Deployment of an unreliable robotic sensor network for spatial estimation

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***Abstract*— This paper studies an optimal deployment problem for a network of robotic sensors moving in the real line. Consider the scenario where each sensor is to take a measurement of a spatial process of interest and send it back to a data fusion center. Assume only a specific fraction of the messages containing the measurements will arrive at the center. We show that, for several fraction values, the optimal deployment configurations have the following features: agents are grouped into clusters, the clusters are deployed optimally as if at least a message from each cluster was guaranteed to reach the center, and for each fraction value, there is a specific optimal cluster size. The technical approach combines convex analysis, nonsmooth analysis, and combinatorics.**

I. Introduction

An important motivation for the use of multiple robots is the robustness that robotic networks can provide against individual malfunctions. This paper is a contribution to the growing body of research in cooperative control that seeks to understand how individual failures affect the network performance and how to best account for these failures in designing robust and adaptive robotic networks.

We consider the following problem. A group of robotic sensors is to be deployed over a region to sample a spatial process. Each sensor will take a point measurement and report it back to a data fusion center. However, because of the features of the medium and the limited communication capabilities of the agents, it is known that only a fraction of these packets will arrive at the center. Because of the stochastic nature of the packet drops, it is not known a priori which measurements will arrive. Our objective is to characterize the deployment configurations that maximize the expected information content of the measurements retrieved at the data fusion center. Specifically, we seek to minimize the expected maximum prediction error of the best linear unbiased predictor of the spatial field computed by the fusion center. We are also interested in quantifying the performance degradation of the network as a function of the fraction of packets that are successfully transmitted.

Literature review: The problem considered in this paper combines elements from facility location [1], [2], optimal estimation of spatial fields in statistics [3], [4], and data loss in communications theory [5], [6]. Without packet drops, our scenario corresponds to the disk-covering geometric optimization problem studied in [7], whose solutions turn out to be optimal for minimizing the posterior predictive variance of the best linear unbiased predictor of a spatial field, see [8], [9]. Our model for the communication between the robotic sensors and the data fusion center can be understood as an erasure channel, where packets are either dropped or received without error. Many works have considered erasure channels in problems of control and estimation, see e.g, [10], [11], [12], [13], [14], and in particular, in the context of sensor networks [15], [16], [17], [18], [19]. The work [20] considers a scenario similar to ours for a network of static sensors that take noisy measurements and characterizes the tradeoff between transmission rate and estimation quality. Finally, the work [21] deals with the optimization of the location of controllers when sensors and actuators are connected by an array of unreliable links.

Statement of contributions: We define an aggregate objective function that, to each configuration of n robot positions, associates the expected performance of the network under $n-k$ stochastic packet drops. Although some of our results could be presented in arbitrary dimensions, we restrict our attention to a closed segment of the real line. We characterize the convexity, smoothness, and invariance properties of the objective function. This study is key as it allows us to restrict our search for minimizers to a subset of the space of network configurations, more specifically, those that are invariant under the symmetric projection around the midpoint of the segment and whose positions are ordered in increasing order according to the agent identifier. We provide closed-form expressions for the minimizers for several subfamilies of problems.

	$n = 2m$ agents				
Successful transmissions	1	2	3	$n-1$	n
Optimal number of clusters	1	2	2	$n/2$	n
Optimal cluster size	n	$n/2$	$n/2$	2	1

TABLE I

Minimizers for networks with even number of agents. The minimizers correspond to deploying the clusters as if at least a packet from each one was guaranteed to reach the center.

A common feature of the minimizers is that agents are grouped into clusters, and the resulting clusters are deployed optimally as if at least a message from each cluster was guaranteed to reach the data fusion center. Our results show that there is an optimal tradeoff between grouping agents in clusters to increase the likelihood of measurements from those location arriving at the center and having as many different clusters as possible to increase the number of distinct measurements. We establish this tradeoff for the pairs $\{(n, k) \mid k \in \{1, 2, 3, n-1, n\}\}$. Table I provides a summary

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of these results when n is even. For space reasons, the proofs of the results are omitted and will appear elsewhere.

Organization: Section II introduces some useful notation and presents some basic facts on nonsmooth analysis. Section III states the problem considered here and introduces the objective function. Section IV studies in detail its smoothness, convexity, and invariance properties. Section V characterizes the solutions to the optimal deployment problem in a range of situations. We conclude by discussing the implications of our results and ideas for future work in Section VI. Some of the proofs are omitted for space reasons.

