Deployment of an unreliable robotic sensor network for spatial estimation

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Abstract

This paper studies an optimal deployment problem for a network of robotic sensors moving in the real line. Given a spatial process of interest, each individual sensor sends a packet that contains a measurement of the process to a data fusion center. We assume that, due to communication limitations or hardware unreliability, only a fraction of the packets arrive at the center. Using convex analysis, nonsmooth analysis, and combinatorics, we show that, for various fractional rates of packet arrival, the optimal deployment configuration has the following features: agents group into clusters, clusters deploy optimally as if at least one packet from each cluster was guaranteed to reach the center, and there is an optimal cluster size for each fractional arrival rate.

1. Introduction

An important motivation for the use of multiple robots in cooperative control is the robustness that robotic networks can provide against individual malfunctions. This paper is a contribution to the growing body of research in cooperative control that seeks to understand how individual failures affect network performance and how to best account for these failures in designing robust and adaptive robotic networks.

We consider the following problem. A group of robotic sensors is to be deployed over a region to sample an environmental process of interest. Each sensor will take a point measurement and report it back to a data fusion center. However, because of the features of the medium and the limited communication capabilities of the agents, it is known that only a fraction of these packets will arrive at the center. Because of the stochastic nature of the packet drops, it is not known a priori which measurements will arrive. Our objective is to characterize deployment configurations that maximize the expected information content of the measurements retrieved at the data fusion center. We are also interested in quantifying the performance degradation of the network as a function of the fraction of packets that are successfully transmitted.

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Literature review. The problem considered in this paper combines elements from facility location [1, 2], optimal estimation of spatial fields in statistics [3, 4], and data loss in communications theory [5]. Without packet drops, our scenario corresponds to the disk-covering geometric optimization problem studied in [6], whose solutions turn out to be optimal for minimizing the posterior predictive variance of the best linear unbiased predictor of a spatial field, see [7, 8]. Our model for the communication between the sensors and the data fusion center can be understood as an erasure channel, where packets are either dropped or received without error. Many works have considered erasure channels in problems of control and estimation, see e.g, [9–12], and in particular, in the context of sensor networks [13–16]. The work [17] considers a scenario similar to the one in this paper for a network of static sensors that take noisy measurements and characterizes the trade-off between transmission rate and estimation quality. Finally, [18] deals with the optimization of the location of controllers when sensors and actuators are connected by an array of unreliable links.

Statement of contributions. We define an aggregate objective function that, to each configuration of n robot positions, associates the expected performance of the network under $n-k$ packet drops (or, alternatively, under k a priori unknown successful packet transmissions). Although some of our results could be presented in arbitrary dimensions, we restrict our attention to a closed segment of the real line. We characterize the convexity, smoothness, and invariance properties of the objective function. This study is key as it allows us to restrict our search for the minimizers to a subset of the space of network configurations, more specifically, those that are invariant under the symmetric projection around the midpoint of the segment and whose positions are sorted in increasing order according to the agent identifier. We provide closed-form expressions for the minimizers for several subfamilies of problems. A common feature of

	$n=2m$ agents				
Successful transmissions	1	2	3	$n-1$	n
Optimal number of clusters	1	2	2	$n/2$	n
Optimal cluster size	n	$n/2$	$n/2$	2	1

Table 1: Minimizers for networks with even number of agents. Minimizers correspond to deploying clusters as if at least one packet from each cluster was guaranteed to reach the center.

the minimizers is that agents are grouped into clusters, and the resulting clusters are deployed optimally as if at least one message from each cluster was guaranteed to reach the data fusion center. Our results show that there is an optimal trade-off between grouping agents in clusters to increase the likelihood of measurements from those location arriving at the center and having as many different clusters as possible to increase the number of distinct measurements. We establish this trade-off for the pairs $\{(n,k) \mid k \in \{1,2,3,n-1,n\}\}$. Table 1 provides a summary of the results when n is even.

Organization. Section 2 introduces some useful notation and presents some basic facts on nonsmooth analysis. Section 3 states the problem considered here and introduces the objective function. Section 4 studies in detail the smoothness, convexity, and invariance properties of the objective function. Section 5 characterizes the solutions to the optimal deployment problem in a range of situations. We conclude by discussing the implications of our results and ideas for future work in Section 6.

2. Preliminaries

We let $\{e_1, \dots, e_d\}$ denote the canonical Euclidean basis of $\mathbb{R}^d$. Let $\operatorname{co}(S)$ denote the convex closure of a set $S \subset \mathbb{R}^d$ and let $B(x, \varepsilon) = \{y \in \mathbb{R}^d \mid \|y - x\| < \varepsilon\}$ denote the open ball in $\mathbb{R}^d$ with center x and radius ε. For $k \leq n$, we let $C(n, k)$ denote the set of k-combinations from $\{1, \dots, n\}$. Given $\{s_1, \dots, s_k\} \in C(n, k)$, we assume without loss of generality that $s_1 < \dots < s_k$.

2.1. Computational geometric notions

The *Voronoi partition* of $Q \subset \mathbb{R}^d$ generated by $p_1, \dots, p_n \in Q$ is the collection of sets $\{V_1, \dots, V_n\}$ defined by

$$V_i = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\| \text{ for } j \neq i\},$$

for $i \in \{1, \dots, n\}$. Note that the union of the Voronoi cells is the whole set Q and that the intersection of the interiors of any two cells is empty. On the real line, $d = 1$, the notion of Voronoi partition is particularly simple. Given $Q = [a, b] \subset \mathbb{R}$ and $(p_1, \dots, p_n) \in Q^n$, let $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ denote a permutation such that $p_{\sigma(1)} \leq \dots \leq p_{\sigma(n)}$. The Voronoi partition of Q determined by $p_1, \dots, p_n$ is given by

$$\begin{aligned} V_{\sigma(1)} &= \Big[a, \frac{p_{\sigma(1)} + p_{\sigma(2)}}{2}\Big], \quad V_{\sigma(n)} = \Big[\frac{p_{\sigma(n-1)} + p_{\sigma(n)}}{2}, b\Big], \\ V_{\sigma(i)} &= \Big[\frac{p_{\sigma(i-1)} + p_{\sigma(i)}}{2}, \frac{p_{\sigma(i)} + p_{\sigma(i+1)}}{2}\Big], \end{aligned}$$

where $i \in \{2, \dots, n-1\}$.

2.2. Nonsmooth analysis

Let f be a function of the form $f : \mathbb{R}^d \to \mathbb{R}$. f is *locally Lipschitz at* $x \in \mathbb{R}^d$ if there exist $L_x, \varepsilon \in \mathbb{R}_{>0}$ such that $|f(y) - f(y')| \leq L_x \|y - y'\|$, for $y, y' \in B(x, \varepsilon)$. f is *locally Lipschitz on* $S \subset \mathbb{R}^d$ if it is locally Lipschitz at x, for all $x \in S$. The *generalized gradient* of a locally Lipschitz function f is defined by

$$\partial f(x) = \operatorname{co}\{\lim_{i \to +\infty} df(x_i) \mid x_i \to x\,,\ x_i \notin S \cup \Omega_f\},$$

where $\Omega_f \subset \mathbb{R}^d$ is the set of points where f fails to be differentiable, and S denotes any other set of measure zero. A point $x \in \mathbb{R}^d$ which verifies that $0 \in \partial f(x)$ is called a *critical point of* f. Minimizers and maximizers of f are of course critical points of f in the sense of this definition. A technical notion that we need to introduce is that of regular function. f is *regular at* $x \in \mathbb{R}^d$

if for all $v \in \mathbb{R}^d$, the right directional derivative of f at x in the direction of v exists and coincides with the generalized directional derivative of f at x in the direction of v. Precise definitions of these directional derivatives can be found in [19]. We end this section with a result that will be useful later.

Lemma 2.1. *Let $f, g, h : \mathbb{R}^d \to \mathbb{R}$ be convex, locally Lipschitz, and regular functions with $h = f + g$. Let $X \subset \mathbb{R}^d$ containing a minimizer of h and assume there exists $P_* \in X$ such that*

$$\partial f(P) \subset \partial f(P_*) \text{ and } \partial g(P) \subset \partial g(P_*), \text{ for all } P \in X.$$

Then, P_* is a minimizer of h.

