

# Evolution of the perception about the opponent in hypergames

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**Abstract**—This paper studies the evolution of the perceptions of players about the game they are involved in using the framework of hypergame theory. The focus is on developing methods that players can implement to modify their perception about other players’ payoffs by incorporating the lessons learned from observing their actions. We introduce a misperception function that measures the mismatch between a player’s perception and the true payoff structure of the other players. Our first update mechanism, called swap learning method, is guaranteed to decrease the value of the misperception function but in general can lead to inconsistencies in the stability properties of the resulting perception. This motivates the introduction of a second mechanism, called modified swap update method, that is guaranteed to produce a consistent perception. Finally, we identify a class of hypergames for which the modified strategy is also guaranteed to decrease the misperception function.

## I. INTRODUCTION

The manipulation of perceptions plays a key role in many strategic situations. Scenarios include military operations, bargaining, negotiation, investment banking, and card games. The objective of this paper is to study the evolution of the perceptions of players about the game they are involved in. A proper understanding of this evolution is critical in order to induce deception and design mechanisms that are robust against deception. Specifically, we focus on developing methods that players can implement to modify their perception about other players’ ultimate objectives by observing the actions they take.

*Literature review:* The works [1], [2], [3], [4] show how deception can arise in games of imperfect observation (i.e., when players do not perfectly observe each others’ actions). Here, instead, we consider the setup of games of incomplete information, where deception may arise because of an imprecise understanding about the objectives and true intentions of other players. We move away from the common approach in game theory, see e.g., [5], that consists of transforming a game of incomplete information into one of imperfect information, e.g., Bayesian games [6]. The reason for this is that such approach does not consider situations when a player believes that other players are of a certain type or have a specific set of actions available to them, and is not considering that this might not be true because she is not aware of other possibilities. [7] shows that, by allowing subjective information structures for players, the inconsistent structure of beliefs may lead to counterintuitive behaviors. Furthermore, differences in beliefs are not necessarily smoothed out if the game is repeated infinitely many times. [8] shows that there exist games with incomplete

information in which players almost never learn to predict their opponents’ behavior. Our technical approach makes use of the framework of hypergames [9], [10], [11], [12]. A hypergame is a generalization of the concept of game that explicitly models the possibility of different agents having different perceptions about the scenario they are involved in. Few works [13], [14], [15] have addressed the study of learning in hypergames.

*Statement of contributions:* The first contribution of the paper is the introduction of the basic notions of partial order, preference vector, rank, and H-digraph. These notions simplify the determination of the equilibria of hypergames and their stability analysis. The second contribution is the introduction of the swap learning method which allows a player to update its own perception based on the information contained in the actions taken by other players. We also introduce the misperception function to provide a measure of the mismatch between a player’s perception and the true payoff structure of the other players. We show that the swap learning method ensures that the misperception function will decrease and that the players’ perceptions will converge if they repeatedly use this strategy. The third and last contribution is the introduction of the notion of inconsistency in perceptions. Specifically, we show that the swap learning method can yield preference vectors that are inconsistent with the stability properties of the outcomes implied by the actions of other players. This leads us to propose a modified version of the swap learning method which is guaranteed to prevent any inconsistency in the perceptions. We establish a class of hypergames for which the modified strategy is also guaranteed to decrease the misperception function. Throughout the paper, we illustrate our discussion with several examples.

## II. A NEW FRAMEWORK FOR HYPERGAMES

In this section, we review the basic notions from hypergame theory [11], [16], [10]. We also introduce the useful concept of H-digraph for analyzing stability in hypergames.

### A. Basic notions

A 0-level hypergame is a finite game, as defined next.

*Definition 2.1:* A (finite) game is a triplet  $\mathbf{G} = (V, \mathbf{S}_{\text{outcome}}, \mathbf{P})$ , where  $V$  is a set of  $n$  players;  $\mathbf{S}_{\text{outcome}} = S_1 \times \dots \times S_n$  is the outcome set, where  $S_i$  is a finite set of strategies available to player  $v_i \in V$ ,  $i \in \{1, \dots, n\}$ ;  $\mathbf{P} = (P_1, \dots, P_n)$ , with  $P_i = (x_1, \dots, x_N)^T \in \mathbf{S}_{\text{outcome}}^N$ ,  $N = |\mathbf{S}_{\text{outcome}}|$  and  $i \in \{1, \dots, n\}$ , is called the *preference vector* of player  $v_i$ .

A nonempty set  $X$  along with a preorder  $\succeq$ , i.e., a reflexive and transitive binary relation, is called a *directed set* if for

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every pair of elements in  $X$  there exists an upper bound with respect to the preorder. Note that the set of outcomes  $\mathbf{S}_{\text{outcome}}$  is a directed set under the preorder  $\succeq_{P_i}$  induced by the preference vector  $P_i$  of player  $v_i$  as follows:  $x \succeq_{P_i} y$  if and only if  $x$  has a lower entry index than  $y$  in  $P_i$ .

*Definition 2.2: (1-level hypergames):* A 1-level hypergame with  $n$  players is a set  $H^1 = \{\mathbf{G}_1, \dots, \mathbf{G}_n\}$ , where  $\mathbf{G}_i = (V, (\mathbf{S}_{\text{outcome}})_i, \mathbf{P}_i)$ , for  $i \in \{1, \dots, n\}$ , is the subjective finite game of player  $v_i \in V$ , and  $V$  is a set of  $n$  players;  $(\mathbf{S}_{\text{outcome}})_i = S_{1i} \times \dots \times S_{ni}$ , where  $S_{ji}$  is the finite set of strategies available to  $v_j$ , as perceived by  $v_i$ ;  $\mathbf{P}_i = (P_{1i}, \dots, P_{ni})$ , where  $P_{ji}$  is the preference vector of  $v_j$ , as perceived by  $v_i$ .

