# When does a digraph admit a doubly stochastic adjacency matrix?

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*Abstract*— **Digraphs with doubly stochastic adjacency matrices play an essential role in a variety of cooperative control problems including distributed averaging, optimization, and gossiping. In this paper, we fully characterize the class of digraphs that admit an edge weight assignment that makes the digraph adjacency matrix doubly stochastic. As a by-product of our approach, we also unveil the connection between weight-balanced and doubly stochastic adjacency matrices. Several examples illustrate our results.**

## I. INTRODUCTION

A digraph is doubly stochastic if, at each vertex, the sum of the weights of the incoming edges as well as the sum of the weights of the outgoing edges are equal to one. Doubly stochastic digraphs play a key role in networked control problems. Examples include distributed averaging [1], [2], [3], [4], distributed convex optimization [5], [6], [7], and gossip algorithms [8], [9]. Because of the numerous algorithms available in the literature that use doubly stochastic interaction topologies, it is an important research question to characterize when a digraph can be given a nonzero edge weight assignment that makes it doubly stochastic. This is the question we investigate in this paper.

We refer to a digraph as doubly stochasticable if it admits a doubly stochastic adjacency matrix. In studying this class of digraphs, we unveil their close relationship with a special class of weight-balanced digraphs. A digraph is weight-balanced if, at each node, the sum of the weights of the incoming edges equals the sum of the weights of the outgoing edges. The notion of weight-balanced digraph is key in establishing convergence results of distributed algorithms for average-consensus [10], [2] and consensus on general functions [11] via Lyapunov stability analysis. Weight-balanced digraphs also appear in the design of leader-follower strategies under time delays [12], virtual leader strategies under asymmetric interactions [13] and stable flocking algorithms for agents with significant inertial effects [14]. We call a digraph weight-balanceable if it admits an edge weight assignment that makes it weight-balanced. A characterization of weight-balanceable digraphs was presented in [15].

In this paper, we provide a constructive a necessary and sufficient condition for a digraph to be doubly stochasticable. This condition involves a very particular structure that the cycles of the digraph must enjoy. As a by-product of our characterization, we unveil the connection between the structure of doubly stochasticable digraphs and a special class

of weight-balanced digraphs. To the authors' knowledge, the establishment of this relationship is also a novel contribution.

The paper is organized as follows. Section II presents some mathematical preliminaries from graph theory. Section III introduces the problem statement. Section IV examines the connection between weight-balanced and doubly stochastic adjacency matrices. Section V gives necessary and sufficient conditions for the existence of a doubly stochastic adjacency matrix assignment for a given digraph. Section VI studies the properties of the topological character associated with strongly connected doubly stochasticable digraphs. We gather our conclusions and ideas for future work in Section VII.

## II. MATHEMATICAL PRELIMINARIES

We adopt some basic notions from [1], [16], [17]. A *directed graph*, or simply *digraph*, is a pair $G = (V, E)$, where $V$ is a finite set called the vertex set and $E \subseteq V \times V$ is the edge set. If $|V| = n$, i.e., the cardinality of $V$ is $n \in \mathbb{Z}_{>0}$, we say that $G$ is of order $n$. We say that an edge $(u, v) \in E$ is *incident away from* $u$ and *incident toward* $v$, and we call $u$ an *in-neighbor* of $v$ and $v$ an *out-neighbor* of $u$. The *in-degree* and *out-degree* of $v$, denoted $d_{\text{in}}(v)$ and $d_{\text{out}}(v)$, are the number of in-neighbors and out-neighbors of $v$, respectively. We call a vertex $v$ *isolated* if it has zero in- and out-degrees.

An *undirected graph*, or simply *graph*, is a pair $G = (V, E)$, where $V$ is a finite set called the vertex set and the edge set $E$ consists of unordered pairs of vertices. In a graph, neighboring relationships are always bidirectional, and hence we simply use neighbor, degree, etc. for the notions introduced above. A graph is *regular* if each vertex has the same number of neighbors. The *union* $G_1 \cup G_2$ of digraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. The intersection of two digraphs can be defined similarly. A digraph $G$ is *generated* by a set of digraphs $G_1, \ldots, G_m$ if $G = G_1 \cup \cdots \cup G_m$. We let $E^- \subseteq V \times V$ denote the set obtained by changing the order of the elements of $E$, i.e., $(v, u) \in E^-$ if $(u, v) \in E$. The digraph $\overline{G} = (V, E \cup E^-)$ is the *mirror* of $G$.