Proof. Let $\overline{P} \in X$ be a minimizer of h. Then, $0 \in \partial h(\overline{P}) = \partial f(\overline{P}) + \partial g(\overline{P}) \subset \partial f(P_*) + \partial g(P_*)$, and hence P_* is a minimizer of h. □

3. Problem statement

Consider a team of robotic sensors with the ability of taking point measurements of a spatial random field of interest. Once taken, the data will be sent to a fusion center that constructs the estimate with the information received. In this scenario, it makes sense to optimize the network deployment in order to construct estimates with minimum uncertainty. Further complicating the problem, assume that because of the features of the environment and the limited capabilities of the sensors, it is known that only a fraction of the measurements will arrive at the center. Our main objective is then to characterize the optimal deployment configurations for this unreliable sensor network. The fact that the identity of the agents whose measurements arrive at the fusion center is not known a priori makes the problem challenging.

The notion of optimal deployment depends of course on the specific objective function. We describe this point in detail next. Let $Q = [a, b] \subset \mathbb{R}$ be a closed interval. Given $m \in \mathbb{Z}$, consider the *disk-covering* function $\mathcal{H}_{\mathrm{DC}} : Q^m \to \mathbb{R}$,

$$\mathcal{H}_{\mathrm{DC}}(q_1, \dots, q_m) = \max_{q \in Q} \min_{i \in \{1,\dots,m\}} \|q - q_i\|. \tag{1}$$

The value of $\mathcal{H}_{\mathrm{DC}}$ corresponds to the smallest radius such that the union of balls centered at the points $q_1, \dots, q_m \in Q$ with radius $\mathcal{H}_{\mathrm{DC}}(q_1, \dots, q_m)$ covers the whole environment Q. Under an asymptotic regime known as near-independence [8], the minimization of $\mathcal{H}_{\mathrm{DC}}$ is equivalent to the minimization of the maximum uncertainty about the estimation of the spatial random field.

Consider now a group of n mobile robotic sensors with positions $p_1, \dots, p_n \in Q$ that can take point measurements of the spatial random field. Let us refer to a sensor as *working* if once the network has been deployed and the measurements have been taken, its message arrives at the data fusion center. Of course, the identity of these sensors is a priori unknown. Assume that only $k \leq n$ sensors are working. If all of the sensors were working properly, then the objective function to optimize would be $(p_1, \dots, p_n) \mapsto \mathcal{H}_{\mathrm{DC}}(p_1, \dots, p_n)$.

However, since only k of the sensors work and their identity is unknown, we consider the expected performance of the overall group. This corresponds to considering all the possibilities of what k sensors might be working, computing its performance, and doing the average, i.e.,

$$\mathcal{H}_{n,k}(p_1,\ldots,p_n) = \frac{1}{\binom{n}{k}} \sum_{\{s_1,\ldots,s_k\}\in C(n,k)} \mathcal{H}_{\mathrm{DC}}(p_{s_1},\ldots,p_{s_k}), \qquad (2)$$

Note that $\mathcal{H}_{n,n}$ is exactly $\mathcal{H}_{\mathrm{DC}}$. The minimizers of $\mathcal{H}_{n,k}$ correspond then to the optimal network configurations of the unreliable sensor network.

Remarks 3.1. (i) The problem described above could also be formulated in arbitrary dimensions. We have found that, even on the real line, the problem is challenging enough to deserve attention on its own. Section 5 below illustrates the challenging analysis involved in its solution;

(ii) The optimization of $\mathcal{H}_{n,k}$ can be given other alternative interpretations that involve the capability of servicing events in the environment, but we do not get into the details here for simplicity. •

4. Analysis of the objective function

Here, we unveil some of the properties of $\mathcal{H}_{n,k}$. Our strategy to characterize the minimizers of $\mathcal{H}_{n,k}$, outlined later in Section 5.2, is based on this analysis. Specifically, the invariance properties will allow us to restrict the search for minimizers to a subspace, the convexity properties will allow us to characterize the minimizers as critical points of a suitable function, and the nonsmooth properties will allow us to obtain a geometric description of these critical points.

4.1. Convexity properties

Here we establish some important facts regarding the convexity properties of the objective function $\mathcal{H}_{n,k}$. We begin by noting that $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ are invariant under permutations.

Lemma 4.1. *For any permutation $\sigma : \{1,\ldots,m\} \to \{1,\ldots,m\}$ and any $q_1,\ldots,q_m \in Q$, $\mathcal{H}_{\mathrm{DC}}(q_{\sigma(1)},\ldots,q_{\sigma(m)}) = \mathcal{H}_{\mathrm{DC}}(q_1,\ldots,q_m)$. Consequently, $\mathcal{H}_{n,k}$ is also invariant under permutations.*

The invariance of $\mathcal{H}_{n,k}$ under permutations allows us to restrict our search for minimizers to configurations that satisfy $p_1 \leq \ldots \leq p_n$. For convenience, we use the shorthand notation $Q^n_{\leq}$ to denote the set of such configurations, i.e.,

$$Q^n_{\leq} = \{(p_1,\ldots,p_n) \in Q^n \mid p_1 \leq \ldots \leq p_n\} \subset Q^n. \qquad (3)$$

Regarding the study of the critical points, it is important to observe that $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ are not convex on the whole spaces Q^m and Q^n, respectively, but only on $Q^m_{\leq}$ and $Q^n_{\leq}$. This fact is related with the invariance of these functions under permutations, cf. Lemma 4.1. Let us illustrate that $\mathcal{H}_{\mathrm{DC}}$ is not convex. Let $q_1, \ldots, q_m \in Q$ with $q_1 < \cdots < q_m$ such that

$$q_1 - a > \max\big\{\frac{q_2 - q_1}{2}, \ldots, \frac{q_m - q_{m-1}}{2}, b - q_m\big\}.$$

Then we have that $\mathcal{H}_{\mathrm{DC}}(q_1, q_2, q_3, \ldots, q_m) = q_1 - a = \mathcal{H}_{\mathrm{DC}}(q_2, q_1, q_3, \ldots, q_m)$. Moreover,

$$\begin{aligned}&\mathcal{H}_{\mathrm{DC}}\big(\frac{1}{2}(q_1 + q_2), \frac{1}{2}(q_1 + q_2), q_3, \ldots, q_m\big) \geq \frac{1}{2}(q_1 + q_2) - a \\ &> q_1 - a = \frac{1}{2}\mathcal{H}_{\mathrm{DC}}(q_1, q_2, q_3, \ldots, q_m) + \frac{1}{2}\mathcal{H}_{\mathrm{DC}}(q_2, q_1, q_3, \ldots, q_m),\end{aligned}$$

and hence $\mathcal{H}_{\mathrm{DC}}$ is not convex on Q^m.

To establish that both $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ are convex on convenient subsets of their domain of definition, let us define the maps $\widetilde{\mathcal{H}_{\mathrm{DC}}} : Q^m \to \mathbb{R}$ and $\widetilde{\mathcal{H}_{n,k}} : Q^n \to \mathbb{R}$,

$$\begin{aligned}\widetilde{\mathcal{H}_{\mathrm{DC}}}(q_1, \ldots, q_m) &= \max\big\{q_1 - a, \frac{q_2 - q_1}{2}, \ldots, \frac{q_m - q_{m-1}}{2}, b - q_m\big\}, \\ \widetilde{\mathcal{H}_{n,k}}(p_1, \ldots, p_n) &= \frac{1}{\binom{n}{k}} \sum_{\{s_1, \ldots, s_k\} \in C(n,k)} \widetilde{\mathcal{H}_{\mathrm{DC}}}(p_{s_1}, \ldots, p_{s_k}).\end{aligned}$$

These maps are not invariant under permutations. Their relationship with $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ is given by

$$\begin{aligned}\mathcal{H}_{\mathrm{DC}}(q_1, \ldots, q_m) &= \widetilde{\mathcal{H}_{\mathrm{DC}}}(q_{\sigma(1)}, \ldots, q_{\sigma(m)}), \\ \mathcal{H}_{n,k}(p_1, \ldots, p_n) &= \widetilde{\mathcal{H}_{n,k}}(p_{\rho(1)}, \ldots, p_{\rho(n)}),\end{aligned}$$

for $q_1, \ldots, q_m, p_1, \ldots, p_n \in Q$, where σ and ρ are permutations such that $q_{\sigma(1)} \leq \ldots \leq q_{\sigma(m)}$ and $p_{\rho(1)} \leq \ldots \leq p_{\rho(n)}$. The following result states the convexity properties of all the functions mentioned so far.