In a 1-level hypergame, each player  $v_i \in V$  plays the game  $\mathbf{G}_i$  with the perception that she is playing a game with complete information, which is not necessarily true. The definition of a 1-level hypergame can be extended to higher level hypergames, where some of the players have access to some additional information that allow them to form perceptions about other players' beliefs, other players' perceptions about them, and so on.

*Definition 2.3 (Higher-level hypergame):* A  $k$ -level hypergame,  $k \geq 1$ , with  $n$  players is a set  $H^k = \{H_1^{k-1}, \dots, H_n^{k-1}\}$  in which each player's game is a  $(k-1)$ -level hypergame.

Throughout the paper, for simplicity, we restrict our attention to 2-level hypergames  $H^2 = (H_A^1, H_B^1)$  with two players  $A$  and  $B$ . We do not consider scenarios in which players are playing in different levels of hypergames.

## B. Equilibria and stability

Here we discuss the notion of equilibria for hypergames making use of the concept of partial order. Let us start by introducing some notation. For a 2-level hypergame  $H^2$ , we denote by  $H_A^1 = \{\mathbf{G}_{AA}, \mathbf{G}_{BA}\}$  the 1-level hypergame for player  $A$ , where  $\mathbf{G}_{AA}$  and  $\mathbf{G}_{BA}$  are, respectively, the games of player  $A$  and player  $B$  perceived by player  $A$ . We also denote by  $P_{\mathbf{G}_{AA}}$  and  $P_{\mathbf{G}_{BA}}$ , respectively, the preference vectors corresponding to  $\mathbf{G}_{AA}$  and  $\mathbf{G}_{BA}$ . Similarly, we define  $H_B^1 = \{\mathbf{G}_{BB}, \mathbf{G}_{AB}\}$ . We denote by  $\mathbf{S}_{\text{outcome}}$  the outcome set of the hypergame  $H^2$ . Here, we assume that players have no misperception in their own preferences. Also we assume that all the 1-level hypergames have the same set of outcomes, i.e., the outcomes set  $\mathbf{S}_{\text{outcome}}$  is observable for both players. This assumption is common in the literature, see e.g., [10]. Throughout the paper, we let  $\mathbf{S}_P \subset \mathbf{S}_{\text{outcome}}^N$ ,  $N = |\mathbf{S}_{\text{outcome}}|$ , denote the set of all elements of  $\mathbf{S}_{\text{outcome}}^N$  with pairwise different entries. Since, by assumption, both players have the same outcome sets, we sometimes use the preference vectors  $(P_{\mathbf{G}_{AA}}, P_{\mathbf{G}_{BA}})$  to refer to the 1-level hypergame  $H_A^1$ . We denote by  $\succeq_{P_{\mathbf{G}_{IJ}}}$  the binary relation on  $\mathbf{S}_{\text{outcome}}$  induced by  $P_{\mathbf{G}_{IJ}}$ , where  $I, J \in \{A, B\}$ . We often use the notation  $P_{\mathbf{G}_{IJ}}^i$  to denote the  $i$ th entry of  $P_{\mathbf{G}_{IJ}}$ . We denote by  $\pi_I$  the natural projection of the outcome set  $\mathbf{S}_{\text{outcome}}$  to the strategy set of player  $I \in \{A, B\}$ . We also find it useful to use  $I'$  to denote the opponent of  $I$  in  $\{A, B\}$ . The next definition introduces two important notions.

### Definition 2.4 (Improvement and rational outcome):

Given two distinct outcomes  $x, y \in \mathbf{S}_{\text{outcome}}$ ,  $y$  is an *improvement* from  $x$  for player  $I \in \{A, B\}$  in  $\mathbf{G}_{IJ}$ , perceived by player  $J \in \{A, B\}$ , if and only if  $\pi_{I'}(y) = \pi_{I'}(x)$  and  $y \succ_{P_{\mathbf{G}_{IJ}}} x$ . An outcome  $x \in \mathbf{S}_{\text{outcome}}$  is called *rational* for player  $I \in \{A, B\}$  in  $\mathbf{G}_{IJ}$ ,  $J \in \{A, B\}$  if there exists no improvement from  $x$  for this player.

The concept of Nash equilibrium is inappropriate in the context of hypergames because it does not take into account the different perceptions of the players. This is best illustrated with an example. Suppose player  $A$  has some perception about player  $B$ 's game and suppose  $A$  has an improvement  $y$  from  $x$ . If  $A$  believes that  $B$  has an improvement  $z$  from an outcome  $y$  such that  $x \succ_{P_{\mathbf{G}_{AA}}} z$ , then  $A$  will decide not to take the action associated with the improvement  $y$ , hence invalidating the basic assumption behind the notion of Nash equilibrium. This problem can be addressed with the concept of sequential rationality [17], [18], [11].

