

A CATALOG OF INVERSE-KINEMATICS PLANNERS FOR UNDERACTUATED SYSTEMS ON MATRIX GROUPS

SONIA MARTÍNEZ AND JORGE CORTÉS

Mechanical and Aerospace Engineering
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0411, USA

FRANCESCO BULLO

Mechanical Engineering
2338 Engineering Building II
University of California, Santa Barbara
Santa Barbara, CA 9310-5070, USA

ABSTRACT. This paper presents motion planning algorithms for underactuated systems evolving on rigid rotation and displacement groups. Motion planning is transcribed into (low-dimensional) combinatorial selection and inverse-kinematics problems. We present a catalog of solutions for all underactuated systems on $SE(2)$, $SO(3)$, and $SE(2) \times \mathbb{R}$ classified according to their controllability properties.

1. Introduction. This paper presents motion planning algorithms for kinematic models of underactuated mechanical systems; i.e. systems with less actuators than degrees of freedom. We consider kinematic (i.e., driftless) models that are switched control systems, that is, dynamical systems described by a family of admissible vector fields and a control strategy that governs the switching between them. In particular, we focus on families of left-invariant vector fields defined on rigid displacements subgroups.

This class of systems arises in the context of kinematic modeling and kinematic reductions for mechanical control systems; see the example works [1, 4, 5, 6, 14, 17], motivated by the following consideration. Previous work in robotics suggests that almost time-optimal trajectories for fully-actuated robots can be obtained by implementing a simple algorithmic decomposition. First, a path-planning problem in the configuration space (where obstacles and limitations are included) is solved; then a fast time-scaling algorithm over the given path can be applied; see [2]. For systems that are kinematically controllable [6], this decomposition can also be performed without violating underactuation constraints. However, in this case, motions need to be restricted to integral curves of kinematic vector fields. The robot can switch among them by coming to rest first.

2000 *Mathematics Subject Classification.* Primary: XXX, XXX; Secondary: XXX.
Key words and phrases. mechanical control systems, kinematics, motion planning.
The first author is supported by NSF grant xx-xxxx, etc.

More detail about how the transcription into kinematic models simplifies the motion planning problem can be found in [6], which discusses 3R planar manipulators, and [3, 11] discussing the snakeboard system.

Literature review. Motion planning for kinematic models, sometimes referred to as driftless or nonholonomic models, is a classic problem in robotics; see [10] and also the references therein. In particular, the algorithms in [12, 13, 18] focus on dynamical aspects and exploit controllability properties.

On the other hand, the class of nonholonomic systems that motivate the work in this paper are special underactuated mechanical systems; see the definition in [4]. In particular, due to the technique of kinematic reductions outlined in the previous subsection (see also [4]), the search for a motion planning algorithm is closely related to the inverse-kinematics problem. Example inverse-kinematics methods include (i) iterative numerical methods for nonlinear optimization, see [9], (ii) geometric and decoupling methods for classes of manipulators, see [21, 23], (iii) the Paden-Kahan subproblems approach, see [18, 19], and (iv) the general polynomial programming approach, see [15]. The latter and more general method is based on tools from algebraic geometry and relies on simultaneously solving systems of algebraic equations.

Despite these efforts, no general methodology is currently available to solve these problems in closed-form. Accordingly, it is common to provide and catalog closed-form solutions for classes of relevant example systems; see [18, 21, 23].

Finally, it is worth mentioning that a number of papers study how to find state-dependent feedback transformations to render a given system into a form for which inverting controls can be found; see e.g., [8, 16, 20, 22], dealing with systems in chained form and control of unicycles. However, it is important to mention that these methods are not of use here since they would lead to trajectories for which the two-stage decoupling described earlier does not respect underactuation constraints.

Problem statement. We consider left-invariant control systems evolving on a matrix Lie subgroup $G \subset SE(3)$. Examples include systems on $SE(2)$, $SO(3)$ and $SE(2) \times \mathbb{R}$. As usual in Lie group theory, we identify left-invariant vector fields with their value at the identity. Given a family of left-invariant vector fields $\{V_1, \dots, V_m\}$ on G , consider the associated driftless control system

$$\dot{g}(t) = \sum_{i=1}^m V_i(g(t))u_i(t), \quad (1)$$

where $g: \mathbb{R} \rightarrow G$ and where the controls (u_1, \dots, u_m) take value in $\{(\pm 1, 0, \dots, 0), (0, \pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\}$. For these systems, controllability can be assessed by algebraic means: it suffices to check the lack of involutivity of the distribution $\text{span}\{V_1, \dots, V_m\}$. Recall that for matrix Lie algebras, Lie brackets are matrix commutators $[A, B] = AB - BA$.

This paper addresses the problem of how to compute feasible motion plans for the control system (1) by concatenating a finite number of flows along the input vector fields. We are interested in finding motion plans that employ a minimum number of flows—in terms of motion planning for mechanical systems using kinematic reductions this will imply that the robots will have to stop a minimum number of times.

We call a flow along any input vector field a *motion primitive* and its duration a *coasting time*. Therefore, motion planning is reduced to the problem of selecting

a finite-length combination of k motion primitives $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ and computing appropriate coasting times $(t_1, \dots, t_k) \in \mathbb{R}^k$ that steer the system from the identity in the group to any target configuration $g_f \in G$. In mathematical terms, we need to solve

$$g_f = \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k}).$$

Hence, motion planning is transcribed into low-dimensional combinatorial selection and inverse-kinematics problems.

Contribution. The contribution of this paper is a catalog of solutions for underactuated example systems defined on $SE(2)$, $SO(3)$, or $SE(2) \times \mathbb{R}$. Based on a controllability analysis, we classify families of underactuated systems that pose qualitatively different planning problems. For each family, we solve the planning problem for any target configuration by providing a combination of k motion primitives and corresponding closed-form expressions for the coasting times. In each case, we attempt to select $k = \dim(G)$: generically, this is the minimum necessary (but sometimes not sufficient) number of motion primitives needed in order obtain a motion plan with a minimum number of stops. If the motion planning algorithm entails exactly $\dim(G)$ motion primitives, i.e., generically minimizes the number of switches, we will refer to it as a *switch-optimal* algorithm. Note that this does not mean that for particular isolated target configurations one cannot find a sequence of $\ell < \dim(G)$ motion primitives. Sections 2, 3, and 4 present switch-optimal planners for $SE(2)$, $SO(3)$, and $SE(2) \times \mathbb{R}$, respectively.

Notation. Here we briefly collect the notation used throughout the paper. Let S be a set, $\text{id}_S: S \rightarrow S$ denote the identity map on S and let $\text{ind}_S: \mathbb{R} \rightarrow \mathbb{R}$ denote the characteristic function of S , i.e., $\text{ind}_S(x) = 1$ if $x \in S$ and $\text{ind}_S(x) = 0$ if $x \notin S$. Let $\text{arctan2}(x, y)$ denote the arctangent of y/x taking into account which quadrant the point (x, y) is in. We make the convention $\text{arctan2}(0, 0) = 0$. Let $\text{sign}: \mathbb{R} \rightarrow \mathbb{R}$ be the sign function, i.e., $\text{sign}(x) = 1$ if $x > 0$, $\text{sign}(x) = -1$ if $x < 0$ and $\text{sign}(0) = 0$. Let A_{ij} be the (i, j) element of the matrix A . Given $v, w \in \mathbb{R}^n$, let $\arg(v, w) \in [0, \pi]$ denote the angle between them. Let $\|\cdot\|$ denote the Euclidean norm.