Lemma 4.2. *The functions $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ and $\widetilde{\mathcal{H}_{n,k}}$ are convex on Q^m and Q^n, respectively. Consequently, the functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{n,k}$ are convex on $Q^m_{\leq}$ and $Q^n_{\leq}$, respectively.*

Proof. The result follows from these basic facts on convex functions [20]: the nonnegative weighted sum of convex functions is convex, the maximum of a set of convex functions is convex, and affine functions are convex. □

The minimizers of $\mathcal{H}_{n,k}$ over $Q^n_{\leq}$ might belong to the boundary of the set, and hence, in spite of Lemma 4.2, not be fully described with gradient information only. As we will explain later in Section 5.2, the following result will be most helpful to overcome this hurdle.

Proposition 4.3. *The minimizers of $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ over $Q^m_{\leq}$ are also minimizers of $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ over Q^m. Likewise, the minimizers of $\widetilde{\mathcal{H}_{n,k}}$ over $Q^n_{\leq}$ are also minimizers of $\widetilde{\mathcal{H}_{n,k}}$ over Q^n.*

Proof. For the result regarding $\widetilde{\mathcal{H}_{\mathrm{DC}}}$, it is enough to show that given $q_1, \ldots, q_m \in Q$ and a permutation σ such that $q_{\sigma(1)} \leq \ldots \leq q_{\sigma(m)}$, then

$$\widetilde{\mathcal{H}_{\mathrm{DC}}}(q_1, \ldots, q_m) \geq \widetilde{\mathcal{H}_{\mathrm{DC}}}(q_{\sigma(1)}, \ldots, q_{\sigma(m)}). \tag{4}$$

We consider three cases, according to whether $\widetilde{\mathcal{H}_{\mathrm{DC}}}(q_{\sigma(1)}, \ldots, q_{\sigma(m)})$ is equal to (a) $q_{\sigma(1)} - a$, (b) $b - q_{\sigma(m)}$, or (c) $\frac{1}{2}(q_{\sigma(i+1)} - q_{\sigma(i)})$ for some $i \in \{1, \ldots, m-1\}$:

- In case (a), note that $q_{\sigma(1)} - a \leq q_1 - a \leq \widetilde{\mathcal{H}_{\mathrm{DC}}}(q_1, \ldots, q_m)$, and (4) follows;
- In case (b), denote $k = \sigma(i)$. Again, two subcases may occur. If $q_{k+1} \geq q_{\sigma(i+1)}$, then the result follows from $q_{\sigma(i+1)} - q_{\sigma(i)} \leq q_{k+1} - q_k$. If instead $q_{k+1} < q_{\sigma(i+1)}$, we look at the position of agent $k+2$. Since $q_{\sigma(i)}$ and $q_{\sigma(i+1)}$ are, by definition, consecutive, we have $q_{k+1} \leq q_{\sigma(i)}$. Again, two subcases may occur. If $q_{k+2} \geq q_{\sigma(i+1)}$, then the result follows from $q_{\sigma(i+1)} - q_{\sigma(i)} \leq q_{k+2} - q_{k+1}$. If instead $q_{k+2} < q_{\sigma(i+1)}$, this argument can be iterated if necessary until $k + (m-k) = m$. In such case, if $q_m \geq q_{\sigma(i+1)}$, then the result follows from $q_{\sigma(i+1)} - q_{\sigma(i)} \leq q_m - q_{m-1}$. If instead $q_m < q_{\sigma(i+1)}$, then $q_m \leq q_{\sigma(i)}$ and the result follows from $q_{\sigma(i+1)} - q_{\sigma(i)} \leq b - q_m$;
- Finally, the treatment of case (c) is analogous to that of case (a).

The result for $\widetilde{\mathcal{H}_{n,k}}$ follows by noting that (4) implies that $\widetilde{\mathcal{H}_{n,k}}(p_1, \ldots, p_n) \geq \widetilde{\mathcal{H}_{n,k}}(p_{\rho(1)}, \ldots, p_{\rho(n)})$ for any permutation ρ such that $p_{\rho(1)} \leq \ldots \leq p_{\rho(n)}$. □

4.2. Nonsmooth properties

Let us start by reviewing some basic facts about the disk-covering function $\mathcal{H}_{\mathrm{DC}}$ following [6]. Given a set $W \subset Q$, define $\mathrm{lg}_W : W \to \mathbb{R}$ by

$$\mathrm{lg}_W(p) = \max_{q \in W} \|q - p\|.$$

Note that $\mathrm{lg}_W(p)$ is the largest distance from p to the boundary of W. This definition allows us to rewrite $\mathcal{H}_{\mathrm{DC}}$ as follows

$$\mathcal{H}_{\mathrm{DC}}(q_1, \ldots, q_m) = \max_{i \in \{1, \ldots, m\}} \mathrm{lg}_{V_i} \circ \pi_i(q_1, \ldots, q_m). \tag{5}$$

where $\pi_i : Q^m \to Q$, $i \in \{1, \ldots, m\}$, denotes the projection $(q_1, \ldots, q_m) \mapsto q_i$. The individual objective functions in (5) can take three different forms. To write them explicitly, assume, without loss of generality that $(q_1, \ldots, q_m) \in Q^m_{\leq}$ (if this is not the case, then the expressions below need only to be rearranged according to the increasing order of $q_1, \ldots, q_m$). Then, we have

$$\begin{aligned}
\mathrm{lg}_{V_1} \circ \pi_1(q_1, \ldots, q_m) &= \max\Big\{q_1 - a, \frac{q_2 - q_1}{2}\Big\}, \\
\mathrm{lg}_{V_i} \circ \pi_i(q_1, \ldots, q_m) &= \max\Big\{\frac{q_i - q_{i-1}}{2}, \frac{q_{i+1} - q_i}{2}\Big\}, \\
\mathrm{lg}_{V_m} \circ \pi_m(q_1, \ldots, q_m) &= \max\Big\{\frac{q_m - q_{m-1}}{2}, b - q_m\Big\},
\end{aligned}$$

with $i \in \{2, \ldots, m-1\}$. Next, we compute their generalized gradients.

Lemma 4.4. *The functions* $\lg_{V_i} \circ \pi_i : Q^n \to \mathbb{R}$, $i \in \{1, \dots, m\}$, *are locally Lipschitz and regular. Furthermore, for* $(q_1, \dots, q_m) \in Q^m_{\leq}$, *their generalized gradients take one of the following forms*

$$\partial(\lg_{V_1} \circ \pi_1)(q_1, \dots, q_m) = \begin{cases} e_1 & \text{if } q_1 - a > \frac{q_2 - q_1}{2}, \\ \operatorname{co}\{e_1, \frac{1}{2}(e_2 - e_1)\} & \text{if } q_1 - a = \frac{q_2 - q_1}{2}, \\ \frac{1}{2}(e_2 - e_1) & \text{if } q_1 - a < \frac{q_2 - q_1}{2}, \end{cases}$$

$$\partial(\lg_{V_i} \circ \pi_i)(q_1, \dots, q_m) = \begin{cases} \frac{1}{2}(e_i - e_{i-1}) & \text{if } q_i > \frac{q_{i+1} + q_{i-1}}{2}, \\ \frac{1}{2}\operatorname{co}\{e_i - e_{i-1}, e_{i+1} - e_i\} & \text{if } q_i = \frac{q_{i+1} + q_{i-1}}{2}, \\ \frac{1}{2}(e_{i+1} - e_i) & \text{if } q_i < \frac{q_{i+1} + q_{i-1}}{2}, \end{cases}$$

$$\partial(\lg_{V_m} \circ \pi_m)(q_1, \dots, q_m) = \begin{cases} \frac{1}{2}(e_m - e_{m-1}) & \text{if } \frac{q_m - q_{m-1}}{2} > b - q_m, \\ \operatorname{co}\{-e_m, \frac{1}{2}(e_m - e_{m-1})\} & \text{if } \frac{q_m - q_{m-1}}{2} = b - q_m, \\ -e_m & \text{if } \frac{q_m - q_{m-1}}{2} < b - q_m, \end{cases}$$

where recall that $\{e_1, \dots, e_m\}$ *denotes the canonical Euclidean basis of* $\mathbb{R}^m$.