*Definition 2.5 (Sequential rationality):* Consider a 2-level hypergame  $H^2$  between players  $A$  and  $B$ . An outcome  $x \in \mathbf{S}_{\text{outcome}}$  is *sequentially rational* for player  $I \in \{A, B\}$  in the game  $\mathbf{G}_{IJ}$  with respect to  $H_J^1$ ,  $J \in \{A, B\}$ , if and only if for all improvements  $y$  for player  $I$  in the game  $\mathbf{G}_{IJ}$ , perceived by player  $J$ , there exists an improvement  $z$  for player  $I'$  in the game  $\mathbf{G}_{I'J}$ , perceived by player  $J$ , such that  $x \succ_{P_{\mathbf{G}_{I'J}}} z$ . Whenever this holds, we say that the improvement  $z$  from  $y$  for player  $I'$  in the game  $\mathbf{G}_{I'J}$  *sanctions* the improvement  $y$  from  $x$  for player  $I$  in the game  $\mathbf{G}_{IJ}$ .

By definition, a rational outcome is also sequentially rational. We denote by  $\text{Seq}(P_{\mathbf{G}_{IJ}}, P_{\mathbf{G}_{JJ}}) \subset \mathbf{S}_{\text{outcome}}$  the set of all sequentially rational outcomes for player  $I \in \{A, B\}$ , as perceived by player  $J \in \{A, B\}$ , in the game  $\mathbf{G}_{IJ}$  with respect to  $H_J^1$ . An outcome  $x \in \mathbf{S}_{\text{outcome}}$  is *unstable* in the game  $\mathbf{G}_{IJ}$  with respect to  $H_J^1$  if  $x \in \text{Seq}^c(P_{\mathbf{G}_{IJ}}, P_{\mathbf{G}_{JJ}})$  (Here, for  $S \subset U$ ,  $S^c = \{x \in U \mid x \notin S\}$  denotes the complement of  $S$  with respect to  $U$ ) and is an *equilibrium of  $H_J^1$*  if  $x \in \text{Seq}(P_{\mathbf{G}_{JJ}}, P_{\mathbf{G}_{J'J}}) \cap \text{Seq}(P_{\mathbf{G}_{J'J}}, P_{\mathbf{G}_{JJ}})$ . For brevity, we sometimes omit the wording 'with respect to  $H_J^1$ ' when it is clear from the context. An outcome  $x$  is an *equilibrium of  $H^2$*  if  $x \in \text{Seq}(P_{\mathbf{G}_{AA}}, P_{\mathbf{G}_{BA}}) \cap \text{Seq}(P_{\mathbf{G}_{BB}}, P_{\mathbf{G}_{AB}})$ . Note that an outcome  $x$  can be an equilibrium for  $H^2$  while it is not an equilibrium of  $H_A^1$ .

The following result plays an important role in the forthcoming discussion. For simplicity, we present it for  $\mathbf{G}_{BA}$ , however, one can easily extend it to  $\mathbf{G}_{IJ}$ , for  $I, J \in \{A, B\}$ .

*Lemma 2.6: (Existence of rational outcomes):* For each  $x \in \mathbf{S}_{\text{outcome}}$ , either  $x$  is rational for player  $B$  in the game  $\mathbf{G}_{BA}$  or there exists an improvement  $y \in \mathbf{S}_{\text{outcome}}$  from  $x$  for player  $B$  in the game  $\mathbf{G}_{BA}$  which is rational for  $B$  in the game  $\mathbf{G}_{BA}$ .

Since all rational outcomes are also sequentially rational, Lemma 2.6 also shows the existence of sequential rational outcomes. The following result guarantees the existence of equilibria in hypergames. A proof can be found in [10].

*Theorem 2.7 (Existence of an equilibrium in hypergames):* Every hypergame has an equilibrium outcome.

### C. The H-digraph

The stability analysis in hypergames is typically done by means of preference tables, see [10], [11]. Here we introduce a novel digraph, termed H-digraph, that contains the information about the possible improvements from an outcome to another outcome, the equilibria, and the sanctions in a hypergame. We start by introducing the notion of rank.

*Definition 2.8 (Rank):* Let  $H^2$  be a 2-level hypergame and consider the game  $\mathbf{G}_{IJ}$ ,  $I, J \in \{A, B\}$ . We assign to each outcome  $x \in \mathbf{S}_{\text{outcome}}$  a positive number  $\text{rank}(x, \mathbf{P}_{\mathbf{G}_{IJ}}) \in \mathbb{R}_{>0}$ , called the *rank of outcome*  $x$ , such that, for each  $\mathbf{S}_{\text{outcome}} \ni y \neq x$ , we have that  $\text{rank}(y, \mathbf{P}_{\mathbf{G}_{IJ}}) \geq \text{rank}(x, \mathbf{P}_{\mathbf{G}_{IJ}})$  if and only if  $x \succ_{\mathbf{P}_{\mathbf{G}_{IJ}}} y$ .

By this definition, players prefer the outcomes with lowest rank. Throughout the paper, we use the set  $\{1, \dots, |\mathbf{S}_{\text{outcome}}|\}$  to rank the outcomes. Note that, by this assignment, two outcomes with the same payoffs will have different ranks.

A digraph by  $G$  is a pair  $(V, E)$ , where  $V$  is a finite set, called the vertex set, and  $E \subseteq V \times V$ , called the edge set. Given an edge  $(u, v) \in E$ ,  $u$  is the *in-neighbor* of  $v$  and  $v$  is the *out-neighbor* of  $u$ . The set of in-neighbors and out-neighbors of  $v$  are denoted, respectively, by  $\mathcal{N}^{\text{in}}(v)$  and  $\mathcal{N}^{\text{out}}(v)$ . The *in-degree* and *out-degree* of  $v$  are the number of in-neighbors and out-neighbors of  $v$ , respectively.

