Stability of stochastic differential equations with additive persistent noise

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***Abstract*— We present a stability result for stochastic differential equations subject to additive persistent noise. Specifically, we propose a Lyapunov test for noise-to-state stability in pth moment with respect to an arbitrary subspace. To check the hypotheses of our result, we develop a method that exploits equivalence relations between positive semidefinite functions and a family of seminorms. With this method, we can translate inequalities between two positive semidefinite functions into separate sets of geometric conditions that relate each of them to a seminorm.**

I. Introduction

Stochastic differential equations (SDEs) go beyond ordinary differential equations (ODEs) to deal with systems subject to stochastic perturbations. Applications are numerous and include option pricing in the stock market, networked systems with noisy communication channels, and, in general, scenarios whose complexity cannot be captured by deterministic models. By additive persistent noise, we refer to the situation where the stochastic perturbations are present even at the equilibria of the underlying ODE and do not decay with time. Such scenarios arise, for instance, in control-affine systems when the input is corrupted by additive persistent noise. Our objective in this paper is to develop notions and tools to study the stability properties under the presence of additive persistent noise.

Literature review: In general, it is very difficult to obtain explicit descriptions of the solutions of SDEs. Fortunately, well-known Lyapunov techniques used to study the qualitative behavior of ODEs [1], [2] can be adapted to show stability properties of SDEs as well [3], [4], [5]. There are several types of stability results in SDEs depending on the notion of stochastic convergence that is being used. The works [4], [5], [6], [7] consider (asymptotic) stability in probability, almost sure (asymptotic) stability, and pth moment (asymptotic) stability. However, none of these notions are appropriate when additive persistent noise is present because they require the effect of the noise on the set of equilibria to vanish, or at least decay, with time. To deal with this, there is a family of related notions that establish a ultimate bound for the state of a system in terms of the size of the "disturbance". Rephrasing them, there exists a neighborhood of the set of equilibria that enjoys some stability property, and the size of this neighborhood depends on the size of the disturbance. This is ideally suited to study the stability properties of interconnected systems and cascade systems. In this context, Lyapunov techniques are widely used in controller design for the stabilization of such systems to the point that some stability concepts are inspired by dissipativity properties of Lyapunov functions. As an example, concepts like input-to-state stability (ISS) [8] go together with concepts like ISS-Lyapunov function, since the existence of the second implies the former (and in many cases a converse result is also true [8]). In this spirit, the notion of practical stochastic input-to-state stability (SISS) in [9] and [10] generalizes the concept of ISS [8] to systems represented by an SDE. As a particular case of SISS, when the input multiplies the covariance of the noise, we get the concept of noise-to-state-stability (NSS) [11], which focuses on the dissipative properties of a system in relation to the magnitude of the covariance of the noise. This is different from the ISS property for systems that are affine in input, because the stochastic integral against Brownian motion has infinite variation, whereas the integral of a legitimate input for ISS has finite variation.

Statement of contributions: The contributions of this paper are twofold. Our first contribution concerns the noise-to-state stability (NSS) properties of systems described by SDEs with additive persistent noise that only exhibit disturbance attenuation outside an arbitrary subspace. We introduce the concepts of NSS in pth moment, and of noise-dissipative and $p$thNSS-Lyapunov functions, all with respect to an arbitrary subspace. We show that noise-dissipative Lyapunov functions have NSS dynamics and, building on this fact, we establish that the existence of a $p$thNSS-Lyapunov function implies noise-to-state stability in pth moment with respect to an arbitrary subspace. Our second contribution is a methodology to check the hypotheses of the above results. We show that certain pairs of inequalities between positive semidefinite functions, which are refinements of the notion of proper function, give rise to equivalence relations. Building on this study, we characterize the relation between a positive semidefinite function and a family of seminorms, thereby allowing us to relate two positive semidefinite functions that are related to the same family of seminorms. These seminorms provide a way to measure the distance to the subspace considered in our first set of contributions. With this method, we can translate inequalities between two positive semidefinite functions, which are common in Lyapunov-type results, into separate sets of conditions that relate each of them to a seminorm. Most proofs are omitted for reasons of space and will appear elsewhere.

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Organization: The paper is organized as follows. Section II introduces preliminaries on seminorms, comparison functions, and SDEs. Section III presents the NSS stability result, and Section IV shows a methodology to help verify its hypotheses. Finally, Section V discusses our conclusions.

II. Preliminaries

This section outlines basic notation and notions on comparison functions and stochastic differential equations that are used throughout the paper.

A. Notational conventions

Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ be the sets of real and nonnegative real numbers, respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space. $\mathbb{R}^n$ is a vector space, i.e., for any $x, y \in \mathbb{R}^n$, and any $\lambda, \mu \in \mathbb{R}$, the linear combination $\lambda x + \mu y$ belongs to $\mathbb{R}^n$. A set $\mathcal{U} \subseteq \mathbb{R}^n$ is called a subspace if it is itself a vector space. Given a matrix $A \in \mathbb{R}^{n\times n}$, its nullspace $\mathcal{N}(A) \triangleq \{x \in \mathbb{R}^n : Ax = 0\}$ is a subspace. We denote by $\mathcal{C}(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ the set of positive semidefinite continuous functions defined on $\mathbb{R}^n$. A seminorm is a function $S : \mathbb{R}^n \to \mathbb{R}$ that is positively homogeneous, i.e., $S(\lambda x) = |\lambda| s(x)$ for any $\lambda \in \mathbb{R}$, and satisfies the triangular inequality, $S(x+y) \leq S(x) + S(y)$ for any $x, y \in \mathbb{R}^n$. From these properties can be deduced that $S \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}_{\geq 0})$. If, moreover, the function S is positive definite, i.e., $S(x) = 0$ implies $x = 0$, then S is called a norm. The Euclidean norm of $x \in \mathbb{R}^n$ is denoted by $\|x\|_2$, and the Frobenius norm of the matrix $A \in \mathbb{R}^{n\times m}$ is $|A|_{\mathcal{F}} \triangleq \sqrt{\operatorname{trace}(A^T A)} = \sqrt{\operatorname{trace}(AA^T)}$. For any matrix $A \in \mathbb{R}^{n\times m}$, the function $\|x\|_A \triangleq \|Ax\|_2$ is a seminorm because $\|\lambda x\|_A = \|A(\lambda x)\|_2 = \|\lambda Ax\|_2 = |\lambda| \|Ax\|_2 = |\lambda| \|x\|_A$ and $\|x + y\|_A = \|A(x + y)\|_2 = \|Ax + Ay\|_2 \leq \|Ax\|_2 + \|Ay\|_2 = \|x\|_A + \|y\|_A$.