II. Preliminaries

We let $e_1, \dots, e_d$ denote the Euclidean basis of $\mathbb{R}^d$. Let $\mathrm{co}(S)$ denote the convex closure of a set $S \subset \mathbb{R}^d$ and let $B(x,\varepsilon) = \{y \in \mathbb{R}^d \mid \|y-x\| < \varepsilon\}$ denote the open ball in $\mathbb{R}^d$ with center x and radius ε. For $k \leq n$, we let $C(n,k)$ denote the set of k-combinations from $\{1,\dots,n\}$. Given $\{s_1,\dots,s_k\} \in C(n,k)$, we assume without loss of generality that $s_1 < \dots < s_k$.

A. Computational geometric notions

The *Voronoi partition* of $Q \subset \mathbb{R}^d$ generated by $p_1,\dots,p_n \in Q$ is the collection of sets $\{V_1,\dots,V_n\}$,

$$V_i = \{q \in Q \mid \|q-p_i\| \leq \|q-p_j\| \text{ for } j \neq i\},$$

for $i \in \{1,\dots,n\}$. Note that the union of the Voronoi cells is the whole set Q and that the intersection of the interiors of any two cells is empty. On the real line, $d=1$, the notion of Voronoi partition is particularly simple. Given $Q=[a,b] \subset \mathbb{R}$ and $(p_1,\dots,p_n) \in Q^n$, let $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ denote a permutation such that $p_{\sigma(1)} \leq \dots \leq p_{\sigma(n)}$. The Voronoi partition of Q determined by $p_1,\dots,p_n$ is given by

$$V_{\sigma(1)} = \Big[a, \frac{p_{\sigma(1)}+p_{\sigma(2)}}{2}\Big],\ V_{\sigma(n)} = \Big[\frac{p_{\sigma(n-1)}+p_{\sigma(n)}}{2}, b\Big],$$
$$V_{\sigma(i)} = \Big[\frac{p_{\sigma(i-1)}+p_{\sigma(i)}}{2}, \frac{p_{\sigma(i)}+p_{\sigma(i+1)}}{2}\Big],$$

where $i \in \{2,\dots,n-1\}$.

B. Nonsmooth analysis

Let f be a function of the form $f : \mathbb{R}^d \to \mathbb{R}$. f is *locally Lipschitz at* $x \in \mathbb{R}^d$ if there exist $L_x, \varepsilon \in \mathbb{R}_{>0}$ such that

$$|f(y) - f(y')| \leq L_x \|y - y'\|,$$

for $y, y' \in B(x,\varepsilon)$. f is *locally Lipschitz on* $S \subset \mathbb{R}^d$ if it is locally Lipschitz at x, for all $x \in S$.

The *generalized gradient* of a locally Lipschitz function f is defined by

$$\partial f(x) = \mathrm{co}\{\lim_{i\to+\infty} df(x_i) \mid x_i \to x\,,\ x_i \notin S \cup \Omega_f\},$$

where $\Omega_f \subset \mathbb{R}^d$ is the set of points where f fails to be differentiable, and S denotes any other set of measure zero. A point $x \in \mathbb{R}^d$ which verifies that $0 \in \partial f(x)$ is called a *critical point of* f. Minimizers and maximizers of f are of course critical points of f in the sense of this definition.

A final technical notion that we need to introduce is that of regular function. f is *regular at* $x \in \mathbb{R}^d$ if for all $v \in \mathbb{R}^d$, the right directional derivative of f at x in the direction of v exists and coincides with the generalized directional derivative of f at x in the direction of v. Precise definitions of these directional derivatives can be found in [22].

III. Problem statement

Consider a scenario where a group of robotic sensors are to be optimally deployed in order to take point measurements of a spatial random field. Assume that, once taken, the measurements will be sent to a data fusion center that will construct the estimate with the information received. Because of the features of the environment and the limited communication capabilities of the sensors, assume that it is known that only a fraction of those measurements will arrive.

