT A POINT

Figure 2. (a) OUTSIDE LEFT-SIDE SELF-INTERSECTION ON A SEGMENT AND (b) INSIDE LEFT-SIDE SELF-INTERSECTION AT A POINT.

Right-side self-intersection If the self-intersection is of right-side type, it is because there exists the possibility of p_i crossing from $H^{rs}_{[p_j,p_{j+1}]}$ to $H^{ls}_{[p_j,p_{j+1}]}$. If $(u_i - v_i)^T n_{[p_j,p_{j+1}]} \leq 0$, then the intersection will happen unless it is resolved.

Self-intersection At A Point. For each $i \neq j \in \{1,\ldots,n\}$ such that $p_i = p_j$, consider the vectors $u_i = \operatorname{sgn}(\mathcal{H}_o(P))\operatorname{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}_o}{\partial p_i}\right)$ and $u_j = \operatorname{sgn}(\mathcal{H}_o(P))\operatorname{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}_o}{\partial p_j}\right)$.

Left-side self-intersection If the self-intersection is of left-side type, it is because there exists the possibility of p_i crossing out from Leftside$_{\gamma_{\mathrm{opc}}}$ to Rightside$_{\gamma_{\mathrm{opc}}}$, see Fig. 2(b). Define

$$v_1 = \begin{cases} u_{[p_{i-1},p_i]} & \text{if } [p_{j-1},p_j] \subset H^{ls}_{[p_{i-1},p_i]}, \\ u_{[p_j,p_{j-1}]} & \text{if } [p_{j-1},p_j] \not\subset H^{ls}_{[p_{i-1},p_i]}, \end{cases}$$
$$v_2 = \begin{cases} u_{[p_{i+1},p_i]} & \text{if } [p_j,p_{j+1}] \subset H^{ls}_{[p_i,p_{i+1}]}, \\ u_{[p_j,p_{j+1}]} & \text{if } [p_j,p_{j+1}] \not\subset H^{ls}_{[p_i,p_{i+1}]}. \end{cases}$$

If $u_i - u_j \notin \mathtt{wedge}(p_j,(v_1,v_1^\perp),(v_2,-v_2^\perp))$, then intersection will happen unless it is resolved.

Right-side self-intersection If the self-intersection is of right-side type, it is because there exists the possibility of p_i crossing out from Rightside$_{\gamma_{\mathrm{opc}}}$ to Leftside$_{\gamma_{\mathrm{opc}}}$. Define

$$v_1 = \begin{cases} u_{[p_j,p_{j-1}]} & \text{if } [p_{j-1},p_j] \subset H^{ls}_{[p_{i-1},p_i]}, \\ u_{[p_{i-1},p_i]} & \text{if } [p_{j-1},p_j] \not\subset H^{ls}_{[p_{i-1},p_i]}, \end{cases}$$
$$v_2 = \begin{cases} u_{[p_j,p_{j+1}]} & \text{if } [p_j,p_{j+1}] \subset H^{ls}_{[p_i,p_{i+1}]}, \\ u_{[p_{i+1},p_i]} & \text{if } [p_j,p_{j+1}] \not\subset H^{ls}_{[p_i,p_{i+1}]}. \end{cases}$$

If $u_i - u_j \notin \mathtt{wedge}(p_j,(v_1,-v_1^\perp),(v_2,v_2^\perp))$, then the self-intersection will happen unless it is resolved.

State Transition. For simplicity, consider only one agent causing the self-intersection. If multiple self-intersections occur at different locations, then the state transitions corresponding to each one of them can be executed simultaneously.

Outside self-intersection In this case, see Fig. 2(a), γ_{opc} can be decomposed into one open curve γ^1_{opc} and one closed curve γ^2_{cpc}. The curve γ^1_{opc} is defined by the concatenation of the segments $\{[p_k,p_{k+1}] \mid k \in \langle 1,\ldots,j-1\rangle \cup \langle i,\ldots,n-1\rangle\} \cup [p_j,p_i]$, if $p_i \in]p_j,p_{j+1}[$, and $\{[p_k,p_{k+1}] \mid k \in \langle 1,\ldots,j-1\rangle \cup \langle i+1,\ldots,n-1\rangle\} \cup [p_j,p_{i+1}]$, if $p_i = p_j$. The curve γ^2_{cpc} is defined by the concatenation of the segments $\{[p_k,p_{k+1}] \mid k \in \langle j+1,\ldots,i-1\rangle\} \cup [p_i,p_{j+1}]$ for both $p_i \in]p_j,p_{j+1}[$, and $p_i = p_j$. When dealing with a curve self-intersection on an open segment, i.e., p_i belongs to $]p_j,p_{j+1}[$, then p_i appears both in the definition of γ^1_{opc} and γ^2_{cpc}. The wombliness is $\mathcal{H}(\gamma_{\mathrm{opc}}) = \mathcal{H}(\gamma^1_{\mathrm{opc}}) + \mathcal{H}(\gamma^2_{\mathrm{cpc}})$.