We also denote by $\mathcal{C}^2(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ the family of all positive semidefinite functions V defined on $\mathbb{R}^n$ that are continuously twice differentiable. Given a function $\mathrm{V} \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}_{\geq 0})$, we denote its gradient by $\nabla \mathrm{V}$, and its Hessian by $\nabla^2 \mathrm{V}$. A function $f \in \mathcal{C}(\mathbb{R}; \mathbb{R}_{\geq 0})$ is said to be in $\mathcal{O}(s)$ as $s \to \infty$ if there exist $s_0 \in \mathbb{R}$ and $\kappa > 0$ such that $f(s) < \kappa s$ for all $s > s_0$. The operator $\sup$ denotes the supremum and $\operatorname{ess\,sup}$ denotes the essential supremum.

B. Class $\mathcal{K}$, $\mathcal{K}_\infty$, and convex and concave functions

We now introduce some classes of functions following [1, Page 144]. We will use them in this paper as a comparison tool, and their domain is required to be $\mathbb{R}_{\geq 0}$. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$, and it belongs to class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. Also, a continuous function $\mu : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{KL}$ if for each fixed $t \geq 0$, the function $\mu(., t)$ belongs to class $\mathcal{K}$, and for each fixed $s \geq 0$, the function $\mu(s, .)$ is decreasing and $\lim_{t\to\infty} \mu(s, t) = 0$. We will need some important facts: if α_1, α_2 belong to class $\mathcal{K}$, then the composition $\alpha_1 \circ \alpha_2$ belongs to class $\mathcal{K}$ (notation: $(\alpha_1 \circ \alpha_2)(x) \triangleq \alpha_1(\alpha_2(x))$); if α_3, α_4 belong to class $\mathcal{K}_\infty$, then the inverse function α_3^{-1} and the composition $\alpha_3 \circ \alpha_4$ belong to class $\mathcal{K}_\infty$.

A real-valued function f defined in a convex set X in a vector space is called convex if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda) f(y)$ for each $x, y \in X$ and for any $\lambda \in [0, 1]$, and is called concave if $-f$ is convex. By [12, Ex. 3.3], if $f : [a, b] \to [f(a), f(b)]$ is an increasing convex (respectively, concave) function, then the inverse function $f^{-1} : [f(a), f(b)] \to [a, b]$ is concave (respectively, convex). Also, following [12, Section 3], if $f, g : \mathbb{R} \to \mathbb{R}$ are convex (respectively, concave) and f is nondecreasing, then the composition $f \circ g$ is also convex (respectively, concave).

C. Stochastic differential equations

Here we review some basic notions on SDEs following the exposition in [6]; other useful references are [3], [13], [14]. Throughout the paper we assume that $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is a complete probability space, which is a probability measure $\mathbb{P}$ defined on the subsets of Ω that belong to the σ-algebra $\mathcal{F}$; those are the measurable events of the outcome space Ω. It is called complete because $\mathcal{F}$ contains all the subsets of Ω of probability 0. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ is a family of sub-σ-algebras of $\mathcal{F}$ satisfying $\mathcal{F}_t \subseteq \mathcal{F}_s \subseteq \mathcal{F}$ for any $0 \leq t < s < \infty$; we assume it is right continuous (i.e., $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$ for any $t \geq 0$) and $\mathcal{F}_0$ contains all the subsets of Ω of probability 0.

A one-dimensional Brownian motion $\mathrm{B} : \Omega \times [t_0, \infty) \to \mathbb{R}$ is an $\{\mathcal{F}_t\}$-adapted process such that $\mathrm{B}(\omega, t_0) = 0$ with probability 1 and the mapping $\mathrm{B}(\omega, .) : [t_0, \infty) \to \mathbb{R}$ is continuous also with probability 1 (the dependence on ω is usually omitted). Moreover, for $t_0 \leq s < t < \infty$, the increment $\mathrm{B}(t) - \mathrm{B}(s) : \Omega \to \mathbb{R}$ is independent of $\mathcal{F}_s$ (i.e., if we define the set $S_b \triangleq \{\omega \in \Omega : \mathrm{B}(\omega, t) - \mathrm{B}(\omega, s) \in (-\infty, b)\}$, for any $b \in \mathbb{R}$, then $\mathbb{P}(A \cap S_b) = \mathbb{P}(A)\mathbb{P}(S_b)$ for all $A \in \mathcal{F}_s$). In addition, $\mathrm{B}(t) - \mathrm{B}(s)$ is normally distributed with zero mean and variance $t - s$. An m-dimensional Brownian motion $\mathrm{B} : \Omega \times [t_0, \infty) \to \mathbb{R}^m$ is defined as $\mathrm{B}(\omega, t) = [\mathrm{B}_1(\omega, t), \ldots, \mathrm{B}_m(\omega, t)]^T$, where each B_i is a one-dimensional Brownian motion and $\{\mathrm{B}_1(t)\}, ..., \{\mathrm{B}_m(t)\}$ are independent random variables for each $t \geq t_0$.