Our main objective is to characterize the optimal deployment configurations for the group of robotic sensors in the scenario described above. The fact that the identity of the agents whose measurements arrive at the data fusion center is not known a priori is what makes the problem challenging. Let us make precise the notion of what a good deployment is. Let $Q = [a,b] \subset \mathbb{R}$ be a closed interval. Given $m \in \mathbb{Z}$, consider the *disk-covering* function $\mathcal{H}_{\mathrm{DC}} : Q^m \to \mathbb{R}$,

$$\mathcal{H}_{\mathrm{DC}}(q_1,\dots,q_m) = \max_{q \in Q} \min_{i \in \{1,\dots,m\}} \|q - q_i\|. \tag{1}$$

The value of $\mathcal{H}_{\mathrm{DC}}$ corresponds to the smallest radius such that the union of balls centered at the points $q_1,\dots,q_m \in Q$ with radius $\mathcal{H}_{\mathrm{DC}}(q_1,\dots,q_m)$ covers the whole environment Q. The reason why we consider is $\mathcal{H}_{\mathrm{DC}}$ is as follows: one can show [9] that, under certain technical conditions, the minimization of this function is equivalent to the minimization of the maximum uncertainty about the estimation of the spatial random field. We emphasize that if all measurements were guaranteed to arrive at the fusion center, then $\mathcal{H}_{\mathrm{DC}}$ would be the objective function to optimize. However, this is in general not the case, as we describe next.

Consider now a group of n mobile robotic sensors with positions $p_1,\dots,p_n \in Q$ that can take point measurements of the spatial random field. Let us refer to a sensor as *working* if once the network has been deployed and the measurements have been taken, its message arrives at the data fusion center. Of course, the identity of these sensors is a priori unknown. Assume that only $k \leq n$ sensors are working. Since only k of the sensors are working and their identity is unknown, instead of the objective function $(p_1,\dots,p_n) \mapsto \mathcal{H}_{\mathrm{DC}}(p_1,\dots,p_n)$ defined above, we consider the objective function

$$\mathcal{H}_{n,k}(p_1,\dots,p_n) = \frac{1}{\binom{n}{k}} \sum_{\{s_1,\dots,s_k\} \in C(n,k)} \mathcal{H}_{\mathrm{DC}}(p_{s_1},\dots,p_{s_k}), \tag{2}$$

which corresponds to the expected performance of the overall group. Note that $\mathcal{H}_{n,n}$ is exactly $\mathcal{H}_{\mathrm{DC}}$, i.e., both functions coincide if all sensors are working.

Remarks 3.1: (i) The problem described above could also be formulated in arbitrary dimensions. We have found that the problem is challenging enough on the real line to deserve attention on its own;

(ii) The optimization of $\mathcal{H}_{n,k}$ can be given alternative interpretations that involve the capability of servicing events in the environment, but we do not get into the details here for simplicity. •

IV. Analysis of the objective function

In this section, we study the properties of $\mathcal{H}_{n,k}$. This analysis will be most useful to characterize its minimizers.

A. Convexity properties

We begin by noting that $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ are invariant under permutations.

Lemma 4.1: For any permutation $\sigma : \{1,\dots,m\} \to \{1,\dots,m\}$ and any $q_1,\dots,q_m \in Q$,

$$\mathcal{H}_{\mathrm{DC}}(q_{\sigma(1)},\dots,q_{\sigma(m)}) = \mathcal{H}_{\mathrm{DC}}(q_1,\dots,q_m).$$

Consequently, $\mathcal{H}_{n,k}$ is also invariant under permutations.

The invariance of $\mathcal{H}_{n,k}$ under permutations allows us to restrict our search for minimizers to

$$Q^n_{\leq} = \{(p_1,\dots,p_n) \in Q^n \mid p_1 \leq \dots \leq p_n\} \subset Q^n. \quad (3)$$

Regarding the study of the critical points, it is important to note that $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ are *not* convex on Q^m and Q^n, respectively. This fact is related with their invariance under permutations, cf. Lemma 4.1. Let us illustrate it for $\mathcal{H}_{\mathrm{DC}}$. Let $q_1,\dots,q_m \in Q$ with $q_1 < \dots < q_m$ such that

$$q_1 - a > \max\big\{\frac{q_2-q_1}{2},\dots,\frac{q_m-q_{m-1}}{2}, b-q_m\big\}.$$

Then we have that $\mathcal{H}_{\mathrm{DC}}(q_1,q_2,q_3,\dots,q_m) = q_1 - a = \mathcal{H}_{\mathrm{DC}}(q_2,q_1,q_3,\dots,q_m)$. Moreover,

$$\begin{aligned}&\mathcal{H}_{\mathrm{DC}}\big(\tfrac{1}{2}(q_1+q_2),\tfrac{1}{2}(q_1+q_2),q_3,\dots,q_m\big)\\ &\geq \frac{1}{2}(q_1+q_2) - a > q_1 - a\\ &= \frac{1}{2}\mathcal{H}_{\mathrm{DC}}(q_1,q_2,q_3,\dots,q_m) + \frac{1}{2}\mathcal{H}_{\mathrm{DC}}(q_2,q_1,q_3,\dots,q_m),\end{aligned}$$

and hence $\mathcal{H}_{\mathrm{DC}}$ is not convex on Q^m.