Given a family of left-invariant vector fields $\{V_1, \dots, V_m\}$ on G , we associate to each multiindex $(i_1, \dots, i_k) \in \{1, \dots, m\}^k$ the forward-kinematics map $\mathcal{FK}^{(i_1, \dots, i_k)}: \mathbb{R}^k \rightarrow G$ given by $(t_1, \dots, t_k) \mapsto \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k})$.

2. **Catalog for $SE(2)$.** Let $\{e_\theta, e_x, e_y\}$ be the basis of $\mathfrak{se}(2)$:

$$e_\theta = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad e_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $[e_\theta, e_x] = e_y$, $[e_y, e_\theta] = e_x$ and $[e_x, e_y] = 0$. For ease of presentation, we write $V \in \mathfrak{se}(2)$ as $V = ae_\theta + be_x + ce_y \equiv (a, b, c)$, and $g \in SE(2)$ as

$$g = \begin{bmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{bmatrix} \equiv (\theta, x, y).$$

With this notation, $\exp: \mathfrak{se}(2) \rightarrow SE(2)$ is

$$\exp(a, b, c) = \left(a, \frac{\sin a}{a} b - \frac{1 - \cos a}{a} c, \frac{1 - \cos a}{a} b + \frac{\sin a}{a} c \right)$$

for $a \neq 0$, and $\exp(0, b, c) = (0, b, c)$.

Lemma 2.1. (*Controllability conditions*). *Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ in $\mathfrak{se}(2)$. Their Lie closure is full rank if and only if $a_1b_2 - b_1a_2 \neq 0$ or $c_1a_2 - a_1c_2 \neq 0$.*

Proof. Given $[V_1, V_2] = (0, c_1a_2 - a_1c_2, a_1b_2 - b_1a_2)$, we have $\text{span}\{V_1, V_2, [V_1, V_2]\} = \mathfrak{se}(2)$ if and only if

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & c_1a_2 - c_2a_1 & b_2a_1 - b_1a_2 \end{bmatrix} = (a_1b_2 - b_1a_2)^2 + (c_1a_2 - a_1c_2)^2 \neq 0.$$

□

Let $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ satisfy the controllability condition in Lemma 2.1. Accordingly, either a_1 or a_2 is different from zero. Without loss of generality, we will assume that $a_1 \neq 0$, and take $a_1 = 1$. As a consequence of Lemma 2.1, there are two qualitatively different cases to be considered:

$$\mathcal{S}_1 = \{(V_1, V_2) \in \mathfrak{se}(2) \times \mathfrak{se}(2) \mid V_1 = (1, b_1, c_1), V_2 = (0, b_2, c_2) \text{ and } b_2^2 + c_2^2 = 1\},$$

$$\mathcal{S}_2 = \{(V_1, V_2) \in \mathfrak{se}(2) \times \mathfrak{se}(2) \mid V_1 = (1, b_1, c_1), V_2 = (1, b_2, c_2) \text{ and } (b_1 \neq b_2 \text{ or } c_1 \neq c_2)\}.$$

Since $\dim(\mathfrak{se}(2)) = 3$, we need at least three motion primitives along the flows of $\{V_1, V_2\}$ to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(1,2,1)}: \mathbb{R}^3 \rightarrow \text{SE}(2)$. In the following propositions, we compute solutions for \mathcal{S}_1 and \mathcal{S}_2 -systems.

Proposition 1. (*Inversion for \mathcal{S}_1 -systems on $\text{SE}(2)$*). *Let $(V_1, V_2) \in \mathcal{S}_1$. Consider the map $\mathcal{IK}[\mathcal{S}_1]: \text{SE}(2) \rightarrow \mathbb{R}^3$,*

$$\mathcal{IK}[\mathcal{S}_1](\theta, x, y) = (\arctan2(\alpha, \beta), \rho, \theta - \arctan2(\alpha, \beta)),$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}[\mathcal{S}_1]$ is a global right inverse of $\mathcal{FK}^{(1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK}[\mathcal{S}_1] = \text{id}_{\text{SE}(2)}: \text{SE}(2) \rightarrow \text{SE}(2)$.

Note that the algorithm provided in the proposition is not only switch-optimal, but also works globally.

Proof. The proof follows from the expression of $\mathcal{FK}^{(1,2,1)}$. Let $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = (\theta, x, y)$,

$$\theta = t_1 + t_3,$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix} t_2.$$

The equation in $[x, y]^T$ can be rewritten as $[\alpha, \beta]^T = [\cos t_1, \sin t_1]^T t_2$. The selection $t_1 = \arctan2(\alpha, \beta)$, $t_2 = \rho$ solves this equation. □

Proposition 2. (*Inversion for \mathcal{S}_2 -systems on $\text{SE}(2)$*). *Let $(V_1, V_2) \in \mathcal{S}_2$. Define the neighborhood of the identity in $\text{SE}(2)$*

$$U = \{(\theta, x, y) \in \text{SE}(2) \mid \|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x, y)\|^2, 2(1 - \cos \theta)\|(b_1, c_1)\|^2\}\}.$$

Consider the map $\mathcal{IK}[\mathcal{S}_2]: U \subset \text{SE}(2) \rightarrow \mathbb{R}^3$ whose components are

$$\begin{aligned}\mathcal{IK}[\mathcal{S}_2]_1(\theta, x, y) &= \arctan2\left(\rho, \sqrt{4 - \rho^2}\right) + \arctan2(\alpha, \beta), \\ \mathcal{IK}[\mathcal{S}_2]_2(\theta, x, y) &= \arctan2\left(2 - \rho^2, \rho\sqrt{4 - \rho^2}\right), \\ \mathcal{IK}[\mathcal{S}_2]_3(\theta, x, y) &= \theta - \mathcal{IK}[\mathcal{S}_2]_1(\theta, x, y) - \mathcal{IK}[\mathcal{S}_2]_2(\theta, x, y),\end{aligned}$$

and $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|^2} \begin{bmatrix} c_1 - c_2 & b_2 - b_1 \\ b_1 - b_2 & c_1 - c_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}[\mathcal{S}_2]$ is a local right inverse of $\mathcal{FK}^{(1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK}[\mathcal{S}_2] = \text{id}_U: U \rightarrow U$.

Proof. If $(\theta, x, y) \in U$, then

$$\rho = \|(\alpha, \beta)\| \leq \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|} \cdot \left(\|(x, y)\| + \left\| \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \right\| \right) \leq 2,$$

and hence $\mathcal{IK}[\mathcal{S}_2]$ is well-defined on U . Let $\mathcal{IK}[\mathcal{S}_2](\theta, x, y) = (t_1, t_2, t_3)$. The components of $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)$ are

$$\begin{aligned}\mathcal{FK}_1^{(1,2,1)}(t_1, t_2, t_3) &= t_1 + t_2 + t_3, \\ \begin{bmatrix} \mathcal{FK}_2^{(1,2,1)}(t_1, t_2, t_3) \\ \mathcal{FK}_3^{(1,2,1)}(t_1, t_2, t_3) \end{bmatrix} &= \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \\ &\quad + \begin{bmatrix} c_1 - c_2 & b_1 - b_2 \\ b_2 - b_1 & c_1 - c_2 \end{bmatrix} \begin{bmatrix} \cos t_1 - \cos(t_1 + t_2) \\ \sin t_1 - \sin(t_1 + t_2) \end{bmatrix}.\end{aligned}$$

In an analogous way to the previous proof, one verifies $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = (\theta, x, y)$. \square

Remark 1. The map $\mathcal{IK}[\mathcal{S}_2]$ in Proposition 2 is a local right inverse to $\mathcal{FK}^{(1,2,1)}$ on a domain that strictly contains U . In other words, our estimate of the domain of $\mathcal{IK}[\mathcal{S}_2]$ is conservative. For instance, for points of the form $(0, x, y) \in \text{SE}(2)$, it suffices to ask for

$$\|(x, y)\| \leq 2\|(c_1 - c_2, b_1 - b_2)\|.$$

For a point $(\theta, 0, 0) \in \text{SE}(2)$, it suffices to ask for

$$(1 - \cos \theta)\|(b_1, c_1)\|^2 \leq 2\|(c_1 - c_2, b_1 - b_2)\|^2.$$

Additionally, without loss of generality, it is convenient to assume that the vector fields V_1, V_2 satisfy $b_1^2 + c_1^2 \leq b_2^2 + c_2^2$, so as to maximize the domain U .