*Definition 2.9 (H-digraph):* The digraph  $\mathcal{G}_{H_A^1} = (\mathbf{S}_{\text{outcome}}, \mathcal{E}_{H_A^1})$  is the *H-digraph* associated to player  $A$ 's hypergame  $H_A^1$ , where  $x \in \mathbf{S}_{\text{outcome}}$  is labeled with

$$(\text{rank}(x, \mathbf{P}_{\mathbf{G}_{AA}}), \text{rank}(x, \mathbf{P}_{\mathbf{G}_{BA}}))$$

and there exists an edge  $(x, y) \in \mathcal{E}_{H_A^1}$  iff one of the following holds with respect to  $H_A^1$ ,

- there exists an improvement  $y$  from  $x$  for player  $A$  in the game  $\mathbf{G}_{AA}$  for which there exists no perceived sanction of  $B$  in the game  $\mathbf{G}_{BA}$ ;
- there exists a perceived improvement  $y$  from  $x$  for player  $B$  in the game  $\mathbf{G}_{BA}$  for which there exists no sanction of  $A$  in the game  $\mathbf{G}_{AA}$ .

It is clear from the definition that an outcome is an equilibrium for  $H_A^1$  iff its out-degree in the associated H-digraph is zero. Similarly, one can associate a H-digraph to  $H_B^1$ . Furthermore, we define a H-digraph associated to  $H^2$  by using the true players' games  $\mathbf{G}_{AA}$  and  $\mathbf{G}_{BB}$ .

### III. THE SETTLEMENT GAME

Here, we describe in detail a hypergame to illustrate the notions introduced in the previous section. The example will also serve us to motivate the questions addressed in the forthcoming discussion. Suppose two teams  $A$  and  $B$  are trying to launch some troops in a field partitioned into four regions, North West (NW), North East (NE), South West (SW) and South East (SE). Each team has its own perception about the conditions in the field and, based on that, has some preferences for launching the troops. Furthermore, each team has a perception about the opponent's intentions. We associate a vector  $\theta = [\theta_{A_1}, \theta_{A_2}, \theta_{B_1}, \theta_{B_2}]^T \in \{0, 1\}^4$  to each outcome of the settlement game, where  $\theta_{A_1}$  (resp.  $\theta_{B_1}$ )

is 0 if  $A$  (resp.  $B$ ) chooses West and 1 otherwise, and  $\theta_{A_2}$  (resp.  $\theta_{B_2}$ ) is 0 if  $A$  (resp.  $B$ ) chooses North and 1 otherwise. We associate a unique identifier  $\text{Ind}(\theta) \in \mathbb{Z}_{\geq 0}$  to each outcome of the settlement game by computing

$$\text{Ind}(\theta) = \theta_{A_1} \times 2^0 + \theta_{A_2} \times 2^1 + \theta_{B_1} \times 2^2 + \theta_{B_2} \times 2^3.$$

Suppose the players' preferences and perceptions about each other's preferences are given by

$$\begin{aligned} \mathbf{P}_{\mathbf{G}_{AA}} &= (12, 9, 6, 3, 8, 4, 13, 1, 14, 2, 11, 7, 0, 5, 10, 15)^T, \\ \mathbf{P}_{\mathbf{G}_{BA}} &= (0, 5, 15, 10, 1, 2, 3, 7, 4, 6, 14, 13, 8, 11, 12, 9)^T, \\ \mathbf{P}_{\mathbf{G}_{BB}} &= (1, 2, 3, 7, 4, 6, 14, 13, 8, 11, 12, 9, 0, 5, 15, 10)^T, \\ \mathbf{P}_{\mathbf{G}_{AB}} &= (12, 9, 6, 3, 8, 4, 13, 1, 14, 2, 11, 7, 0, 5, 10, 15)^T, \end{aligned}$$

where entries with lower index are preferred. Recall that the preferences of player  $J \in \{A, B\}$ , as perceived by player  $I \in \{A, B\}$ , induce a preorder on the set of outcomes  $\mathbf{S}_{\text{outcome}}$ .

We rank the set of outcomes with the integers  $\{1, \dots, |\mathbf{S}_{\text{outcome}}|\}$ . The H-digraph associated to each team's hypergame are shown in Figure 1. For instance, the out-

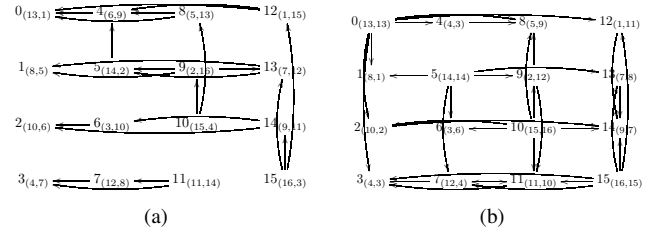


Fig. 1. H-digraphs for the hypergame (a)  $H_A^1$  with  $(\mathbf{P}_{\mathbf{G}_{AA}}, \mathbf{P}_{\mathbf{G}_{BA}})$  and (b)  $H_B^1$  with  $(\mathbf{P}_{\mathbf{G}_{BB}}, \mathbf{P}_{\mathbf{G}_{AB}})$ .

come 0 is ranked 13 in  $\mathbf{G}_{AA}$  and 1 in  $\mathbf{G}_{BA}$  (this corresponds to the subindex  $(13, 1)$  for the node 0 in Figure 1(a)). Moreover, there is no outgoing edge from this outcome to outcomes 4, 8, and 12, which means that 0 is sequentially rational for player  $B$  in  $\mathbf{G}_{BA}$  with respect to  $H_A^1$ .