A *weighted digraph* is a triplet $G = (V, E, A)$, where $(V, E)$ is a digraph and $A \in \mathbb{R}_{\geq 0}^{n \times n}$ is the *adjacency matrix*. We denote the entries of $A$ by $a_{ij}$, where $i, j \in \{1, \ldots, n\}$. The adjacency matrix has the property that the entry $a_{ij} > 0$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$, otherwise. If a matrix $A$ satisfies this property, we say that $A$ can be assigned to the digraph $G = (V, E)$. Note that any digraph can be trivially seen as a weighted digraph by assigning weight 1 to each one of its edges. We will find it useful to extend the

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definition of union of digraphs to weighted digraphs. The union $G_1 \cup G_2$ of weighted digraphs $G_1 = (V_1, E_1, A_1)$ and $G_2 = (V_2, E_2, A_2)$ is defined by $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2, A)$, where

$$A|_{V_1 \cap V_2} = A_1|_{V_1 \cap V_2} + A_2|_{V_1 \cap V_2},$$
$$A|_{V_1 \setminus V_2} = A_1, \ A|_{V_2 \setminus V_1} = A_2.$$

For a weighted digraph, the weighted out-degree and in-degree are, respectively,

$$d_{\text{out}}^{\text{w}}(v_i) = \sum_{j=1}^{n} a_{ij}, \qquad d_{\text{in}}^{\text{w}}(v_i) = \sum_{j=1}^{n} a_{ji}.$$

### A. Graph connectivity notions

A *directed path* in a digraph, or in short path, is an ordered sequence of vertices so that any two consecutive vertices in the sequence are an edge of the digraph. A *cycle* in a digraph is a directed path that starts and ends at the same vertex and has no other repeated vertex. Two cycles are *disjoint* if they do not have any vertex in common.

A digraph is *strongly connected* if there is a path between each pair of distinct vertices and is *strongly semiconnected* if the existence of a path from $v$ to $w$ implies the existence of a path from $w$ to $v$, for all $v, w \in V$. Clearly, strongly connectedness implies strongly semiconnectedness, but the converse is not true. The *strongly connected components* of a directed graph $G$ are its maximal strongly connected subdigraphs.

### B. Basic notions from linear algebra

A matrix $A \in \mathbb{R}_{\geq 0}^{n \times n}$ is *weight-balanced* if $\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ji}$, for all $i \in \{1, \ldots, n\}$. A matrix $A \in \mathbb{R}_{\geq 0}^{n \times n}$ is *row-stochastic* if each of its rows sums 1. One can similarly define a column-stochastic matrix. We denote the set of all row-stochastic matrices on $\mathbb{R}_{\geq 0}^{n \times n}$ by $\text{RStoc}(\mathbb{R}_{\geq 0}^{n \times n})$. A non-zero matrix $A \in \mathbb{R}_{\geq 0}^{n \times n}$ is *doubly stochastic* if it is both row-stochastic and column-stochastic. A matrix $A \in \{0, 1\}^{n \times n}$ is a *permutation matrix*, where $n \in \mathbb{Z}_{\geq 1}$, if $A$ has exactly one entry 1 in each row and each column. A matrix $A \in \mathbb{R}_{\geq 0}^{n \times n}$ is *irreducible* if, for any nontrivial partition $J \cup K$ of the index set $\{1, \ldots, n\}$, there exist $j \in J$ and $k \in K$ such that $a_{jk} \neq 0$. We denote by $\text{Irr}(\mathbb{R}_{\geq 0}^{n \times n})$ the set all irreducible matrices on $\mathbb{R}_{\geq 0}^{n \times n}$. Note that a weighted digraph $G$ is strongly connected if and only if its adjacency matrix is irreducible [17].