We illustrate the performance of the algorithms in Fig. 1.

3. Catalog for $\text{SO}(3)$. Let $\{\hat{e}_x, \hat{e}_y, \hat{e}_z\}$ be the basis of $\mathfrak{so}(3)$:

$$\hat{e}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \hat{e}_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \hat{e}_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

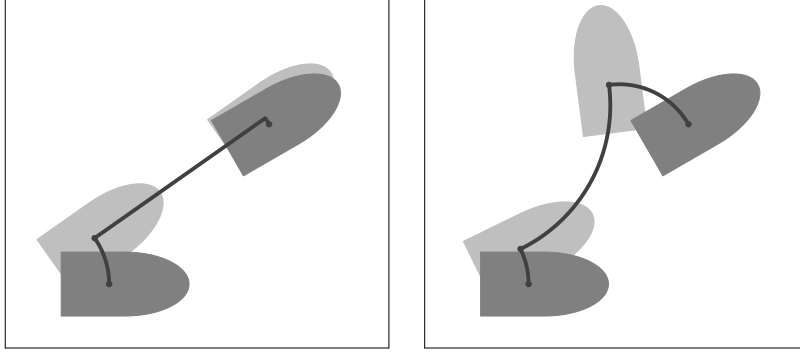


FIGURE 1. We illustrate the inverse-kinematics planners for \mathcal{S}_1 and \mathcal{S}_2 -systems. The parameters of both systems are $(b_1, c_1) = (0, .5)$, $(b_2, c_2) = (1, 0)$. The target location is $(\pi/6, 1, 1)$. Initial and target locations are depicted in dark gray.

Here we make use of the notation $\widehat{V} = a\widehat{e}_x + b\widehat{e}_y + c\widehat{e}_z \equiv \widehat{(a, b, c)}$ based on the Lie algebra isomorphism $\widehat{\cdot} : (\mathbb{R}^3, \times) \rightarrow (\mathfrak{so}(3), [\cdot, \cdot])$. Rodrigues formula [18] for the exponential $\exp : \mathfrak{so}(3) \rightarrow \text{SO}(3)$ is

$$\exp(\widehat{\eta}) = I_3 + \frac{\sin \|\eta\|}{\|\eta\|} \widehat{\eta} + \frac{1 - \cos \|\eta\|}{\|\eta\|^2} \widehat{\eta}^2.$$

The commutator relations are $[\widehat{e}_x, \widehat{e}_z] = -\widehat{e}_y$, $[\widehat{e}_y, \widehat{e}_z] = \widehat{e}_x$ and $[\widehat{e}_x, \widehat{e}_y] = \widehat{e}_z$.

Lemma 3.1. (*Controllability conditions*). *Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ in $\mathfrak{so}(3)$. Their Lie closure is full rank if and only if $c_1 a_2 - a_1 c_2 \neq 0$ or $b_1 c_2 - c_1 b_2 \neq 0$ or $b_1 a_2 - a_1 b_2 \neq 0$.*

Proof. Given the equality $[\widehat{V}_1, \widehat{V}_2] = \widehat{V_1 \times V_2}$, with $V_1 \times V_2 = (b_1 c_2 - b_2 c_1, c_1 a_2 - c_2 a_1, a_1 b_2 - a_2 b_1)$, one can see that $\text{span}\{V_1, V_2, [V_1, V_2]\} = \mathfrak{so}(3)$ if and only if

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ b_1 c_2 - b_2 c_1 & c_1 a_2 - c_2 a_1 & a_1 b_2 - a_2 b_1 \end{bmatrix} = (b_1 c_2 - b_2 c_1)^2 + (c_1 a_2 - c_2 a_1)^2 + (a_1 b_2 - a_2 b_1)^2 \neq 0.$$

□

Let V_1, V_2 satisfy the controllability condition in Lemma 3.1. Without loss of generality, we can assume $V_1 = e_z$ (otherwise we perform a suitable change of coordinates), and $\|V_2\| = 1$. In what follows, we let $V_2 = (a, b, c)$. Since e_z and V_2 are linearly independent, necessarily $a^2 + b^2 \neq 0$ and $c \neq \pm 1$. Since $\dim(\mathfrak{so}(3)) = 3$, we need at least three motion primitives to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(1,2,1)} : \mathbb{R}^3 \rightarrow \text{SO}(3)$, that is

$$\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = \exp(t_1 \widehat{e}_z) \exp(t_2 \widehat{V}_2) \exp(t_3 \widehat{e}_z). \quad (2)$$

Observe that equation (2) is similar to the standard formula for Euler angles (see [18]) with the difference that here the rotation axes are not orthogonal. The following result is related to the problem of how to decompose a rotation by means of non-orthogonal Euler angles; see the papers [7, 24] for related results.

Proposition 3. (*Inversion for systems on $\text{SO}(3)$*). Let $V_1 = (0, 0, 1)$ and $V_2 = (a, b, c)$, with $a^2 + b^2 \neq 0$ and $c \neq \pm 1$. Define the neighborhood of the identity in $\text{SO}(3)$

$$U = \{R \in \text{SO}(3) \mid R_{33} \in [2c^2 - 1, 1]\}.$$

Consider the map $\mathcal{IK}: U \subset \text{SO}(3) \rightarrow \mathbb{R}^3$ whose components are

$$\mathcal{IK}_1(R) = \arctan2(w_1 R_{13} + w_2 R_{23}, -w_2 R_{13} + w_1 R_{23}),$$

$$\mathcal{IK}_2(R) = \arccos\left(\frac{R_{33} - c^2}{1 - c^2}\right),$$

$$\mathcal{IK}_3(R) = \arctan2(v_1 R_{31} + v_2 R_{32}, v_2 R_{31} - v_1 R_{32}),$$

where, for $z = (1 - \cos(\mathcal{IK}_2(R)), \sin(\mathcal{IK}_2(R)))^T$,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} ac & b \\ cb & -a \end{bmatrix} z, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} ac & -b \\ cb & a \end{bmatrix} z.$$

Then, \mathcal{IK} is a local right inverse of $\mathcal{FK}^{(1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1)} \circ \mathcal{IK} = \text{id}_U : U \rightarrow U$.