Let us analyze what happens if players play this hypergame. Team  $A$  will hope for the equilibrium 3 and so they move to SE. Team  $B$  also perceives 3 as their best equilibrium and so will move to NW. The result does not reveal any new information about the misperceptions, in the sense that none of the players would do anything differently if she got the chance to play this hypergame again knowing the previous outcome.

Next, consider the same setup as above with a new set of preferences for the game  $\mathbf{G}_{BB}$ ,

$$\mathbf{P}'_{\mathbf{G}_{BB}} = (13, 14, 12, 8, 9, 11, 2, 1, 3, 4, 7, 6, 15, 10, 0, 5)^T.$$

The new H-digraph associated to player  $B$ 's 1-level hypergame is shown in Figure 2. Team  $A$  hopes for the outcome 3 and so plays the action  $\pi_A(3)$ . Similarly, player  $B$  hopes for the perceived equilibrium 12 and thus plays the action  $\pi_B(12)$ . Thus the result of a one-stage play of this hypergame is outcome 15, which is unstable for both players in their games  $\mathbf{G}_{AA}$  and  $\mathbf{G}_{BB}$ . If players got the chance to move away from this unstable outcome, they could indeed find an

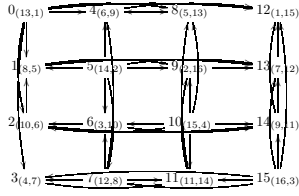


Fig. 2. H-digraph for the hypergame  $H_B^1$  with  $(P'_{G_{BB}}, P_{G_{AB}})$ .

improvement to a sequentially rational outcome and hence select the action associated to it. One can compute the sequence  $\{15, 8, 12\}$  as the result of these improvements. The final outcome 12 is an equilibrium for the corresponding 2-level hypergame.

We are interested in understanding what each of the players could have observed, at each round of playing the hypergame, about her misperception. For example, consider player  $B$ 's game as perceived by player  $A$ . Player  $A$  originally thinks that the outcome 15 is rational for player  $B$ . This can be observed in Figure 1(a), where outcome 15 has no incoming edges from 3, 7, or 11. Based on the action  $\pi_B(8)$ , player  $A$  could learn that

- 1) outcome 15 is not sequentially rational for player  $B$ ;
- 2) player  $B$  prefers outcome 11 to outcome 15, i.e.,  $15 \prec_{G_{BB}} 11 = (\pi_A(15), \pi_B(8))$ .

Thus player  $A$  wants to include these observations to improve her perception about  $B$ 's game.

#### IV. DECREASING MISPERCEPTION BY OBSERVATIONS

In this section, we develop a method that allows a player to update its own perception based on the information contained in the actions taken by other players.

##### A. Swap learning method

Let  $H^2$  be a 2-level hypergame with two players  $A$  and  $B$ . In most of the following, we analyze the hypergame from the view point of player  $A$ . An analogous discussion can be carried out for player  $B$ . Suppose players take some actions which changes the outcome from  $x \in \mathbf{S}_{\text{outcome}}$  to  $y \in \mathbf{S}_{\text{outcome}}$  with  $x \neq y$ . Then if player  $A$  believes that the opponent is rational, concludes that player  $B$  prefers the outcome  $(\pi_A(x), \pi_B(y))$  over the outcome  $x$ . Therefore, player  $A$  can incorporate this information into her hypergame and update her perception. This section introduced the swap update method to incorporate this information.

*Definition 4.1 (swap map):* Let  $V$  be a set of cardinality  $N$  and let  $W$  be the subset of  $V^N$  with pairwise different elements. For  $x_1, x_2 \in V$ , define  $g_{x_1 \mapsto x_2}^{\text{swap}} : W \rightarrow W$  by

$$\begin{aligned} (g_{x_1 \mapsto x_2}^{\text{swap}}(v))_k &= v_k \quad \text{if } v_k \neq x_1, x_2 \\ (g_{x_1 \mapsto x_2}^{\text{swap}}(v))_i &= \begin{cases} v_j & \text{if } v_i = x_1, v_j = x_2 \text{ and } i < j \\ v_i & \text{if } v_i = x_1, v_j = x_2 \text{ and } i > j \end{cases} \\ (g_{x_1 \mapsto x_2}^{\text{swap}}(v))_j &= \begin{cases} v_i & \text{if } v_i = x_1, v_j = x_2 \text{ and } i < j \\ v_j & \text{if } v_i = x_1, v_j = x_2 \text{ and } i > j \end{cases} \end{aligned}$$

We refer to  $g_{x_1 \mapsto x_2}^{\text{swap}}$  as the  $x_1$  to  $x_2$  swap map.

The swap learning map acts on the preference vectors.