### C. Weight-balanced and doubly stochastic digraphs

A weighted digraph $G$ is *weight-balanced* (resp. *doubly stochastic*) if its adjacency matrix is weight-balanced (resp. doubly-stochastic). Note that $G$ is weight-balanced if and only if $d_{\text{out}}^{\text{w}}(v) = d_{\text{in}}^{\text{w}}(v)$, for all $v \in V$. A digraph is called *weight-balanceable* (resp. *doubly stochasticable*) if it admits a weight-balanced (resp. doubly stochastic) adjacency matrix. The following two results establish a constructive *centralized* approach for determining whether a digraph is weight-balanceable, see [15].
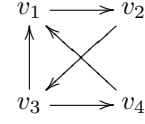


Fig. 1. A weight-balanceable digraph for which their exists no doubly stochastic adjacency assignment.

*Theorem 2.1:* A digraph $G = (V, E)$ is weight-balanceable if and only if the edge set $E$ can be decomposed into $k$ subsets $E_1, \ldots, E_k$ such that

(i) $E = E_1 \cup E_2 \cup \ldots \cup E_k$ and
(ii) every subgraph $G = (V, E_i)$, for $i = \{1, \ldots, k\}$, is a weight-balanceable digraph.

*Theorem 2.2:* Let $G = (V, E)$ be a directed digraph. The following statements are equivalent.

(i) Every element of $E$ lies in a cycle.
(ii) $G$ is weight-balanceable.
(iii) $G$ is strongly semiconnected.

## III. PROBLEM STATEMENT

Our main goal in this paper is to obtain necessary and sufficient conditions that characterize when a digraph is doubly stochasticable. Note that strongly semiconnectedness is a necessary, and sufficient, condition for a digraph to be weight-balanceable. All doubly stochastic digraphs are weight-balanced; thus a necessary condition for a digraph to be doubly stochasticable is strongly semiconnectedness. Moreover, weight-balanceable digraphs that are doubly stochasticable do not have any isolated vertex. However, none of these conditions is sufficient. A simple example illustrates this. Consider the digraph $G$ shown in Figure 1. Note that this digraph is strongly connected; thus there exists a set of positive weights which makes the digraph weight-balanced. However, there exists no set of nonzero weights that makes this digraph doubly stochastic. Suppose

$$A = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ \alpha_3 & 0 & 0 & \alpha_4 \\ \alpha_5 & 0 & 0 & 0 \end{pmatrix},$$

where $\alpha_i \in \mathbb{R}_{>0}$, for all $i \in \{1, \ldots, 5\}$, is a doubly stochastic adjacency assignment for this digraph. Then a simple computation shows that the only solution that makes the digraph doubly stochastic is by choosing $\alpha_3 = 0$, which is not possible by assumption. Thus this digraph is not doubly stochasticable.

The following result will simplify our analysis by allowing us to restrict our attention to strongly connected digraphs.

*Lemma 3.1:* A strongly semiconnected digraph is doubly stochasticable if and only if all of its strongly connected components are doubly stochasticable.

*Proof:* Let $G_1$ and $G_2$ be two strongly connected components of the digraph. Note that there can be no edges from $v_1 \in G_1$ to $v_2 \in G_2$ (or vice versa). If this were the case, then the strongly semiconnectedness of the digraph would imply

that there is a path from $v_2$ to $v_1$ in the digraph, and hence $G_1 \cup G_2$ would be strongly connected, contradicting the fact that $G_1$ and $G_2$ are maximal. Therefore, the adjacency matrix of the digraph is a block-diagonal matrix, where each block corresponds to the adjacency matrix of a strongly connected component, and the result follows. $\blacksquare$

As a result of Lemma 3.1, we are interested in characterizing the class of strongly connected digraphs which are doubly stochasticable.

## IV. THE RELATIONSHIP BETWEEN WEIGHT-BALANCED AND DOUBLY STOCHASTIC ADJACENCY MATRICES

As an intermediate step of the characterization of doubly stochasticable digraphs, we will find it useful to study the relationship between weight-balanced and doubly stochastic digraphs. The example in Figure 1 underscores the importance of characterizing the set of weight-balanceable digraphs that are also doubly-stochasticable.

We start by introducing the *row-stochastic normalization map* $\phi : \mathrm{Irr}(\mathbb{R}_{\geq 0}^{n \times n}) \to \mathrm{RStoc}(\mathbb{R}_{\geq 0}^{n \times n})$ defined by

$$\phi : a_{ij} \mapsto \frac{a_{ij}}{\sum_{l=1}^{n} a_{il}}.$$

Note that, for $A \in \mathrm{Irr}(\mathbb{R}_{\geq 0}^{n \times n})$, $\phi(A)$ is doubly stochastic if and only if

$$\sum_{i=1}^{n} \frac{a_{ij}}{\sum_{l=1}^{n} a_{il}} = 1,$$

for all $j \in \{1, \ldots, n\}$. The following result characterizes when the digraph associated with an irreducible weight-balanced adjacency matrix is doubly stochasticable.