In this paper we consider the n-dimensional SDE

$$\mathrm{d}x(t) = f(x(t), t)\mathrm{d}t + G(x(t), t)\Sigma(t)\mathrm{dB}(t), \qquad (1)$$

for all $t \in [t_0, \infty)$, that is notation for the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)\mathrm{d}s + \int_{t_0}^{t} G(x(s), s)\Sigma(s)\mathrm{dB}(s),$$

where the second integral is an stochastic integral [6, P. 18].

Assumption 2.1. The initial condition $x(t_0) = x_0$ belongs to $\mathbb{R}^n$; the functions $f : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$ and $G : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^{n\times q}$ are measurable; the function $\Sigma : [t_0, \infty) \to \mathbb{R}^{q\times m}$ is measurable and essentially locally bounded; and $\{\mathrm{B}(t)\}_{t\geq t_0}$ is an m-dimensional Brownian motion defined on the probability space. •

Given Assumption 2.1, the next regularity conditions [6, Page 58] guarantee global existence and uniqueness of solutions, with an additional justification regarding Σ, for each deterministic initial condition of the SDE (1).

Assumption 2.2. For every real number $T > t_0$ and each integer $n \geq 1$, there exists a positive constant $K_{T,n}$ such that for almost every $t \in [t_0, T]$ and all $x, y \in \mathbb{R}^n$ with $\max\{\|x\|_2, \|y\|_2\} \leq n$, the following bound holds:

$$\max\{\|f(x,t) - f(y,t)\|_2^2, |G(x,t) - G(y,t)|_{\mathcal{F}}^2\} \leq K_{T,n}\|x-y\|_2^2.$$

Assume also that for every $T > t_0$, there exists $K_T > 0$ such that for almost every $t \in [t_0, T]$ and all $x \in \mathbb{R}^n$,

$$x^T f(x,t) + \tfrac{1}{2}|G(x,t)|_{\mathcal{F}}^2 \leq K_T(1 + \|x\|_2^2). \qquad \bullet$$

Next we present a useful operator in stability analysis. Given $\mathrm{V} \in \mathcal{C}^2(\mathbb{R}^n; \mathbb{R}_{\geq 0})$, define the generator of the SDE (1) acting on the function V as the mapping $\mathcal{L}[\mathrm{V}] : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}$ given by

$$\mathcal{L}[\mathrm{V}](x,t) \triangleq \nabla \mathrm{V}(x)^T f(x,t) + \tfrac{1}{2}\operatorname{trace}\Big(\Sigma(t)^T G(x,t)^T \nabla^2 \mathrm{V}(x) G(x,t)\Sigma(t)\Big). \quad (2)$$

It can be shown that $\mathcal{L}[\mathrm{V}](x,t)$ gives the expected rate of change of V along a solution of the SDE (1) that passes through the point x at time t, so it is a generalization of the Lie derivative. According to [6, Th. 6.4], if we evaluate V along the solution $\{x(t)\}_{t \geq t_0}$ of the SDE (1) (that exists and is unique under Assumptions 2.1 and 2.2), then the process $\{\mathrm{V}(x(t))\}_{t \geq t_0}$ satisfies the SDE

$$\mathrm{V}(x(t)) = \mathrm{V}(x(t_0)) + \int_{t_0}^t \mathcal{L}[\mathrm{V}](x(s), s)\mathrm{d}s + I(t), \quad (3)$$

with $\mathrm{V}(x(t_0)) = \mathrm{V}(x_0)$, where

$$I(t) \triangleq \int_{t_0}^t \nabla \mathrm{V}(x(s))^T G(x(s), s)\Sigma(s)\mathrm{dB}(s). \quad (4)$$

Equation (3) is known as Itô formula and is the stochastic version of the chain rule.

III. Noise-to-state stability via noise-dissipative Lyapunov functions

The next definition generalizes the concept of noise-to-state stability in [11].

Definition 3.1. (Noise-to-state stability with respect to a subspace): The system (1) is *noise-to-state stable* (NSS) *in probability with respect to the subspace* $\mathcal{U}$ if for any $\epsilon > 0$ there exist a class $\mathcal{KL}$ function μ and a class $\mathcal{K}$ function θ, that might depend on ϵ, and a matrix $A \in \mathbb{R}^{n \times n}$ with $\mathcal{N}(A) = \mathcal{U}$, such that

$$\mathbb{P}\Big\{\|x(t)\|_A > \mu(\|x_0\|_A, t - t_0) + \theta\Big(\operatorname*{ess\,sup}_{t_0 \leq s \leq t} |\Sigma(s)|_{\mathcal{F}}\Big)\Big\} \leq \epsilon, \quad (5)$$

for all $t > t_0$ and any $x_0 \in \mathbb{R}^n$. Similarly, the system (1) is *pth moment noise-to-state stable* (*p*thNSS) *with respect to the subspace* $\mathcal{U}$ if there exists a class $\mathcal{KL}$ function μ, a class $\mathcal{K}$ function θ and a matrix $A \in \mathbb{R}^{n \times n}$ with $\mathcal{N}(A) = \mathcal{U}$ such that

$$\mathbb{E}\big[\|x(t)\|_A^p\big] \leq \mu(\|x_0\|_A, t - t_0) + \theta\Big(\operatorname*{ess\,sup}_{t_0 \leq s \leq t} |\Sigma(s)|_{\mathcal{F}}\Big) \quad (6)$$

for all $t > t_0$ and any $x_0 \in \mathbb{R}^n$.