*Definition 4.2 (swap learning):* Let  $H^2$  be a 2-level hypergame with two players  $A$  and  $B$  and suppose players take some actions that changes the outcome from  $x \in \mathbf{S}_{\text{outcome}}$  to  $y \in \mathbf{S}_{\text{outcome}}$ . Then the swap learning map for player  $A$  is the map  $\mathbf{T}_{x \mapsto (\pi_A(x), \pi_B(y))}^{\text{A,swap}} : \mathbf{S}_{\mathbf{P}} \rightarrow \mathbf{S}_{\mathbf{P}}$ , given by

$$\mathbf{T}_{x \mapsto (\pi_A(x), \pi_B(y))}^{\text{A,swap}}(\mathbf{P}) = g_{x \mapsto (\pi_A(x), \pi_B(y))}^{\text{swap}}(\mathbf{P}),$$

where  $g_{x \mapsto (\pi_A(x), \pi_B(y))}^{\text{swap}}$  is the  $x$  to  $(\pi_A(x), \pi_B(y))$  swap map.

Similarly, one can define  $\mathbf{T}_{x \mapsto (\pi_A(y), \pi_B(x))}^{\text{B,swap}}$  for player  $B$ . In the rest of this section, for simplicity and since  $\pi_A(y)$  has no impact on player  $A$ 's swap learning map, we assume that  $\pi_A(x) = \pi_A(y)$ , i.e., player  $B$  is the only one changing her action. Thus we can write  $\mathbf{T}_{x \mapsto (\pi_A(x), \pi_B(y))}^{\text{A,swap}} = \mathbf{T}_{x \mapsto y}^{\text{A,swap}}$ .

Next, our objective is to understand whether the effect caused by the swap learning map on the perception of the player is positive or negative. To that goal, let us introduce the following function to compare the rank of each outcome in the perceived game  $\mathbf{G}_{BA}$  to its rank in the true game  $\mathbf{G}_{BB}$ .

*Definition 4.3 (Misperception function):* Let  $H^2$  be a hypergame with outcome set  $\mathbf{S}_{\text{outcome}}$ . The function  $\mathcal{L}_{BA} : \mathbf{S}_{\mathbf{P}} \times \mathbf{S}_{\mathbf{P}} \rightarrow \mathbb{R}_{\geq 0}$ , given by

$$\mathcal{L}_{BA}(P_1, P_2) = \sum_{i=1}^N |\text{rank}(x_i, P_2) - \text{rank}(x_i, P_1)|,$$

where  $P_2$  is the preference vector of  $B$  and  $P_1$  is the preference vector of player  $B$  perceived by player  $A$ , is the misperception function of player  $A$  about player  $B$ 's game.

The following result demonstrates that the swap learning map decreases the misperception.

*Theorem 4.4: (The misperception does not increase under swap learning):* Consider a 2-level hypergame  $H^2$  between players  $A$  and  $B$ . Suppose player  $B$  takes a rational action such that the outcome of the hypergame changes from  $x_i$  to  $x_j$ , where  $\pi_A(x_i) = \pi_A(x_j)$ . Then

$$\mathcal{L}_{BA}(\mathbf{T}_{x_i \mapsto x_j}^{\text{A,swap}}(\mathbf{P}_{\mathbf{G}_{BA}}), \mathbf{P}_{\mathbf{G}_{BB}}) \leq \mathcal{L}_{BA}(\mathbf{P}_{\mathbf{G}_{BA}}, \mathbf{P}_{\mathbf{G}_{BB}}).$$

##### B. Convergence of the perceptions

Here we investigate the behavior of the hypergame when players repeatedly use the swap update map to update their perceptions. Consider a 2-level hypergame  $H^2$  between two players  $A$  and  $B$ . Suppose at round  $l \in \mathbb{Z}_{\geq 0}$ , each player takes an action that she believes will shift the outcome to a sequentially rational one for her. Suppose that the outcome changes from  $x(l)$  to  $x(l+1)$  by an action of player  $B$ . Suppose player  $A$  uses the swap learning map to update her perception about player  $B$ 's game. Then,

$$\mathbf{P}_{\mathbf{G}_{BA}}(l+1) = \mathbf{T}_{x_l \mapsto x_{l+1}}^{\text{A,swap}}(\mathbf{P}_{\mathbf{G}_{BA}}(l)),$$

defines an evolution on the perceptions of  $A$  about  $B$ , which we denote by  $(\mathbf{P}_{\mathbf{G}_{BA}}, \mathbf{T}^{\text{A,swap}})$ . Here  $\mathbf{P}_{\mathbf{G}_{BA}}(0) = \mathbf{P}_{\mathbf{G}_{BA}}$  is the initial perception of player  $A$  about player  $B$ 's game. A similar equation characterizes the evolution  $(\mathbf{P}_{\mathbf{G}_{AB}}, \mathbf{T}^{\text{B,swap}})$  for player  $B$ . We have the following result.

*Theorem 4.5: (Convergence of evolutions under swap learning):* Suppose players  $A$  and  $B$  are playing a 2-level hypergame, are rational, and are using the swap learning method to update their perceptions. Then, the evolutions defined by  $(P_{G_{BA}}, T^{A, \text{swap}})$  and  $(P_{G_{AB}}, T^{B, \text{swap}})$  for the games  $G_{BA}$  and  $G_{AB}$  converge to some preference vectors  $P_{G_{BA}}^*$  and  $P_{G_{AB}}^*$ , respectively. Furthermore, the induced sequences  $\{\mathcal{L}_{BA}(l) = \mathcal{L}_{BA}(P_{G_{BA}}(l), P_{G_{BB}})\}_{l \geq 0}$  and  $\{\mathcal{L}_{AB}(l) = \mathcal{L}_{AB}(P_{G_{AB}}(l), P_{G_{AA}})\}_{l \geq 0}$  are monotonically convergent.

