Distributed estimation of internal wave parameters via inter-drogue distances

Michael Ouimet   Jorge Cortés

Abstract— **This paper considers a group of drogues estimating the physical parameters of the ocean's linear internal waves. While underwater, individual drogues do not have access to position information and instead rely on inter-drogue distance measurements. We introduce the PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD for determining the parameters of a horizontally propagating internal ocean wave and show that, for noiseless measurements, our strategy determines the parameters exactly. In the presence of additive Gaussian noise, we characterize precisely the robustness of our strategy and bound the error in the estimated parameters. Finally, we define two strategies for aggregating estimates obtained across multiple time instants. Several simulations illustrate our results.**

I. INTRODUCTION

Internal waves are waves that propagate within a fluid, rather than on its surface. They correspond to moving sinusoidal oscillations in the boundary surface between two layers of a stratified fluid. In this paper, we consider internal waves along an ocean pycnocline, which is the surface of constant density where the vertical rate of change in density is largest. Because pycnoclines are typically deep below the ocean surface, collecting data about internal waves from the ocean surface is difficult. Our aim is to design an algorithm that runs on a group of drogues drifting underwater near the internal wave's interface to determine the wave's physical parameters. A drogue is a Lagrangian drifter capable of actuating its depth by changing its buoyancy. Because exact location information is unavailable below the surface, we rely only on inter-drogue distance measurements. Figure 1 presents a pictorial illustration of our problem setup.

Literature review: Internal waves are capable of displacing mass, such as plankton, as they travel, and this makes studying them relevant to oceanographers [1], [2], [3]. Scientists widely use drogues drifting passively as monitoring platforms to gather relevant ocean data [4], [5], [6]. Recent work [7] explores the possibility of actively selecting tidal currents so that drogues can autonomously reach a desired destination. An increasing body of work in the controls literature deals with cooperative networks of agents estimating spatial natural phenomena, including ocean [8], [9], river [10], and hurricane sampling [11]. In these scenarios, agents with limited actuation capabilities are subject to strong flowfields. In the problem considered in this paper, drogues are able to actuate their depth through buoyancy changes, but are completely subject to the force of the internal wave in the flow-wise direction. Because of this, the task of determining the wave parameters can be seen as a data fitting

The authors are with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA 92093, USA, {miouimet,cortes}@ucsd.edu

problem [12], [13]. Due to the periodic nature of the inter-drogue distance trajectories, our problem has connections with least-squares spectral analysis problems [14], [15]. In general, however, the fact that the wave parameters appear nonlinearly makes determining them challenging.
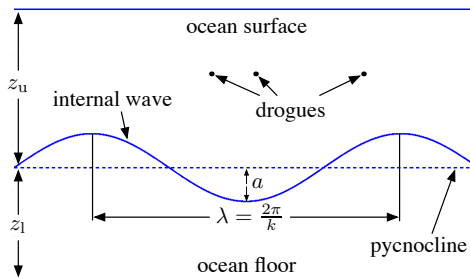

Fig. 1. Horizontally propagating ocean internal wave. The plot shows a vertical cross-section of the ocean perpendicular to the wave propagation direction. A group of drogues float at a constant depth (but not necessarily in a straight line) and do not have access to exact location information. Our objective is to provide drogues with mechanisms that rely only on relative distances to determine the parameters that uniquely define the internal wave.

Statement of contributions: We design a provably correct algorithm for estimating the physical parameters that define an internal wave. This algorithm, PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD, relies on waiting until an inter-drogue distance derivative is momentarily close to zero. We prove that for an interval in times near the zero derivative time, each parameter's value can be precisely captured. This also allows us to give a bound on the minimum sampling rate needed for the algorithm to correctly identify the parameters. Next, for the horizontal wavenumber k, we bound the difference between the estimated and true value when the inter-drogue distances are corrupted by additive Gaussian noise. We also derive a second-order approximation of the k estimates that PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD provides for noisy measurements. We use this approximation to create aggregation strategies, FIRST-ORDER and SECOND-ORDER k FUSION, that fuse estimates obtained across multiple time instants. Finally, simulations illustrate the evolution of these two schemes.