Interestingly, both $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ are convex on convenient subsets of their domain of definition. To show this, let us define the maps $\widetilde{\mathcal{H}_{\mathrm{DC}}} : Q^m \to \mathbb{R}$ and $\widetilde{\mathcal{H}_{n,k}} : Q^n \to \mathbb{R}$,

$$\begin{aligned}&\widetilde{\mathcal{H}_{\mathrm{DC}}}(q_1,\dots,q_m) = \\ &\qquad \max\big\{q_1 - a, \frac{q_2-q_1}{2},\dots,\frac{q_m-q_{m-1}}{2}, b-q_m\big\},\\ &\widetilde{\mathcal{H}_{n,k}}(p_1,\dots,p_n) = \\ &\qquad \frac{1}{\binom{n}{k}} \sum_{\{s_1,\dots,s_k\}\in C(n,k)} \widetilde{\mathcal{H}_{\mathrm{DC}}}(p_{s_1},\dots,p_{s_k}).\end{aligned}$$

These maps are not invariant under permutations. Their relationship with $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ is given by

$$\begin{aligned}\mathcal{H}_{\mathrm{DC}}(q_1,\dots,q_m) &= \widetilde{\mathcal{H}_{\mathrm{DC}}}(q_{\sigma(1)},\dots,q_{\sigma(m)}),\\ \mathcal{H}_{n,k}(p_1,\dots,p_n) &= \widetilde{\mathcal{H}_{n,k}}(p_{\rho(1)},\dots,p_{\rho(n)}),\end{aligned}$$

for $q_1,\dots,q_m,p_1,\dots,p_n \in Q$, where σ and ρ are permutation such that $q_{\sigma(1)} \leq \dots \leq q_{\sigma(m)}$ and $p_{\rho(1)} \leq \dots \leq p_{\rho(n)}$. The following result states the convexity properties of all the functions mentioned so far.

Lemma 4.2: The functions $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ and $\widetilde{\mathcal{H}_{n,k}}$ are convex on Q^m and Q^n, respectively. Consequently, the functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ are convex on $Q^m_{\leq}$ and $Q^n_{\leq}$, respectively.

The minimizers of $\mathcal{H}_{n,k}$ over $Q^n_{\leq}$ might belong to the boundary of the set, and hence, in spite of Lemma 4.2, not be fully described with gradient information only. As we will explain later in Section V-B, the following result will be most helpful to overcome this hurdle.

Proposition 4.3: The minimizers of $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ over $Q^m_{\leq}$ are also minimizers of $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ over Q^m. Likewise, the minimizers of $\widetilde{\mathcal{H}_{n,k}}$ over $Q^n_{\leq}$ are also minimizers of $\widetilde{\mathcal{H}_{n,k}}$ over Q^n.

B. Nonsmooth properties

We review some basic facts about the disk-covering function $\mathcal{H}_{\mathrm{DC}}$ following [7]. Given a set $W \subset Q$ and $p \in W$, let $\mathrm{lg}_W : W \to \mathbb{R}$ be the largest distance from p to W,

$$\mathrm{lg}_W(p) = \max_{q\in W} \|q-p\|.$$

This definition allows us to rewrite $\mathcal{H}_{\mathrm{DC}}$ as follows

$$\mathcal{H}_{\mathrm{DC}}(q_1,\dots,q_m) = \max_{i\in\{1,\dots,m\}} \mathrm{lg}_{V_i} \circ \pi_i(q_1,\dots,q_m). \quad (4)$$

where $\pi_i : Q^m \to Q$, $i \in \{1,\dots,m\}$, denotes the projection $(q_1,\dots,q_m) \mapsto q_i$. The individual objective functions in (4) can take three different forms. To write them explicitly, assume, without loss of generality that $(q_1,\dots,q_m) \in Q^m_{\leq}$ (if this is not the case, then the expressions below need only to be rearranged according to the increasing order of $q_1,\dots,q_m$). Then, we have

$$\begin{aligned}\mathrm{lg}_{V_1} \circ \pi_1(q_1,\dots,q_m) &= \max\Big\{q_1 - a, \frac{q_2-q_1}{2}\Big\},\\ \mathrm{lg}_{V_i} \circ \pi_i(q_1,\dots,q_m) &= \max\Big\{\frac{q_i-q_{i-1}}{2}, \frac{q_{i+1}-q_i}{2}\Big\},\\ \mathrm{lg}_{V_m} \circ \pi_m(q_1,\dots,q_m) &= \max\Big\{\frac{q_m-q_{m-1}}{2}, b-q_m\Big\},\end{aligned}$$

where $i \in \{2,\dots,m-1\}$. The following result states explicit expressions for the generalized gradients of these functions.