In general, the final value of the misperception in Theorem 4.5 is not necessarily zero. This is typical of hypergames whose outcome set has a large cardinality, because the evolution of the hypergame may finish in an equilibrium where none of the players is willing to change her action any more, whereas parts of the outcome set remain unexplored.

## V. DETECTING THE INCONSISTENCIES IN PERCEPTION

Remarkably, even though the swap update method introduced in Section IV is guaranteed to decrease the misperception of a player, it could lead to inconsistencies in its belief about the other players' preferences. To make this point clear, consider the game  $G_{BA}$  and suppose player  $B$  takes an action which shifts the outcome from  $x_i$  to  $x_j$ . If we assume that player  $B$  is rational, moving from  $x_i$  to  $x_j$  implies that  $x_i$  is unstable in player  $B$ 's true game  $G_{BB}$ . Furthermore,  $x_j$  should be sequentially rational for player  $B$ . These two pieces of information are not captured in general by the swap update method, that only takes care of asserting that  $B$  prefers  $x_j$  to  $x_i$ . Our objective in this section is to introduce a modified learning procedure that takes into account these observations.

### A. Inconsistency in perception

Here we formalize the different inconsistencies that might arise when  $A$  and  $B$  are playing a 2-level hypergame. As before, we assume that player  $B$  takes an action that shifts the outcome from  $x_i$  to  $x_j$ , and that  $\text{rank}(x_i, P_{G_{BA}}) < \text{rank}(x_j, P_{G_{BA}})$ . The following results show two cases for which the swap learning does not create inconsistencies.

*Lemma 5.1: (Preservation of correct perception under swap learning):* Suppose player  $B$  takes an action which changes the outcome from  $x_i$  to  $x_j$ , where  $x_i \succ_{G_{BA}} x_j$ .

- 1) If  $x_i$  is perceived by player  $A$  as an unstable outcome for player  $B$  in the game  $G_{BA}$  then it is also perceived as unstable in the game  $(P_{G_{AA}}, T_{x_i \rightarrow x_j}^{A, \text{swap}}(P_{G_{BA}}))$ .
- 2) If  $x_j$  is perceived by player  $A$  as a sequentially rational outcome for player  $B$  in the game  $G_{BA}$  then it is also perceived as sequentially rational for player  $B$  in the game  $(P_{G_{AA}}, T_{x_i \rightarrow x_j}^{A, \text{swap}}(P_{G_{BA}}))$ .

The next result captures an interesting situation in which  $x_j$  is perceived to be unstable in the game  $G_{BA}$  and it remains unstable after imposing the swap learning map, thus giving rise to a contradiction in the belief of  $A$  about the game of player  $B$ .

*Lemma 5.2: (Inconsistency in perceptions under swap learning):* Suppose player  $B$  takes an action which changes the outcome from  $x_i$  to  $x_j$ , where  $x_i \succ_{G_{BA}} x_j$ .

- 1) The outcome  $x_j$  is perceived as unstable in the game  $(P_{G_{AA}}, T_{x_i \rightarrow x_j}^{A, \text{swap}}(P_{G_{BA}}))$  if and only if  $x_i$  is unstable in the game  $G_{BA}$ .
- 2) If  $x_i$  is perceived as sequentially rational in the game  $G_{BA}$  and that there exists a sequentially rational outcome  $y$  with  $\text{rank}(y, P_{G_{BA}}) > \text{rank}(x_j, P_{G_{BA}})$ . Then  $x_i$  remains sequentially rational in the game  $(P_{G_{AA}}, T_{x_i \rightarrow x_j}^{A, \text{swap}}(P_{G_{BA}}))$ .

The following lemma captures another scenario which leads to contradiction in beliefs of player  $A$  about the game played by player  $B$ .

*Lemma 5.3: (Correction of perceptions under swap learning):* Suppose player  $B$  takes an action which changes the outcome from  $x_i$  to  $x_j$ . Suppose that  $x_i$  is perceived as sequentially rational in the game  $G_{BA}$  and there exists an unstable outcome  $y$  with  $\text{rank}(y, P_{G_{BA}}) < \text{rank}(x_j, P_{G_{BA}})$ . Then  $x_i$  is unstable in the game  $(P_{G_{AA}}, T_{x_i \rightarrow x_j}^{A, \text{swap}}(P_{G_{BA}}))$ .

### B. Modified swap learning method

In this section, we investigate how a player can include the information gathered from the contradictions in her belief update under swap learning (cf. Lemma 5.2) to learn more about the other player's true game. Note that the contradictions in the perception of player  $A$  about  $B$ 's game are not necessarily due to the misperception of player  $A$  and may entirely be due to the misperception of  $B$  about  $A$ 's game. Consequently, in spite of the inconsistencies that may arise, player  $A$  may still decide to employ the swap learning method described in Section IV.

Here, we introduce a modified version of the swap learning method that prevents any inconsistency in perceptions from happening. Under this learning method, player  $A$  assumes that player  $B$  has perfect information about her game and thus is convinced that any misperception is due to her lack of knowledge about the game played by  $B$  (interestingly, this assumption can be modeled in a third-level hypergame by specifying  $G_{ABA} = G_{AAA}$ ). In order to formally define the method, we need to discuss the existence of two outcomes with a particular set of properties. This is what we do next.