Inside self-intersection In this case, see Fig. 2(b), the decomposed curves intersection each other, we can split the curve γ_{opc} into three curves, two open curves γ^1_{opc}, γ^2_{opc} and one closed curve γ^3_{cpc}. The curve γ^1_{opc} is defined by the concatenation of the segments $\{[p_k,p_{k+1}] \mid k \in \langle 1,\ldots,j-1\rangle\} \cup [p_j,p_i]$, if $p_i \in]p_j,p_{j+1}[$, and $\{[p_k,p_{k+1}] \mid k \in \langle 1,\ldots,j-1\rangle\}$, if $p_i = p_j$. The curve γ^2_{opc} is defined by the concatenation of the segments $\{[p_k,p_{k+1}] \mid k \in \langle i,\ldots,n-1\rangle\}$ for both $p_i \in]p_j,p_{j+1}[$ and $p_i = p_j$. The curve γ^3_{cpc} is defined by the concatenation of the segments $\{[p_k,p_{k+1}] \mid k \in \langle j+1,\ldots,i-1\rangle\} \cup [p_i,p_{j+1}]$, if $p_i \in]p_j,p_{j+1}[$, and $\{[p_k,p_{k+1}] \mid k \in \langle j+1,\ldots,i-1\rangle\} \cup [p_i,p_{j+1}]$, if $p_i = p_j$. Likewise, if a self-intersection is on an open segment, then p_i appears in the definition of γ^1_{opc}, γ^2_{opc}, and γ^3_{cpc}. If a self-intersection is at a point, i.e., $p_i = p_j$, then p_i appears both in the definition of γ^2_{opc} and γ^3_{cpc}. The wombliness is summed up as $\mathcal{H}(\gamma_{\mathrm{opc}}) = \mathcal{H}(\gamma^1_{\mathrm{opc}}) + \mathcal{H}(\gamma^2_{\mathrm{opc}}) + \mathcal{H}(\gamma^3_{\mathrm{cpc}})$.

We then deal with the curves if a self-intersection occurs:

Agent re-positioning: For the outside self-intersection, if $\mathcal{H}(\gamma^1_{\mathrm{opc}})$ and $\mathcal{H}(\gamma^2_{\mathrm{cpc}})$ have a different sign, we only keep the curve whose wombliness has the same sign as $\mathcal{H}(\gamma_{\mathrm{opc}})$. Assume we keep γ^2_{cpc}. Then, we re-position the agents in γ^1_{opc} along the boundary of γ^2_{cpc}, see Fig. 3. This does not affect the value of the wombliness of γ^2_{cpc}, and can be made in an arbitrary way. The absolute value of the wombliness of the resulting non-self-intersecting curve is strictly larger than that of the original self-intersecting curve.

For the inside self-intersection, if $\mathcal{H}(\gamma^1_{\mathsf{opc}})$ has a different sign from $\mathcal{H}(\gamma^2_{\mathsf{opc}})$ and $\mathcal{H}(\gamma^3_{\mathsf{cpc}})$, either $\mathcal{H}(\gamma^1_{\mathsf{opc}})$ and $\mathcal{H}(\gamma_{\mathsf{opc}})$ have the same sign or not. For the former one, we re-position the agents in γ^2_{opc} and γ^3_{cpc} along the boundary of γ^1_{opc}, while for the latter one, we re-position the agents in γ^1_{opc} along the boundary of γ^2_{opc} or γ^3_{cpc} and then keep γ^2_{opc} and γ^3_{cpc} connecting at point p_i, see Fig. 4. We treat analogously the case when $\mathcal{H}(\gamma^2_{\mathsf{opc}})$ has a different sign from $\mathcal{H}(\gamma^1_{\mathsf{opc}})$ and $\mathcal{H}(\gamma^3_{\mathsf{cpc}})$. If $\mathcal{H}(\gamma^3_{\mathsf{cpc}})$ has a different sign from $\mathcal{H}(\gamma^1_{\mathsf{opc}})$ and $\mathcal{H}(\gamma^2_{\mathsf{opc}})$, the only difference is that we need to merge γ^1_{opc} and γ^2_{opc} into one open curve after we re-position the agents in γ^3_{cpc} along the boundary of γ^1_{opc} and γ^2_{opc}. We detail later the procedure for curve merging, when discussing transitions for intersecting open and closed curves.

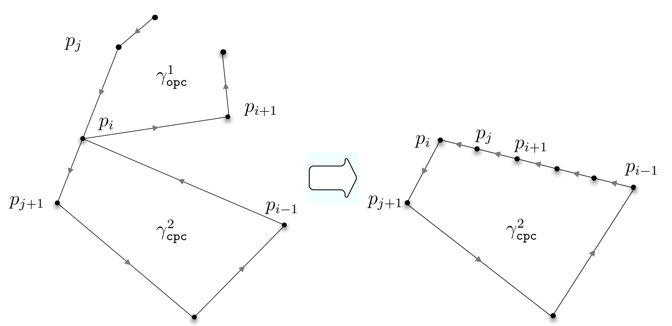
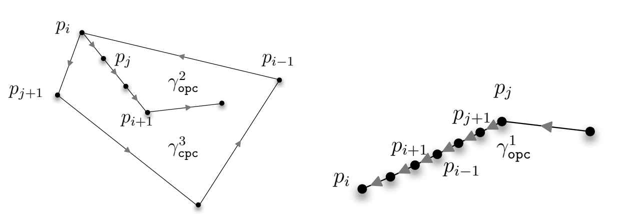


Figure 3. OUTSIDE SELF-INTERSECTION, AGENT RE-POSITIONING. AGENTS γ^1_{OPC} GET RE-POSITIONED ONTO γ^2_{CPC}.

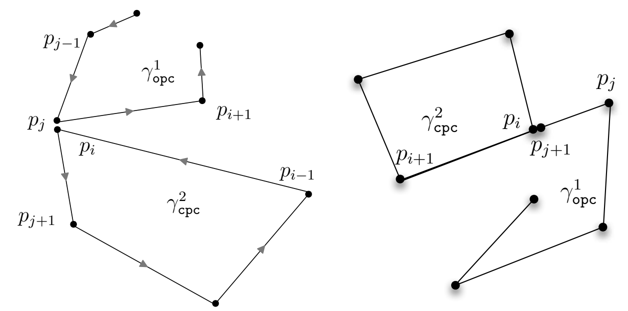


(a) AGENTS IN γ^1_{OPC} GET RE-POSITIONED

(b) AGENTS IN γ^2_{OPC} AND γ^3_{CPC} GET RE-POSITIONED

Figure 4. INSIDE SELF-INTERSECTION, AGENT RE-POSITIONING. $\mathcal{H}(\gamma^1_{\mathsf{OPC}})$ HAS A DIFFERENT SIGN FROM $\mathcal{H}(\gamma^2_{\mathsf{OPC}})$ AND $\mathcal{H}(\gamma^3_{\mathsf{CPC}})$.