Given the fact that the maximum of locally Lipschitz and regular functions is itself locally Lipschitz and regular, see e.g., [19], one has the following result.

Lemma 4.5. *The functions* $\mathcal{H}_{\mathrm{DC}}$ *and* $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ *are locally Lipschitz and regular, and their generalized gradients are*

$$\begin{aligned} \partial\mathcal{H}_{\mathrm{DC}}(q_1, \dots, q_m) &= \operatorname{co}\{\partial(\lg_{V_i} \circ \pi_i)(q_1, \dots, q_m) \mid \\ &\qquad i \text{ such that } \mathcal{H}_{\mathrm{DC}}(q_1, \dots, q_m) = \lg_{V_i} \circ \pi_i(q_1, \dots, q_m)\}, \\ \partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(q_1, \dots, q_m) &= \operatorname{co}\{S(q_1, \dots, q_m)\}, \end{aligned}$$

where $S = S(q_1, \dots, q_m)$ *is the set defined by*

$$\begin{aligned} e_1 \in S &\text{ iff } q_1 - a = \widetilde{\mathcal{H}_{\mathrm{DC}}}(q_1, \dots, q_m), \\ -e_m \in S &\text{ iff } b - q_m = \widetilde{\mathcal{H}_{\mathrm{DC}}}(q_1, \dots, q_m), \\ \tfrac{1}{2}(e_{i+1} - e_i) \in S &\text{ iff } \tfrac{1}{2}(q_{i+1} - q_i) = \widetilde{\mathcal{H}_{\mathrm{DC}}}(q_1, \dots, q_m), \end{aligned}$$

for $i \in \{1, \dots, m-1\}$.

Since the sum of locally Lipschitz functions is locally Lipschitz and a linear combination of regular functions with positive coefficients is also regular, see e.g., [19], we have the next result.

Lemma 4.6. *The functions* $\mathcal{H}_{n,k}$ *and* $\widetilde{\mathcal{H}_{n,k}}$ *are locally Lipschitz and regular, and their generalized gradients are*

$$\begin{aligned} \partial\mathcal{H}_{n,k}(p_1, \dots, p_n) &= \frac{1}{\binom{n}{k}} \sum_{\{s_1, \dots, s_k\} \in C(n,k)} \partial\mathcal{H}_{\mathrm{DC}}(p_{s_1}, \dots, p_{s_k}), \\ \partial\widetilde{\mathcal{H}_{n,k}}(p_1, \dots, p_n) &= \frac{1}{\binom{n}{k}} \sum_{\{s_1, \dots, s_k\} \in C(n,k)} \partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_{s_1}, \dots, p_{s_k}). \end{aligned}$$

4.3. Invariance properties

Next, let us study the invariance properties of $\mathcal{H}_{n,k}$ for the symmetric projection with respect to the midpoint of $Q = [a, b]$. Consider the bijective map $\mathfrak{i} : Q \to Q$ given by $\mathfrak{i}(q) = b + a - q$. The following result makes precise our claim of invariance.

Lemma 4.7. *The functions $\mathcal{H}_{\mathrm{DC}} : Q^m \to \mathbb{R}$ and $\mathcal{H}_{n,k} : Q^n \to \mathbb{R}$ are invariant under $\mathfrak{i}$, that is, for all $q_1, \dots, q_m, p_1, \dots, p_n \in Q$,*

$$\mathcal{H}_{\mathrm{DC}}(\mathfrak{i}(q_1), \dots, \mathfrak{i}(q_m)) = \mathcal{H}_{\mathrm{DC}}(q_1, \dots, q_m),$$
$$\mathcal{H}_{n,k}(\mathfrak{i}(p_1), \dots, \mathfrak{i}(p_n)) = \mathcal{H}_{n,k}(p_1, \dots, p_n).$$

Proof. Note that $\mathfrak{i}^2(q) = q$, and therefore, if $z = \mathfrak{i}(q)$, we have $q = \mathfrak{i}(z)$. We use this fact in the following set of equalities

$$\begin{aligned}\mathcal{H}_{\mathrm{DC}}(\mathfrak{i}(q_1), \dots, \mathfrak{i}(q_m)) &= \max_{q \in Q} \min_{i \in \{1,\dots,m\}} \|q - \mathfrak{i}(q_i)\| = \max_{z \in Q} \min_{i \in \{1,\dots,m\}} \|\mathfrak{i}(z) - \mathfrak{i}(q_i)\| \\ &= \max_{z \in Q} \min_{i \in \{1,\dots,m\}} \|z - q_i\| = \mathcal{H}_{\mathrm{DC}}(q_1, \dots, q_m).\end{aligned}$$

This fact implies the invariance for $\mathcal{H}_{n,k}$. □

Inspired by Lemma 4.7, we define the set of configurations in Q^n that are invariant under $\mathfrak{i}$ as,

$$Q^n_{\mathrm{inv}} = \{P \in Q^n \mid \{p_1, \dots, p_n\} = \{\mathfrak{i}(p_1), \dots, \mathfrak{i}(p_n)\}\}.$$

Note that a configuration $(p_1, \dots, p_n) \in Q^n_{\leq}$ belongs to Q^n_{inv} if $p_1 = \mathfrak{i}(p_n), \dots, p_n = \mathfrak{i}(p_1)$. If $n = 2m + 1$, $m \in \mathbb{Z}_{\geq 0}$, is odd, then this implies that $p_{m+1} = \frac{a+b}{2}$.

The following result narrows down the search for the solutions to our optimal deployment problem.

Proposition 4.8. *The minimizers of $\mathcal{H}_{\mathrm{DC}}$ over $Q^m_{\mathrm{inv}} \cap Q^m_{\leq}$ are also minimizers of $\mathcal{H}_{\mathrm{DC}}$ over $Q^m_{\leq}$. Likewise, the minimizers of $\mathcal{H}_{n,k}$ over $Q^n_{inv} \cap Q^n_{\leq}$ are also minimizers of $\mathcal{H}_{n,k}$ over $Q^n_{\leq}$.*

Proof. We show the result for $\mathcal{H}_{n,k}$ (the proof for $\mathcal{H}_{\mathrm{DC}}$ is analogous). Given $(p_1, \dots, p_n) \in Q^n_{\leq}$, note that

$$\frac{1}{2}(p_1, \dots, p_n) + \frac{1}{2}(\mathfrak{i}(p_n), \dots, \mathfrak{i}(p_1)) \in Q^n_{\mathrm{inv}} \cap Q^n_{\leq}.$$

Then, we have

$$\begin{aligned}&\mathcal{H}_{n,k}\big(\frac{1}{2}(p_1, \dots, p_n) + \frac{1}{2}(\mathfrak{i}(p_n), \dots, \mathfrak{i}(p_1))\big) \\ &\leq \frac{1}{2}\big(\mathcal{H}_{n,k}(p_1, \dots, p_n) + \mathcal{H}_{n,k}(\mathfrak{i}(p_n), \dots, \mathfrak{i}(p_1))\big) = \mathcal{H}_{n,k}(p_1, \dots, p_n),\end{aligned}$$

where the inequality follows from the convexity of $\mathcal{H}_{n,k}$ on $Q^n_{\leq}$, cf. Lemma 4.2, and the equality follows from the invariance of $\mathcal{H}_{n,k}$ under permutations, cf. Lemma 4.1, and $\mathfrak{i}$, cf. Lemma 4.7. The result now follows from the above inequality. □

5. Characterization of the optimal deployment configurations

In this section, we characterize the solutions to the optimal deployment problem formulated in Section 3 for a range of situations depending on the number of working agents, k, with respect to the total number of agents, n.

5.1. Performance bounds

Here, we formalize the intuition that, from a performance viewpoint, it is worse to have a network composed of n agents with k of them working whose identity is unknown than to have a network composed of k working agents. Let us start with a useful result that characterizes the minimizers of $\mathcal{H}_{n,n}$, i.e., when all agents are working.