Proof. Let $R \in U$. Then, $|\frac{R_{33} - c^2}{1 - c^2}| \leq 1$, and hence $\mathcal{IK}(R)$ is well-defined. Denote $t_i = \mathcal{IK}_i(R)$ and let us show $R = \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)$. Recall that the rows (resp. the columns) of a rotation matrix consist of orthonormal vectors in \mathbb{R}^3 . Therefore, the matrix $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) \in \text{SO}(3)$ is determined by its third column $\mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)e_z$ and its third row $e_z^T \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)$. The factors in (2) admit the following closed-form expressions. For $c_t = \cos t$ and $s_t = \sin t$,

$$\exp(t\hat{e}_z) = \begin{bmatrix} c_t & -s_t & 0 \\ s_t & c_t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $\exp(t\hat{V}_2)$ equals

$$\begin{bmatrix} a^2 + (1 - a^2)c_t & ba(1 - c_t) - cs_t & ca(1 - c_t) + bs_t \\ ab(1 - c_t) + cs_t & b^2 + (1 - b^2)c_t & cb(1 - c_t) - ast \\ ac(1 - c_t) - bs_t & bc(1 - c_t) + ast & c^2 + (1 - c^2)c_t \end{bmatrix}.$$

Now, using the fact that $\exp(t\hat{e}_z)e_z = e_z$, we get

$$\begin{aligned} \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3)e_z &= \exp(t_1\hat{e}_z) \exp(t_2\hat{V}_2) \exp(t_3\hat{e}_z)e_z \\ &= \exp(t_1\hat{e}_z) \exp(t_2\hat{V}_2)e_z = \exp(t_1\hat{e}_z) \begin{bmatrix} w_1 \\ w_2 \\ R_{33} \end{bmatrix} = Re_z. \end{aligned}$$

A similar computation shows that $e_z^T \mathcal{FK}^{(1,2,1)}(t_1, t_2, t_3) = e_z^T R$, which concludes the proof. \square

Remark 2. If \hat{e}_z and V_2 are perpendicular, then $U = \text{SO}(3)$ and the map \mathcal{IK} is a global right inverse of $\mathcal{FK}^{(1,2,1)}$. Otherwise, let us provide an equivalent formulation of the constraint $R_{33} \in [2c^2 - 1, 1]$ in terms of the axis/angle representation of the rotation matrix R . Recall that there always exist a, possibly non-unique, rotation angle $\theta \in [0, \pi]$ and an unit-length axis of rotation $\omega \in \mathbb{S}^2$ such that $R = \exp(\hat{\omega}\theta)$. Because $\hat{\omega}^2 = \omega^T \omega - I_3$, an equivalent statement of Rodrigues formula is

$$R = I_3 + \hat{\omega} \sin \theta + (1 - \cos \theta)(\omega^T \omega - I_3).$$

From $e_z^T \omega = \cos(\arg(e_z, \omega))$, we compute

$$\begin{aligned} e_z^T R e_z &= e_z^T e_z + (1 - \cos \theta)((e_z^T \omega)^2 - e_z^T e_z) \\ &= 1 + (1 - \cos \theta)((e_z^T \omega)^2 - 1) \\ &= 1 - \sin^2(\arg(e_z, \omega))(1 - \cos \theta). \end{aligned} \quad (3)$$

Therefore, $R_{33} \in [2c^2 - 1, 1]$ if and only if

$$\begin{aligned} 1 - \sin^2(\arg(e_z, \omega))(1 - \cos \theta) &\geq 2c^2 - 1 \\ \iff \sin^2(\arg(e_z, \omega))(1 - \cos \theta) &\leq 2(1 - c^2). \end{aligned}$$

Two sufficient conditions are also meaningful. In terms of the rotation angle, if $|\theta| \leq \arccos(2c^2 - 1)$ then $1 - \cos \theta \leq 2(1 - c^2)$, and in turn equation (3) is satisfied. In terms of the axis of rotation, a sufficient condition for equation (3) is $\sin^2(\arg(e_z, \omega)) \leq \sin^2(\arg(e_z, V_2)) = 1 - c^2$.

We illustrate the performance of the algorithm in Fig. 2.

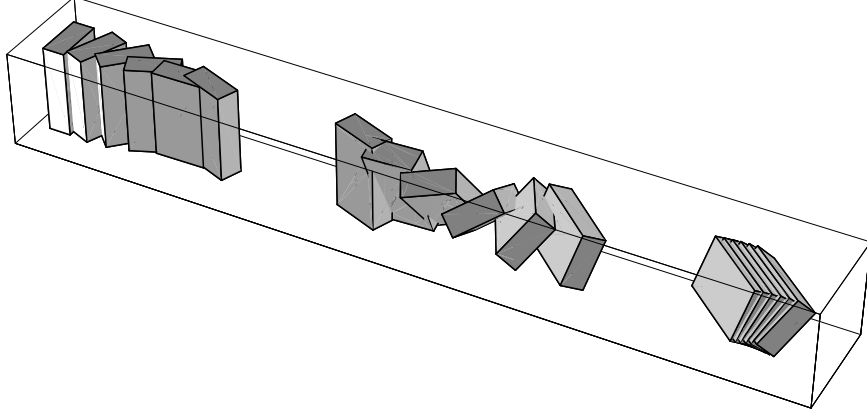


FIGURE 2. We illustrate the inverse-kinematics planner on $SO(3)$. The system parameters are $(a, b, c) = (0, 1/\sqrt{2}, 1/\sqrt{2})$. The target final rotation is $\exp(\pi/3, \pi/3, 0)$. To render the sequence of three rotations visible, the body is translated along the inertial x -axis.

4. Catalog for $SE(2) \times \mathbb{R}$. Let $\{(e_\theta, 0), (e_x, 0), (e_y, 0), (0, 0, 0, 1)\}$ be a basis of $\mathfrak{se}(2) \times \mathbb{R}$, where $\{e_\theta, e_x, e_y, \}$ stands for the basis of $\mathfrak{se}(2)$ introduced in Section 2. With a slight abuse of notation, we let e_θ denote $(e_\theta, 0)$, and we similarly redefine e_x and e_y . We also let $e_z = (0, 0, 0, 1)$. The only non-vanishing Lie algebra commutators are $[e_\theta, e_x] = e_y$ and $[e_\theta, e_y] = -e_x$.

A left-invariant vector field V in $\mathfrak{se}(2) \times \mathbb{R}$ is written as $V = ae_\theta + be_x + ce_y + de_z \equiv (a, b, c, d)$, and $g \in SE(2) \times \mathbb{R}$ as $g = (\theta, x, y, z)$. The exponential map, $\exp : \mathfrak{se}(2) \times \mathbb{R} \rightarrow SE(2) \times \mathbb{R}$, is given component-wise by the exponential on $\mathfrak{se}(2)$ and \mathbb{R} , respectively. That is, $\exp(V)$ is equal to

$$\left(a, \frac{\sin a}{a}b - \frac{1 - \cos a}{a}c, \frac{1 - \cos a}{a}b + \frac{\sin a}{a}c, d \right)$$

if $a \neq 0$, and $\exp(V) = (0, b, c, d)$ if $a = 0$.

Lemma 4.1. *(Controllability conditions for systems in $\text{SE}(2) \times \mathbb{R}$ with 2 inputs). Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1, d_1)$ and $V_2 = (a_2, b_2, c_2, d_2)$ in $\mathfrak{se}(2) \times \mathbb{R}$. Their Lie closure is full rank if and only if $a_2d_1 - d_2a_1 \neq 0$, and either $c_1a_2 - a_1c_2 \neq 0$ or $a_1b_2 - b_1a_2 \neq 0$.*

Proof. Since $[V_1, V_2] = (0, c_1a_2 - a_1c_2, a_1b_2 - b_1a_2, 0) \neq 0$, we deduce that either $c_1a_2 - a_1c_2 \neq 0$ or $a_1b_2 - b_1a_2 \neq 0$. In particular, this implies that necessarily $a_1 \neq 0$ or $a_2 \neq 0$. Assume $a_1 \neq 0$. Now,

$$[V_1, [V_1, V_2]] = (0, a_1(-b_2a_1 + b_1a_2), a_1(c_1a_2 - c_2a_1), 0),$$

and note that $[V_2, [V_1, V_2]] = (a_2/a_1)[V_1, [V_1, V_2]]$. Finally, $\overline{\text{Lie}}(\{V_1, V_2\}) = \mathfrak{se}(2) \times \mathbb{R}$ if and only if

$$\det \begin{bmatrix} b_1 & c_1 & d_1 & a_1 \\ b_2 & c_2 & d_2 & a_2 \\ c_1a_2 - c_2a_1 & b_2a_1 - b_1a_2 & 0 & 0 \\ a_1(-b_2a_1 + b_1a_2) & a_1(c_1a_2 - c_2a_1) & 0 & 0 \end{bmatrix} \\ = a_1(a_2d_1 - d_2a_1) [(c_1a_2 - c_2a_1)^2 + (-b_2a_1 + b_1a_2)^2] \neq 0.$$

