

# Coalition formation and motion coordination for optimal deployment

Michael Ouimet      Jorge Cortés

**Abstract**—This paper presents a distributed algorithmic solution to achieve network configurations where agents cluster into coincident groups that are distributed optimally over the environment. The motivation for this problem comes from spatial estimation tasks executed with unreliable sensors. We propose a probabilistic strategy that combines a repeated game governing the formation of coalitions with a spatial motion component governing their location. We establish the convergence of the agents to coincident groups of a desired size in finite time and the asymptotic convergence of the overall network to the optimal deployment, both with probability 1. The algorithm is robust to agent addition and subtraction. From a game perspective, the algorithm is novel in that the players’ information is limited to neighboring clusters. From a motion coordination perspective, the algorithm is novel because it brings together two basic tasks, rendezvous (individual agents into clusters) and deployment (clusters in the environment). The technical approach draws on tools from computational geometry, game theory, and probability theory.

## I. INTRODUCTION

This paper is motivated by optimal spatial sampling problems under possibly failing communications. Consider a group of mobile robotic sensors that take point measurements of a random field over an environment and relay them back to a data fusion center. Assume that because of the features of the medium and the limited agent communication capabilities, it is known that only a fraction of these packets will arrive at the center, but it is not a priori known which ones will. Given that some sensors are not working and their identity is unknown, a reasonable strategy consists of grouping sensors together into clusters so that the likelihood of obtaining a measurement from the position of each cluster is higher. In this paper, our aim is to design a distributed algorithm that makes the network autonomously create groups of a desired size such that (i) members of each individual group become coincident, and (ii) the groups deploy in an optimal way with regards to the spatial estimation objective.

*Literature review:* There is an increasing body of research that deals with spatial estimation problems with possibly failing communications where packets are either received without corruption or not received at all, see e.g., [1], [2], [3], [4]. In particular, [4] shows that, for the problem motivating our algorithm design, the clustering strategy outlined above is not only reasonable but optimal in some cases: the configurations that maximize the expected information content of the measurements retrieved at the center correspond to agents grouping into clusters, and the resulting clusters being deployed optimally. Achieving such desirable configurations

is challenging because of the spatially distributed nature of the problem and the agent mobility. Our technical approach combines elements of spatial facility location [5], rendezvous and deployment of multi-agent systems [6], and coalition formation games [7], [8]. From a game-theoretic perspective, our analysis of the coalition formation dynamics is novel because of the consideration of evolving and partial interaction network topologies. From a motion coordination perspective, the novelty relies on the coupled dynamics between the coalition formation, the clustering, and the network deployment. Other works in the cooperative control literature that have employed game-theoretic ideas to solve coordination tasks such as formation control, target assignment, self-organization for efficient communication, consensus, and sensor coverage include [9], [10], [11], [12].

*Statement of contributions:* The main contribution of the paper is the design and analysis of the COALITION FORMATION AND DEPLOYMENT ALGORITHM. The aim of this synchronous and distributed strategy is to allow robotic agents to autonomously form groups of a given desired size while clustering together and deploying optimally in the environment. The deployment objective is encoded through a locational optimization function whose optimizers correspond to circumcenter Voronoi configurations. The algorithm design combines a repeated game component that governs the dynamics of coalition formation with a spatial motion component that determines how agents’ positions evolve. In the game, agents seek to join a neighboring coalition that most closely resembles one with the desired size. According to the motion coordination law, agents not yet in a well-formed coalition cluster together while agents in a coalition of the desired size also move towards the circumcenter of their Voronoi cell. We establish that the executions of the COALITION FORMATION AND DEPLOYMENT ALGORITHM converge in finite time to a configuration where agents are coincident with their own coalition and these coalitions are the desired size, and asymptotically converge to an optimal deployment configuration, each with probability 1. The algorithm does not require the agents to have a common reference frame, and is robust to agent addition and deletion. Finally, we illustrate these properties in simulation. Some proofs are omitted for reasons of space.

## II. PRELIMINARIES

In this section, we collect some basic facts on geometry, spatial deployment, probability, and coalition games.

### A. Basic geometric notions

We denote by  $\mathbb{R}$  and  $\mathbb{Z}$  the sets of real and integer numbers, respectively. Let  $\|\cdot\|$  be the Euclidean distance. Given a

The authors are with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA 92093, USA, {mouimet,cortes}@ucsd.edu

set  $S \subset X$ , let  $\mathbb{F}(S)$  denote the collection of finite subsets of  $S$  and  $S^c = X \setminus S$  its complement. Let  $|S|$  denote the cardinality of the set  $S$ . Let  $\text{vr} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be defined by  $\text{vr}(u) = u/\|u\|$  for  $u \in \mathbb{R}^d \setminus \{0\}$ , and  $\text{vr}(0) = 0$ . We denote the closed ball centered at  $x \in \mathbb{R}^d$  of radius  $r \in \mathbb{R}_{>0}$  by  $B(x, r) = \{p \in \mathbb{R}^d \mid \|x - p\| \leq r\}$ . The circumcenter of a set of points  $P$ , denoted  $\text{CC}(P)$ , is the center of the ball of minimum radius, denoted  $\text{CR}(P)$ , which encloses all points in  $P$ . Next, we introduce the get-together-toward-goal function  $\text{gttg} : S \times \mathbb{F}(S) \times S \rightarrow S$  by

$$\text{gttg}(p, P, q) = p + w_1 + w_2,$$

where we use the shorthand notation  $P_0 = P \cup \{p\}$ ,

$$w_1 = \min\{\|\text{CC}(P_0) - p\|, d_1(r)\} \text{vr}(\text{CC}(P_0) - p),$$

$$w_2 = \min\{\|q - (p + w_1)\|, d_2(r)\} \text{vr}(q - (p + w_1)),$$

and  $r = \text{CR}(P_0)/\|q - \text{CC}(P_0)\|$ . Here,  $d_1 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is a continuous, increasing function on  $(0, \infty)$  that satisfies

$$d_1(0) = 0, \quad \lim_{s \rightarrow \infty} d_1(s) = d_{\max}, \quad \lim_{s \rightarrow 0^+} d_1(s) = d_{\min},$$

for  $d_{\max} > d_{\min} > 0$ , and  $d_2 : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is defined by  $d_2(s) = d_{\max} - d_1(s)$ . Figure 1 illustrates the definition of  $\text{gttg}$ . Appendix A gathers some relevant properties of  $\text{gttg}$ .