The quantity $|\Sigma(t)|_{\mathcal{F}} = \sqrt{\operatorname{trace}\big(\Sigma(t)\Sigma(t)^T\big)}$ is a measure of the size of the noise because it is related to the infinitesimal covariance $\Sigma(t)\Sigma(t)^T$. Also, the definition above does not depend on the choice of the matrix $A \in \mathbb{R}^{n \times n}$ because for two matrices $A, B \in \mathbb{R}^{n \times n}$ with $\mathcal{N}(A) = \mathcal{N}(B) = \mathcal{U}$, the seminorms are equivalent, i.e., there are constants $\gamma_1, \gamma_2 > 0$ such that $\gamma_1\|x\|_A \leq \|x\|_B \leq \gamma_2\|x\|_A$ for all $x \in \mathbb{R}^n$.

Remark 3.2. (NSS is not a particular case of ISS): The concept of NSS is not a particular case of input-to-state stability (ISS) for systems that are affine in the input, namely,

$$\dot{x} = f(x,t) + G(x,t)u(t), \quad (7)$$

where $u : [t_0, \infty) \to \mathbb{R}^q$ is measurable and essentially locally bounded. The reason is the following: on the one hand, the integral form of (7) is driven by the function $\int_{t_0}^t G(x(s), s)u(s)\,\mathrm{d}s$, whose coordinates are differentiable almost everywhere by the Lebesgue fundamental theorem of calculus [15, P. 289], and thus they are absolutely continuous [15, P. 292] and have bounded variation [15, Prop. 8.5]. On the other hand, the driving disturbance of the system (1), at any time $t_k(t)$ previous to the first exit of $x(t)$ from any arbitrarily large ball $\{x \in \mathbb{R}^n : \|x\|_2 \leq k\}$, is the function $\int_{t_0}^{t_k(t)} G(x(s), s)\Sigma(s)\mathrm{dB}(s)$ whose ith coordinate has quadratic variation [6, Th. 5.14] equal to

$$\int_{t_0}^{t_k(t)} \sum_{j=1}^m |\sum_{l=1}^q G(x(s), s)_{il}\Sigma(s)_{lj}|^2 \mathrm{d}s,$$

which is larger than zero, and so does not have bounded variation. That is, the driving disturbance that we are considering is of a different kind. $\bullet$

Our first goal is to derive the noise-to-state stability properties in Definition 3.1 via a Lyapunov-type result. To do this we look at the dissipativity properties of a special kind of energy functions along the solutions of the SDE (1).

Definition 3.3. (Noise-dissipative Lyapunov functions): We say that V in $\mathcal{C}^2(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ is a *noise-dissipative Lyapunov function* for the SDE (1) if there exist $\mathrm{W} \in \mathcal{C}(\mathbb{R}^n; \mathbb{R}_{\geq 0})$ and $\sigma \in \mathcal{K}$ such that the following dissipation inequality holds:

$$\mathcal{L}[\mathrm{V}](x,t) \leq -\mathrm{W}(x) + \sigma\big(|\Sigma(t)|_{\mathcal{F}}\big) \quad (8)$$

for all $(x,t) \in \mathbb{R}^n \times [t_0, \infty)$, where

$$\mathrm{V}(x) \leq \tilde{\alpha}(\mathrm{W}(x)) \qquad \forall x \in \mathbb{R}^n, \quad (9)$$

for some concave function $\tilde{\alpha} \in \mathcal{K}_\infty$.

Remark 3.4. (Itô formula and exponential dissipativity): Interestingly, the conditions (8) and (9) are equivalent to

$$\mathcal{L}[\mathrm{V}](x,t) \leq -\tilde{\alpha}^{-1}(\mathrm{V}(x)) + \sigma\big(|\Sigma(t)|_{\mathcal{F}}\big), \qquad (10)$$

for all $x \in \mathbb{R}^n$, where $\tilde{\alpha}^{-1} \in \mathcal{K}_\infty$ is convex. Note that, since $\mathcal{L}[\mathrm{V}]$ is not the Lie derivative of V (because it has a term with the Hessian of V), one cannot directly conclude from (10) that there exists some continuously twice differentiable function $\tilde{\mathrm{V}}$ such that

$$\mathcal{L}[\tilde{\mathrm{V}}](x,t) \leq -c\tilde{\mathrm{V}}(x) + \tilde{\sigma}\big(|\Sigma(t)|_{\mathcal{F}}\big),$$

as it is the case in [16] in the context of ISS. •

The next result is motivated by [17, Th. 4.1]: we extend it to positive semidefinite Lyapunov functions and relax the condition on $\mathcal{L}[\mathrm{V}]$.

Theorem 3.5. (Noise-dissipative Lyapunov functions have an NSS dynamics): Under Assumptions 2.1 and 2.2, suppose that V is a noise-dissipative Lyapunov function for the SDE (1). Then there exists a class $\mathcal{KL}$ function $\tilde{\mu}$ such that

$$\begin{aligned}\mathbb{E}\big[\mathrm{V}(x(t))\big] \ &\leq \tilde{\mu}\big(\mathrm{V}(x_0), t-t_0\big) \\ &\quad + \tilde{\alpha}\Big(2\,\sigma\big(\operatorname*{ess\,sup}_{t_0\leq s\leq t}|\Sigma(s)|_{\mathcal{F}}\big)\Big) \qquad (11)\end{aligned}$$

for all $t \geq t_0$, where σ and α are as in Definition 3.3.