Lemma 5.1. *The function $\mathcal{H}_{n,n}$ has a unique minimizer given by*

$$p_1^* = \frac{(2n-1)a+b}{2n}, \quad p_2^* = \frac{(2n-3)a+3b}{2n}, \quad \dots$$
$$p_{n-1}^* = \frac{3a+(2n-3)b}{2n}, \quad p_n^* = \frac{a+(2n-1)b}{2n},$$

with value $\mathcal{H}_{n,n}(p_1^*, \dots, p_n^*) = \frac{b-a}{2n}$.

Proof. Recall that $\mathcal{H}_{n,n}$ is equal to $\mathcal{H}_{\mathrm{DC}}$. Let $(p_1, \dots, p_n)$ be a minimizer. Note that, by definition of $\mathcal{H}_{\mathrm{DC}}$, all agents must be active, i.e., $\mathrm{lg}_{V_i}(p_i) = \mathcal{H}_{\mathrm{DC}}(p_1, \dots, p_n)$ for all $i \in \{1, \dots, n\}$ (if this is not the case, then one can find arbitrarily close configurations where the value of $\mathcal{H}_{\mathrm{DC}}$ is strictly smaller, contradicting the definition of minimum). Moreover, each agent must be at the midpoint of its own Voronoi cell (if this is not the case, then moving one non-centered agent to the midpoint of its own cell does not modify the value of $\mathcal{H}_{\mathrm{DC}}$, and hence yields a minimizer configuration where not all agents are active, which is a contradiction). Both these facts imply that the minimizer of $\mathcal{H}_{\mathrm{DC}}$ is necessarily $(p_1^*, \dots, p_n^*)$. □

The following result shows that the location of the minimizers of the problem when all agents are working (cf. Lemma 5.1) bounds the location of the minimizers when only a fraction of the agents work.

Theorem 5.2. *For $n \geq 2$ and $k \leq \lfloor n/2 \rfloor + 1$, let $(\overline{p}_1, \dots, \overline{p}_n) \in Q^n_{inv} \cap Q^n_{\leq}$ be a minimizer of $\mathcal{H}_{n,k}$ and let $(p_1^*, \dots, p_k^*)$ be the minimizer of $\mathcal{H}_{k,k}$. Then*

$$\overline{p}_1 \geq p_1^*, \qquad \overline{p}_n \leq p_k^*.$$

Proof. Note that, given the results in Lemma 4.1 and Proposition 4.8, we can restrict our attention to the set $Q^n_{\mathrm{inv}} \cap Q^n_{\leq}$ without loss of generality. Therefore, both inequalities in the statement are either simultaneously true or simultaneously false. We proceed by contradiction. Assume $\overline{p}_1 < p_1^*$ and $\overline{p}_n > p_k^*$. We divide the proof in two steps. First, we show that (a) a simultaneous infinitesimal motion of all agents located at $\overline{p}_1$ (respectively, $\overline{p}_n$) in the positive (respectively, negative) direction does not increase the value of $\mathcal{H}_{n,k}$. We prove this statement for $\overline{p}_1$ (the proof for $\overline{p}_n$ is analogous). Let $\mathcal{N}$ denote the set of agents at $\overline{p}_1$ at the optimal configuration, i.e., $\mathcal{N} = \{j \in \{1, \dots, n\} \mid \overline{p}_j = \overline{p}_1\}$, and let $\mathcal{N}^c = \{1, \dots, n\} \setminus \mathcal{N}$. Consider the decomposition

$$\binom{n}{k} \mathcal{H}_{n,k}(p_1, \dots, p_n) = \sum_{\substack{\{s_1,\dots,s_k\} \in C(n,k) \\ \{s_1,\dots,s_k\} \not\subset \mathcal{N}^c}} \mathcal{H}_{\mathrm{DC}}(p_{s_1}, \dots, p_{s_k}) + \sum_{\substack{\{s_1,\dots,s_k\} \in C(n,k) \\ \{s_1,\dots,s_k\} \subset \mathcal{N}^c}} \mathcal{H}_{\mathrm{DC}}(p_{s_1}, \dots, p_{s_k}). \tag{6}$$

Note that the second sum is independent of the value of p_1, and hence we can concentrate on the first sum. Consider then $\{s_1, \dots, s_k\} \in C(n,k)$ with $\{s_1, \dots, s_k\} \not\subset \mathcal{N}^c$. Since by definition $s_1 < \dots < s_k$, note that this is equivalent to saying that at least one element in $\{s_1, \dots, s_k\}$ belongs to $\mathcal{N}$. We claim that

$$\overline{p}_1 - a < \mathcal{H}_{\mathrm{DC}}(\overline{p}_{s_1}, \dots, \overline{p}_{s_k}). \tag{7}$$

Assume this was not the case. Then, we would have

$$\begin{aligned} &\overline{p}_1 - a \geq \frac{1}{2}(\overline{p}_{s_{j+1}} - \overline{p}_{s_j}),\ j \in \{1, \dots, k-1\}, \\ &\overline{p}_1 - a \geq b - \overline{p}_{s_k}. \end{aligned}$$

Summing up these inequalities, we deduce $(2k-1)(\overline{p}_1 - a) \geq b - \overline{p}_1$, where we have used $\overline{p}_{s_1} = \overline{p}_1$. In turn, this implies that $2k(\overline{p}_1 - a) \geq b - a$, i.e., $\overline{p}_1 \geq p_1^*$, which contradicts $\overline{p}_1 < p_1^*$. Fact (a) now follows from (6) and (7).

Next, we show that (b) there is at least one combination $(s_1, \dots, s_k) \in C(n,k)$ for which a simultaneous infinitesimal motion of the agents located at $\overline{p}_1$ in the positive direction and the agents located at $\overline{p}_n$ in the negative direction causes a strict decrease in the value of $(p_{s_1}, \dots, p_{s_k}) \mapsto \mathcal{H}_{\mathrm{DC}}(p_{s_1}, \dots, p_{s_k})$. Take $\{s_2, \dots, s_k\} \subset \{\lceil n/2 \rceil, \dots, n\}$ (this is possible because $k \leq \lfloor n/2 \rfloor + 1$) and $s_1 = 1$. We claim that, for $j \in \{2, \dots, k-1\}$,

$$\overline{p}_{s_{j+1}} - \overline{p}_{s_j} \leq \overline{p}_{s_2} - \overline{p}_1, \tag{8}$$

and that equality can only hold if $\overline{p}_{s_j} = \overline{p}_{s_2} = (a+b)/2$ and $\overline{p}_{s_{j+1}} = b + a - \overline{p}_1$. Note that, by symmetry, for all $j \in \{\lceil n/2 \rceil, \dots, n\}$, we have $\overline{p}_j \geq (a+b)/2$ and, moreover, there exists $t \in \{1, \dots, \lfloor n/2 \rfloor\}$ such that $\mathfrak{i}(\overline{p}_t) = b + a - \overline{p}_t = \overline{p}_j$. Therefore, $\overline{p}_{s_{j+1}} - \overline{p}_{s_j} = \overline{p}_{t'} - \overline{p}_t$, for some $1 \leq t < t' \leq \lfloor n/2 \rfloor$. Since $\overline{p}_{t'} - \overline{p}_t \leq \overline{p}_{s_2} - \overline{p}_1$, we conclude (8). If equality holds, then necessarily $\overline{p}_{t'} = \overline{p}_{s_2} = (a+b)/2$ and $\overline{p}_t = \overline{p}_1$, and hence $\overline{p}_{s_j} = \overline{p}_{s_2} = (a+b)/2$, and $\overline{p}_{s_{j+1}} = \mathfrak{i}(p_1)$, as claimed. From (8), we deduce that fact (b) holds. Finally, the combination of (a) and (b) is a contradiction with $(\overline{p}_1, \dots, \overline{p}_n)$ being a global minimizer of $\mathcal{H}_{n,k}$, and therefore the statement of the result follows. □

We believe the result in Theorem 5.2 holds for any $n \geq 2$ and $k \leq n$, but the proof of this general case remains open. We are now ready to formally establish the result that a network composed of n agents with k of them working whose identity is unknown performs worse than a network composed of k working agents.