Since $[V_1, V_2] \neq 0$, this reduces to $a_2d_1 - d_2a_1 \neq 0$. \square

Let V_1, V_2 satisfy the controllability condition in Lemma 4.1. Without loss of generality, we can assume $a_1 = 1$. As in the case of $\text{SE}(2)$, there are two qualitatively different situations to be considered:

$$\mathcal{T}_1 = \{(V_1, V_2) \in (\mathfrak{se}(2) \times \mathbb{R})^2 \mid V_1 = (1, b_1, c_1, d_1),$$

$$V_2 = (0, b_2, c_2, 1) \text{ and } b_2^2 + c_2^2 \neq 0\},$$

$$\mathcal{T}_2 = \{(V_1, V_2) \in (\mathfrak{se}(2) \times \mathbb{R})^2 \mid V_1 = (1, b_1, c_1, d_1),$$

$$V_2 = (1, b_2, c_2, d_2), d_1 \neq d_2 \text{ and } (b_1 \neq b_2 \text{ or } c_1 \neq c_2)\}.$$

Lemma 4.2. *(Controllability conditions for $\text{SE}(2) \times \mathbb{R}$ systems with 3 inputs). Consider three left-invariant vector fields $V_i = (a_i, b_i, c_i, d_i)$, $i = 1, 2, 3$ in $\mathfrak{se}(2) \times \mathbb{R}$. Assume $\overline{\text{Lie}}(\{V_{i_1}, V_{i_2}\}) \subsetneq \mathfrak{se}(2) \times \mathbb{R}$, for $i_j \in \{1, 2, 3\}$ and $\overline{\text{Lie}}(\{V_1, V_2, V_3\}) = \mathfrak{se}(2) \times \mathbb{R}$. Then, possibly after a reordering of the vector fields, they must fall in one of the following cases:*

$$\mathcal{T}_3 = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1),$$

$$V_2 = (0, b_2, c_2, 0), V_3 = (1, b_1, c_1, d_3), d_1 \neq d_3 \text{ and } b_2^2 + c_2^2 \neq 0\},$$

$$\mathcal{T}_4 = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1),$$

$$V_2 = (0, b_2, c_2, 0), V_3 = (0, 0, 0, d_3), 0 \neq d_3 \neq d_1 \text{ and } b_2^2 + c_2^2 \neq 0\},$$

$$\mathcal{T}_5 = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \mid V_1 = (1, b_1, c_1, d_1),$$

$$V_2 = (1, b_2, c_2, d_1), V_3 = (0, 0, 0, d_3), d_3 \neq 0 \text{ and } (b_2 \neq b_1 \text{ or } c_1 \neq c_2)\}.$$

Proof. Without loss of generality, we can assume that $[V_1, V_2] \neq 0$ and $a_1 = 1$. Since $\overline{\text{Lie}}(\{V_1, V_2\}) \neq \mathfrak{se}(2) \times \mathbb{R}$, then $a_2d_1 = d_2$. Given that the Lie closure of $\{V_1, V_2, V_3\}$ is full-rank, and $\dim(\text{span}\{V_1, V_2, [V_1, V_2]\}) = 3$, we have that $d_3 \neq a_3d_1$. This latter fact, together with $\overline{\text{Lie}}(\{V_1, V_3\}) \subsetneq \mathfrak{se}(2) \times \mathbb{R}$, implies that $[V_1, V_3] = 0$, and therefore $b_3 = a_3b_1$, $c_1a_3 = c_3$. We now distinguish two situations depending on $[V_2, V_3]$ being zero or not.

(a) $[V_2, V_3] \neq 0$. Necessarily, $a_3 \neq 0$. Therefore, we can assume $a_3 = 1$. Since $\overline{\text{Lie}}(\{V_2, V_3\})$ is not full-rank, then $a_2 = 0$. We then have a \mathcal{T}_3 -system.

(b) $[V_2, V_3] = 0$. Necessarily, $b_3a_2 = b_2a_3$ and $c_2a_3 = c_3a_2$. Depending on the values of a_2 and a_3 , we consider:

- (i) If $a_2 = a_3 = 0$, then $d_2 = 0$, $d_3 \neq 0$, $b_3 = c_3 = 0$. Then, this is a \mathcal{T}_4 -system.
- (ii) If $a_2 = 0$, and $a_3 = 1$, then $b_2 = b_3a_2 = 0$, $c_2 = c_3a_2 = 0$ and also $d_2 = d_1a_2 = 0$. This is not possible as it would make $V_2 = 0$.
- (iii) If $a_2 = 1$ and $a_3 = 0$, then $b_3 = c_3 = 0$, and $d_2 = d_1$. Therefore, this is a \mathcal{T}_5 -system.
- (iv) Finally, if $a_2 = 1$ and $a_3 = 1$, then $b_1 = b_2$, $c_1 = c_2$, and $d_1 = d_2$, which makes V_1 and V_2 linearly dependent.

□

4.1. Two-dimensional input distribution. Let V_1, V_2 satisfy the controllability condition in Lemma 4.1. Since $\dim(\mathfrak{se}(2) \times \mathbb{R}) = 4$, we need at least four motion primitives to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(2,1,2,1)}: \mathbb{R}^4 \rightarrow \text{SE}(2) \times \mathbb{R}$.

Proposition 4. *(Lack of switch-optimal inversion for \mathcal{T}_1 -systems on $\text{SE}(2) \times \mathbb{R}$). Let $(V_1, V_2) \in \mathcal{T}_1$. Then, the map $\mathcal{FK}^{(2,1,2,1)}$ is not invertible at any neighborhood of the origin.*

Proof. Let $\mathcal{FK}^{(2,1,2,1)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$. Then,

$$\begin{aligned} \theta &= t_2 + t_4, \\ z &= t_1 + t_3 + d_1(t_2 + t_4) = t_1 + t_3 + d_1\theta, \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -c_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} c_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} + \begin{bmatrix} b_2 \\ c_2 \end{bmatrix} t_1 + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos t_2 \\ \sin t_2 \end{bmatrix} t_3. \end{aligned}$$

Consider a configuration with $\theta = z = 0$. Then, the equation in (x, y) is invertible if and only if the map $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\begin{bmatrix} t_2 \\ t_3 \end{bmatrix} \mapsto \begin{bmatrix} \cos t_2 - 1 \\ \sin t_2 \end{bmatrix} t_3$$

is invertible. But f can not be inverted in $(0, \beta)$, $\beta \neq 0$. □

Remark 3. An identical negative result holds if we start taking motion primitives along the flow of V_1 instead of V_2 , i.e., if we consider the map $\mathcal{FK}^{(1,2,1,2)}: \mathbb{R}^4 \rightarrow \text{SE}(2) \times \mathbb{R}$.

Consider the map $\mathcal{FK}^{(1,2,1,2,1)}: \mathbb{R}^5 \rightarrow \text{SE}(2) \times \mathbb{R}$.