Curve splitting: For the outside self-intersection, if $\mathcal{H}(\gamma^1_{\mathsf{opc}})$ and $\mathcal{H}(\gamma^2_{\mathsf{cpc}})$ have the same sign as $\mathcal{H}(\gamma_{\mathsf{opc}})$, we keep both curves. If the self-intersection occurs at a point, we split the curve. After the splitting, each curve evolves independently according to (2). The transition is illustrated in Fig. 5(a).



(a) (b)

Figure 5. (a) SELF-INTERSECTION AT A POINT, γ_{OPC} IS SPLIT INTO γ^1_{OPC} AND γ^2_{CPC}. (b) CURVE SPLITTING UNDER DISCONTINUOUS LAW.

If the self-intersection occurs on a segment, we constrain the agent motion to remain along the segment, $p_i \in [p_j, p_{j+1}]$, and project its control law (2) along the segment,

$$\dot{p}_i = \mathrm{pr}_{]p_j,p_{j+1}[}\left(\mathrm{sgn}(\mathcal{H}_o(P_0))\,\mathrm{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}_o}{\partial p_i}\right)\right). \qquad (3)$$

This does not affect the wombliness of γ_{opc}. This control law defines a discontinuous dynamical system, and we understand its solution in the Krasovskii sense [9]. If the ensuing evolution leads the agent to intersect with any of the extreme points of the segment, then we treat this case as a self-intersection at a point with an additional consideration. If the criterium to split is satisfied, we also evaluate the criterium as if the two curves were split and we were considering a merging event (see below the discussion on intersection between open and closed curves). If the criterium to merge is not satisfied, then there is a network splitting. If the criterium to merge is satisfied, then there is no transition and the two agents evolve together with the same law. If the two intersecting points p_{i+1} and p_{j+1} stay together, the evolution may cause the point p_i next to the intersecting-position intersect with the segment $]p_j, p_{j+1}[$. In this case, we move one agent p_{j+1} to the position p_i and consider the open curve γ^1_{opc} and the closed curve γ^2_{cpc}, see Fig. 5(b), and recalculate the wombliness $\mathcal{H}(\gamma^1_{\mathsf{opc}})$ and $\mathcal{H}(\gamma^2_{\mathsf{cpc}})$. Depending on the values of the new wombliness, agents reposition or the curve splits. Note that the sum of the absolute value of $\mathcal{H}(\gamma^1_{\mathsf{opc}})$ and $\mathcal{H}(\gamma^2_{\mathsf{cpc}})$ decreases because we do not calculate the wombliness of the segment $]p^-_{j+1}, p^+_{j+1}[$ when we move p_{j+1} to p_i, where p^-_{j+1} and p^+_{j+1} denote the position of $j+1$ agent before and after the movement, respectively. Therefore, we add the absolute value of the wombliness of this segment back to make sure the wombliness of the curve is monotonically nondecreasing. For the inside self-intersection, we also use this discontinuous control law.

Remark 0.2 The implementation of the state transitions described above requires the agents to be able to detect the self-intersection, determine its type and the values of the wombliness of the curves involved. Although for space reasons we do not go into detail here, this information can be computed by the network using distributed algorithms [17, 18]. This observation is also valid for the transitions described in the next sections. •

Intersection Between Open Curves

As a result of the curve splitting transition, there might be more than one curve moving in $\mathcal{D}$. It is conceivable that along the ensuing evolution these curves intersect each other. For simplicity, we only consider the case when there are two curves evolving in $\mathcal{D}$. The case with more than two curves can be treated in an analogous way. Here, we discuss the intersection between two open curves. Let $\gamma^{\alpha}_{\mathsf{opc}}$ and $\gamma^{\beta}_{\mathsf{opc}}$ be the open curves determined by n_1 agents at positions $P^{\alpha} = (p^{\alpha}_1, \dots, p^{\alpha}_{n_1})$ and n_2 agents at positions $P^{\beta} = (p^{\beta}_1, \dots, p^{\beta}_{n_2})$, respectively. Let $\mathcal{H}^{\alpha}_o(P^{\alpha}) = \mathcal{H}(\gamma^{\alpha}_{\mathsf{opc}})$ and $\mathcal{H}^{\beta}_o(P^{\beta}) = \mathcal{H}(\gamma^{\beta}_{\mathsf{opc}})$. If $\gamma^{\alpha}_{\mathsf{opc}}$ and $\gamma^{\beta}_{\mathsf{opc}}$ are both in the left-side or right-side of each other, we name this as a same-side intersection, see Fig. 6, otherwise as a different-side intersection.

Intersection On An Open Segment. For each $i \in \{1, \dots, n_1\}$ such that $p^{\alpha}_i \in]p^{\beta}_j, p^{\beta}_{j+1}[$ for some $j \in \{1, \dots, n_2\}$, define $\lambda \in [0, 1)$ by $p^{\alpha}_i = (1-\lambda)p^{\beta}_j + \lambda p^{\beta}_{j+1}$. For $k \in \{j, j+1\}$, consider the vectors $v_i = (1-\lambda)u_j + \lambda u_{j+1}$, $u_i = \mathrm{sgn}(\mathcal{H}^{\alpha}_o(P^{\alpha}))\frac{\partial \mathcal{H}^{\alpha}_o}{\partial p^{\alpha}_i}$ and $u_k = \mathrm{sgn}(\mathcal{H}^{\beta}_o(P^{\beta}))\frac{\partial \mathcal{H}^{\beta}_o}{\partial p^{\beta}_k}$.