II. PRELIMINARIES

Here we present some basic concepts, starting with notational conventions. Let $\mathbb{R}$ and $\mathbb{Z}$ denote the sets of real and integer numbers, respectively. For $x \in \mathbb{R}$, let $\lfloor x \rfloor \in \mathbb{Z}$ denote the largest integer that satisfies $\lfloor x \rfloor \leq x$. For $f : \mathbb{R}^d \to \mathbb{R}$, let $\partial_k f$ denote the partial derivative of f with respect to the kth component. We let $\Sigma_{\text{fixed}} = (p_{\text{fixed}}, \{e_{x_{\text{fixed}}}, e_{y_{\text{fixed}}}, e_{z_{\text{fixed}}}\})$ be a reference frame in $\mathbb{R}^3$ fixed at point p_{fixed}. A point q and a vector v expressed with respect to the frame

Σ_{fixed} are denoted by q^{fixed} and $v^{\text{fixed}} = (v^{x_{\text{fixed}}}, v^{y_{\text{fixed}}}, v^{z_{\text{fixed}}})$, respectively. Next, we let $\Sigma_b = (p_b, \{e_{x_b}, e_{y_b}, e_{z_b}\})$ be a reference frame fixed to a moving body with origin p_b. When expressed with respect to Σ_{fixed}, we denote this point by p_b^{fixed}. The orientation of Σ_b is defined by the rotation matrix R_b^{fixed}, whose columns are $\{e_{x_b}, e_{y_b}, e_{z_b}\}$ expressed with respect to Σ_{fixed}. Finally, a change of reference frame is given by

$$q^{\text{fixed}} = R_b^{\text{fixed}} q^b + p_b^{\text{fixed}}, \quad v^{\text{fixed}} = R_b^{\text{fixed}} v^b.$$

A. Derivative estimation from noisy data

We estimate an analytic function f and $\dot{f}$ from evenly sampled measurements corrupted by additive Gaussian noise

$$y_i = f(t_i) + \epsilon(t_i), \quad \epsilon \sim \mathcal{N}(0, \sigma^2).$$

We choose a polynomial smoothing filter approach [16]. This allows us to justify that the derivative estimates are unbiased and Gaussian. For $p \ll m \in \mathbb{Z}_{\geq 1}$, we construct a p-th order polynomial filter from m evenly spaced noisy measurements over the sampling window T, $\{(\frac{-i}{m-1}T, y_i)\}_{i \in \{0, \ldots, m-1\}}$.

$$
\begin{bmatrix} y_0 \\ \vdots \\ y_{m-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & t_1 & \cdots & t_1^p \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_{m-1} & \cdots & t_{m-1}^p \end{bmatrix} \begin{bmatrix} f(0) \\ f'(0) \\ \vdots \\ \frac{f^{(p)}(0)}{p!} \end{bmatrix}
$$
$$
+ \begin{bmatrix} 0 \\ \sum_{j=p+1}^{\infty} \frac{f^{(j)}(0)}{j!} t_1^j \\ \vdots \\ \sum_{j=p+1}^{\infty} \frac{f^{(j)}(0)}{j!} t_{m-1}^j \end{bmatrix} + \begin{bmatrix} \epsilon_0 \\ \vdots \\ \epsilon_{m-1} \end{bmatrix}
$$

More compactly, this can be written as

$$Y = VF + \epsilon_{\text{bias}} + \epsilon_{\text{random}}.$$

Thus, the least-squares estimate for $\dot{f}(0)$ is the second component of the vector F:

$$\dot{f}'(0) = ((V^T V)^{-1} V^T (Y + \epsilon_{\text{bias}} + \epsilon_{\text{random}}))_2$$
$$= \dot{f}(0) + (V^T V)^{-1} V^T (\epsilon_{\text{bias}} + \epsilon_{\text{random}})_2.$$

Remark 2.1 (Bias): We ignore the bias which arises from considering only the p-th order expansion of f because for a fixed p, the bias can be made arbitrarily close to zero by choosing the sampling window T small enough. •

After accounting for the bias, the estimate of $\dot{f}'(0)$ is an unbiased Gaussian variable with variance $\sigma_{\dot{f}}^2 = ||((V^T V)^{-1} V^T)_{i=2, j=1:p}||^2 \sigma^2$.

III. PROBLEM STATEMENT

In this section we describe the basics of linear internal waves, our model for drogues' capabilities and their dynamics when their motion is governed by internal waves, and the description of the problem we seek to solve.