*Lemma 5.4: (Existence of  $y$ ):* Consider a 2-level hypergame between two players  $A$  and  $B$ . Suppose player  $B$  takes an action such that the outcome changes from  $x_i$  to  $x_j$  and suppose both  $x_i$  and  $x_j$  are perceived as unstable in  $T_{x_i \rightarrow x_j}^{A, \text{swap}}(P_{G_{BA}})$ . Then there exists an outcome  $y \in \text{S}_{\text{outcome} | \pi_A(x_j)}$  such that  $y$  is sequentially rational in the hypergame  $(P_{G_{AA}}, T_{x_i \rightarrow x_j}^{A, \text{swap}}(P_{G_{BA}}))$ .

The proof of this lemma follows from Lemma 2.6.

*Lemma 5.5: (Existence of  $z$ ):* Consider a 2-level hypergame between two players  $A$  and  $B$ . Suppose player  $B$  takes an action such that the outcome changes from  $x_i$  to  $x_j$  and suppose both  $x_i$  and  $x_j$  are perceived as sequentially rational in  $T_{x_i \rightarrow x_j}^{A, \text{swap}}(P_{G_{BA}})$ . Then there exists an improvement  $z \in \text{S}_{\text{outcome} | \pi_B(x_j)}$  from  $x_j$  for player  $A$  in  $G_{AA}$ .

Next, we introduce the modified swap learning method.

*Definition 5.6 (Modified swap learning):* Consider a 2-level hypergame between two players  $A$  and  $B$ . Suppose

players take actions such that the outcome changes from  $x_i \in \mathbf{S}_{\text{outcome}}$  to  $x_j \in \mathbf{S}_{\text{outcome}}$ . The *modified-swap learning map*  $\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,mod-swap}} : \mathbf{S}_P \rightarrow \mathbf{S}_P$  is defined by

- if  $x_i \in \text{Seq}^c(\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}})(P), P_{G_{AA}})$  and  $x_j \in \text{Seq}(\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}})(P), P_{G_{AA}})$  then

$$\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,mod-swap}}(P) = \mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}}(P).$$

- if  $x_i, x_j \in \text{Seq}^c(\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}})(P), P_{G_{AA}})$  then

$$\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,mod-swap}}(P) = \mathbf{T}_{(\pi_A(x_i), \pi_B(x_j)) \mapsto y}^{\text{A,swap}} \circ \mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}}(P),$$

where  $y \in \mathbf{S}_{\text{outcome}}|_{\pi_A(x_i)}$  is the outcome with the lowest rank, with respect to  $\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}}(P)$ , which satisfies the conditions of Lemma 5.4.

- if  $x_i, x_j \in \text{Seq}(\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}})(P), P_{G_{AA}})$  then

$$\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,mod-swap}}(P) = \mathbf{T}_{x_i \mapsto z}^{\text{A,swap}} \circ \mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}}(P),$$

where  $z \in \mathbf{S}_{\text{outcome}}|_{\pi_B(x_j)}$  is the outcome with the highest rank, with respect to  $\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,swap}}(P)$ , which satisfies the conditions of Lemma 5.5.

The following result follows from Lemmas 5.4 and 5.5 and highlights an important property of modified swap learning.

*Proposition 5.7: (Modified swap learning results in no inconsistency):* Consider a 2-level hypergame between two players  $A$  and  $B$ . Suppose players  $A$  and  $B$  take actions which shift the outcome from  $x_i$  to  $x_j$ . Then under the modified-swap learning, outcomes  $x_i$  and  $(\pi_A(x_i), \pi_B(x_j))$  are perceived by  $A$ , respectively, as unstable and sequentially rational in  $\mathbf{T}_{x_i \mapsto (\pi_A(x_i), \pi_B(x_j))}^{\text{A,mod-swap}}(P_{G_{BA}})$ . Similarly, outcomes  $x_i$  and  $(\pi_A(x_j), \pi_B(x_i))$  are perceived by  $B$ , respectively, as unstable and sequentially rational in  $\mathbf{T}_{x_i \mapsto (\pi_A(x_j), \pi_B(x_i))}^{\text{B,mod-swap}}(P_{G_{AB}})$ .

### C. Decreasing the misperception via modified swap learning

In general, the modified swap learning method is not guaranteed to decrease the misperception function. The following result shows that, under the assumption that  $B$  has perfect information about  $A$ 's game and always chooses the sequentially rational outcome with the best payoff, the modified swap learning method for  $A$  decreases her misperception function in the sense of Definition 4.3, while preventing inconsistency in her beliefs.

*Theorem 5.8: (Misperception function and modified swap learning):* Consider a 2-level hypergame between two players  $A$  and  $B$  where  $G_{AB} = G_{AA}$ . Suppose player  $B$  takes an action which changes the outcome from  $x_i$  to her best sequentially rational outcome  $x_j$ . Then, under the modified-swap learning method, the misperception function  $\mathcal{L}_{BA}$  does not increase.

## VI. CONCLUSIONS

We have studied how players' perceptions about the game they are involved in change in adversarial situations adopting the framework of hypergames. We have introduced the swap

learning method to allow players to incorporate into their beliefs the information gained from observing the opponents' actions. We have shown that a player that uses this method decreases her misperception at the cost of potentially incurring in inconsistencies in her beliefs. This has motivated the introduction of a modified version which always yields consistent beliefs, and, under the assumption that the opponent has perfect information and plays her best strategy, also decreases the misperception. The methods discussed here put all the burden of the misperception on the player doing the update. Future work will relax this assumption and explore the setup of higher-level hypergames.

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