*Theorem 4.1:* Let $A \in \mathrm{Irr}(\mathbb{R}_{\geq 0}^{n \times n})$ be an adjacency matrix associated to a weight-balanced digraph. Then $\phi(A)$ is doubly stochastic if and only if $\sum_{l=1}^{n} a_{il} = C$, for all $i \in \{1, \ldots, n\}$, for some $C \in \mathbb{R}_{>0}$.

*Proof:* The implication from right to left is immediate. Suppose then that $A$ is associated to a strongly connected weight-balanced digraph. Then we need to show that if $A$ satisfies the following set of equations

$$\sum_{l=1}^{n} a_{jl} = \sum_{l=1}^{n} a_{lj},$$

$$\sum_{i=1}^{n} \frac{a_{ij}}{\sum_{l=1}^{n} a_{il}} = 1,$$

for all $j \in \{1, \ldots, n\}$, there exists $C \in \mathbb{R}_{>0}$ such that $\sum_{l=1}^{n} a_{il} = C$, for all $i \in \{1, \ldots, n\}$. Let $C_k = \sum_{l=1}^{n} a_{kl}$, $k \in \{1, \ldots, n\}$. Then the doubly stochastic conditions can be written as

$$\frac{a_{1j}}{C_1} + \frac{a_{2j}}{C_2} + \cdots + \frac{a_{nj}}{C_n} = 1, \quad (1)$$

for all $j \in \{1, \ldots, n\}$. Note that $C_k \neq 0$, for all $k \in \{1, \ldots, n\}$, since $A$ is irreducible. By the weight-balanced assumption, we have

$$a_{1j} + a_{2j} + \cdots + a_{nj} = C_j,$$

for all $j \in \{1, \ldots, n\}$. Thus

$$\frac{a_{1j}}{C_j} + \frac{a_{2j}}{C_j} + \cdots + \frac{a_{nj}}{C_j} = 1, \quad (2)$$

From Equations (1) and (2), we have

$$a_{1j}\left(\frac{1}{C_1} - \frac{1}{C_j}\right) + \cdots + a_{nj}\left(\frac{1}{C_n} - \frac{1}{C_j}\right) = 0, \quad (3)$$

for all $j \in \{1, \ldots, n\}$. Suppose that, up to rearranging,

$$C_1 = \min_{k}\{C_k \mid k \in \{1, \ldots, n\}\},$$

and, $0 < C_1 < C_i$, for all $i \in \{2, \ldots, n\}$. Then (3) gives

$$a_{21}\left(\frac{1}{C_2} - \frac{1}{C_1}\right) + \cdots + a_{n1}\left(\frac{1}{C_n} - \frac{1}{C_1}\right) = 0;$$

thus $a_{j1} = 0$, for all $j \in \{2, \ldots, n\}$, which contradicts the irreducibility assumption. If the set $\{C_k\}_{k=1}^{n}$ has more than one element giving the minimum, the proof follows a similar argument. Suppose

$$C_1 = C_2 = \min_{k}\{C_k \mid k \in \{1, \ldots, n\}\},$$

and suppose that $0 < C_1 = C_2 < C_i$, for all $i \in \{3, \ldots, n\}$. Then we have

$$a_{31}\left(\frac{1}{C_3} - \frac{1}{C_1}\right) + \cdots + a_{n1}\left(\frac{1}{C_n} - \frac{1}{C_1}\right) = 0,$$

and

$$a_{32}\left(\frac{1}{C_3} - \frac{1}{C_2}\right) + \cdots + a_{n2}\left(\frac{1}{C_n} - \frac{1}{C_2}\right) = 0,$$

and thus $a_{j1} = 0 = a_{j2}$, for all $j \in \{3, \ldots, n\}$, which contradicts the irreducibility assumption. The same argument holds for an arbitrary number of minima. $\blacksquare$

*Corollary 4.2:* Any strongly connected digraph is doubly stochasticable after adding enough number of self-loops.