Corollary 5.3. *For $n \geq 2$ and $k \leq \lfloor n/2 \rfloor$, let $(\overline{p}_1, \ldots, \overline{p}_n)$ be a minimizer of $\mathcal{H}_{n,k}$ and let $(p_1^*, \ldots, p_k^*)$ be the minimizer of $\mathcal{H}_{k,k}$. Then*

$$\mathcal{H}_{k,k}(p_1^*, \ldots, p_k^*) < \mathcal{H}_{n,k}(\overline{p}_1, \ldots, \overline{p}_n).$$

Proof. If $k = 1$, then the result follows from Lemma 5.4 below. Assume then $k \geq 2$. For any $(s_1, \ldots, s_k) \in C(n, k)$,

$$\mathcal{H}_{\mathrm{DC}}(\overline{p}_{s_1}, \ldots, \overline{p}_{s_k}) \geq \overline{p}_{s_1} - a \geq \overline{p}_1 - a \geq p_1^* - a,$$

where we have used Theorem 5.2 in the last inequality. From Lemma 5.1, $p_1^* - a = \mathcal{H}_{k,k}(p_1^*, \ldots, p_k^*)$, and we deduce $\mathcal{H}_{\mathrm{DC}}(\overline{p}_{s_1}, \ldots, \overline{p}_{s_k}) \geq \mathcal{H}_{k,k}(p_1^*, \ldots, p_k^*)$. To conclude the result, we show that this inequality is strict for any $(s_1, \ldots, s_k) \in C(n, k)$ with $\{s_1, \ldots, s_k\} \subset \{1, \ldots, \lfloor n/2 \rfloor\}$. This can be seen as follows. By symmetry, $\overline{p}_j \leq (a + b)/2$ for $j \in \{1, \ldots, \lfloor n/2 \rfloor\}$, and hence

$$\mathcal{H}_{\mathrm{DC}}(\overline{p}_{s_1}, \ldots, \overline{p}_{s_k}) \geq \frac{b-a}{2} > \frac{b-a}{2k} = \mathcal{H}_{k,k}(p_1^*, \ldots, p_k^*),$$

as claimed. □

5.2. Proof strategy for the characterization of minimizers

Our next step is to study the location of the minimizers of $\mathcal{H}_{n,k}$. In this section, we describe our strategy to do so. Because of the invariance under permutations, it is sufficient to characterize the minimizers of $\mathcal{H}_{n,k}$ over $Q^n_{\leq}$. These minimizers, however, are not necessarily described properly by the equation $0 \in \partial\mathcal{H}_{n,k}(p_1, \ldots, p_n)$ if they belong to the boundary of $Q^n_{\leq}$.

The combination of the results presented in Section 4 allows us to adopt the following strategy to find these solutions. On the one hand, Proposition 4.8 states that we can restrict our search to $Q^n_{\mathrm{inv}} \cap Q^n_{\leq}$. On the other hand, since $\mathcal{H}_{n,k}$ and $\widetilde{\mathcal{H}_{n,k}}$ are the same function over $Q^n_{\leq}$, they have the same minimizers over $Q^n_{\leq}$. Noting that minimizers of $\widetilde{\mathcal{H}_{n,k}}$ must belong to the interior of Q^n, and using Lemma 4.2 and Proposition 4.3, we deduce that the minimizers of $\widetilde{\mathcal{H}_{n,k}}$ over $Q^n_{\leq}$ are described by

$$0 \in \partial\widetilde{\mathcal{H}_{n,k}}(p_1, \ldots, p_n). \tag{9}$$

Therefore, our strategy to find the solutions of the optimal deployment problem is to look for $(p_1, \ldots, p_n) \in Q^n_{\mathrm{inv}} \cap Q^n_{\leq}$ that satisfy (9). Lemma 4.6 provides us with the tools to characterize these minimizers. The idea is to understand the geometric conditions on the critical configurations imposed by this description. We make extensive use of this strategy in the following section.

5.3. Optimally-sized-and-positioned agent clusters are minimizers

For the deployment problem formulated in Section 3, given that some sensors are not working and their identity is unknown, it seems reasonable to group sensors together into clusters so that the likelihood of obtaining a message from the position covered by the cluster is higher. In this section, we show that, somehow surprisingly, this 'play-it-safe' clustering strategy turns out to be optimal, and specify the optimal size of the clusters in a range of situations. Our proof strategy in each case follows the discussion in Section 5.2. We start with the case of a single working agent.

Lemma 5.4. *For $n \in \mathbb{N}$, the minimizer of $\mathcal{H}_{n,1}$ is*

$$p_1^* = \cdots = p_n^* = \frac{a+b}{2}.$$

Proof. Note that

$$\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i) = \begin{cases} e_i & \text{if } p_i - a > b - p_i, \\ \mathrm{co}\{e_i, -e_i\} & \text{if } p_i - a = b - p_i, \\ -e_i & \text{if } p_i - a < b - p_i. \end{cases}$$

From this expression, we deduce that $0 \in \partial\widetilde{\mathcal{H}_{n,1}}(p_1, \ldots, p_n)$ if and only if

$$p_i = \frac{a+b}{2}, \quad \text{for all } i \in \{1, \ldots, n\},$$

as claimed. □

Next, we examine the case when there are two working agents.

Proposition 5.5. *For $n \geq 2$, the minimizer of $\mathcal{H}_{n,2}$ is*

$$p_1^* = \cdots = p_m^* = \frac{3a+b}{4}, \quad p_{m+1}^* = \cdots = p_{2m}^* = \frac{a+3b}{4}, \tag{10a}$$

if $n = 2m$ is even, and

$$p_1^* = \cdots = p_m^* = \frac{3a+b}{4}, \quad p_{m+1}^* = \frac{a+b}{2}, \quad p_{m+2}^* = \cdots = p_{2m+1}^* = \frac{a+3b}{4}, \tag{10b}$$

if $n = 2m+1$ is odd.

Proof. Using Lemma 4.6, we can express the generalized gradient as

$$\binom{n}{2} \partial\widetilde{\mathcal{H}_{n,2}}(p_1^*, \ldots, p_n^*) = \sum_{\{s_1,s_2\} \in C(n,2)} \partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_{s_1}^*, p_{s_2}^*) = \sum_{1 \leq i < j \leq n} \partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*, p_j^*).$$

Step 1) Consider the case $n = 2m$ even. Then,

$$\binom{n}{2}\,\partial\widetilde{\mathcal{H}_{n,2}}(p_1^*,\ldots,p_n^*) = \sum_{1\le i<j\le m}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*,p_j^*) + \sum_{m+1\le i<j\le 2m}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*,p_j^*) + \sum_{\substack{1\le i\le m\\ m+1\le j\le 2m}}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*,p_j^*).$$

Now, using Lemma 4.5, we deduce

$$\binom{n}{2}\,\partial\widetilde{\mathcal{H}_{n,2}}(p_1^*,\ldots,p_n^*) = \sum_{1\le i<j\le m} -e_j + \sum_{m+1\le i<j\le 2m} e_i + \sum_{\substack{1\le i\le m\\ m+1\le j\le 2m}} \mathrm{co}\{e_i, \frac{1}{2}(e_j-e_i), -e_j\}.$$

We claim that $0 \in \partial\widetilde{\mathcal{H}_{n,2}}(p_1^*,\ldots,p_n^*)$. To show this, we argue that the terms in the third sum can be used to offset the first and second sums. We prove the assertion by induction. Given that $0 = \frac{1}{4}e_i + \frac{1}{2}\frac{1}{2}(e_j - e_i) + \frac{1}{4}(-e_j) \in \mathrm{co}\{e_i, \frac{1}{2}(e_j - e_i), -e_j\}$, this is trivially true if $m = 1$. Assume the result is true for $m-1$ and let us show it for m. Note that

$$\begin{aligned}
&\binom{n}{2}\,\partial\widetilde{\mathcal{H}_{n,2}}(p_1^*,\ldots,p_n^*)\\
&= -\sum_{j=2}^{m}(j-1)e_j + \sum_{i=m+1}^{2m-1}(2m-i)e_i + \sum_{\substack{1\le i\le m\\ m+1\le j\le 2m}} \mathrm{co}\{e_i, \frac{1}{2}(e_j-e_i), -e_j\}\\
&= -(m-1)e_m + (m-1)e_{m+1} + \mathrm{co}\{e_m, \frac{1}{2}(e_{m+1}-e_m), -e_{m+1}\}\\
&\quad + \sum_{m+2\le j\le 2m}\mathrm{co}\{e_m, \frac{1}{2}(e_j-e_m), -e_j\} + \sum_{1\le i\le m-1}\mathrm{co}\{e_i, \frac{1}{2}(e_{m+1}-e_i), -e_{m+1}\}\\
&\quad - \sum_{j=2}^{m-1}(j-1)e_j + \sum_{i=m+2}^{2m-1}(2m-i)e_i + \sum_{\substack{1\le i\le m-1\\ m+2\le j\le 2m}} \mathrm{co}\{e_i, \frac{1}{2}(e_j-e_i), -e_j\}.
\end{aligned}$$