Lemma 4.4: The functions $\mathrm{lg}_{V_i} \circ \pi_i : Q^n \to \mathbb{R}$, $i \in \{1,\dots,m\}$, are locally Lipschitz and regular. Furthermore, for $(q_1,\dots,q_m) \in Q^m_{\leq}$, their generalized gradients take one of the following forms

$$\partial(\lg_{V_1} \circ \pi_1)(q_1, \dots, q_m) = \begin{cases} e_1 & \text{if } q_1 - a > \frac{q_2 - q_1}{2}, \\ \text{co}\{e_1, \frac{1}{2}(e_2 - e_1)\} & \text{if } q_1 - a = \frac{q_2 - q_1}{2}, \\ \frac{1}{2}(e_2 - e_1) & \text{if } q_1 - a < \frac{q_2 - q_1}{2}, \end{cases}$$

$$\partial(\lg_{V_i} \circ \pi_i)(q_1, \dots, q_m) = \begin{cases} \frac{1}{2}(e_i - e_{i-1}) & \text{if } q_i > \frac{q_{i+1} + q_{i-1}}{2}, \\ \frac{1}{2}\text{co}\{e_i - e_{i-1}, e_{i+1} - e_i\} & \text{if } q_i = \frac{q_{i+1} + q_{i-1}}{2}, \\ \frac{1}{2}(e_{i+1} - e_i) & \text{if } q_i < \frac{q_{i+1} + q_{i-1}}{2}, \end{cases}$$

$$\partial(\lg_{V_m} \circ \pi_m)(q_1, \dots, q_m) = \begin{cases} \frac{1}{2}(e_m - e_{m-1}) & \text{if } \frac{q_m - q_{m-1}}{2} > b - q_m, \\ \text{co}\{-e_m, \frac{1}{2}(e_m - e_{m-1})\} & \text{if } \frac{q_m - q_{m-1}}{2} = b - q_m, \\ -e_m & \text{if } \frac{q_m - q_{m-1}}{2} < b - q_m. \end{cases}$$

Given the fact that the maximum of locally Lipschitz and regular functions is itself locally Lipschitz and regular, see e.g., [22], one has the following result.

Lemma 4.5: The functions $\mathcal{H}_{\text{DC}}$ and $\widetilde{\mathcal{H}_{\text{DC}}}$ are locally Lipschitz and regular, and their generalized gradients are

$$\partial \mathcal{H}_{\text{DC}}(q_1, \dots, q_m) = \text{co}\{\partial(\lg_{V_i} \circ \pi_i)(q_1, \dots, q_m) \mid i \text{ such that } \mathcal{H}_{\text{DC}}(q_1, \dots, q_m) = \lg_{V_i} \circ \pi_i(q_1, \dots, q_m)\},$$

$$\partial \widetilde{\mathcal{H}_{\text{DC}}}(q_1, \dots, q_m) = \text{co}\{S(q_1, \dots, q_m)\},$$

where $S = S(q_1, \dots, q_m)$ is the set defined by

$$e_1 \in S \text{ iff } q_1 - a = \widetilde{\mathcal{H}_{\text{DC}}}(q_1, \dots, q_m),$$
$$-e_m \in S \text{ iff } b - q_m = \widetilde{\mathcal{H}_{\text{DC}}}(q_1, \dots, q_m),$$
$$\tfrac{1}{2}(e_{i+1} - e_i) \in S \text{ iff } \tfrac{1}{2}(q_{i+1} - q_i) = \widetilde{\mathcal{H}_{\text{DC}}}(q_1, \dots, q_m),$$

for $i \in \{1, \dots, m-1\}$.

Using the fact that a sum of locally Lipschitz functions is locally Lipschitz and the fact that a linear combination of regular functions with positive coefficients is also regular, see e.g., [22], we deduce the following useful result.