*Proof:* Any strongly connected digraph is weight-balanceable. The result follows from noting that, for any weight-balanced matrix, it is enough to add self-loops with appropriate weights to the vertices of the digraph to make the conditions of Theorem 4.1 hold. $\blacksquare$

Regular undirected graphs trivially satisfy the conditions of Theorem 4.1 and hence the following result.

*Corollary 4.3:* All undirected regular graphs are doubly stochasticable.

## V. NECESSARY AND SUFFICIENT CONDITIONS FOR DOUBLY STOCHASTICABILITY

In this section, we provide a characterization of the structure of digraphs that are doubly stochasticable. The main contributions are Theorem 5.4 and Corollary 5.5.

Let $G = (V, E)$ be a strongly semiconnected digraph. Let $G_{\mathrm{cyc}}$ denote a union of some disjoint cycles of $G$ (note that $G_{\mathrm{cyc}}$ can be just one cycle). One can extend the adjacency matrix associated to $G_{\mathrm{cyc}}$ to a matrix $A_{\mathrm{cyc}} \in \mathbb{R}^{n \times n}$, by adding zero rows and columns for the vertices of $G$ that are not included in $G_{\mathrm{cyc}}$. Note that the matrix $A_{\mathrm{cyc}}$ is the adjacency matrix for a subdigraph of $G$. We call $A_{\mathrm{cyc}}$ *the extended*

*adjacency matrix associated to* $G_{\text{cyc}}$. We have the following result.

*Lemma 5.1:* The extended adjacency matrix associated to $G_{\text{cyc}}$ is a permutation matrix if and only if $G_{\text{cyc}}$ contains all the vertices of $G$.

*Proof:* It is clear that if $G_{\text{cyc}}$ is a union of some disjoint cycles and contains all the vertices of $G$, then the adjacency matrix associated to $G_{\text{cyc}}$ is a permutation matrix. Conversely, suppose that $G_{\text{cyc}}$ does not contain one of the vertices of $G$. Then the adjacency matrix associated to $G_{\text{cyc}}$ has a zero row and thus is not a permutation matrix. ∎

By Theorem 2.2, any strongly semiconnected digraph can be generated by the cycles contained in it. Thus it makes sense to define a minimal set of such cycles that can generate the digraph. That is what we define next.

*Definition 5.2:* Let $G = (V, E)$ be a strongly semiconnected digraph. Let $\mathcal{C}(G)$ denote the set of all subdigraphs of $G$ that are either isolated vertices, cycles of $G$, or a union of disjoint cycles of $G$. $P(G) \subseteq \mathcal{C}(G)$ is a *principal cycle set* of $G$ if its elements generate $G$, and there is no subset of $\mathcal{C}(G)$ with strictly smaller cardinality that satisfies this property.

Note that there might exist more than one principal set, however, by definition, the cardinalities of all principal cycle sets are the same. We denote this cardinality by $p(G)$. Principal cycle sets give rise to weight-balanced assignments.

*Proposition 5.3:* Let $G = (V, E)$ be a strongly semiconnected digraph. Then, the union of the elements of a principal cycle set of $G$, considered as subdigraphs with trivial weight assignment, gives a set of positive integer weights which make the digraph weight-balanced.

*Proof:* Since each element of a principal cycle $P(G)$ is either an isolated vertex, a cycle, or union of disjoint cycles, it is weight-balanced. By definition, $G$ can be written as the union of the elements of $P(G)$. Thus by Theorem 2.1, the weighted union of the elements of $P(G)$ gives a set of weights makes the digraph $G$ weight-balanced. ∎

Note that, in general, the assignment in Proposition 5.3 uses fewer number of cycles than the ones used in Theorem 2.2. Note that a cycle, or a union of disjoint cycles, that contains all the vertices has the maximum number of edges that an element of $\mathcal{C}(G)$ can have. Thus these elements are the obvious candidates for constructing a principal cycle set. Next, we state a necessary condition for a digraph to be doubly stochasticable.

*Theorem 5.4:* Let $G$ be a strongly semiconnected digraph. Suppose that one can assign a doubly stochastic adjacency matrix $A$ to $G$. Then

$$A = \sum_{i=1}^{\xi} \lambda_i A_{\text{cyc}}^i,$$

where

- $\{\lambda_i\}_{i=1}^{\xi} \subset \mathbb{R}_{>0}$, $\sum_{i=1}^{\xi} \lambda_i = 1$, and $\xi \geq p(G)$.
- $A_{\text{cyc}}^i$, $i \in \{1, \ldots, \xi\}$, is the extended adjacency matrix associated to an element of $\mathcal{C}(G)$ that contains all the vertices.