Proposition 5. *(Inversion for \mathcal{T}_1 -systems on $\text{SE}(2) \times \mathbb{R}$). Let $(V_1, V_2) \in \mathcal{T}_1$. Consider the map $\mathcal{IK}[\mathcal{T}_1]: \text{SE}(2) \times \mathbb{R} \rightarrow \mathbb{R}^5$ whose components are*

$$\begin{aligned} \mathcal{IK}[\mathcal{T}_1]_1(\theta, x, y, z) &= \pi \text{ind}_{]-\infty, 0[}(\gamma - \rho) + \arctan 2(\alpha, \beta) + \arctan 2((\rho + \gamma)/2, 0), \\ \mathcal{IK}[\mathcal{T}_1]_2(\theta, x, y, z) &= (\gamma - \rho)/2, \\ \mathcal{IK}[\mathcal{T}_1]_3(\theta, x, y, z) &= \arctan 2((\rho^2 - \gamma^2)/4, 0) \\ &\quad + \pi (\text{ind}_{]-\infty, 0[}(\gamma + \rho) - \text{ind}_{]-\infty, 0[}(\gamma - \rho)), \\ \mathcal{IK}[\mathcal{T}_1]_4(\theta, x, y, z) &= (\gamma + \rho)/2, \\ \mathcal{IK}[\mathcal{T}_1]_5(\theta, x, y, z) &= \theta - \mathcal{IK}[\mathcal{T}_1]_1(\theta, x, y, z) - \mathcal{IK}[\mathcal{T}_1]_3(\theta, x, y, z), \end{aligned}$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{aligned} \gamma &= z - d_1\theta, \\ \begin{bmatrix} \alpha \\ \beta \end{bmatrix} &= \frac{1}{b_2^2 + c_2^2} \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \right). \end{aligned}$$

Then, $\mathcal{IK}[\mathcal{T}_1]$ is a global right inverse of $\mathcal{FK}^{(1,2,1,2,1)}$, i.e. it satisfies $\mathcal{FK}^{(1,2,1,2,1)} \circ \mathcal{IK}[\mathcal{T}_1] = \text{id}_{\text{SE}(2) \times \mathbb{R}}: \text{SE}(2) \times \mathbb{R} \rightarrow \text{SE}(2) \times \mathbb{R}$.

Proof. The proof follows from the expression of $\mathcal{FK}^{(1,2,1,2,1)}$. First, let us denote $\mathcal{FK}^{(1,2,1,2,1)}(t_1, t_2, t_3, t_4, t_5) = (\theta, x, y, z)$,

$$\begin{aligned} \theta &= t_1 + t_3 + t_5, \\ z &= t_2 + t_4 + d_1\theta, \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix} t_2 + \begin{bmatrix} \cos(t_1 + t_3) \\ \sin(t_1 + t_3) \end{bmatrix} t_4 \right). \end{aligned}$$

The equation in $[x, y]^T$ can be rewritten as

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix} t_2 + \begin{bmatrix} \cos(t_1 + t_3) \\ \sin(t_1 + t_3) \end{bmatrix} t_4,$$

which is solved by the selection of coasting times given by the components of the map $\mathcal{IK}[\mathcal{T}_1]$. \square

Proposition 6. (*Inversion for \mathcal{T}_2 -systems on $\text{SE}(2) \times \mathbb{R}$*). Let $(V_1, V_2) \in \mathcal{T}_2$. Define the neighborhood of the identity in $\text{SE}(2) \times \mathbb{R}$

$$\begin{aligned} U &= \left\{ (\theta, x, y, z) \in \text{SE}(2) \times \mathbb{R} \mid \right. \\ &\quad \left. 4 \|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x, y)\|^2, 2(1 - \cos \theta)\|(b_1, c_1)\|^2\}, \right. \\ &\quad \left. |z - d_1\theta| \leq 2|d_2 - d_1| \arccos\left(-1 + \frac{\|(x, y)\| + \|(b_1, c_1)\|\sqrt{2(1 - \cos \theta)}}{\|(c_1 - c_2, b_1 - b_2)\|}\right) \right\}. \end{aligned}$$

Consider the map $\mathcal{IK}[\mathcal{T}_2]: \text{SE}(2) \times \mathbb{R} \rightarrow \mathbb{R}^5$ whose components are

$$\begin{aligned} \mathcal{IK}[\mathcal{T}_2]_1(\theta, x, y, z) &= \arctan2\left(l, \sqrt{4 - l^2}\right) + \arctan2(\alpha, \beta), \\ \mathcal{IK}[\mathcal{T}_2]_2(\theta, x, y, z) &= 2 \arctan2\left(\sqrt{4 - l^2}, l\right), \\ \mathcal{IK}[\mathcal{T}_2]_3(\theta, x, y, z) &= -\arctan2\left(\rho - l, \sqrt{4 - (\rho - l)^2}\right) \\ &\quad - \mathcal{IK}[\mathcal{T}_2]_1(\theta, x, y, z) - \mathcal{IK}[\mathcal{T}_2]_2(\theta, x, y, z), \\ \mathcal{IK}[\mathcal{T}_2]_4(\theta, x, y, z) &= \gamma - \mathcal{IK}[\mathcal{T}_2]_2(\theta, x, y, z), \\ \mathcal{IK}[\mathcal{T}_2]_5(\theta, x, y, z) &= \theta - \sum_{i=1}^4 \mathcal{IK}[\mathcal{T}_2]_i(\theta, x, y, z), \end{aligned}$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$, $s = \sin(\gamma/2)$, $c = \cos(\gamma/2)$ and

$$\gamma = (z - d_1\theta)/(d_2 - d_1),$$

$$l = \frac{\rho(1+c) + \text{sign}(\gamma)\sqrt{\rho^2(1+c)^2 - (1+c)(2\rho^2 - 8s^2)}}{2(1+c)},$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\|(d_1 - d_2, c_1 - c_2)\|^2} \begin{bmatrix} d_1 - d_2 & c_2 - c_1 \\ c_1 - c_2 & d_1 - d_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -d_1 & c_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}[\mathcal{T}_2]$ is a local right inverse of $\mathcal{FK}^{(1,2,1,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1,2,1)} \circ \mathcal{IK}[\mathcal{T}_2] = \text{id}_U: U \rightarrow U$.

Proof. If $(\theta, x, y, z) \in U$, then $\rho \leq 4$ and $|\gamma| \leq 2 \arccos(-1 + \rho/2)$. This in turn implies that

$$c = \cos\left(\frac{\gamma}{2}\right) \geq -1 + \frac{\rho}{2} \geq -1 + \frac{\rho^2}{8}$$

over $\rho \leq 4$. The second inequality guarantees that l is well-defined. The first one implies $l \in [\rho - 2, 2]$, which makes $\mathcal{IK}[\mathcal{T}_2]$ well-defined on U . Let $\mathcal{IK}[\mathcal{T}_2](\theta, x, y, z) = (t_1, t_2, t_3, t_4, t_5)$. The components of $\mathcal{FK}^{(1,2,1,2,1)}(t_1, t_2, t_3, t_4, t_5)$ are the following

$$\theta = t_1 + t_2 + t_3 + t_4 + t_5,$$

$$z = d_1\theta + (d_2 - d_1)(t_2 + t_4),$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} c_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} c_1 - c_2 & b_1 - b_2 \\ b_2 - b_1 & c_1 - c_2 \end{bmatrix} \cdot \begin{bmatrix} \cos t_1 - \cos(t_1 + t_2) + \cos(t_1 + t_2 + t_3) - \cos(\sum_{i=1}^4 t_i) \\ \sin t_1 - \sin(t_1 + t_2) + \sin(t_1 + t_2 + t_3) - \sin(\sum_{i=1}^4 t_i) \end{bmatrix}$$

After some involved computations, one can verify $\mathcal{FK}^{(1,2,1,2,1)}(t_1, t_2, t_3, t_4, t_5) = (\theta, x, y, z)$. \square

4.2. Three-dimensional input distribution. Let V_1, V_2, V_3 satisfy the controllability condition in Lemma 4.2. Consider $\mathcal{FK}^{(1,3,2,1)}: \mathbb{R}^4 \rightarrow \text{SE}(2) \times \mathbb{R}$.