Lemma 4.6: The functions $\mathcal{H}_{n,k}$ and $\widetilde{\mathcal{H}_{n,k}}$ are locally Lipschitz and regular, and their generalized gradients are

$$\partial \mathcal{H}_{n,k}(p_1, \dots, p_n) = \frac{1}{\binom{n}{k}} \sum_{\{s_1, \dots, s_k\} \in C(n,k)} \partial \mathcal{H}_{\text{DC}}(p_{s_1}, \dots, p_{s_k}),$$

$$\partial \widetilde{\mathcal{H}_{n,k}}(p_1, \dots, p_n) = \frac{1}{\binom{n}{k}} \sum_{\{s_1, \dots, s_k\} \in C(n,k)} \partial \widetilde{\mathcal{H}_{\text{DC}}}(p_{s_1}, \dots, p_{s_k}).$$

C. Invariance properties

Next, we study the invariance properties of $\mathcal{H}_{n,k}$ for the symmetric projection with respect to the midpoint of $Q = [a, b]$. Consider the bijective map $\mathfrak{i} : Q \to Q$, $\mathfrak{i}(q) = b + a - q$. The following result makes precise our claim of invariance.

Lemma 4.7: The functions $\mathcal{H}_{\text{DC}} : Q^m \to \mathbb{R}$ and $\mathcal{H}_{n,k} : Q^n \to \mathbb{R}$ are invariant under $\mathfrak{i}$, that is,

$$\mathcal{H}_{\text{DC}}(\mathfrak{i}(q_1), \dots, \mathfrak{i}(q_m)) = \mathcal{H}_{\text{DC}}(q_1, \dots, q_m),$$
$$\mathcal{H}_{n,k}(\mathfrak{i}(p_1), \dots, \mathfrak{i}(p_n)) = \mathcal{H}_{n,k}(p_1, \dots, p_n),$$

for all $q_1, \dots, q_m, p_1, \dots, p_n \in Q$.

Inspired by Lemma 4.7, we define the set of configurations in Q^n that are invariant under $\mathfrak{i}$ as,

$$Q^n_{\text{inv}} = \{P \in Q^n \mid \{p_1, \dots, p_n\} = \{\mathfrak{i}(p_1), \dots, \mathfrak{i}(p_n)\}.$$

Note that a configuration $(p_1, \dots, p_n) \in Q^n_{\leq}$ belongs to Q^n_{inv} if $p_1 = \mathfrak{i}(p_n), \dots, p_n = \mathfrak{i}(p_1)$. If $n = 2m + 1$, $m \in \mathbb{Z}_{\geq 0}$, is odd, then this implies that $p_{m+1} = \frac{a+b}{2}$.

The following result narrows down the search for the solutions to our optimal deployment problem.

Proposition 4.8: The minimizers of $\mathcal{H}_{\text{DC}}$ over $Q^m_{\text{inv}} \cap Q^m_{\leq}$ are minimizers of $\mathcal{H}_{\text{DC}}$ over $Q^m_{\leq}$. Likewise, the minimizers of $\mathcal{H}_{n,k}$ over $Q^n_{\text{inv}} \cap Q^n_{\leq}$ are minimizers of $\mathcal{H}_{n,k}$ over $Q^n_{\leq}$.

V. Characterization of the optimal deployment configurations

Here, we characterize the solutions to the optimal deployment problem formulated in Section III for a range of situations depending on the relative value of the number k of working agents with respect to the total number n of agents.

A. Performance bounds

Here, we formalize the intuition that, from a performance viewpoint, a network composed of n agents with k of them working whose identity is unknown is worse than a network composed of k working agents. We have only been able to prove this result for $k \leq \lfloor n/2 \rfloor + 1$, although we suspect it to be true in general. We start with a characterization of the minimizers when all agents are working.

Lemma 5.1: The function $\mathcal{H}_{n,n}$ has as unique minimizer the configuration

$$p_1^* = \frac{(2n-1)a + b}{2n},$$
$$p_2^* = \frac{(2n-3)a + 3b}{2n},$$
$$\vdots$$
$$p_{n-1}^* = \frac{3a + (2n-3)b}{2n},$$
$$p_n^* = \frac{a + (2n-1)b}{2n},$$

with value $\mathcal{H}_{n,n}(p_1^*, \dots, p_n^*) = \frac{b-a}{2n}$.

The following result shows that the minimizers of the problem when all agents are working bound the location of the minimizers when only a fraction of the agents work.

Theorem 5.2: For $n \geq 2$ and $k \leq \lfloor n/2 \rfloor + 1$, let $(\overline{p}_1, \dots, \overline{p}_n)$ be a minimizer of $\mathcal{H}_{n,k}$ and let $(p_1^*, \dots, p_k^*)$ be the minimizer of $\mathcal{H}_{k,k}$. Then

$$\overline{p}_1 \geq p_1^*, \qquad \overline{p}_n \leq p_k^*.$$

We suspect that Theorem 5.2 holds for any $n \geq 2$ and $k \leq n$. The next result states that a network composed of n agents with k of them working whose identity is unknown performs worse than a network composed of k working agents.