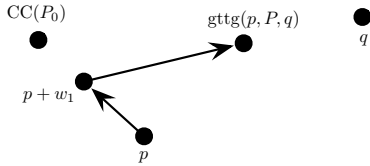


Fig. 1. Illustration of the action of the function  $\text{gttg}$ .

## B. Voronoi partitions and deployment objective

Here, we introduce some computational geometric notions that play an important role in the formalization of the deployment problem. Given  $Q \subset \mathbb{R}^d$  and a finite set of points  $P = \{p_1, \dots, p_n\} \subset Q$ , the Voronoi partition  $V(P) = \{V_1(P), \dots, V_n(P)\}$  of  $Q$  is defined by

$$V_i(P) = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\|, \forall p_j \in P\}.$$

The set  $V_i(P)$  is the Voronoi cell of  $p_i$ . The points  $p_i$  and  $p_j$  are Voronoi neighbors if the boundaries of their Voronoi cells intersect. To compute the Voronoi cell of  $p_i$ , all that is required is the location of its Voronoi neighbors in  $P$ . The work [13] introduces a procedure, that we term the ADJUST RADIUS strategy, which does the following: starting from  $r = 0$ , it repeatedly grows  $r$  until all Voronoi neighbors of  $p_i$  are guaranteed to be contained in the ball  $B(p_i, r)$ .

Given a partition  $\{W_1, \dots, W_n\}$  of  $Q$ , the disk-covering function  $\mathcal{H}_{\text{DC},n}$  is defined by

$$\mathcal{H}_{\text{DC},n}(p_1, \dots, p_n, W_1, \dots, W_n) = \max_{i \in \{1, \dots, n\}} \max_{q \in W_i} \|q - p_i\|_2.$$

The value of  $\mathcal{H}_{\text{DC},n}$  solves the following problem: cover the whole environment with balls centered at the points in  $P = \{p_1, \dots, p_n\}$  with minimum common radius such that  $W_i \subset$

$B(p_i, r)$ , for  $i \in \{1, \dots, n\}$ . For convenience, we use the notation  $\mathcal{H}_{\text{DC},n}(p_1, \dots, p_n) = \mathcal{H}_{\text{DC},n}(p_1, \dots, p_n, V_1, \dots, V_n)$ . Two properties are worth noting [6]: for a fixed configuration, the Voronoi partition is optimal among all partitions,

$$\mathcal{H}_{\text{DC},n}(p_1, \dots, p_n, V_1(P), \dots, V_n(P)) \leq \mathcal{H}_{\text{DC},n}(p_1, \dots, p_n, W_1, \dots, W_n),$$

and, for a fixed partition, the circumcenter locations of the cells are optimal,

$$\mathcal{H}_{\text{DC},n}(\text{CC}(W_1), \dots, \text{CC}(W_n), W_1, \dots, W_n) \leq \mathcal{H}_{\text{DC},n}(p_1, \dots, p_n, W_1, \dots, W_n).$$

Under certain technical conditions, optimizing  $\mathcal{H}_{\text{DC},n}$  corresponds to minimizing the maximum error variance in the estimation of a random spatial field [14]. The deployment objective function that motivates our algorithm is given by

$$\mathcal{H}_{n,k}(p_1, \dots, p_n) = \frac{1}{\binom{n}{k}} \sum_{\{s_1, \dots, s_k\} \in C(n,k)} \mathcal{H}_{\text{DC},k}(p_{s_1}, \dots, p_{s_k}), \quad (1)$$

where  $C(n, k)$  denotes the set of unique  $k$ -sized combinations of elements in  $\{1, \dots, n\}$ . This function corresponds to the expected disk-covering performance of a network of  $n$  agents where only  $k$  of them are working and their identity is unknown. Optimizers of  $\mathcal{H}_{n,k}$  correspond to grouping agents into coincident clusters, say  $m$ , that themselves are optimally deployed according to  $\mathcal{H}_{\text{DC},m}$ , see [4].

## C. Probability notions

Here we gather some probability notions from [15], [16]. Let  $X$  be a random variable that has outcomes  $\{x_1, x_2, \dots\}$  with probabilities  $\{p_1, p_2, \dots\} \subset \mathbb{R}_{\geq 0}$ . An event  $E$  is a set of outcomes of  $X$ . For brevity, we use  $P(E) = P(X \in E)$ . Given a sequence of events  $\{E_n\}_{n=1}^{\infty}$ , let

$$\limsup_n E_n = \{E_n \text{ i.o.}\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k,$$

$$\liminf_n E_n = \{E_n \text{ a.a.}\} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k.$$

Here ‘i.o.’ stands for infinitely often, and ‘a.a.’ stands for almost always. Note that  $\{E_n \text{ i.o.}\}^c = \{E_n^c \text{ a.a.}\}$ .

**Lemma II.1 (Borel-Cantelli Lemma)** *Given a sequence of events  $\{E_n\}_{n=1}^{\infty}$  satisfying  $\sum_{n=1}^{\infty} P(E_n) < \infty$ . Then  $P(\limsup_n E_n) = 0$ .*

## D. Hedonic coalition games

Hedonic coalition formation games [7] are  $N$ -player noncooperative games [17], [18] where players attempt to join/stay in preferable coalitions. Each player is hedonic because the utility it assigns to a given network coalition partitioning is only a function of its own coalition. Each player’s action set is finite: the only actions are to stay in the current coalition or join another coalition. For a finite set of players  $\mathcal{A} =$