Of particular interest is the case when the function V is lower and upper bounded by class $\mathcal{K}_\infty$ functions of a seminorm.

Definition 3.6. (NSS-Lyapunov functions): We say that V in $\mathcal{C}^2(\mathbb{R}^n;\mathbb{R}_{\geq 0})$ is a *NSS-Lyapunov function in probability with respect to* $\mathcal{U}$ for the SDE (1) if V is a noise dissipative Lyapunov function and, in addition, there is a matrix $A \in \mathbb{R}^{n\times n}$ with $\mathcal{N}(A) = \mathcal{U}$ such that, for some $\alpha_1,\, \alpha_2 \in \mathcal{K}_\infty$,

$$\alpha_1(\|x\|_A^p) \leq \mathrm{V}(x) \leq \alpha_2(\|x\|_A^p) \qquad \forall x \in \mathbb{R}^n. \qquad (12)$$

(The power p is irrelevant when we just care about class $\mathcal{K}_\infty$ functions, but it does make a difference if we impose α_1 to be convex as we are about to do next.) If, moreover, α_1 is convex in (12), we say that V is a *pth moment NSS-Lyapunov function with respect to* $\mathcal{U}$.

As in Definition 3.1, the particular choice of the matrix A is irrelevant. The dissipativity property dictated by the above Lyapunov function gives rise to the following stability result for SDEs with additive persistent noise.

Corollary 3.7. (NSS-Lyapunov function implies NSS with respect to a subspace): Under Assumptions 2.1 and 2.2, suppose that V is a NSS-Lyapunov function in probability with respect to the subspace $\mathcal{U}$ for the SDE (1). Then,

(i) the system (1) is NSS in probability with respect to the subspace $\mathcal{U}$ with gain functions

$$\begin{aligned}\mu(r,s) \ &\triangleq \alpha_1^{-1}\big(\tfrac{2}{\epsilon}\tilde{\mu}(\alpha_2(r),s)\big), \\ \theta(r) \ &\triangleq \alpha_1^{-1}\big(\tfrac{2}{\epsilon}\tilde{\alpha}(2\sigma(r))\big). \qquad (13)\end{aligned}$$

(ii) Moreover, if V is a pth moment NSS-Lyapunov function, then the system (1) is pth moment NSS with respect to the subspace $\mathcal{U}$ with gain functions μ and θ as in (13) setting $\epsilon = 1$.

Proof. (i) Since $\alpha_1(\|x\|_A^p) \leq \mathrm{V}(x)$ for all $x \in \mathbb{R}^n$, with $\alpha_1 \in \mathcal{K}_\infty$, then for any $\hat{\rho} > 0$ and $t \geq t_0$ we have that

$$\begin{aligned}&\mathbb{P}\Big\{\|x(t)\|_A^p > \hat{\rho}\Big\} = \mathbb{P}\Big\{\alpha_1(\|x(t)\|_A^p) > \alpha_1(\hat{\rho})\Big\} \\ &\leq \mathbb{P}\Big\{\mathrm{V}(x(t)) > \alpha_1(\hat{\rho})\Big\} \leq \frac{\mathbb{E}\big[\mathrm{V}(x(t))\big]}{\alpha_1(\hat{\rho})} \\ &\leq \frac{1}{\alpha_1(\hat{\rho})}\Bigg(\tilde{\mu}\Big(\alpha_2(\|x_0\|_A),\, t-t_0\Big) \\ &\qquad + \tilde{\alpha}\Big(2\sigma\big(\operatorname*{ess\,sup}_{t_0\leq s\leq t}|\Sigma(s)|_{\mathcal{F}}\big)\Big)\Bigg), \qquad (14)\end{aligned}$$

where we have used the strict monotonicity of α_1 in the first equation, Chebyshev's inequality [18, Chapter 3] in the second inequality, and the upper bound for $\mathbb{E}\big[\mathrm{V}(x(t))\big]$ obtained in (11) in the last inequality as well as the monotonicity of $\tilde{\mu}$ in the first argument and the fact that $\mathrm{V}(x) \leq \alpha_2(\|x\|_A)$. Also, for any function $\alpha \in \mathcal{K}$ we have that $\alpha(2r) + \alpha(2s) \geq \alpha(r+s)$ for all $r, s \geq 0$. Thus,

$$\begin{aligned}\rho(\epsilon, x_0, t) \ &\triangleq \mu\big(\|x_0\|_A, t-t_0\big) + \theta\Big(\operatorname*{ess\,sup}_{t_0\leq s\leq t}|\Sigma(s)|_{\mathcal{F}}\Big) \\ &\geq \alpha_1^{-1}\Bigg(\frac{1}{\epsilon}\tilde{\mu}\Big(\alpha_2(\|x_0\|_A), t-t_0\Big) \\ &\qquad + \frac{1}{\epsilon}\tilde{\alpha}\Big(2\sigma\big(\operatorname*{ess\,sup}_{t_0\leq s\leq t}|\Sigma(s)|_{\mathcal{F}}\big)\Big)\Bigg) \triangleq \hat{\rho}(\epsilon), \quad (15)\end{aligned}$$

and the result follows substituting in (14) $\hat{\rho} \triangleq \hat{\rho}(\epsilon)$ because, by (15), $\mathbb{P}\big\{\|x(t)\|_A^p > \rho(\epsilon, x_0, t)\big\} \leq \mathbb{P}\big\{\|x(t)\|_A^p > \hat{\rho}(\epsilon)\big\}$.