Corollary 5.3: For $n \geq 2$ and $k \leq \lfloor n/2 \rfloor$, let $(\overline{p}_1, \dots, \overline{p}_n)$ be a minimizer of $\mathcal{H}_{n,k}$ and let $(p_1^*, \dots, p_k^*)$ be the minimizer of $\mathcal{H}_{k,k}$. Then $\mathcal{H}_{k,k}(p_1^*, \dots, p_k^*) < \mathcal{H}_{n,k}(\overline{p}_1, \dots, \overline{p}_n)$.

B. Proof strategy for the characterization of minimizers

Our next step is to study the location of the minimizers of $\mathcal{H}_{n,k}$. In this section, we describe our strategy to do so. Because of the invariance under permutations, it is sufficient to characterize the minimizers of $\mathcal{H}_{n,k}$ over $Q^n_{\leq}$. These minimizers, however, are not necessarily described properly by the equation $0 \in \partial \mathcal{H}_{n,k}(p_1, \dots, p_n)$ if they belong to the boundary of $Q^n_{\leq}$.

The combination of the results presented in Section IV allows us to adopt the following strategy to find these solutions. On the one hand, Proposition 4.8 states that we can restrict our search to $Q^n_{\text{inv}} \cap Q^n_{\leq}$. On the other hand, since $\mathcal{H}_{n,k}$ and $\widetilde{\mathcal{H}_{n,k}}$ are the same function over $Q^n_{\leq}$, they have the same minimizers over $Q^n_{\leq}$. Noting that minimizers of $\widetilde{\mathcal{H}_{n,k}}$ must belong to the interior of Q^n, and using Lemma 4.2 and Proposition 4.3, we deduce that the minimizers of $\widetilde{\mathcal{H}_{n,k}}$ over $Q^n_{\leq}$ are described by

$$0 \in \partial \widetilde{\mathcal{H}_{n,k}}(p_1, \dots, p_n). \tag{5}$$

Therefore, our strategy to find the solutions is to look for $(p_1, \dots, p_n) \in Q^n_{\text{inv}} \cap Q^n_{\leq}$ that satisfy (5). Lemma 4.6 provides us with the tools to characterize these minimizers. The idea is to understand the geometric conditions on the critical configurations imposed by this description. We make extensive use of this strategy next.

C. Optimally-sized-and-positioned agent clusters

Here, we show that the agent clustering turns out to be optimal for our deployment problem, and specify the optimal size of the clusters in a range of situations. Our proof strategy is that of Section V-B. We start with the case of a single working agent.

Lemma 5.4: For $n \in \mathbb{N}$, the minimizer of $\mathcal{H}_{n,1}$ is

$$p_1^* = \cdots = p_n^* = \frac{a+b}{2}.$$

Next, we examine the case of two working agents.

Proposition 5.5: For $n \geq 2$, the minimizer of $\mathcal{H}_{n,2}$ is

$$\begin{aligned} p_1^* = \cdots = p_m^* &= \frac{3a+b}{4}, \\ p_{m+1}^* = \cdots = p_{2m}^* &= \frac{a+3b}{4}, \end{aligned} \tag{6a}$$

if $n = 2m$ is even, and

$$\begin{aligned} p_1^* = \cdots = p_m^* &= \frac{3a+b}{4}, \qquad p_{m+1}^* = \frac{a+b}{2}, \\ p_{m+2}^* = \cdots = p_{2m+1}^* &= \frac{a+3b}{4}, \end{aligned} \tag{6b}$$

if $n = 2m+1$ is odd.

The following result shows that the solution with three working agents is the same as when there are only two.

Proposition 5.6: For $n \geq 4$, the minimizer of $\mathcal{H}_{n,3}$ is given by (6).

Our final result shows that, when all agents but one are working, clustering in groups of two is optimal.