$\{1, \dots, N\}$ , a finite coalition partition is a set  $\Pi = \{S_k\}_{k=1}^K$ ,  $K \in \mathbb{Z}_{\geq 1}$ , that partitions  $\mathcal{A}$ . The subsets  $S_k$  are called coalitions. For player  $i$  and partition  $\Pi$ , let  $S_\Pi(i)$  be the set  $S_k \in \Pi$  such that  $i \in S_k$ . Agent  $i$ 's preference is defined by an ordering  $\succeq_i$  over the set  $\mathfrak{S}_i = \{S \in \mathbb{F}(\mathcal{A}) \mid i \in S\}$ . A coalition partition  $\Pi$  is called Nash stable if, for each  $i \in \mathcal{A}$ ,

$$S_\Pi(i) \succeq_i S_k \cup \{i\}, \quad \forall S_k \in \Pi \cup \emptyset. \quad (2)$$

A game's purpose is to study the stable coalition partitions based on the players' action sets and preferences. Finally, we define the subset of coalitions that player  $i$  can join as  $\tau_i \subseteq \{S_\Pi(j)\}_{j \in \mathcal{A} \setminus \{i\}} \cup S_\Pi(0)$ , where  $S_\Pi(0) = \emptyset$ . The function best-set defines the set of players that  $i$  can join that maximizes its coalition preference:

$$\begin{aligned} \text{best-set}(\succeq_i, \tau_i) &= \{j \in \mathcal{A} \setminus \{i\} \cup \{0\} \mid S_\Pi(j) \in \tau_i, \\ &S_\Pi(j) \cup \{i\} \succeq_i S_\Pi(k) \cup \{i\}, \forall S_\Pi(k) \in \tau_i\}. \end{aligned}$$

An important observation is that, in coalition formation games, an agent has information about which coalitions all other agents are in and may join any of them. This is in contrast to our scenario, where coalition information is only partial due to the limited capabilities of individual agents.

### III. PROBLEM STATEMENT

Consider a group of robotic sensors with unique identifiers  $\mathcal{A} = \{1, \dots, N\}$  moving in a convex polygon  $Q \subset \mathbb{R}^2$ . Let  $p_i$  denote the location of agent  $i$  and  $P = (p_1, \dots, p_N)$  denote the overall network configuration. We consider arbitrary agent dynamics, assuming each agent can move up to a distance  $d_{\max} \in \mathbb{R}_{>0}$  within one timestep,

$$p_i(\ell + 1) \in B(p_i(\ell), d_{\max}), \quad \ell \in \mathbb{Z}.$$

Through either sensing or communication, we assume each agent  $i$  can get the relative position and identity of agents within distance  $r_i \in \mathbb{R}_{>0}$ . During the coalition formation process, agents can communicate with other agents within this radius. Agent  $i$  can adjust  $r_i$  but the cost of acquiring information is an increasing function of it. Inter-agent communication occurs instantaneously.

The group's objective is dual. On the one hand, agents want to cluster into groups of a predefined size  $\kappa$ . Equivalently, the network wants to self-assemble into  $m = \lfloor \frac{N}{\kappa} \rfloor$  clusters of size  $\kappa$ , with possibly one additional cluster of size  $z$ ,  $0 \leq z < \kappa$ , with  $N = m\kappa + z$ . We call this the *goal coalition partition*. On the other hand, the resulting clusters should be positioned in the environment so as to minimize  $\mathcal{H}_{\text{DC}, \bar{m}}$ , where the final number of coalitions  $\bar{m}$  is given by

$$\bar{m} = \begin{cases} m, & \text{if } \text{mod}(N, \kappa) = 0, \\ m + 1, & \text{otherwise,} \end{cases}$$

where  $\text{mod}(N, \kappa)$  is the remainder of  $N/\kappa$ . As discussed in Section II-B, such deployments correspond to optimizers of a spatial estimation problem with unreliable data transmission. Our aim is to create a distributed algorithm that accomplishes the dual network objective in a robust and efficient way.

### IV. COALITION FORMATION AND DEPLOYMENT ALGORITHM

In this section, we solve the spatial deployment problem posed in Section III with the COALITION FORMATION AND DEPLOYMENT ALGORITHM. This distributed, synchronous strategy specifies for each agent the dynamics of coalition formation and spatial motion. Section IV-A outlines the logic used by agents to determine which coalition to join as well as the supporting inter-agent communication and Section IV-B discusses how agents decide how to move depending on their coalition size and the deployment objective.

#### A. Coalition formation game

The formation of coalitions evolves according to a simultaneous-action hedonic coalition game with partial information. Let us start with an informal description.

*[Informal description]:* The agents' objective is to be in a  $\kappa$ -sized coalition. There are two rounds of communication per timestep. In the first one, each agent acquires information to determine if any neighboring coalition is more attractive than its current one. In the second one, the agents involved in a coalition change (either because they have decided to switch or because someone else decided to join their coalition) exchange information to update the coalition membership.

Next, we formally describe the hedonic coalition formation game. The agent  $i$ 's preference ordering  $\succeq_i$  over  $\mathfrak{S}_i$  is

$$\begin{aligned} \{S \in \mathfrak{S}_i \mid |S| = \kappa\} &\succ \{S \in \mathfrak{S}_i \mid |S| = \kappa - 1\} \succ \dots \\ &\succ \{S \in \mathfrak{S}_i \mid |S| = 1\} \succ \{S \in \mathfrak{S}_i \mid |S| = \kappa + 1\} \succ \dots \\ &\succ \{S \in \mathfrak{S}_i \mid |S| = N\}. \end{aligned} \quad (3)$$

According to (3), the agents most prefer to be in  $\kappa$ -sized coalitions. The memory  $\mathcal{M}_i$  of agent  $i$  is composed of

- the *coalition set*  $C_i$ . Elements of this set are of the form  $(j, p_j)$ , i.e., identity and position of the member. For convenience, we set  $(i, p_i) \in C_i$  and  $C_0 = \emptyset$ ;
- the *communication radius*  $r_i$  at which the agent interacts with other agents not necessarily in its coalition set;
- the *neighboring set*  $\mathcal{N}_i$  corresponding to agents within distance  $r_i$ , i.e.,  $(j, p_j) \in \mathcal{N}_i$  iff  $p_j \in B(p_i, r_i)$ ;
- the *farthest-away radius*  $\tau_i$ , corresponding to the maximum distance to members of its coalition set.
- the *flag last*, which indicates if an agent belongs to the single final coalition not of size  $\kappa$  when  $\bar{m} \neq m$ .