$\gamma^{\alpha}_{\mathsf{opc}}$ belongs to Leftside$_{\gamma^{\beta}_{\mathsf{opc}}}$ If the intersection is of this type, see Fig. 6(a), there exists the possibility of p^{α}_i crossing from Leftside$_{\gamma^{\beta}_{\mathsf{opc}}}$ to Rightside$_{\gamma^{\beta}_{\mathsf{opc}}}$. If $(u_i - v_i)^T n_{[p^{\beta}_j, p^{\beta}_{j+1}]} > 0$, then p^{α}_i will cross unless the intersection is resolved.

$\gamma^{\alpha}_{\mathsf{opc}}$ belongs to Rightside$_{\gamma^{\beta}_{\mathsf{opc}}}$ If the intersection is of this type, there exists the possibility of p^{α}_i crossing from Rightside$_{\gamma^{\beta}_{\mathsf{opc}}}$ to Leftside$_{\gamma^{\beta}_{\mathsf{opc}}}$. If $(u_i - v_i)^T n_{[p^{\beta}_j, p^{\beta}_{j+1}]} < 0$, p^{α}_i will cross unless the intersection is resolved.

Intersection At A Point. For each $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_2\}$ such that $p^{\alpha}_i = p^{\beta}_j$, consider the vectors $u_i = \mathrm{sgn}(\mathcal{H}_o(P^{\alpha}))\,\mathrm{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}^{\alpha}_o}{\partial p^{\alpha}_i}\right)$ and $u_j = \mathrm{sgn}(\mathcal{H}_o(P^{\beta}))\,\mathrm{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}^{\beta}_o}{\partial p^{\beta}_j}\right)$.

$\gamma^{\alpha}_{\mathsf{opc}}$ belongs to Leftside$_{\gamma^{\beta}_{\mathsf{opc}}}$ In this case, define

$$v_1 = \begin{cases} u_{[p^{\alpha}_{i-1}, p^{\alpha}_i]} & \text{if } [p^{\beta}_{j-1}, p^{\beta}_j] \subset H^{ls}_{[p^{\alpha}_{i-1}, p^{\alpha}_i]}, \\ u_{[p^{\beta}_j, p^{\beta}_{j-1}]} & \text{if } [p^{\beta}_{j-1}, p^{\beta}_j] \not\subset H^{ls}_{[p^{\alpha}_{i-1}, p^{\alpha}_i]}, \end{cases} \tag{4a}$$

$$v_2 = \begin{cases} u_{[p^{\alpha}_{i+1}, p^{\alpha}_i]} & \text{if } [p^{\beta}_j, p^{\beta}_{j+1}] \subset H^{ls}_{[p^{\alpha}_i, p^{\alpha}_{i+1}]}, \\ u_{[p^{\beta}_j, p^{\beta}_{j+1}]} & \text{if } [p^{\beta}_j, p^{\beta}_{j+1}] \not\subset H^{ls}_{[p^{\alpha}_i, p^{\alpha}_{i+1}]}. \end{cases} \tag{4b}$$

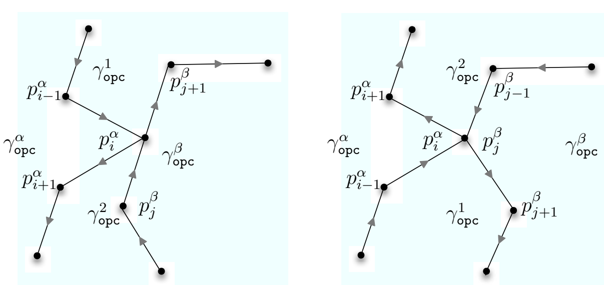
If $u_i - u_j \not\in \mathtt{wedge}(p^{\beta}_j, (v_1, v_1^{\perp}), (v_2, -v_2^{\perp}))$, then the intersection needs to be resolved.

$\gamma^{\alpha}_{\mathsf{opc}}$ belongs to Rightside$_{\gamma^{\beta}_{\mathsf{opc}}}$ In this case, see Fig. 6(b), define

$$v_1 = \begin{cases} u_{[p^{\beta}_j, p^{\beta}_{j-1}]} & \text{if } [p^{\beta}_{j-1}, p^{\beta}_j] \subset H^{ls}_{[p^{\alpha}_{i-1}, p^{\alpha}_i]}, \\ u_{[p^{\alpha}_{i-1}, p^{\alpha}_i]} & \text{if } [p^{\beta}_{j-1}, p^{\beta}_j] \not\subset H^{ls}_{[p^{\alpha}_{i-1}, p^{\alpha}_i]}, \end{cases} \tag{5a}$$

$$v_2 = \begin{cases} u_{[p^{\beta}_j, p^{\beta}_{j+1}]} & \text{if } [p^{\beta}_j, p^{\beta}_{j+1}] \subset H^{ls}_{[p^{\alpha}_i, p^{\alpha}_{i+1}]}, \\ u_{[p^{\alpha}_{i+1}, p^{\alpha}_i]} & \text{if } [p^{\beta}_j, p^{\beta}_{j+1}] \not\subset H^{ls}_{[p^{\alpha}_i, p^{\alpha}_{i+1}]}. \end{cases} \tag{5b}$$

If $u_i - u_j \not\in \mathtt{wedge}(p^{\beta}_j, (v_1, -v_1^{\perp}), (v_2, v_2^{\perp}))$, then the intersection needs to be resolved.



(a) $\gamma^{\alpha}_{\mathsf{OPC}}$ IN Leftside$_{\gamma^{\beta}_{\mathsf{OPC}}}$ (b) $\gamma^{\alpha}_{\mathsf{OPC}}$ IN Rightside$_{\gamma^{\beta}_{\mathsf{OPC}}}$

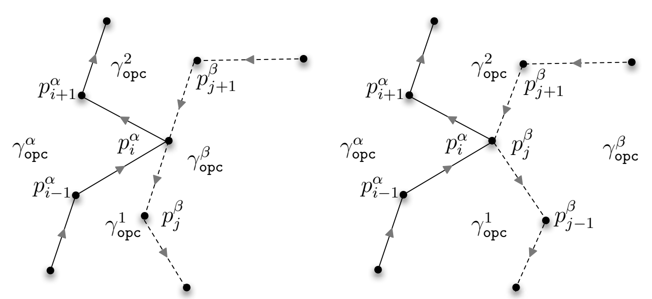
Figure 6. SAME-SIDE INTERSECTION HAPPENS (a) ON A SEGMENT AND (b) AT A POINT.