Proposition 7. (*Inversion for \mathcal{T}_3 -systems on $\text{SE}(2) \times \mathbb{R}$*). Let $(V_1, V_2, V_3) \in \mathcal{T}_3$. Consider the map $\mathcal{IK}[\mathcal{T}_3]: \text{SE}(2) \times \mathbb{R} \rightarrow \mathbb{R}^4$ whose components are

$$\mathcal{IK}[\mathcal{T}_3]_1(\theta, x, y, z) = \arctan2(\alpha, \beta) - \mathcal{IK}[\mathcal{T}_3]_2(\theta, x, y, z),$$

$$\mathcal{IK}[\mathcal{T}_3]_2(\theta, x, y, z) = \frac{z - d_1\theta}{d_3 - d_1},$$

$$\mathcal{IK}[\mathcal{T}_3]_3(\theta, x, y, z) = \rho,$$

$$\mathcal{IK}[\mathcal{T}_3]_4(\theta, x, y, z) = \theta - \arctan2(\alpha, \beta),$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{b_2^2 + c_2^2} \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}[\mathcal{T}_3]$ is a global right inverse of $\mathcal{FK}^{(1,3,2,1)}$, that is, it satisfies $\mathcal{FK}^{(1,3,2,1)} \circ \mathcal{IK}[\mathcal{T}_3] = \text{id}_{\text{SE}(2) \times \mathbb{R}}: \text{SE}(2) \times \mathbb{R} \rightarrow \text{SE}(2) \times \mathbb{R}$.

Proof. The proof follows from the expression of $\mathcal{FK}^{(1,3,2,1)}$. first, if we let denote $\mathcal{FK}^{(1,3,2,1)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$, then

$$\begin{aligned}\theta &= t_1 + t_2 + t_4, \\ z &= d_1 t_1 + d_3 t_2 + d_1 t_4 = d_1 \theta + (d_3 - d_1) t_2, \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos(t_1 + t_2) \\ \sin(t_1 + t_2) \end{bmatrix} t_3.\end{aligned}$$

The equation in $[x, y]^T$ can be rewritten as

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \cos(t_1 + t_2) \\ \sin(t_1 + t_2) \end{bmatrix} t_3,$$

which is solved by the selection given by $(t_1, t_2, t_3, t_4) = \mathcal{IK}[\mathcal{T}_3](\theta, x, y, z)$. \square

Consider the map $\mathcal{FK}^{(1,2,1,3)}: \mathbb{R}^4 \rightarrow \text{SE}(2) \times \mathbb{R}$.

Proposition 8. (*Inversion for \mathcal{T}_4 -systems on $\text{SE}(2) \times \mathbb{R}$*). Let $(V_1, V_2, V_3) \in \mathcal{T}_4$. Consider the map $\mathcal{IK}[\mathcal{T}_4]: \text{SE}(2) \times \mathbb{R} \rightarrow \mathbb{R}^4$ given by

$$\mathcal{IK}[\mathcal{T}_4](\theta, x, y, z) = \left(\arctan2(\alpha, \beta), \rho, \theta - \arctan2(\alpha, \beta), \frac{z - d_1 \theta}{d_3} \right),$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{b_2^2 + c_2^2} \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}[\mathcal{T}_4]$ is a global right inverse of $\mathcal{FK}^{(1,2,1,3)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1,3)} \circ \mathcal{IK}[\mathcal{T}_4] = \text{id}_{\text{SE}(2) \times \mathbb{R}}: \text{SE}(2) \times \mathbb{R} \rightarrow \text{SE}(2) \times \mathbb{R}$.

Proof. If $\mathcal{FK}^{(1,2,1,3)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$, then

$$\begin{aligned}\theta &= t_1 + t_3, \\ \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix} t_2, \\ z &= d_1(t_1 + t_3) + d_3 t_4.\end{aligned}$$

The equation in $[x, y]^T$ can be rewritten as $[\alpha, \beta]^T = [\cos t_1, \sin t_1]^T t_2$. As in the proof of Proposition 1, the selection $t_1 = \arctan2(\alpha, \beta)$, $t_2 = \rho$ solves it. \square

Proposition 9. (*Inversion for \mathcal{T}_5 -systems on $\text{SE}(2) \times \mathbb{R}$*). Let $(V_1, V_2, V_3) \in \mathcal{T}_5$. Define the neighborhood of the identity in $\text{SE}(2) \times \mathbb{R}$

$$U = \{(\theta, x, y) \in \text{SE}(2) \times \mathbb{R} \mid \|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x, y)\|^2, 2(1 - \cos \theta)\|(b_1, c_1)\|^2\}\}.$$

Consider the map $\mathcal{IK}[\mathcal{T}_5]: U \subset \text{SE}(2) \times \mathbb{R} \rightarrow \mathbb{R}^4$ whose components are

$$\begin{aligned}\mathcal{IK}[\mathcal{T}_5]_1(\theta, x, y, z) &= \arctan2\left(\rho, \sqrt{4 - \rho^2}\right) + \arctan2(\alpha, \beta), \\ \mathcal{IK}[\mathcal{T}_5]_2(\theta, x, y, z) &= \arctan2\left(2 - \rho^2, \rho\sqrt{4 - \rho^2}\right), \\ \mathcal{IK}[\mathcal{T}_5]_3(\theta, x, y, z) &= \theta - \mathcal{IK}[\mathcal{T}_5]_1(\theta, x, y) - \mathcal{IK}[\mathcal{T}_5]_2(\theta, x, y), \\ \mathcal{IK}[\mathcal{T}_5]_4(\theta, x, y, z) &= \frac{z - d_1 \theta}{d_3},\end{aligned}$$

and $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|^2} \begin{bmatrix} c_1 - c_2 & b_2 - b_1 \\ b_1 - b_2 & c_1 - c_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}[\mathcal{T}_5]$ is a local right inverse of $\mathcal{FK}^{(1,2,1,3)}$, that is, it satisfies $\mathcal{FK}^{(1,2,1,3)} \circ \mathcal{IK}[\mathcal{T}_5] = \text{id}_U: U \rightarrow U$.

Proof. If $(\theta, x, y, z) \in U$, then one can see that $\rho = \|(\alpha, \beta)\| \leq 2$, and therefore $\mathcal{IK}[\mathcal{T}_5]$ is well-defined on U . Let $\mathcal{IK}[\mathcal{T}_5](\theta, x, y, z) = (t_1, t_2, t_3, t_4)$. The components of $\mathcal{FK}^{(1,2,1,3)}(t_1, t_2, t_3, t_4)$ are

$$\begin{aligned} \mathcal{FK}_1^{(1,2,1,3)}(t_1, t_2, t_3, t_4) &= t_1 + t_2 + t_3, \\ \begin{bmatrix} \mathcal{FK}_2^{(1,2,1,3)}(t_1, t_2, t_3, t_4) \\ \mathcal{FK}_3^{(1,2,1,3)}(t_1, t_2, t_3, t_4) \end{bmatrix} &= \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos \theta \\ \sin \theta \end{bmatrix} \\ &\quad + \begin{bmatrix} c_1 - c_2 & b_1 - b_2 \\ b_2 - b_1 & c_1 - c_2 \end{bmatrix} \begin{bmatrix} \cos t_1 - \cos(t_1 + t_2) \\ \sin t_1 - \sin(t_1 + t_2) \end{bmatrix}, \\ \mathcal{FK}_4^{(1,2,1,3)}(t_1, t_2, t_3, t_4) &= d_1(t_1 + t_2 + t_3) + d_3 t_4. \end{aligned}$$

One can verify that $\mathcal{FK}^{(1,2,1,3)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$. \square

We illustrate the performance of the algorithms in Fig. 3.