For convenience, the operators  $\text{id}(\cdot)$  and  $\text{pos}(\cdot)$  extract identities and positions, respectively, from sets whose elements are of the form  $(i, p_i)$ . A *consistent partitioning* is a collection  $\{C_1, \dots, C_N\} \subset \mathbb{F}(\{(1, p_1), \dots, (N, p_N)\})$  with  $(i, p_i) \in C_i$  and  $C_i = C_j$ , for each  $j \in \text{id}(C_i)$  and  $i \in \mathcal{A}$ . Initially, for some  $\delta \in \mathbb{R}_{>0}$ , we require the agents be in a consistent partitioning,  $r_i = \delta$ , and  $\text{last} = \text{False}$ .

Next, we specify the two rounds of communication that take place per timestep. Agents who already are in a coalition of size  $\kappa$  do not actively take part in this process; they only

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**Algorithm 1: BEST NEIGHBOR COALITION DETECTION**

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**Executed by:** Agents  $i$  with  $|C_i| \neq \kappa$

```
1 Acquire  $\mathcal{N}_i$            % get location of neighbors
2 Set  $j^* := i$            % reset switching flag
3 if  $\mathcal{N}_i \setminus C_i \neq \emptyset$  then
4   Send (query,  $r_i$ ) at  $r_i$  to  $\text{id}(\mathcal{N}_i \setminus C_i)$ 
5   % request coalition sizes
6   Receive  $\text{id}(C_j)$  from all  $j \in \text{id}(\mathcal{N}_i \setminus C_i)$ 
7   % receive coalition sizes
8    $\mathcal{C}_i := \{(m, p_m) \in \mathcal{N}_i \setminus C_i \mid |C_m| \neq \kappa\} \cup \{(0, 0)\}$ 
9   % candidate agents to join
10  % better coalitions exist
11  if  $\exists j \in \text{id}(\mathcal{C}_i)$  s.t.  $\text{id}(C_j) \cup \{i\} \succ_i \text{id}(C_i)$  then
12    % better coalitions exist
13    with probability  $P(|C_i|, \kappa)$  do
14      Set  $j^*$  from best-set( $\succeq_i, \{\text{id}(C_k)\}_{k \in \text{id}(\mathcal{C}_i)}$ )
15      % identify best coalition to join
16      if  $j^* \neq 0$  then  $r_i := \|p_{j^*} - p_i\|$ 
17    end
18  end
19 end
```

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respond to other agents' messages. First, agents execute the BEST NEIGHBOR COALITION DETECTION strategy described as Algorithm 1. According to this strategy (cf. step 8), an agent that finds a neighboring coalition better than its own will decide to join it with probability given by

$$P(|C_i|, \kappa) = 1 - (1 - b)^{1/|C_i|} \quad \text{if } |C_i| \neq \kappa. \quad (4)$$

If  $|C_i| = \kappa$ , the player  $i$  will surely not switch coalitions. The design parameter  $b \in (0, 1)$  corresponds to the probability that at least one agent in a non- $\kappa$  coalition has the ability to act. The choice of  $b$  influences the rate of coalition changes.

**Remark IV.1 (Justification for probabilistic actions)** The probabilistic model for actions described in (4) helps avoid deadlock situations that may result from the decentralized nature of the game. As an example, in a situation with two groups of size  $\kappa - 1$ , all agents will desire to join the other group. If this were the case, a group of size  $\kappa$  would never form. Instead, under (4), there is a positive probability  $2b(1-b)$  that agents in only one of the groups act, breaking the deadlock. In contrast with a one-agent-acting-per-timestep policy, the model (4) allows multiple agents to switch coalitions at the same timestep. •

Next, all agents execute the COALITION SWITCHING strategy described in Algorithm 2.

### B. Motion control law

Here, we describe how agents move at each timestep, beginning with an informal description:

*[Informal description]:* At each timestep, agents adjust their communication radius and move. Both of these actions are dependent on the size of their coalition. Agents not yet in a coalition of size  $\kappa$

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**Algorithm 2: COALITION SWITCHING**

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**Executed by:** All agents  $i$

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1 if  $j^* \neq i$  then
2   Send (leave,  $i$ ) at  $r_i$  to  $\text{id}(C_i)$ 
3   if  $j^* \neq 0$  then
4     | Send (join,  $i, r_i$ ) at  $r_i$  to  $j^*$ 
5   end
6   % alert old and new coalitions
7 end
8  $M := \{k \in \mathcal{A} \mid i \text{ received join from } k\}$ 
9 % agents relying on  $i$  to aid switching
10 foreach  $m \in M$  do Send (join,  $m, r_m$ ) to  $\text{id}(C_i)$ 
11 % alert other coalition members via  $r_i$ 
12  $L := \{k \in \mathcal{A} \mid i \text{ received leave from } k\}$ 
13  $J := \{k \in \mathcal{A} \mid \text{an } m \in \text{id}(C_i) \text{ got join from } k\}$ 
14 % agents leaving/joining  $i$ 's coalition
15  $\text{id}(C_i) := (\text{id}(C_i) \cup J) \setminus L$  and  $r_i := r_i + \max\{r_j\}_{j \in J}$ 
16 % update current coalition and radius
17 foreach  $m \in M$  do Send ( $r_i, \text{id}(C_i)$ ) at  $r_m$  to  $m$ 
18 % update agents joining  $i$ 's coalition
19 end
20 if  $j^* \neq i$  then
21   if  $j^* = 0$  then
22     |  $C_i = \{(i, p_i)\}$  % form a new coalition
23   else
24     |  $\text{id}(C_i) := \text{id}(C_{j^*})$  and  $r_i := \|p_{j^*} - p_i\| + r_{j^*}$ 
25     % update coalition and radius
26   end
27 end
28 if  $J \neq \emptyset \vee j^* \neq i$  then
29   Acquire  $\mathcal{N}_i$ ,  $\text{pos}(C_i)$ , recompute  $\mathcal{C}_i$ 
30    $j^* := i$  % reset switching variable
31 end
```

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increase their radius to improve the chances of finding a better coalition and move towards their coalition members. Agents in a coalition of size  $\kappa$  adjust their radius to ensure they can calculate their Voronoi cell and move towards both their coalition members and the circumcenter of their cell.

Formally, the RADIUS ADJUSTMENT AND MOTION strategy is described as Algorithm 3. Its interaction with the coalition formation dynamics is described in steps 9-15, which governs the set of agents that a robot not yet in a  $\kappa$ -sized coalition may interact with. The next result ensures that the agent communication radius is kept at the smallest value guaranteeing a successful completion of coalitions.

**Lemma IV.2** For each  $i \in \mathcal{A}$ , let  $k_i$  be the closest agent which is in a coalition different from  $i$ 's with size different from  $\kappa$  and define  $r_i(P, (C_1, \dots, C_N)) = \|p_i - p_{k_i}\|$ . Such radii guarantee the property that at least one agent has an incentive to switch coalitions when the configuration is not in the goal coalition partition. Furthermore, if the communication radii were set according to  $\{r'_i(P, (C_1, \dots, C_N))\}_{i \in \mathcal{A}}$  with  $r'_i(P, (C_1, \dots, C_N)) < r_i(P, (C_1, \dots, C_N))$  for some  $i$



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**Algorithm 3:** RADIUS ADJUSTMENT AND MOTION

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**Executed by:** All agents  $i$