State Transition. For simplicity, consider only one agent causing the intersection. The two intersecting open curves can be rearranged into two different open curves γ^1_{opc} and γ^2_{opc} depending on the types of the intersections, as we discuss next.

Same-side intersection In this case, see Fig. 6. γ^1_{opc} is defined by the concatenation of the segments $\{[p^{\alpha}_k, p^{\alpha}_{k+1}] \mid k \in \langle 1, \dots, i-1 \rangle\} \cup [p^{\alpha}_i, p^{\beta}_{j+1}] \cup \{[p^{\beta}_k, p^{\beta}_{k+1}] \mid k \in \langle j+1, \dots, n_2 - 1 \rangle\}$, if $p^{\alpha}_i \in]p^{\beta}_j, p^{\beta}_{j+1}[$ and $\{[p^{\alpha}_k, p^{\alpha}_{k+1}] \mid k \in \langle 1, \dots, i-1 \rangle\} \cup [p^{\alpha}_i, p^{\beta}_{j+1}] \cup$

$\{[p_k^\beta, p_{k+1}^\beta] \mid k \in \langle j+1, \ldots, n_2-1\rangle\}$, if $p_i^\alpha = p_j^\beta$. γ_{opc}^2 is defined by the concatenation of the segments $\{[p_k^\beta, p_{k+1}^\beta] \mid k \in \langle 1, \ldots, j-1\rangle\} \cup [p_j^\beta, p_i^\alpha] \cup \{[p_k^\alpha, p_{k+1}^\alpha] \mid k \in \langle i+1, \ldots, n_1-1\rangle\}$, if $p_i^\alpha \in]p_j^\beta, p_{j+1}^\beta[$ and by $\{[p_k^\beta, p_{k+1}^\beta] \mid k \in \langle 1, \ldots, j-1\rangle\} \cup [p_j^\beta, p_{i+1}^\alpha] \cup \{[p_k^\alpha, p_{k+1}^\alpha] \mid k \in \langle i+1, \ldots, n_1-1\rangle\}$, if $p_i^\alpha = p_j^\beta$. When intersection happens on an open segment, then p_i appears both in the definition of γ_{opc}^1 and γ_{opc}^2. The wombliness of γ_{opc}^1 and γ_{opc}^2 is summed up as $\mathcal{H}(\gamma_{\mathsf{opc}}) = \mathcal{H}(\gamma_{\mathsf{opc}}^1) + \mathcal{H}(\gamma_{\mathsf{opc}}^2)$.

Different-side intersection For this case, we change the direction of one open curve, say $\gamma_{\mathsf{opc}}^\beta$, then the resulting curves can be rearranged into two open curves γ_{opc}^1 and γ_{opc}^2, see Fig. 7. The γ_{opc}^1 is defined by the concatenation of the segments $\{[p_k^\alpha, p_{k+1}^\alpha] \mid k \in \langle 1, \ldots, i-1\rangle\} \cup [p_i^\alpha, p_j^\beta] \cup \{[p_k^\beta, p_{k-1}^\beta] \mid k \in \langle j, \ldots, 2\rangle\}$, if $p_i^\alpha \in]p_j^\beta, p_{j+1}^\beta[$ and $\{[p_k^\alpha, p_{k+1}^\alpha] \mid k \in \langle 1, \ldots, i-1\rangle\} \cup [p_i^\alpha, p_{j-1}^\beta] \cup \{[p_k^\beta, p_{k-1}^\beta] \mid k \in \langle j-1, \ldots, 2\rangle\}$, if $p_i^\alpha = p_j^\beta$. γ_{opc}^2 is defined by the concatenation of the segments $\{[p_k^\beta, p_{k-1}^\beta] \mid k \in \langle n_2, \ldots, j+2\rangle\} \cup [p_{j+1}^\beta, p_i^\alpha] \cup \{[p_k^\alpha, p_{k+1}^\alpha] \mid k \in \langle i+1, \ldots, n_1-1\rangle\}$, if $p_i^\alpha \in]p_j^\beta, p_{j+1}^\beta[$ and by $\{[p_k^\beta, p_{k-1}^\beta] \mid k \in \langle n_2, \ldots, j+1\rangle\} \cup [p_j^\beta, p_{i+1}^\alpha] \cup \{[p_k^\alpha, p_{k+1}^\alpha] \mid k \in \langle i+1, \ldots, n_1-1\rangle\}$, if $p_i^\alpha = p_j^\beta$. Likewise, when intersection happens on an open segment, then p_i appears both in the definition of γ_{opc}^1 and γ_{opc}^2. The wombliness of γ_{opc}^1 and γ_{opc}^2 is summed as $\mathcal{H}(\gamma_{\mathsf{opc}}) = \mathcal{H}(\gamma_{\mathsf{opc}}^1) + \mathcal{H}(\gamma_{\mathsf{opc}}^2)$. We need to change the sign of the wombliness of the segments whose directions are changed.



(a) INTERSECTION ON A SEGMENT (b) INTERSECTION AT A POINT

Figure 7. DIFFERENT-SIDE INTERSECTIONS.