*Proof:* Since one can assign a doubly stochastic adjacency matrix to $G$, the digraph cannot have any isolated vertex. Let $A$ be a doubly stochastic matrix associated to $G$. By the Birkhoff–von Neumann theorem [18], a square matrix is doubly stochastic if and only if it is a convex combination of permutation matrices. Therefore,

$$A = \sum_{i=1}^{n!} \bar{\lambda}_i A_{\text{perm}}^i,$$

where $\bar{\lambda}_i \in \mathbb{R}_{\geq 0}$, $\sum_{i=1}^{n!} \bar{\lambda}_i = 1$, and $A_{\text{perm}}^i$ is a permutation matrix for each $i \in \{1, \ldots, n!\}$. By Lemma 5.1, for all $\bar{\lambda}_i > 0$, one can associate to the corresponding $A_{\text{perm}}^i$ a union of disjoint cycles that contains all the vertices. Thus each $A_{\text{perm}}^i$ is an extended adjacency matrix associated to an element of $\mathcal{C}(G)$. Let us rename all the nonzero coefficients $\bar{\lambda}_i > 0$ by $\lambda_i$. In order to complete the proof, we need to show that at least $p(G)$ of the $\lambda_i$'s are nonzero. Suppose otherwise. Since each $A_{\text{perm}}^i$ with nonzero coefficient is associated to an element of $\mathcal{C}(G)$, this means that the digraph $G$ can be generated by fewer elements than $p(G)$, which contradicts Definition 5.2. ∎

The following result fully characterizes the set of strongly connected doubly stochasticable digraphs.

*Corollary 5.5:* A strongly connected digraph $G$ is doubly stochasticable if and only if there exists a set $\{G_{\text{cyc}}^i\}_{i=1}^{\xi} \subseteq \mathcal{C}(G)$, where $\xi \geq p(G)$, that generates $G$ and such that $G_{\text{cyc}}^i$ contains all the vertices of $G$, for each $i \in \{1, \ldots, \xi\}$.

*Proof:* Suppose $G$ is doubly stochasticable. By Theorem 5.4, $A$ can be written as the union of at least $p(G)$ elements of $\mathcal{C}(G)$ which contain all the vertices and generate $G$. This proves the implication from left to right. Suppose $G = \cup_{i=1}^{\xi} G_{\text{cyc}}^i$, where $G_{\text{cyc}}^i \in \mathcal{C}(G)$ contain all the vertices, for all $i \in \{1, \ldots, \xi\}$. Consider the adjacency matrix

$$A = \sum_{i=1}^{\xi} A_{\text{cyc}}^i,$$

where $A_{\text{cyc}}^i$ is the extended adjacency matrix associated to $G_{\text{cyc}}^i$. Note that $A$ is weight-balanced and satisfies the conditions of Theorem 4.1 (the sum of each row is equal to $\xi$). Thus $G$ is doubly stochasticable. ∎

Corollary 5.5 suggests the definition of the following notion. Given a strongly connected doubly stochasticable digraph $G$, $DS(G) \subseteq \mathcal{C}(G)$ is a *DS-cycle set* of $G$ if all its elements contain all the vertices of $G$, they generate $G$, and there is no subset of $\mathcal{C}(G)$ with strictly smaller cardinality that satisfies these properties. Corollary 5.5 implies that DS-cycle sets exist for any strongly connected doubly stochasticable digraph. The cardinality of any DS-cycle set of $G$ is the *DS-character* of $G$, denoted $\text{ds}(G)$. If a doubly stochastic digraph is not strongly connected, one can use this notion on each strongly connected component.