Now, for $m+2 \le j \le 2m$, we choose the convex combination $e_m = 1e_m + 0\frac{1}{2}(e_j - e_m) + 0(-e_j)$ and for $1 \le i \le m-1$, we choose the convex combination $-e_{m+1} = 0e_i + 0\frac{1}{2}(e_{m+1} - e_i) - e_{m+1}$. Since $0 \in \mathrm{co}\{e_m, \frac{1}{2}(e_{m+1} - e_m), -e_{m+1}\}$, we deduce that $0 \in \partial\widetilde{\mathcal{H}_{n,2}}(p_1^*,\ldots,p_n^*)$ if 0 belongs to

$$-\sum_{j=2}^{m-1}(j-1)e_j + \sum_{i=m+2}^{2m-1}(2m-i)e_i + \sum_{\substack{1\le i\le m-1\\ m+2\le j\le 2m}} \mathrm{co}\{e_i, \frac{1}{2}(e_j-e_i), -e_j\},$$

which follows from the induction hypothesis.

Step 2) Next, consider the case $n = 2m+1$ odd. In this case, the generalized gradient of $\widetilde{\mathcal{H}_{n,2}}$ can be written as

$$\binom{n}{2}\,\partial\widetilde{\mathcal{H}_{n,2}}(p_1^*,\dots,p_n^*) = \sum_{1\le i<j\le m}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*,p_j^*) + \sum_{m+2\le i<j\le 2m+1}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*,p_j^*)$$
$$+\sum_{\substack{1\le i\le m\\ m+2\le j\le 2m+1}}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*,p_j^*) + \sum_{1\le i\le m}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*,p_{m+1}^*) + \sum_{m+2\le j\le 2m+1}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_{m+1}^*,p_j^*).$$

Given the result for n even, in order to show that $0 \in \partial\widetilde{\mathcal{H}_{n,2}}(p_1^*,\dots,p_n^*)$ it is sufficient to establish that

$$0 \in \sum_{1\le i\le m}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i^*,p_{m+1}^*) + \sum_{m+2\le j\le 2m+1}\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_{m+1}^*,p_j^*).$$

The result now follows from noting that if $p < \frac{a+b}{2}$, $\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}\big(p,p_{m+1}^*\big) = -e_{m+1}$ and, if $p > \frac{a+b}{2}$, $\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}\big(p_{m+1}^*,p\big) = e_{m+1}$. □

The following result shows that the solution when there are three working agents is the same as when there are only two.

Proposition 5.6. *For $n \ge 4$, the minimizer of $\mathcal{H}_{n,3}$ is given by* (10)*.*

Proof. We only prove the result for n even. The proof for n odd is analogous and we omit it for reasons of space. Let $n = 2m$. We decompose $\binom{n}{3}\widetilde{\mathcal{H}_{n,3}}$ into the sum of the convex functions f and g defined by

$$f(p_1,\dots,p_n) = \sum_{1\le i<j<k\le m}\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i,p_j,p_k) + \sum_{m+1\le i<j<k\le 2m}\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i,p_j,p_k),$$
$$g(p_1,\dots,p_n) = \sum_{\substack{1\le i<j\le m\\ m+1\le k\le 2m}}\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i,p_j,p_k) + \sum_{\substack{1\le i\le m\\ m+1\le l<k\le 2m}}\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_i,p_l,p_k).$$

Our proof strategy consists of verifying the hypotheses of Lemma 2.1. Define

$$X = \{(p_1,\dots,p_n) \in \left[\frac{5a+b}{6},\frac{a+5b}{6}\right]^n \mid p_m < (a+b)/2 < p_{m+1}\} \cap Q^n_{\mathrm{inv}} \cap Q^n_{\le},$$

and let us show that a minimizer of $\widetilde{\mathcal{H}_{n,3}}$ is contained in X. The continuous function $\widetilde{\mathcal{H}_{n,3}}$ attains its minimum over the compact set $Q^n_{\mathrm{inv}} \cap Q^n_{\le}$. Because of Theorem 5.2, we deduce that there is at least a minimizer $\overline{P} \in Q^n_{\mathrm{inv}} \cap Q^n_{\le}$ of $\widetilde{\mathcal{H}_{n,3}}$, with $\overline{p}_1 = b + a - \overline{p}_n \ge (5a+b)/6$. Let us show that $\overline{P}$ belongs to X. By symmetry, it suffices to justify that $\overline{p}_m < (a+b)/2$. We reason by contradiction. Assume $\overline{p}_m = (a+b)/2$. We consider two cases: (i) $\overline{p}_1 > (5a+b)/6$ and (ii) $\overline{p}_1 = (5a+b)/6$. In case (i), we claim that an infinitesimal motion of agent m in the negative direction will strictly decrease the value of $\widetilde{\mathcal{H}_{n,3}}$. Note that m only appears in the definition of $\widetilde{\mathcal{H}_{n,3}}$ in combinations of type (a) (i,j,m), (b) (i,m,l), and (c) (m,l,k).

- For a combination of type a), the value of $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ is given by $b - p_m$, and hence a motion of agent m in the negative direction strictly increases it.
- For a combination of type b), the value of $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ does not depend on the position of agent m in a neighborhood of $\overline{P}$ because of $\overline{p}_1 > (5a+b)/6$.
- For a combination of type c), the value of $\widetilde{\mathcal{H}_{\mathrm{DC}}}$ is given by $p_m - a$, hence a motion of agent m in the negative direction strictly decreases it.

Since there are $\binom{m-1}{2}$ different combinations of type a) and $\binom{m}{2}$ different combinations of type c), our claim follows, contradicting the fact that $\overline{P}$ is a minimizer.

In case (ii), assume there are A agents at position $(5a+b)/6$ (and hence, by symmetry, A agents at position $(a+5b)/6$) and $2B$ agents at position $(a+b)/2$, with $A+B \leq m$. After some computations, one can see that an infinitesimal motion dx of all A agents at $(5a+b)/6$ in the positive direction and of all A agents at $(a+5b)/6$ in the negative direction leads to a change in the value of $\binom{n}{3}\widetilde{\mathcal{H}_{n,3}}$ of

$$2\left(A^2B - \binom{A}{3} - A\binom{A}{2}\right)dx.$$

After this motion has taken place, an infinitesimal motion dy of B agents at $(a+b)/2$ in the negative direction and the other B agents at $(a+b)/2$ in the positive direction leads to a change in the value of $\binom{n}{3}\widetilde{\mathcal{H}_{n,3}}$ of $-2AB^2dy$, so long as $dy \leq 2dx$. Putting these two changes together with $dy = 2dx$, we deduce that the values of $\binom{n}{3}\widetilde{\mathcal{H}_{n,3}}$ at the new configuration and at $\overline{P}$ differ by

$$-\frac{2}{3}A\big(1 + 2A^2 + 6B^2 - 3A(1+B)\big)dx.$$

It is not difficult to see that this quantity is negative for any $A, B \in \mathbb{Z}_{>0}$, and hence we reach a contradiction with the fact that $\overline{P}$ is a minimizer.

Therefore, we conclude that the minimizer $\overline{P}$ is contained in X. Moreover, using Lemma 4.5, we find that

$$\partial f(p_1^*, \dots, p_n^*) = \{\sum_{1 \leq i < j < k \leq m} -e_k + \sum_{m+1 \leq i < j < k \leq 2m} e_i\},$$

and $\partial f(p_1, \dots, p_n) \subset \partial f(p_1^*, \dots, p_n^*)$ for all $(p_1, \dots, p_n) \in X$. On the other hand, one can compute

$$\partial g(p_1^*, \dots, p_n^*) = \sum_{\substack{1 \leq i < j \leq m \\ m+1 \leq k \leq 2m}} \mathrm{co}\{e_i, \frac{1}{2}(e_k - e_j), -e_k\} + \sum_{\substack{1 \leq i \leq m \\ m+1 \leq l < k \leq 2m}} \mathrm{co}\{e_i, \frac{1}{2}(e_l - e_i), -e_k\},$$

and verify that $\partial g(p_1, \dots, p_n) \subset \partial g(p_1^*, \dots, p_n^*)$ for all $(p_1, \dots, p_n) \in X$. The application of Lemma 2.1 now concludes the result. □

Our final result shows that, when all agents but one are working, clustering in groups of two is optimal.