Agent re-positioning: If $\mathcal{H}(\gamma_{\mathsf{opc}}^\alpha)$ and $\mathcal{H}(\gamma_{\mathsf{opc}}^\beta)$ have a different sign, we only keep the curve whose wombliness is larger in absolute value. Without loss of generality, assume the curve we keep is $\gamma_{\mathsf{opc}}^\beta$. Then, we re-position the agents in $\gamma_{\mathsf{opc}}^\alpha$ along the boundary of $\gamma_{\mathsf{opc}}^\beta$. This process does not affect the value of the wombliness of $\gamma_{\mathsf{opc}}^\beta$, and can be made in an arbitrary way. Note that the absolute value of the wombliness of the resulting non-self-intersecting curve is strictly larger than the value of the wombliness of $\gamma_{\mathsf{opc}}^\alpha$ and $\gamma_{\mathsf{opc}}^\beta$.

Curve rearrangement: When the intersection happens at a point, if $\mathcal{H}(\gamma_{\mathsf{opc}}^\alpha)$ and $\mathcal{H}(\gamma_{\mathsf{opc}}^\beta)$ have the same sign, then we rearrange the original curves $\gamma_{\mathsf{opc}}^\alpha$ and $\gamma_{\mathsf{opc}}^\beta$ into curves γ_{opc}^1 and γ_{opc}^2, see Fig. 7. After the rearrangement, the curves evolve according to (2). If the intersection occurs on an open segment, we use the discontinuous control law in (3).

Intersection Between Open And Closed Curves

For simplicity, we only treat the case of two curves in $\mathcal{D}$. Let $\gamma_{\mathsf{cpc}}^\beta$ be a closed curve determined by n_2 agents at positions $P^\beta = (p_1^\beta, \ldots, p_{n_2}^\beta)$ and wombliness $\mathcal{H}_c^\beta(P^\beta) = \mathcal{H}(\gamma_{\mathsf{cpc}}^\beta)$. There are three different kinds of intersections. The first case is when a point of the open curve intersects on a segment of the closed curve. The second case is when a point of the closed curve intersects on a segment of the open curve. The last case is when the intersection happens at a point of the two curves.

Intersection At A Point Of $\gamma_{\mathsf{opc}}^\alpha$ And On An Open Segment Of $\gamma_{\mathsf{cpc}}^\beta$. For each $i \in \{1, \ldots, n_1\}$ such that $p_i^\alpha \in]p_j^\beta, p_{j+1}^\beta[$ for some $j \in \{1, \ldots, n_2\}$, define $\lambda \in [0,1)$ by $p_i^\alpha = (1-\lambda)p_j^\beta + \lambda p_{j+1}^\beta$. For $k \in \{j, j+1\}$, consider $v_i = (1-\lambda)u_j + \lambda u_{j+1}$, $u_i = \mathrm{sgn}(\mathcal{H}_o^\alpha(P^\alpha))\frac{\partial \mathcal{H}_o^\alpha}{\partial p_i^\alpha}$ and $u_k = \mathrm{sgn}(\mathcal{H}_c^\beta(P^\beta))\frac{\partial \mathcal{H}_c^\beta}{\partial p_k^\beta}$.

$\gamma_{\mathsf{opc}}^\alpha$ belongs to Inside$_{\gamma_{\mathsf{cpc}}^\beta}$ In this case, there exists the possibility of p_i^α crossing from Inside$_{\gamma_{\mathsf{cpc}}^\beta}$ to Outside$_{\gamma_{\mathsf{cpc}}^\beta}$. The criterium to identify if a transition is needed in the network configuration is the same as that for the intersection between open curves on a segment when $\gamma_{\mathsf{opc}}^\alpha$ belongs to Leftside$_{\gamma_{\mathsf{opc}}^\beta}$.

$\gamma_{\mathsf{opc}}^\alpha$ belongs to Outside$_{\gamma_{\mathsf{cpc}}^\beta}$ In this case, there exists the possibility of p_i^α crossing from Outside$_{\gamma_{\mathsf{cpc}}^\beta}$ to Inside$_{\gamma_{\mathsf{cpc}}^\beta}$. The criterium to identify a transition is the same as that for the intersection between open curves on a segment when $\gamma_{\mathsf{opc}}^\alpha$ belongs to Rightside$_{\gamma_{\mathsf{opc}}^\beta}$. The condition when a vertex of a closed curve intersects on a segment of an open curve is the same as above.

Intersection At A Point Of $\gamma_{\mathsf{opc}}^\alpha$ And $\gamma_{\mathsf{cpc}}^\beta$. For each $i \in \{1, \ldots, n_1\}$ and $j \in \{1, \ldots, n_2\}$ such that $p_i^\alpha = p_j^\beta$, consider $u_i = \mathrm{sgn}(\mathcal{H}_o(P^\alpha))\,\mathrm{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}_o^\alpha}{\partial p_i^\alpha}\right)$ and $u_j = \mathrm{sgn}(\mathcal{H}_c(P^\beta))\,\mathrm{pr}_{T\mathcal{D}}\left(\frac{\partial \mathcal{H}_c^\beta}{\partial p_j^\beta}\right)$.

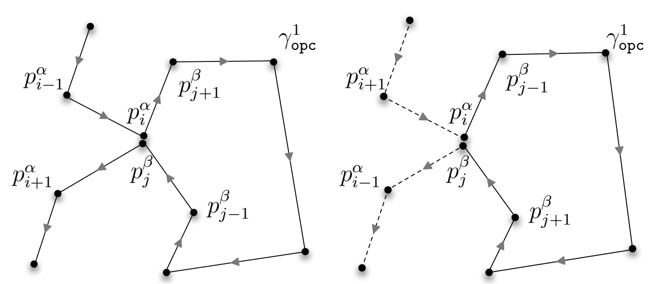
$\gamma_{\mathsf{opc}}^\alpha$ belongs to Inside$_{\gamma_{\mathsf{cpc}}^\beta}$ For v_1 and v_2 as in (4), the criterium to identify a transition is the same as that for the intersection between open curves at a point when $\gamma_{\mathsf{opc}}^\alpha$ belongs to Leftside$_{\gamma_{\mathsf{opc}}^\beta}$.