*Example 5.6: (Weight-balanceable, not doubly stochasticable digraph):* Consider the digraph $G$ shown in Figure 2(a). It is shown in [19] that there exists a set of weights which makes this digraph weight-balanced. We show that the
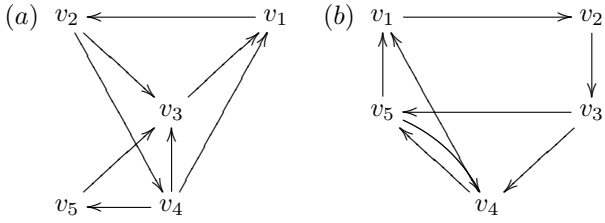
Fig. 2. The digraph of Examples 5.6 and 5.7 are shown in plots (a) and (b), respectively.
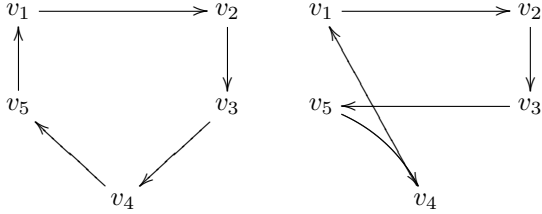


Fig. 3. The only principal cycle set for the digraph of Example 5.7 contains the above cycles.

digraph is not doubly stochasticable. The edge $(v_2, v_3)$ only appears in the cycle $G_{\text{cyc}} = \{v_1, v_2, v_3\}$. Thus this cycle appears in any set of elements of $\mathcal{C}(G)$ that generates $G$. Since $G_{\text{cyc}}$ does not include all the vertices, by Corollary 5.5, there exists no doubly stochastic adjacency assignment for this digraph. One can verify this by trying to find such assignment explicitly, i.e., by seeking $\alpha_i \in \mathbb{R}_{>0}$, where $i \in \{1, \ldots, 8\}$, such that

$$A = \begin{pmatrix} 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & \alpha_3 & 0 \\ \alpha_4 & 0 & 0 & 0 & 0 \\ \alpha_5 & 0 & \alpha_6 & 0 & \alpha_7 \\ 0 & 0 & \alpha_8 & 0 & 0 \end{pmatrix}$$

is doubly stochastic. A simple computation shows that such an assignment is not possible unless $\alpha_2 = \alpha_5 = \alpha_6 = 0$, which is a contradiction. ●

*Example 5.7 (Doubly stochasticable digraph):* Consider the digraph $G$ shown in Figure 2(b). One can observe that the only principal cycle set of $G$ contains the two cycles shown in Figure 3. Both of these cycles pass through all the vertices of the digraph and thus, using Corollary 5.5, this digraph is doubly stochasticable. Note that this digraph has another three cycles, shown in Figure 4, none of which is in the principal cycle set. The adjacency matrix assignment

$$A = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

obtained by the sum of the elements of the principal cycle set, is weight-balanced and satisfies the conditions of Theorem 4.1 and thus is doubly stochasticable. Also note that
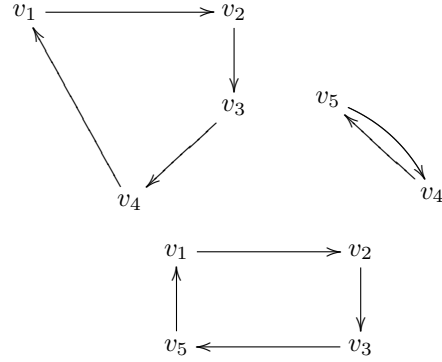


Fig. 4. Cycles of the digraph $G$ of Example 5.7 which are not in the principal cycle set.

not all the weight-balanced adjacency assignments become doubly stochastic under the row-stochastic normalization map. An example is given by the adjacency matrix

$$A = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \\ 2 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 2 & 0 \end{pmatrix}. \qquad \bullet$$

An alternative question to the one considered above would be to find a set of edge weights (some possibly zero) that make the digraph doubly stochastic. Such assignments exist for the digraph in Figure 1. However, such weight assignments are not guaranteed, in general, to preserve the connectivity of the digraph. The following result gives a sufficient condition for the existence of such an edge weight assignment.

*Proposition 5.8:* A strongly connected digraph $G$ admits an edge weight assignment (where some entries might be zero) such that the resulting weighted digraph is strongly connected and doubly stochastic if there exists a cycle containing all the vertices of $G$.

Regarding Proposition 5.8, note that even if connectivity is preserved, fewer edges lead to smaller algebraic connectivity [20], which in turn affects negatively the rate of convergence of the consensus, optimization, and gossip algorithms executed over doubly stochastic digraphs, see e.g., [8], [9], [21], [22].