```
1 if  $|C_i| = \kappa \vee \text{last} = \text{True}$  then
2   Update  $r_i$  with ADJUST RADIUS strategy
3   Acquire  $\mathcal{N}_i$ 
4    $A_i := (\{\text{CC}(\text{pos}(C_i))\} \cup \text{pos}(\mathcal{N}_i)) \setminus \text{pos}(C_i)$ 
5    $V_i := V_1(A_i)$  % compute Voronoi cell
6   goal = CC( $V_i$ )
7 else
8   goal = CC( $\text{pos}(C_i)$ )
9   if  $\mathcal{C}_i \setminus \{(0,0)\} \neq \emptyset$  then
10     $r_i := \min_{p_k \in \text{pos}(\mathcal{C}_i \setminus \{(0,0)\})} \|p_k - p_i\| + 2d_{\max}$ 
11    % guarantees a neighbor after motion
12  else
13    if  $\text{id}(\mathcal{N}_i) = \mathcal{A}$  then
14      last := True % one non- $\kappa$  coalition
15    else
16       $r_i := r_i + \delta$  % increase radius
17    end
18  end
19 foreach  $j \in \text{id}(C_i)$  do  $p_j := \text{gttg}(p_j, \text{pos}(C_i), \text{goal})$ 
20 % compute next position
21  $\text{pos}(C_i) := \{p_j\}_{j \in \text{id}(C_i)}$  % update positions
22  $r_i := \max_{p_j \in \text{pos}(C_i)} \|p_j - p_i\|$  % recompute radius
```

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and  $P$ , then this property is no longer guaranteed.

*Proof:* In the case that there exists at least one coalition of size greater than  $\kappa$ , all agents in this coalition have an incentive to start their own coalition. Consider instead, the case where all coalitions are of size at most  $\kappa$ . An agent  $i$  in the smallest coalition has an incentive to join its neighbor  $k_i$  and the claimed property follows. Next, we show the minimality property. It is enough to show that there is one consistent partition different from the goal coalition partition for which a smaller communication radius assignment would not work. Consider a consistent partitioning at configuration  $P$  where all coalitions but one have been formed, and the remaining agents are in two coalitions, one with the single agent,  $i$ , and the other one,  $C$ , with the rest. Since  $r'_i(P, (C_1, \dots, C_N)) < r_i(P, (C_1, \dots, C_N)) = \|p_i - p_{k_i}\|$ , agent  $i$  has no agents in  $\mathcal{N}_i$  that it has incentive to join. Furthermore, given the coalition partition, agent  $i$  is the only one who could have an incentive to switch coalitions, which finalizes the proof. ■

Steps 9-15 in Algorithm 3 implement the result described in Lemma IV.2. If agent  $i$  is not within  $r_i$  of a non-coalition agent that is not in a  $\kappa$ -sized coalition, increase  $r_i$ . If agent  $i$  is within  $r_i$  of such an agent, change  $r_i$  to the distance between the two agents plus a constant that ensures that they remain within communication range after moving.

**Remark IV.3 (Voronoi cell computation)** In the Voronoi cell computation of step 5 in Algorithm 3, the coalition's

circumcenter replaces all the locations of the individual agents. This ensures that all members in a coalition compute the same Voronoi cell. However, this also implies that, in general, the collection of cells computed by the coalition is not a partition of the environment. This issue gets resolved when the members within each coalition are coincident and will be treated in the proof of Theorem V.1. •

**Remark IV.4 (Choice of parameter  $\delta$ )** In step 15 of Algorithm 3, the parameter  $\delta$  describes the amount that an agent  $i$ 's communication radius  $r_i$  increases if it does not have any neighboring candidate agents to join. Several choices for  $\delta$  are possible. For instance, when agents are roughly uniformly distributed across  $Q$ , choosing  $\delta \propto \frac{\text{diam}(Q)}{\sqrt{N}}$  makes it likely that agent  $i$  will discover at least one new agent. •

We refer to the composition of Algorithms 1-3 as the COALITION FORMATION AND DEPLOYMENT ALGORITHM. We note that this strategy does not require the agents to share a common reference frame.

**Remark IV.5 (Robustness to agent addition and subtraction)** The COALITION FORMATION AND DEPLOYMENT ALGORITHM is robust to agents joining or leaving the network under the following assumptions: (i) new agents alert the network of their presence by sending a query message, (ii) when an agent fails, the other members of its coalition detect this fact, and (iii) when agents receive a query message they set  $\text{last} := \text{False}$ . •

## V. CONVERGENCE ANALYSIS

This section analyzes the convergence properties of the COALITION FORMATION AND DEPLOYMENT ALGORITHM. Our main objective is to establish the following result.

**Theorem V.1** Consider a network of  $N$  agents executing the COALITION FORMATION AND DEPLOYMENT ALGORITHM. The following holds,

- (i) there exists a finite time after which all agents are in the goal coalition partition and each is coincident with its coalition members, with probability 1;
- (ii) the network asymptotically converge towards the set of minimizers of  $\mathcal{H}_{\text{DC}, \bar{m}}$ , with probability 1.