$\gamma^{\alpha}_{\mathrm{opc}}$ belongs to Outside$_{\gamma^{\beta}_{\mathrm{cpc}}}$ For v_1 and v_2 as in (5), the criterium for a transition is the same as that for the intersection between open curves at a point when $\gamma^{\alpha}_{\mathrm{opc}}$ belongs to Rightside$_{\gamma^{\beta}_{\mathrm{opc}}}$.

State Transition. We have encountered above the necessity to deal with intersections between the curves $\gamma^{\alpha}_{\mathrm{opc}}$ and $\gamma^{\beta}_{\mathrm{cpc}}$. For simplicity, we begin by considering the case where there is only one agent causing the intersection.

Same-side intersection In this case, the two original curves can be merged into one open curve $\gamma^{1}_{\mathrm{opc}}$, defined by the concatenation of $\{[p^{\alpha}_k, p^{\alpha}_{k+1}] \mid k \in \langle 1,\dots,i-1\rangle\} \cup [p^{\alpha}_i, p^{\beta}_{j+1}] \cup \{[p^{\beta}_k, p^{\beta}_{k+1}] \mid k \in \langle j+1,\dots,j-1\rangle\} \cup [p^{\beta}_j, p^{\alpha}_i] \cup \{[p^{\alpha}_k, p^{\alpha}_{k+1}] \mid k \in \langle i,\dots,n_1-1\rangle\}$, if $p^{\alpha}_i \in]p^{\beta}_j, p^{\beta}_{j+1}[$ and $\{[p^{\alpha}_k, p^{\alpha}_{k+1}] \mid k \in \langle 1,\dots,i-1\rangle\} \cup [p^{\alpha}_i, p^{\beta}_{j+1}] \cup \{[p^{\beta}_k, p^{\beta}_{k+1}] \mid k \in \langle j+1,\dots,j-1\rangle\} \cup [p^{\beta}_j, p^{\alpha}_{i+1}] \cup \{[p^{\alpha}_k, p^{\alpha}_{k+1}] \mid k \in \langle i+1,\dots,n_1-1\rangle\}$, if $p^{\alpha}_i = p^{\beta}_j$.

Different-side intersection Here, we change the direction of $\gamma^{\alpha}_{\mathrm{opc}}$, and the resulting curves are merged into an open curve $\gamma^{1}_{\mathrm{opc}}$, defined by the concatenation of $\{[p^{\alpha}_k, p^{\alpha}_{k-1}] \mid k \in \langle n_1,\dots,i+1\rangle\} \cup [p^{\alpha}_i, p^{\beta}_{j+1}] \cup \{[p^{\beta}_k, p^{\beta}_{k+1}] \mid k \in \langle j+1,\dots,j-1\rangle\} \cup [p^{\beta}_j, p^{\alpha}_i] \cup \{[p^{\alpha}_k, p^{\alpha}_{k-1}] \mid k \in \langle i,\dots,2\rangle\}$, if $p^{\alpha}_i \in]p^{\beta}_j, p^{\beta}_{j+1}[$ and $\{[p^{\alpha}_k, p^{\alpha}_{k-1}] \mid k \in \langle n_1,\dots,i+1\rangle\} \cup [p^{\alpha}_i, p^{\beta}_{j+1}] \cup \{[p^{\beta}_k, p^{\beta}_{k+1}] \mid k \in \langle j+1,\dots,j-1\rangle\} \cup [p^{\beta}_j, p^{\alpha}_{i-1}] \cup \{[p^{\alpha}_k, p^{\alpha}_{k-1}] \mid k \in \langle i-1,\dots,2\rangle\}$, if $p^{\alpha}_i = p^{\beta}_j$.



(a) INTERSECTION AT A POINT OF $\gamma^{\alpha}_{\mathrm{OPC}}$, ON A SEGMENT OF $\gamma^{\beta}_{\mathrm{CPC}}$ (b) INTERSECTION AT A POINT OF $\gamma^{\alpha}_{\mathrm{OPC}}$ AND $\gamma^{\beta}_{\mathrm{CPC}}$

Figure 8. CURVES MERGING:$\gamma^{\alpha}_{\mathrm{OPC}}$ AND $\gamma^{\beta}_{\mathrm{CPC}}$ MERGE INTO ONE OPEN CURVE. (a) SHOWS SAME-SIDE INTERSECTION ON A SEGMENT. (b) SHOWS DIFFERENT-SIDE INTERSECTION AT A POINT.

For both same-side and different-side intersections, when dealing with a curve intersection at an open segment, i.e., p_i in $]p_j, p_{j+1}[$, the node p_i appears both in the definition of $\gamma^{1}_{\mathrm{opc}}$. The wombliness of $\gamma^{1}_{\mathrm{opc}}$ is as $\mathcal{H}(\gamma^{1}_{\mathrm{opc}}) = \mathcal{H}(\gamma^{\alpha}_{\mathrm{opc}}) + \mathcal{H}(\gamma^{\beta}_{\mathrm{opc}})$. Note that for this type of intersections, we need to change the sign of the wombliness of the segments whose directions are changed.

Agent re-positioning: If $\mathcal{H}(\gamma^{\alpha}_{\mathrm{opc}})$ and $\mathcal{H}(\gamma^{\beta}_{\mathrm{cpc}})$ have a different sign, we only keep the curve whose wombliness is larger in absolute value. Assume we keep $\gamma^{\beta}_{\mathrm{cpc}}$. Then, we re-position the agents in $\gamma^{\alpha}_{\mathrm{opc}}$ along the boundary of $\gamma^{\beta}_{\mathrm{cpc}}$. This process does not affect the value of the wombliness of $\gamma^{\beta}_{\mathrm{cpc}}$, and can be made arbitrarily. The absolute value of the wombliness of the resulting non-self-intersecting curve is strictly larger than the value of the wombliness of $\gamma^{\alpha}_{\mathrm{opc}}$ and $\gamma^{\beta}_{\mathrm{cpc}}$.