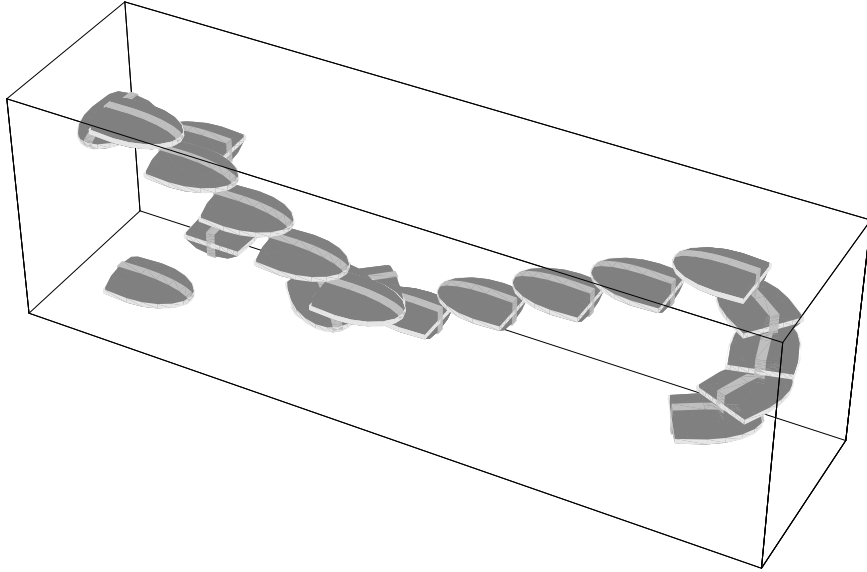


FIGURE 3. We illustrate the inverse-kinematics planner for a \mathcal{T}_1 -system on $\text{SE}(2) \times \mathbb{R}$. The system parameters are $b_1 = 1$, $c_1 = 0$, $d_1 = .5$, $b_2 = -2$, and $c_2 = 0$. The target location is $(\pi/6, 10, 0, 1)$.

5. Conclusions. We have presented a catalog of feasible motion planning algorithms for underactuated controllable systems on $\text{SE}(2)$, $\text{SO}(3)$ and $\text{SE}(2) \times \mathbb{R}$. Future directions of research include (i) considering other relevant classes of underactuated systems on $\text{SE}(3)$, (ii) computing catalogs of optimal sequences of motion

primitives, and (iii) developing hybrid feedback schemes that rely on the proposed open-loop planners to achieve point stabilization and trajectory tracking.

Acknowledgments. The authors would like to thank Professor Kevin M. Lynch for insightful discussions.

REFERENCES

- [1] H. Arai, K. Tanie, and N. Shiroma. Nonholonomic control of a three-DOF planar underactuated manipulator. *IEEE Transactions on Robotics and Automation*, 14(5):681–695, 1998.
- [2] J. E. Bobrow, S. Dubowsky, and J. S. Gibson. Time-optimal control of robotic manipulators along specified paths. *International Journal of Robotics Research*, 4(3):3–17, 1985.
- [3] F. Bullo and A. D. Lewis. Kinematic controllability and motion planning for the snakeboard. *IEEE Transactions on Robotics and Automation*, 19(3):494–498, 2003.
- [4] F. Bullo and A. D. Lewis. *Geometric Control of Mechanical Systems*, volume 49 of *Texts in Applied Mathematics*. Springer, 2004.
- [5] F. Bullo and A. D. Lewis. Low-order controllability and kinematic reductions for affine connection control systems. *SIAM Journal on Control and Optimization*, 44(3):885–908, 2005.
- [6] F. Bullo and K. M. Lynch. Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems. *IEEE Transactions on Robotics and Automation*, 17(4):402–412, 2001.
- [7] P. B. Davenport. Rotations about nonorthogonal axes. *AIAA Journal*, 11(6):853–857, 1973.
- [8] S. S. Ge, S. Zhendong, T. H. Lee, and M. W. Spong. Feedback linearization and stabilization of second-order non-holonomic chained systems. *International Journal of Control*, 74(14):1383–92, 2001.
- [9] A. Goldenberg, B. Benhabib, and R. Fenton. A complete generalized solution to the inverse kinematics of robots. *IEEE Journal of Robotics and Automation*, 1(1):14–20, 1985.
- [10] D. Halperin, L. E. Kavradi, and J.-C. Latombe. Robotics. In J. E. Goodman and J. O’Rourke, editors, *Handbook of Discrete and Computational Geometry*, chapter 41, pages 755–778. CRC Press, 1997.
- [11] S. Iannitti and K. M. Lynch. Minimum control-switch motions for the snakeboard: A case study in kinematically controllable underactuated systems. *IEEE Transactions on Robotics*, 20(6):994–1006, 2004.
- [12] G. Lafferriere and H. J. Sussmann. A differential geometric approach to motion planning. In Z. Li and J. F. Canny, editors, *Nonholonomic Motion Planning*, pages 235–270. Kluwer Academic Publishers, 1993.
- [13] N. E. Leonard and P. S. Krishnaprasad. Motion control of drift-free, left-invariant systems on Lie groups. *IEEE Transactions on Automatic Control*, 40(9):1539–1554, 1995.
- [14] K. M. Lynch, N. Shiroma, H. Arai, and K. Tanie. Collision-free trajectory planning for a 3-DOF robot with a passive joint. *International Journal of Robotics Research*, 19(12):1171–1184, 2000.
- [15] D. Manocha and J. F. Canny. Efficient inverse kinematics for general 6R manipulators. *IEEE Transactions on Robotics and Automation*, 10(5):648–657, 1994.
- [16] P. Martin and P. Rouchon. Feedback linearization and driftless systems. *Mathematics of Control, Signals and Systems*, 7(3):235–254, 1995.
- [17] M. C. Muñoz-Lecanda and F.J. Yáñez. Mechanical control systems and kinematic systems. *IEEE Transactions on Automatic Control*, 53(5):1297–1302, 2008.
- [18] R. M. Murray, Z. X. Li, and S. S. Sastry. *A Mathematical Introduction to Robotic Manipulation*. CRC Press, 1994.
- [19] B. Paden. *Kinematics and Control of Robot Manipulators*. PhD thesis, UC Berkeley, 1986.
- [20] C. Samson. Control of chained systems. Application to path following and time-varying point-stabilization of mobile robots. *IEEE Transactions on Automatic Control*, 40(1):64–77, 1995.
- [21] L. Sciavicco and B. Siciliano. *Modeling and Control of Robot Manipulators*. Springer, 2 edition, 2000.
- [22] O. J. Sordalen and O. Egeland. Exponential stabilization of nonholonomic chained systems. *IEEE Transactions on Automatic Control*, 40(1):35–49, 1995.
- [23] M. W. Spong and M. Vidyasagar. *Robot Dynamics and Control*. John Wiley, 1989.
- [24] J. Wittenburg and L. Lilov. Decomposition of a finite rotation into three rotations about given axes. *Multibody System Dynamics*, 9(4):353–375, 2003.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: `soniamd@ucsd.edu`

E-mail address: `jcortes@ucsd.edu`

E-mail address: `bullo@engineering.ucsb.edu`