In particular, note that this result states that, with probability 1, the network will not converge to a coalition partition other than the desired one. Agents may be stuck for some time in a different partition but, in finite time, they will reach the desired coalition partition with probability 1.

To prove Theorem V.1, we first establish several intermediate results. We begin by showing that the coalition formation game gives rise to the desired partition.

**Lemma V.2** In the  $N$ -agent simultaneous-action game where agents have preference orderings that satisfy (3), complete knowledge about all other coalition memberships and their action set is to stay or join any other coalition, the only Nash stable partition is the goal coalition partition.

As stated in Theorem V.1, the COALITION FORMATION AND DEPLOYMENT ALGORITHM achieves the same goal coalition partition even though agents have partial coalition information. Before continuing our discussion, we define here the collection of actions of all agents at a given timestep as a *timestep-event*. The next result determines a strictly positive lower bound on the probability of any possible timestep-event happening.

**Lemma V.3** *Let  $E$  be a timestep-event with  $P(E) > 0$ . Then  $P(E) \geq \min\{(1-b)^N, (1-(1-b)^{\frac{1}{N}})^N\}$ .*

*Proof:* Note that the probability that an agent switches coalitions is lower bounded by  $1 - (1-b)^{\frac{1}{N}}$  and the probability that an agent wishes to switch coalitions but is not able to is lower bounded by  $1-b$ . Moreover, agents in coalitions of size  $\kappa$  or with no incentive to switch coalitions will surely stay in the same coalition. The result now follows by noting that all agents' probabilistic actions are independent. ■

The next result establishes that one agent joining a coalition of at least its own current coalition's size is a timestep-event with a positive effect on the convergence of the overall network towards the goal coalition partition.

**Lemma V.4** *When exactly one agent joins a new coalition of at least its current coalition's size, this action strictly increases the function  $\Xi$  defined by*

$$\Xi(|C_1|, \dots, |C_N|) = \sum_{i \in \mathcal{A}} \frac{N^{2|C_i|}}{|C_i|}.$$

*Proof:* Let  $j$  be the agent changing coalitions,  $C_1$  be the coalition being joined,  $C_2$  the one being left, and so  $|C_1| \geq |C_2|$ . The net effect on  $\Xi$  of all of the agents in  $\text{id}(C_i)$  is  $N^{2|C_i|}$ , so the change in  $\Xi$  when  $j$  switches is

$$\Delta = \begin{cases} N^{2|C_1|+2} + N^{2|C_2|-2} - N^{2|C_1|} - N^{2|C_2|}, & |C_2| > 1, \\ N^{2|C_1|+2} - N^{2|C_1|} - N^{2|C_2|}, & |C_2| = 1. \end{cases}$$

In either case, using  $|C_1| \geq |C_2|$ , one can lower bound  $\Delta \geq N^{2|C_1|}(N^2 - 2)$ , which is strictly positive for all  $N > 1$ . ■

Our next step is to show that there exists a finite sequence of timestep-events leading to the goal coalition partition starting from any consistent partitioning.

**Proposition V.5** *From any consistent partitioning, there exists a finite sequence of timestep-events, each having a positive probability of occurring under the COALITION FORMATION AND DEPLOYMENT ALGORITHM, leading to the goal coalition partition. Furthermore, the length of this sequence is bounded by  $N^{2N}(\frac{\text{diam}(Q)}{\delta} + 1) + m$ .*

*Proof:* Initially, if any coalitions are larger than size  $\kappa$ , let the first timestep-event  $E_1$  be one where the correct number of agents leave one of these large coalitions and all other agents do not switch, creating a coalition of size  $\kappa$ . From Lemma V.3,  $P(E_1)$  is bounded away from zero. There can be at most  $m - 1$  more coalitions larger than size  $\kappa$ , and so  $E_2, \dots, E_m$  are defined similarly. From Step 15 in

Algorithm 3, within at most  $\frac{\text{diam}(Q)}{\delta}$  timesteps, each agent  $i$  will have a radius  $r_i$  satisfying Lemma IV.2, so at least one agent has an incentive to change coalitions. In the timesteps in which no agents wish to change coalitions, the corresponding timestep-events,  $E_{m+1}, \dots, E_{m+\alpha}$ ,  $\alpha \leq \frac{\text{diam}(Q)}{\delta}$ , occur with probability 1. Define  $E_{m+\alpha+1}$  to be a timestep-event where exactly one agent joins a coalition it has an incentive to and all others do not switch. By Lemma V.3, the probability of this event is bounded away from zero. Additionally, because all coalitions are at most size  $\kappa$ , the function  $\Xi$  increases by Lemma V.4. If the configuration is not in the goal coalition partition, within at most  $\frac{\text{diam}(Q)}{\delta}$  timesteps, at least one agent will have an incentive to switch coalitions. Because the integer-valued and upper-bounded function  $\Xi$  monotonically increases each time this sequence of timestep-events occurs, the number of times this can occur is at most  $N^{2N}$ . Within  $N^{2N}(\frac{\text{diam}(Q)}{\delta} + 1) + m$  timesteps, the agents will be in the goal coalition partition. ■

The following result shows that in finite time all agents are coincident with their coalitions and these coalitions form the goal coalition partition, with probability 1.