(ii) Since α_1^{-1} is concave, applying Jensen's inequality [18, Chapter 3] we get that

$$\begin{aligned}\mathbb{E}\big[\|x(t)\|_A^p\big] \ &\leq \mathbb{E}\big[\alpha_1^{-1}\big(\mathrm{V}(x(t))\big)\big] \leq \alpha_1^{-1}\Big(\mathbb{E}\big[\mathrm{V}(x(t))\big]\Big) \\ &\leq \hat{\rho}(1) \leq \rho(1, x_0, t),\end{aligned}$$

where in the last two inequalities we have used the bound for $\mathbb{E}\big[\mathrm{V}(x(t))\big]$ in (14) and the definition of $\hat{\rho}(\epsilon)$ in (15). □

IV. Positive semidefinite functions and seminorms

We have two goals in this section: first, we show that the inequalities in (12) can be regarded as an equivalence relation between the candidate function V and $\|.\|_A^p$. This is true in general when two positive semidefinite functions are related in this way by class $\mathcal{K}_\infty$ functions. Second, we characterize the functions V that verify (12), including the case in which the class $\mathcal{K}_\infty$ functions are convex and concave, respectively. These ideas culminate in a framework to help verify the conditions (9) and (12) that appear in the definition of noise-dissipative Lyapunov function, NSS-Lyapunov function in probability, and pth moment NSS-Lyapunov function.

A. Equivalence classes that are useful in stability analysis

The following relations point towards some refinements of the notion of proper function which plays an important role in stability properties like input-to-state stability (ISS) and integral input-to-state stability (iISS) [19].

Definition 4.1. (Proper functions with respect to each other): Consider a set $\mathcal{D} \subseteq \mathbb{R}^n$ (which can be thought to be $\mathbb{R}^n$) and let V and W in $\mathcal{C}(\mathcal{D}\,;\mathbb{R}_{\geq 0})$ be such that

$$\alpha_1(\mathrm{W}(x)) \leq \mathrm{V}(x) \leq \alpha_2(\mathrm{W}(x)) \qquad \forall x \in \mathcal{D}, \tag{16}$$

for some functions α_1 and α_2 in $\mathcal{C}(\mathbb{R}_{\geq 0};\mathbb{R}_{\geq 0})$.

(i) When $\alpha_1, \alpha_2 \in \mathcal{K}$, we say that V is *$\mathcal{K}$-dominated by* W, and we write $\mathrm{V} \lhd^{\mathcal{K}} \mathrm{W}$ in $\mathcal{D}$.
(ii) When $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, we say that V and W are *$\mathcal{K}_\infty$-proper with respect to each other*, and we write $\mathrm{V} \sim^{\mathcal{K}_\infty} \mathrm{W}$ in $\mathcal{D}$.
(iii) When $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ can be taken convex and concave respectively, we say that V and W are *$\mathcal{K}_\infty^{cc}$-proper with respect to each other*, and we write $\mathrm{V} \sim^{\mathcal{K}_\infty^{cc}} \mathrm{W}$ in $\mathcal{D}$.
(iv) When $\alpha_1(s) \triangleq \gamma_1 s$ and $\alpha_2(s) \triangleq \gamma_2 s$ in $\mathbb{R}_{\geq 0}$ for some constants $\gamma_1, \gamma_2 > 0$, we say that V and W are *equivalent*, and we write $\mathrm{V} \sim \mathrm{W}$ in $\mathcal{D}$.

Remark 4.2. (Refinements of the notion of proper function): When $\mathcal{D}$ is a neighborhood of 0, W is the two-norm, and α_1, α_2 belong to the classes $\mathcal{K}$ or $\mathcal{K}_\infty$, we recover the well-known concept of V being a *proper* function [1]. Whereas the relation $\sim^{\mathcal{K}_\infty}$ is relevant for ISS, iISS and NSS in probability, the relation $\sim^{\mathcal{K}_\infty^{cc}}$ is important for NSS in pth moment. As we saw in Section III, this is because the inequalities in $\sim^{\mathcal{K}_\infty^{cc}}$ are preserved if we evaluate V and W along a stochastic process and take expectations. •

Remark 4.3. (The relations are nested): Given V and W in $\mathcal{C}(\mathcal{D}\,;\mathbb{R}_{\geq 0})$, the following chain of implications hold in $\mathcal{D}$:

$$\mathrm{V} \sim \mathrm{W} \;\Rightarrow\; \mathrm{V} \sim^{\mathcal{K}_\infty^{cc}} \mathrm{W} \;\Rightarrow\; \mathrm{V} \sim^{\mathcal{K}_\infty} \mathrm{W} \;\Rightarrow\; \mathrm{V} \lhd^{\mathcal{K}} \mathrm{W}. \quad \bullet$$

In the case when $\mathcal{D} = \mathbb{R}^n$, and V and W are seminorms, the relationship $\sim$ gives rise to the well-known concept of equivalent seminorms.

Lemma 4.4. (Powers of seminorms with the same nullspace): Let A and B in $\mathbb{R}^{n\times n}$ be two nonzero matrices with the same nullspace, $\mathcal{N}(A) = \mathcal{N}(B)$. Then $\|.\|_A^p \sim \|.\|_B^p$ and $\|.\|_A^p \sim^{\mathcal{K}_\infty} \|.\|_B^q$ in $\mathbb{R}^n$ for any real numbers $p, q > 0$.

Inspired by the fact that the relationship $\sim$ is an equivalence relation, we present the next generalization.

Proposition 4.5. (Equivalence relations): The relationships $\sim^{\mathcal{K}_\infty}$ and $\sim^{\mathcal{K}_\infty^{cc}}$ in any set $\mathcal{D} \subseteq \mathbb{R}^n$ are both equivalence relations.