Proposition 5.7. *For $n \geq 2$, the minimizer of $\mathcal{H}_{n,n-1}$ is*

$$p_k^* = p_{k+1}^* = \frac{(2m-k)a + kb}{2m},$$

for $k \in \{1, 3, \ldots, 2m-1\}$, if $n = 2m$ is even,

$$p_k^* = p_{k+1}^* = \frac{(2(m+1)-k)a + kb}{2(m+1)},$$
$$p_m^* = \frac{(m+2)a + mb}{2(m+1)}, \quad p_{m+1}^* = \frac{a+b}{2}, \quad p_{m+2}^* = \frac{ma + (m+2)b}{2(m+1)},$$

for $k \in \{1, 3, \ldots, m-2, m+4, m+6, \ldots, 2m+1\}$, if $n = 2m+1$ is odd with m odd, and

$$p_k^* = p_{k+1}^* = \frac{(2(m+1)-k)a + kb}{2(m+1)}, \quad p_{m+1}^* = \frac{a+b}{2},$$

for $k \in \{1, 3, \ldots, m-1, m+1, \ldots, 2m+1\}$, if $n = 2m+1$ is odd with m even.

Proof. For brevity, we only prove the result for $n = 2m$. The proof for the case when n is odd proceeds in an analogous way. Using Lemma 4.6, we can write

$$\binom{n}{n-1} \partial\widetilde{\mathcal{H}_{n,n-1}}(p_1^*, \ldots, p_n^*) =$$
$$\sum_{\{s_1,\ldots,s_{n-1}\} \in C(n,n-1)} \partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_{s_1}^*, \ldots, p_{s_{n-1}}^*) = \sum_{i=1}^{n} \partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_1^*, \ldots, \widehat{p_i^*}, \ldots, p_n^*),$$

where the notation $\widehat{p_i^*}$ means that the point p_i^* is removed from the tuple. Next, we compute each of the individual generalized gradients in the sum using Lemma 4.5. To simplify the presentation, let us introduce the following notation: given vectors $v_1, \ldots, v_n$, define

$$\mathcal{A}(v_1, \ldots, v_n) = \{v_1, \frac{1}{2}(v_3 - v_2), \frac{1}{2}(v_5 - v_4), \ldots, \frac{1}{2}(v_{n-1} - v_{n-2}), -v_n\}.$$

Then, we have

$$\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_1^*, \ldots, \widehat{p_i^*}, \ldots, p_n^*) = \begin{cases} \mathrm{co}(\mathcal{A}(e_1, \ldots, e_{i-1}, e_{i-1}, e_{i+1}, \ldots, e_n)) & i \text{ even}, \\ \mathrm{co}(\mathcal{A}(e_1, \ldots, e_{i-1}, e_{i+1}, e_{i+1}, \ldots, e_n)) & i \text{ odd}. \end{cases}$$

Now we are ready to show that $0 \in \partial\mathcal{H}_{n,n-1}(p_1^*, \ldots, p_n^*)$. For $i \in \{1, \ldots, n\}$, take the following convex combination in $\partial\widetilde{\mathcal{H}_{\mathrm{DC}}}(p_1^*, \ldots, \widehat{p_i^*}, \ldots, p_n^*)$,

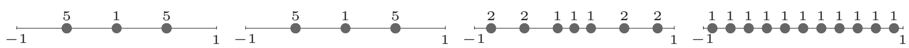
$$\begin{cases} \frac{1}{2}e_2 - \frac{1}{2}e_n & i = 1, \\ \frac{1}{2}(e_{i+1} - e_{i-1}) & 1 < i < n, \\ \frac{1}{2}e_1 - \frac{1}{2}e_{n-1} & i = n. \end{cases}$$

The sum of this convex combination belongs to $\partial\widetilde{\mathcal{H}_{n,n-1}}(p_1^*, \dots, p_n^*)$ and equals

$$\begin{aligned}
&\frac{1}{2}(e_2 - e_n) + \frac{1}{2}(e_1 - e_{n-1}) + \sum_{i=2}^{n-1} \frac{1}{2}(e_{i+1} - e_{i-1}) \\
&= \frac{1}{2}(e_2 - e_n) + \frac{1}{2}(e_1 - e_{n-1}) + \frac{1}{2}(e_{n-1} + e_n - e_1 - e_2) = 0,
\end{aligned}$$

as claimed. □

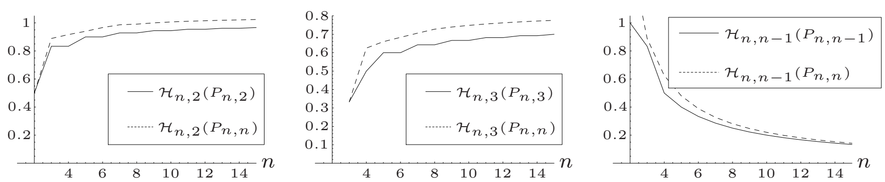
Figure 1 shows the minimizer of $\mathcal{H}_{n,k}$, denoted $P_{n,k}$, over the interval $Q = [-1, 1]$, for $n = 11$ and $k = 2$, 3, 10, and 11.



(a) $P_{11,2}$ (b) $P_{11,3}$ (c) $P_{11,10}$ (d) $P_{11,11}$

Figure 1: From left to right, minimizers of $\mathcal{H}_{11,k}$ over $[-1, 1]$ for $k = 2$, 3, 10, and 11. The superindex of each point represents the number of agents at the specific location.

Figure 2 illustrates the performance of the minimizer of $\mathcal{H}_{n,k}$ against the solution with all agents working, cf. Lemma 5.1, for $k = 2$, 3, and $n - 1$.



(a) $k = 2$ (b) $k = 3$ (c) $k = n - 1$

Figure 2: Performance comparison between deployment solutions that take and do not take into account, respectively, packet drops. The solid curve corresponds to the optimal value of $\mathcal{H}_{n,k}$ and the dashed curve corresponds to the value of $\mathcal{H}_{n,k}$ at the optimizer of $\mathcal{H}_{n,n}$.

Remark 5.8. *(Conjecture for arbitrary number of working agents):* The results presented in this section point in the same direction: grouping agents into clusters and optimally deploying the resulting clusters is the solution to the deployment problem. The strategy of grouping agents increases the chances of each cluster having at least a working agent, and hence being able to produce a measurement. The larger the size of the clusters, the more likely that they will work. However, the larger the size, the smaller the number of clusters, and hence the worse the performance. So there is a trade-off between grouping agents to increase the likelihood of them working and having as few groups as possible to ensure good performance. These observations lead us to conjecture that the minimizers of $\mathcal{H}_{n,k}$ for arbitrary $k \leq n$ correspond to agent clusters optimally deployed according to its number, and that a precise formula exists that determines the number of clusters and their size. Table 1 is a partial realization of this formula. We believe the conjecture to hold in higher dimensions since the trade-off between size and number of clusters is independent of the dimension, although the specific formula might be different for each environment. •

6. Conclusions

We have analyzed a deployment problem for an unreliable robotic sensor network taking point measurements of a spatial process and relaying them back to a data fusion center. We have shown that grouping sensors not only makes sense as a 'play-it-safe' strategy, but that, surprisingly, turns out to be optimal. We have conjectured that the right amount of grouping depends upon the proportion of working sensors with respect to the total number of agents. We have shown this conjecture to be true in a number of cases. Our analysis has required a blend of nonsmooth analysis, convex analysis, and combinatorics. The results presented here offer guidance as to how to design coordination algorithms to achieve optimal deployment with failing sensors. Future work will be devoted to this topic as well as to establish the validity of the conjecture and examine the connection of the solutions presented here with dynamic scenarios with randomly failing sensors taking multiple measurements.

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