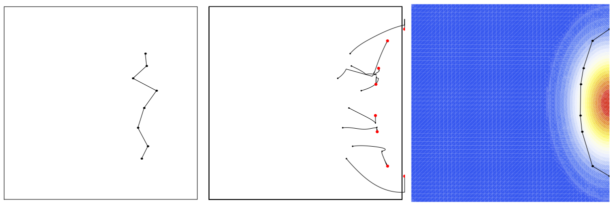
Curve merging: When the intersection occurs at a point, if $\mathcal{H}(\gamma^{\alpha}_{\mathrm{opc}})$ and $\mathcal{H}(\gamma^{\beta}_{\mathrm{cpc}})$ have the same sign, we merge the original curves $\gamma^{\alpha}_{\mathrm{opc}}$ and $\gamma^{\beta}_{\mathrm{cpc}}$ into a new open curves $\gamma^{1}_{\mathrm{opc}}$. see Fig. 8. After the merging, the curve evolves according to (2). When the intersection occurs on an open segment, we use the same discontinuous control law described in (3).

CONVERGENCE ANALYSIS

Here, we characterize the convergence properties of the wombling algorithm. Before stating the main result, we introduce some necessary notation. Let Σ be the set of piecewise constant signals $\sigma(t) : [0,+\infty[\to \mathbb{Z}_{>0}$. For $h > 0$, let Σ_{dwell} be the set of all switching signals with dwell time h. Consider the constrained switched system pair $(\mathscr{F}, \Upsilon)$, where $\mathscr{F}$ is a finite family of continuous vector fields and Υ is a map from $\mathcal{D}^n$ to $\mathscr{P}(\Sigma)$, where $\mathscr{P}(\Sigma)$ is the power set of Σ.

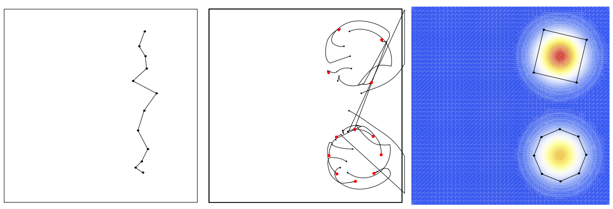
Theorem 0.3 *The evolution of a robotic network under the wombling algorithm monotonically optimizes the total wombliness of the spatial field. Moreover, assume $\Upsilon(P_0) \subset \Sigma_{dwell}$ for all $P_0 \in \mathcal{S}$. Then each of the subnetworks that are solutions of $(\mathscr{F}, \Upsilon(P_0))$ converges to a critical configuration of the spatial wombliness.*

We only provide a proof sketch for space reasons. Given n agents, there is a finite number of possibilities to divide the network into subgroups of two or more agents. In each case, there exists a region of $\mathcal{D}^n$ where the wombling algorithm is a well-defined vector field. With this information, we use the multiple weak Lyapunov functions method in [15] to analyze the convergence of the robotic network. Given a subdivision $\mathfrak{a}$ of the network into groups $\{g_1,\dots,g_k\}$, we consider the wombliness measure $\mathcal{H}_{g_\ell}$ associated to each group of agents, $\ell \in \{1,\dots,k\}$, and associate to $\mathfrak{a}$ the Lyapunov function $V_{\mathfrak{a}} = \sum_{\ell=1}^{k} |\mathcal{H}_{g_\ell}|$. According to wombling algorithm, the absolute value of the wombliness of the subnetworks can only but increase when the network undergoes curve-splitting and curve-merging or when the agents in the network are re-positioned. Hence, the algorithm monotonically optimizes the wombliness of the robotic network, and the collection of functions $V_{\mathfrak{a}}$ is a set of multiple weak Lyapunov functions. Using [15, Theorem 2] and [14, Theorem 4.3], the solutions of the system converge to the set of critical points of the wombliness of the spatial field.



(a) (b) (c)

Figure 9. SIMULATION OF ROBOTIC NETWORK WITH 8 AGENTS.



(a) (b) (c)

Figure 10. SIMULATION OF ROBOTIC NETWORK WITH 12 AGENTS.

Remark 0.4 Regarding Theorem 0.3, note that, as the number of agents increase, the network obtains increasingly accurate approximations of the corresponding critical curve of $\mathcal{H}$. Secondly, the convergence result is local. This means that, depending on the initial agent configuration, some areas of abrupt change in the environment might not be detected by the network. •

Fig. 9 and Fig. 10 present illustrations of the execution of the wombling algorithm with 8 and 12 agents, resp. The domain is $\mathcal{D} = [-4,4] \times [-4,4]$. Fig. 9-10(a) show the initial configuration, (b) show the robot trajectories, and (c) show the final configuration. In Fig. 9, the spatial field is $Y(x_1,x_2) = e^{-(x_1-4)^2 - x_2^2/5}$. The algorithm triggers an outside right-side self-intersection that results in agents repositioning. In Fig. 10, the spatial field is $Y(x_1,x_2) = e^{-(x_1-2)^2-(x_2+2)^2/5} + 1.25e^{-(x_1-2)^2-(x_2-2)^2/5}$. The algorithm first triggers two outside right-side self-intersections where the discontinuous law 3 kicks in. The ensuing evolution leads to two additional self-intersections at a point, resulting in curve splittings. After this, the lower curve undergoes an outside right-side self-intersection that results in agents repositioning.

CONCLUSIONS

We have studied robotic sensor networks whose objective is to detect areas of rapid change in spatial phenomena. After formulating the aggregate wombliness function and characterizing its smoothness properties, we have combined ideas from discontinuous dynamics and hybrid modeling to synthesize a provably correct distributed algorithm. The wombling algorithm allows for network splitting, merging, and agent re-positioning while monotonically increasing the wombliness. We have characterized its asymptotic convergence properties. Future work will explore the extension of the results to scenarios with noise, dynamic boundaries, evolutions in three dimensions, and the incorporation of ideas from active contours.

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