Proof. We have to show three properties. For convenience, we represent both relations by $\sim^*$. We first derive the algebraic constructions regarding each of the three properties, and after that we justify how the specific requirements for both relations $\sim^{\mathcal{K}_\infty}$ and $\sim^{\mathcal{K}_\infty^{cc}}$ are met.

It is *reflexive*, i.e., $\mathrm{V} \sim^* \mathrm{V}$. This follows taking $\alpha_1(s) = \alpha_2(s) = s$ in $\mathbb{R}_{\geq 0}$, which belongs to $\mathcal{K}_\infty$ and is both convex and concave.

It is *symmetric*, i.e., $\mathrm{V} \sim^* \mathrm{W} \Leftrightarrow \mathrm{W} \sim^* \mathrm{V}$. This follows because if the inequalities $\alpha_1 \circ \mathrm{W} \leq \mathrm{V} \leq \alpha_2 \circ \mathrm{W}$ hold in $\mathcal{D}$, then $\alpha_2^{-1} \circ \mathrm{V} \leq \mathrm{W} \leq \alpha_1^{-1} \circ \mathrm{V}$ also hold in $\mathcal{D}$.

It is *transitive*, i.e., $\mathrm{U} \sim^* \mathrm{V}$ and $\mathrm{V} \sim^* \mathrm{W} \Rightarrow \mathrm{U} \sim^* \mathrm{W}$. This follows because if the inequalities $\alpha_1 \circ \mathrm{V} \leq \mathrm{U} \leq \alpha_2 \circ \mathrm{V}$ and $\tilde{\alpha}_1 \circ \mathrm{W} \leq \mathrm{V} \leq \tilde{\alpha}_2 \circ \mathrm{W}$ hold in $\mathcal{D}$, then the inequalities $\alpha_1 \circ \tilde{\alpha}_1 \circ \mathrm{W} \leq \mathrm{U} \leq \alpha_2 \circ \tilde{\alpha}_2 \circ \mathrm{W}$ also hold in $\mathcal{D}$.

Taking the inverse of a function or the composition of two functions preserves the class $\mathcal{K}_\infty$ as explained in Section II-B. Thus, the constructions above are valid for the relation $\sim^{\mathcal{K}_\infty}$. On the other hand, convexity and concavity that are required for the relation $\sim^{\mathcal{K}_\infty^{cc}}$ are also guaranteed because, as a consequence of Section II-B, if $\alpha \in \mathcal{K}_\infty$ is convex (respectively, concave), then $\alpha^{-1} \in \mathcal{K}_\infty$ is concave (respectively, convex). Also, if $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ are both convex (respectively, concave), then the compositions $\alpha_1 \circ \alpha_2$ and $\alpha_2 \circ \alpha_1$ belong to $\mathcal{K}_\infty$ and are convex (respectively, concave). □

Remark 4.6. ($\lhd^{\mathcal{K}}$ is also transitive): The proof above also shows that the relation $\lhd^{\mathcal{K}}$ is reflexive and transitive. •

B. Characterizing $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{K}_\infty^{cc}$-proper functions with respect to seminorms

Here we generalize the characterization of proper function in [1, Lemma 4.3]. Let $\mathrm{V} \in \mathcal{C}(\mathcal{D}\,;\mathbb{R}_{\geq 0})$, where $\mathcal{D} \subset \mathbb{R}^n$. Given a real number $p \geq 1$ and a nonzero matrix $A \in \mathbb{R}^{n\times n}$, consider the functions $\phi_{p,A}, \psi_{p,A} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$\phi_{p,A}(s) \triangleq \sup_{\{x\in\mathcal{D}\,:\,\|x\|_A^p \leq s\}} \mathrm{V}(x),$$
$$\psi_{p,A}(s) \triangleq \inf_{\{x\in\mathcal{D}\,:\,\|x\|_A^p \geq s\}} \mathrm{V}(x).$$

Using these auxiliary functions, we make a list with the hypotheses that appear in our characterizations of $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{K}_\infty^{cc}$ - proper functions with respect to the seminorm $\|.\|_A$.

H0$_A$. The set $\mathcal{D}$ contains $\mathcal{N}(A)$ and the set $\{x \in \mathcal{D} : \|x\|_A \geq s\}$ is nonempty for every $s \geq 0$.

H1$_A$. The nullset of V is equal to $\mathcal{N}(A)$.

H2$_A$. The function $\phi_{1,A}$ is locally bounded and right continuous at 0, and $\psi_{1,A}$ is positive definite.

H3$_A$. The next limit holds: $\lim_{s\to\infty} \psi_{1,A} = \infty$.

H4$_{A,p}$. The asymptotic behavior of $\phi_{p,A}$ and $\psi_{p,A}$ is such that $\phi_{p,A}(s)$ and $s^2/\psi_{p,A}(s)$ are both in $\mathcal{O}(s)$ as $s \to \infty$.

The next result generalizes [1, Lemma 4.3].

Proposition 4.7. (Characterizations): Consider a function V in $\mathcal{C}(\mathcal{D}\,;\mathbb{R}_{\geq 0})$ that satisfies H0$_A$. Then

(i) V satisfies $\{\mathrm{H}i_A\}_{i=1}^2 \Leftrightarrow \mathrm{V} \lhd^{\mathcal{K}} \|.\|_A$ in $\mathcal{D}$;
(ii) V satisfies $\{\mathrm{H}i_A\}_{i=1}^3 \Leftrightarrow \mathrm{V} \sim^{\mathcal{K}_\infty} \|.\|_A$ in $\mathcal{D}$;
(iii) V satisfies $\{\mathrm{H}i_{A,p}\}_{i=1}^4 \Leftrightarrow \mathrm{V} \sim^{\mathcal{K}_\infty^{cc}} \|.\|_A^p$ in $\mathcal{D}$.