## VI. PROPERTIES OF THE TOPOLOGICAL CHARACTER OF DOUBLY STOCHASTICABLE DIGRAPHS

In this section, we investigate the properties of DS-cycle sets and of their cardinality $\mathrm{ds}(G)$. We start with the following definition.

*Definition 6.1:* Let $G = (V, E)$ be a strongly connected digraph and let $A$ be a weight-balanced adjacency matrix which satisfies the conditions of Theorem 4.1 with $C \in \mathbb{R}_{>0}$. Then we call the weighted digraph $G = (V, E, A)$ a $C$-*regular digraph.*

We have the following result.

*Theorem 6.2:* Let $G$ be a strongly connected doubly stochasticable digraph of order $n \in \mathbb{Z}_{>0}$ with DS-character $\mathrm{ds}(G)$. Then the following statements hold,

- There exists a weight assignment $A_{\mathrm{wb}} \in \mathbb{Z}_{\geq 0}^{n \times n}$ that makes $G$ a $C$-regular digraph with $C \geq \mathrm{ds}(G)$.
- There exist no integer weight assignment $A_{\mathrm{wb}} \in \mathbb{Z}_{\geq 0}^{n \times n}$ that makes $G$ a $C$-regular digraph with $C < \mathrm{ds}(G)$.

*Proof:* By Corollary 5.5, it is clear that one can generate $A_{\mathrm{wb}} \in \mathbb{Z}_{\geq 0}^{n \times n}$ that makes $G$ weight-balanced and also satisfies the conditions of Theorem 4.1 for $C = \mathrm{ds}(G)$, just by taking the weighted union of the members of a DS-cycle set. Let $C > \mathrm{ds}(G)$. Choose a set of integer numbers $\lambda_i \in \mathbb{Z}_{>0}$, for $i \in \{1, \dots, \mathrm{ds}(G)\}$, such that $\sum_{i=1}^{\mathrm{ds}(G)} \lambda_i = C$. Consider the adjacency matrix

$$A = \sum_{i=1}^{\mathrm{ds}(G)} \lambda_i A_{\mathrm{cyc}}^i,$$

where $A_{\mathrm{cyc}}^i$ is the extended adjacency matrix associated to the $i$th element of the DS-cycle set. The matrix $A$ is weight-balanced adjacency and satisfies the conditions of Theorem 4.1. This proves the first part of the theorem.

Now assume that there exists a weight-balanced adjacency matrix $A \in \mathbb{Z}_{\geq 0}^{n \times n}$ that makes $G$ a $C$-regular digraph with $C < \mathrm{ds}(G)$. Then $\frac{1}{C} A$ is a doubly stochastic adjacency matrix for $G$. Thus using Theorem 5.4

$$\frac{1}{C} A = \sum_{i=1}^{C} \lambda_i A_{\mathrm{cyc}}^i,$$

where $A_{\mathrm{cyc}}^i$, $i \in \{1, \dots, C\}$, is the extended adjacency matrix associated to an element of $\mathcal{C}(G)$ which contains all the vertices. But this contradicts the minimality in the definition of a DS-cycle set. ∎

We finish this section by bounding the DS-character of a digraph.

*Lemma 6.3:* Let $G = (V, E)$ be a strongly connected doubly stochasticable digraph. Then

$$\max_{v \in V} d_{\mathrm{out}}(v) \leq \mathrm{ds}(G) \leq |E| - |V| + 1.$$

*Proof:* The first inequality follows from the fact that none of the out-edges of the vertex $v$ with maximum out-degree are contained in the same element of any DS-cycle set $DS(G)$. To show the second inequality, take any element of $DS(G)$. This element must contain $|V|$ edges. The rest of the edges of the digraph can be represented by at most $|E| - |V|$ elements of $\mathcal{C}(G)$, and hence the bound follows. ∎

## VII. CONCLUSIONS

We have provided necessary and sufficient conditions for the existence of an edge weight assignment that makes the adjacency matrix of a given digraph doubly stochastic. We have unveiled the particular connection of this class of digraphs with a special subset of weight-balanced digraphs. The characterization provided here enlarges the range of network interconnection topologies for which one can run a variety of distributed algorithms for consensus and optimization. Future work will investigate the design of distributed algorithms that a network of agents can run in order to obtain the set of weights that makes the adjacency matrix doubly stochastic.

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