Proposition 5.7: For $n \geq 2$, the minimizer of $\mathcal{H}_{n,n-1}$ is

$$p_k^* = p_{k+1}^* = \frac{(2m-k)a + kb}{2m},$$

for $k \in \{1, 3, \dots, 2m-1\}$, if $n = 2m$ is even,

$$\begin{aligned} p_k^* = p_{k+1}^* &= \frac{(2(m+1)-k)a + kb}{2(m+1)}, \\ p_m^* &= \frac{(m+2)a + mb}{2(m+1)}, \\ p_{m+1}^* &= \frac{a+b}{2}, \\ p_{m+2}^* &= \frac{ma + (m+2)b}{2(m+1)}, \end{aligned}$$

for $k \in \{1, 3, \dots, m-2, m+4, m+6, \dots, 2m+1\}$, if $n = 2m+1$ is odd with m odd, and

$$\begin{aligned} p_k^* = p_{k+1}^* &= \frac{(2(m+1)-k)a + kb}{2(m+1)}, \\ p_{m+1}^* &= \frac{a+b}{2}, \end{aligned}$$

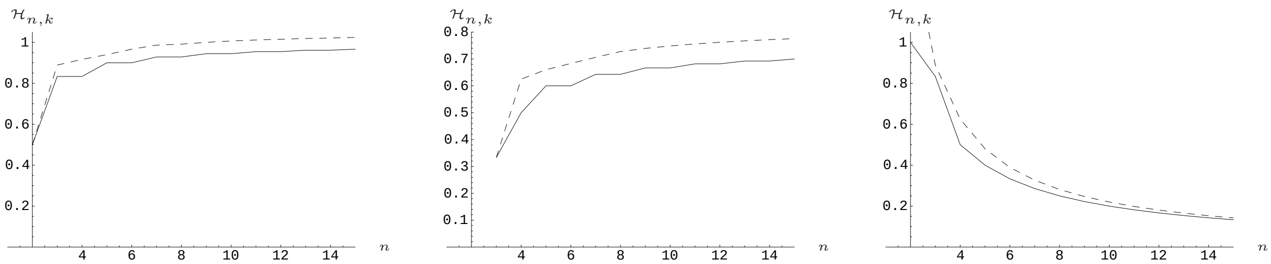
for $k \in \{1, 3, \dots, m-1, m+1, \dots, 2m+1\}$, if $n = 2m+1$ is odd with m even.

Figure 1 illustrates the performance with regards to $\mathcal{H}_{n,k}$ of its minimizer against the solution with all agents working, cf. Lemma 5.1, for $k = 2$, 3, and $n-1$.

Remark 5.8: (Conjecture for arbitrary number of working agents): Interestingly, all the results in this section point in the same direction: grouping agents into clusters and optimally deploying the resulting clusters is the solution to our optimal deployment problem. Grouping agents increases the chances of each cluster having at least a working agent, and hence being able to produce a measurement. The larger the size of the clusters is, the most likely it is that they will work. However, the larger the size, the smaller the number of clusters, and hence the worse the performance. These observations lead us to conjecture that, in general, the minimizers of $\mathcal{H}_{n,k}$ for arbitrary $k \leq n$ correspond to agent clusters optimally deployed according to its number, and that there is a precise formula that determines the number of clusters and their size for given n and k. We also believe the conjecture to hold in higher dimensions since the tradeoff between size and number of clusters is independent of the dimension, although the specific formula might be different depending of the environment. •

VI. Conclusions

We have analyzed a deployment problem for an unreliable robotic sensor network taking point measurements of a spatial process and relying them back to a data fusion center. We have shown that grouping sensors not only makes sense as a 'play-it-safe' strategy, but that, surprisingly, turns out



(a) $k = 2$ (b) $k = 3$ (c) $k = n - 1$

Fig. 1. Performance comparison between deployment solutions that take (solid line) and do not take (dashed line) into account, respectively, packet drops. In each plot, the solid curve corresponds to the optimal value of $\mathcal{H}_{n,k}$, while the dashed curve corresponds to the value of $\mathcal{H}_{n,k}$ at the optimal configuration of $\mathcal{H}_{n,n}$. Since k is constant in (a) and (b) (i.e., a fixed number of sensors work), the difference in performance becomes larger as n grows. However, since $k = n - 1$ in (c) (i.e., all but one sensor work), the difference in performance decreases as n grows.

to be optimal. From our study, we have conjectured that the right amount of grouping depends upon the proportion of working sensors with respect to the total number of agents. We have shown this conjecture to be true in a number of cases. Our analysis has required a blend of nonsmooth analysis, convex analysis, and combinatorics. The results presented here offer guidance as to how to design coordination algorithms to achieve optimal deployment with failing sensors. Future work will be devoted to this topic as well as to establish the validity of the conjecture in all cases and examine the connection of the solutions presented here with dynamic scenarios with randomly failing sensors taking multiple measurements.

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