**Theorem V.6** *There exists a finite time after which  $N$  agents using the COALITION FORMATION AND DEPLOYMENT ALGORITHM are in the goal coalition partition with probability 1.*

*Proof:* Lemma V.3 asserts that the probability of a timestep-event occurring is lower bounded by  $\gamma = \min\{(1-b)^N, (1-(1-b)^{\frac{1}{N}})^N\}$ . Given an initial consistent partitioning, Proposition V.5 guarantees that there exists a finite sequence of timestep-events, whose length is upper bounded by  $L = N^{2N}(\frac{\text{diam}(Q)}{\delta} + 1) + m$ , leading to the goal coalition partition. If the length of this sequence is smaller than  $L$ , this sequence can be extended to one of exactly length  $L$  by considering additional timestep-events where no agents wish to change coalitions. The latter occur with probability 1. Therefore, the sequence of timestep-events leading to the goal coalition partition has a probability of occurring of at least  $\gamma^L$ , independent of the initial partitioning.

Define a sequence of events  $\{A_1, A_2, \dots\}$ , where  $A_n$  is the event that the coalitions do not exist after  $nL$  timesteps. The probability of  $A_n$  occurring is at most  $(1 - \gamma^L)^n$ . Now,

$$\sum_{n=1}^{\infty} A_n \leq \sum_{n=1}^{\infty} (1 - \gamma^L)^n < \infty,$$

since it corresponds to a convergent geometric series. Thus, by the Borel-Cantelli Lemma, cf. Lemma II.1,  $P(\{A_n \text{ i.o.}\}) = 0$ . This means  $P(\{A_n \text{ i.o.}\}^c) = 1$  or, equivalently,  $P(\{A_n^c \text{ a.a.}\}) = 1$ . The result follows by noting that  $A_n^c$  is the event that the coalitions occur at some point in  $nL$  timesteps and  $\{A_n^c \text{ a.a.}\}$  is the event that all but a finite number of events  $A_n^c$  occur. ■

We are now ready to prove Theorem V.1.

*Proof of Theorem V.1:* In statement (i), the fact that there exists, with probability 1, a finite time after which all agents are in the goal coalition partition follows from

Theorem V.6. Proposition A.2 allows to upper bound the number of timesteps it takes for the circumradius of one of these coalitions to vanish by  $\lceil \frac{\text{diam}(Q)}{d_{\min}} \rceil$ . This implies the fact that in finite time agents become coincident with its coalition members. Once coalitions form and all individual agents are coincident with the members of their respective coalitions, the collection of Voronoi cells that the agents compute correspond to a correct Voronoi partition with  $\overline{m}$  generators. Statement (ii) then follows from [6, Theorem 5.5]. ■

## VI. SIMULATIONS

This section presents several simulations of the COALITION FORMATION AND DEPLOYMENT ALGORITHM. We illustrate the convergence to the desired goal coalition partition and the achievement of the deployment task, and the robustness against agent addition and subtraction. We also pay attention to the number of timesteps required on average for coalition formation. We investigate the average coalition formation time as functions of  $N$ ,  $k$ , and  $b$ . Regarding (4), in all simulations where  $b$  is constant, we have chosen  $b = 0.5$ . In all simulations,  $\delta = d_{\max} = \frac{.2}{\sqrt{2}} \frac{\text{diam}(Q)}{\sqrt{N}}$ . We use the function

$$\phi(C_1, \dots, C_N) = \frac{1}{N(\kappa - 1)} \sum_{i \in \mathcal{A}} ||C_i| - \kappa|, \quad (5)$$

to illustrate the dynamics of coalition formation. This function is zero if and only if all agents are in  $\kappa$ -sized coalitions. Figure 2 shows an execution of the COALITION FORMATION AND DEPLOYMENT ALGORITHM with 20 agents forming coalitions of size 2. The network converges to both cor-

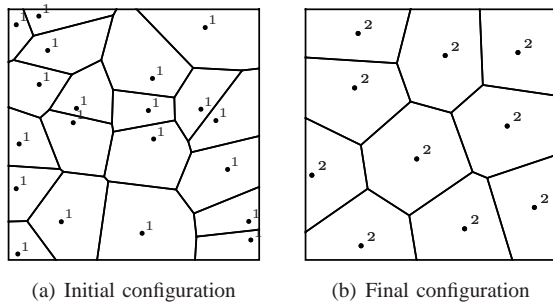


Fig. 2. Execution of the COALITION FORMATION AND DEPLOYMENT ALGORITHM with 20 agents and  $\kappa = 2$ . The network converges to a configuration where all agents are in correctly-sized coalitions and these coalitions are optimally deployed.

rectly sized groups and coalitions optimally deployed at their Voronoi cell's circumcenters. From Theorem V.1, the final configuration optimizes  $\mathcal{H}_{DC,10}$ . Figure 3(a) shows the number of coalition switches at each timestep for the same run. Many switches happen early, but decrease in frequency as agents form correctly sized coalitions. The evolution of  $\phi$  depicted in Figure 3(b) confirms this by showing how agents join more desirable coalitions over time. Figure 3(b) also shows the evolution of the objective function  $\mathcal{H}_{N,N-1}$  that, in the language of Section II-B, corresponds to the situation where  $N - 1$  of the sensors are working. This choice of function is motivated by the fact that, in one dimension, it is known that in such a case, forming coalitions of size 2

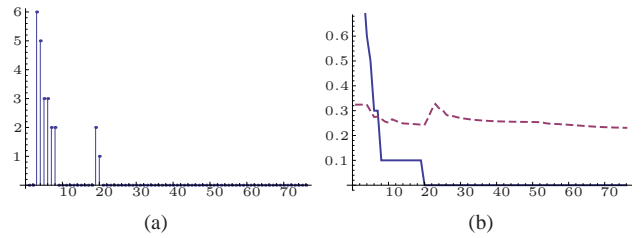


Fig. 3. For the execution in Figure 2, (a) shows the number of agents switching coalitions at each timestep, and (b) shows the evolution of  $\phi$  (solid line) as defined in (5) and  $\mathcal{H}_{20,19}$  (dashed line) as defined in (1).