The following result culminates our efforts to show a route to guarantee the inequalities in (9) and (12) in the definition

of dissipative Lyapunov function, NSS-Lyapunov function in probability, and pth moment NSS-Lyapunov function.

Theorem 4.8. (A bridge between positive semidefinite functions): Let V and W in $\mathcal{C}(\mathcal{D};\mathbb{R}_{\geq 0})$ verify the hypotheses H0-H3 for two matrices $A, \tilde{A} \in \mathbb{R}^{n\times n}$, respectively, with $\mathcal{N}(A) = \mathcal{N}(\tilde{A})$. Then for any $q > 0$ the following relations hold in $\mathcal{D}$:

$$\mathrm{V} \sim^{\mathcal{K}_\infty} \mathrm{W}, \quad \text{and} \quad \mathrm{V} \sim^{\mathcal{K}_\infty} \|.\|_A^q, \quad \text{and} \quad \mathrm{W} \sim^{\mathcal{K}_\infty} \|.\|_{\tilde{A}}^q.$$

If, moreover, V and W verify H4 for the same matrices A and $\tilde{A}$, respectively, and some $p \geq 1$, then the following relations hold in $\mathcal{D}$:

$$\mathrm{V} \sim^{\mathcal{K}_\infty^{cc}} \mathrm{W}, \quad \text{and} \quad \mathrm{V} \sim^{\mathcal{K}_\infty^{cc}} \|.\|_A^p, \quad \text{and} \quad \mathrm{W} \sim^{\mathcal{K}_\infty^{cc}} \|.\|_{\tilde{A}}^p.$$

Remark 4.9. (Quadratic forms in a constrained domain): Sometimes is more convenient to re-write the functions V and W in Definition 3.3 as $\hat{\mathrm{V}} \in \mathcal{C}^2(\mathcal{D};\mathbb{R}_{\geq 0})$ and $\hat{\mathrm{W}} \in \mathcal{C}(\mathcal{D};\mathbb{R}_{\geq 0})$ for an appropriate set $\mathcal{D} \subseteq \mathbb{R}^m$ with $m \geq n$. For instance, this scenario arises if the functions V and W can be re-written as quadratic forms in a constrained set in an extended Euclidean space. In that case, the equivalence $\hat{\mathrm{V}} \sim^{\mathcal{K}_\infty^{cc}} \hat{\mathrm{W}}$ in $\mathcal{D}$ implies condition (9) of Definition 3.3, and $\hat{\mathrm{V}} \sim^{\mathcal{K}_\infty^{cc}} \|.\|_A^p$ in $\mathcal{D}$ implies condition (12) of Definition 3.6.•

Based on the previous observation, we present another version of Corollary 3.7 with assumptions that might be easier to verify in some scenarios.

Corollary 4.10. (Sufficient conditions for pth moment NSS): Consider the SDE (1) under Assumptions 2.1 and 2.2. Suppose that there exist $\mathrm{V} \in \mathcal{C}^2(\mathbb{R}^n;\mathbb{R}_{\geq 0})$, $\mathrm{W} \in \mathcal{C}(\mathbb{R}^n;\mathbb{R}_{\geq 0})$ and $\sigma \in \mathcal{K}$ such that the dissipation inequality (8) holds. Further assume that V and W can be written as $\hat{\mathrm{V}} \in \mathcal{C}^2(\mathcal{D};\mathbb{R}_{\geq 0})$ and $\hat{\mathrm{W}} \in \mathcal{C}(\mathcal{D};\mathbb{R}_{\geq 0})$, respectively, for some set $\mathcal{D} \subseteq \mathbb{R}^m$ with $m \geq n$. Then the following implications hold:

(i) If $\hat{\mathrm{V}}$ and $\hat{\mathrm{W}}$ satisfy the hypotheses H0-H3 for two matrices A and $\tilde{A}$ in $\mathbb{R}^{m\times m}$ with $\mathcal{N}(A) = \mathcal{N}(\tilde{A})$, then the system (1) is NSS in probability with respect to the subspace $\mathcal{N}(A)$.

(ii) If, moreover, $\hat{\mathrm{V}}$ and $\hat{\mathrm{W}}$ satisfy H4 for the same matrices A and $\tilde{A}$, respectively, and some $p \geq 1$, then the system (1) is pth moment NSS with respect to the subspace $\mathcal{N}(A)$.

V. Conclusions

We have proposed the concept of pth moment NSS-Lyapunov function, with respect to an arbitrary subspace, and shown its usefulness to establish a stability property of a class of systems represented by SDEs subject to additive persistent noise. Our noise-to-state stability result provides an ultimate bound, depending on the size of the disturbance, for the expectation of the pth power of a seminorm of the state. This bound is achieved regardless of the possibility that some subspace of the states accumulates the variance of the noise. This is a meaningful stability property for the aforementioned class of systems because the presence of persistent noise makes it impossible to establish a stochastic notion of asymptotic stability for the set of equilibria of the underlying differential equation. We have also identified the inequalities in the assumptions of our stability result as pieces of equivalence relations, and we have characterized those relations with respect to a family of seminorms. This has allowed us to reformulate part of the hypotheses required by our noise-to-state stability in terms of seminorms. Future work will include characterizing the overshoot gain and considering the effect of delays and impulsive right hand sides in SDEs under this framework.

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