is optimal [4]. The bumps in the evolution of  $\mathcal{H}_{20,19}$  in the plot occur when an agent has no nearby coalitions to join and increases its radius until it joins a group far away from it.  $\mathcal{H}_{20,19}$  temporarily increases while these agents get together. Figure 4 illustrates the robustness of the COALITION FORMATION AND DEPLOYMENT ALGORITHM. After agents have achieved the final optimal configuration seen in Figure 2(b), we let one agent fail and two new agents come into the picture. The agents adapt to the new network composition and optimally deploy according to the available resources.

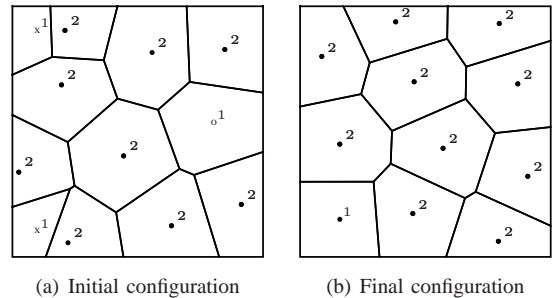


Fig. 4. Execution of the COALITION FORMATION AND DEPLOYMENT ALGORITHM from the configuration in Figure 2(b) where an agent has failed in the coalition marked as 'o' and two agents, marked as 'x', have been added. After these agent additions and subtractions, coalitions adapt and the network re-converges to an optimal deployment configuration.

Finally, Figure 5 illustrates the dependency of the average number of timesteps required for all coalitions to form on  $N$ ,  $\kappa$ , and  $b$ . Each point is the average of 200 runs, where the agents were initially randomly placed with uniform distribution in a unit square. The error bars correspond to plus and minus one standard deviation. Figure 5(a) shows the average coalition formation convergence time for different  $N$  for cases of fixed  $\kappa = 4$  and changing  $\kappa = \lfloor \frac{N}{2} \rfloor$ . In both cases, the completion time appears linear in  $N$  and each take a similar amount of time. The latter is corroborated in Figure 5(b), which shows the average coalition formation convergence time for fixed  $N = 20$  and varying  $\kappa$ . The coalition formation time is roughly equal for all desired coalition sizes, until nearly all agents are joining one coalition, which takes less time on average. Figure 5(c) shows the average coalition formation time for 20 agents forming coalitions of size 4 with various values for  $b$ . The completion time is roughly constant for values of  $b$  away from 0 and 1.

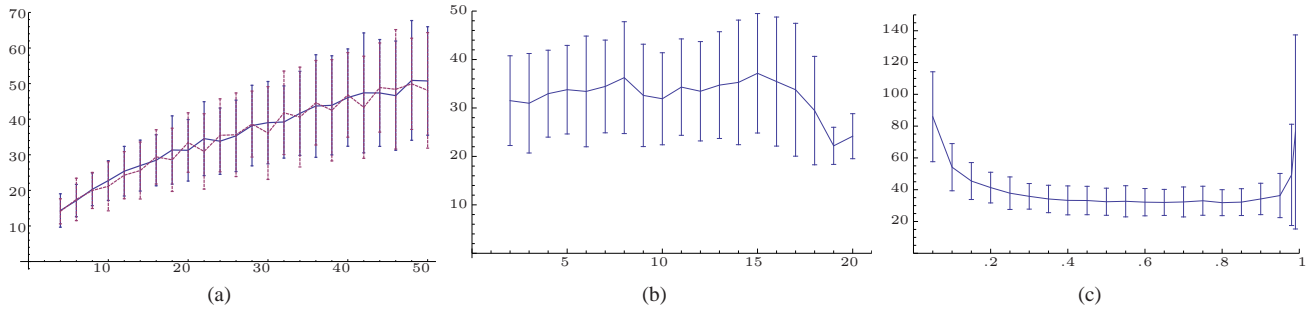


Fig. 5. (a) shows the average coalition formation time as a function of the number of agents  $N$ , for  $\kappa = 4$  (dashed line) and  $\kappa = \lfloor \frac{N}{2} \rfloor$  (solid line). (b) shows the average coalition formation time as a function of desired coalition size  $\kappa$  for  $N = 20$  agents. (c) shows the average coalition formation time for 20 agents forming coalitions of size 4 as a function of  $b$ . In all plots, the error bars correspond to plus and minus one standard deviation.

## VII. CONCLUSIONS

Motivated by a spatial estimation problem, we have designed a synchronous, distributed algorithm for a network of robotic agents to autonomously deploy over a given region in groups. Our strategy allows agents to autonomously form coalitions of a desired size, cluster together within finite time, and asymptotically reach an optimal deployment, each with probability 1. The algorithm design is a combination of a hedonic coalition formation game where agents only have partial information about other coalition memberships with motion coordination strategies for group clustering and deployment. The coalition formation game has probabilistic actions to avoid deadlock situations that may arise when agents act synchronously. The proposed algorithmic solution is provably correct, does not rely on a common reference frame and is robust to agents joining or leaving the environment. Simulations have illustrated these features along with the dependency of the average coalition formation time on  $N$ ,  $\kappa$ , and  $b$ . Future work will be devoted to analytically characterizing this time complexity, as well as investigating  $\delta$  policies which optimize the coalition formation process. We also plan to further explore the impact of noncooperative game-theoretic ideas in other motion coordination problems.

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## APPENDIX A PROPERTIES OF gttg

This appendix contains useful properties of gttg. Before stating these, we begin with a basic geometric fact.

**Lemma A.1** *Given  $d > 0$  and  $p_1, p_2, q \in \mathbb{R}^n$ , for  $i \in \{1, 2\}$ , let  $p_i^+ = \min\{\|q - p_i\|, d\} \text{ vr}(q - p_i) + p_i$ . Then  $\|p_1^+ - p_2^+\| \leq \|p_1 - p_2\|$ .*

Lemma A.1 is used in determining how much the circumradius of a coalition decreases and how much they get closer to the goal point  $q$  after moving according to gttg.

**Proposition A.2** *Given  $P = (p_1, \dots, p_k)$  and  $q \in Q$ , let  $P^+ = (p_1^+, \dots, p_k^+)$  be given by  $p_i^+ = \text{gttg}(p_i, P, q)$ ,  $i \in \{1, \dots, k\}$ . Then  $\text{CR}(P^+) \leq \text{CR}(P) - \delta_1$  and*

$$P^+ \subset B(q, \| \text{CC}(P) - q \| + \text{CR}(P) - \delta_1 - \delta_2),$$

*with  $\delta_1 = \max_{i \in \{1, \dots, k\}} \min\{\| \text{CC}(P) - p_i \|, d_1(r)\}$  and  $\delta_2 = \min\{\|q - \text{CC}(P)\|, d_2(r)\}$ .*