A. Internal wave model

We define the global reference frame as follows: $\Sigma_g = (p, \{e_x, e_y, e_z\})$. The origin p corresponds to any point at the surface of the water. The basis vector e_x corresponds to the direction of wave propagation, which is parallel to the ocean bottom, and e_z is perpendicular to the ocean bottom, pointing from ocean bottom to surface. As depicted in Figure 1, we consider an internal wave with frequency ω, propagating horizontally in the e_x-direction with horizontal wavenumber k, with amplitude a, and with the pycnocline at mean depth $-z_u$. The depth of the wave as a function of x and t is

$$z_w(t, x) = -z_u - a \sin(kx - \omega t + \phi),$$

where the phase of the wave ϕ defines the wave relative to the reference $(x, t) = (0, 0)$. From [1], [2], the horizontal and vertical velocities (u, w) of the upper layer are

$$u_u(t, x) = \frac{ca}{z_u} \sin(kx - \omega t + \phi),$$
$$w_u(t, x, z) = -\frac{za\omega}{z_u} \cos(kx - \omega t + \phi),$$

where c is the phase speed given by $c = \frac{\omega}{k}$. Additionally, we denote the total phase as $v = kx - \omega t + \phi$. This model comes from the assumption that vertical velocity varies linearly with depth, coupled with the conservation of mass law for an incompressible fluid. Likewise, the lower-layer velocities are

$$u_l(t, x) = -\frac{ca}{z_l} \sin(kx - \omega t + \phi),$$
$$w_l(t, x, z) = \frac{z + z_u + z_l}{z_l} a\omega \cos(kx - \omega t + \phi).$$

Remark 3.1 (Assumptions on wave parameters): The linear internal wave model is only meaningful for $|\frac{a}{z_u}| < 1$. Additionally, the spatial wavelengths of internal waves range from hundreds of meters to tens of kilometers [3]. Since k is inversely proportional to the spatial wavelength, we assume that there exists $[k_{\min}, k_{\max}]$ containing k. •

B. Drogue model and dynamics

A drogue is an untethered, submersible buoy which drifts freely in the ocean. The drogues we consider can change their depth by controlling their buoyancy. A drogue measures the relative distance and orientation to other drogues using acoustics and an onboard compass. However, it cannot measure absolute position because GPS is unavailable underwater. For drogue i, we define the local coordinate system $\Sigma_i = (p_i, \{e_{x_i}, e_{y_i}, e_{z_i}\}$. The origin p_i corresponds to the location of the ith drogue. Like the global coordinate frame, $e_{z_i} = e_z$. However, the vectors e_{x_i} and e_{y_i} are parallel to the ocean floor, but neither is necessarily oriented in the direction of e_x, the wave propagation direction. Thus, each drogue i must determine θ_i, the angle between e_{x_i} and e_x. We assume each drogue can measure inter-drogue distances at a sampling rate of f_s. At time t drogue i has measurements of $\{(d_{i,z}^{x_i}(\frac{\kappa}{f_s}), d_{i,z}^{y_i}(\frac{\kappa}{f_s}))\}_{\kappa \in \{0, \ldots, \lfloor f_s t \rfloor\}}$ and $z \in \{j, l, m, n\}$ where drogues j, l, m, and n are the ones closest to i.

Given distance d, we let d' denote the measurement of d by an drogue, with the following Gaussian error model

$$d' = d + \epsilon_d, \quad \epsilon_d \sim \mathcal{N}(0, \sigma^2).$$

Consider the scenario where drogues move in the upper layer of the internal wave at a constant depth. There is no loss of generality here, since drogues can control their depth through buoyancy changes. We also assume that the drogue dynamics under the linear internal wave is Lagrangian. In other words, the dynamics of the drogue position $\mathbf{p} = (p^x, p^y, p^z)$ in the global reference frame is given by

$$\dot{p} = (\dot{p}^x, \dot{p}^y, \dot{p}^z) = (u_\mathrm{u}(t, p^x), 0, 0). \tag{1}$$

The lack of motion in the z-direction is due to the drogue's buoyancy control, which we assume is capable of counteracting the vertical forcing of the internal wave. Figure 2 illustrates the evolution of the x-component of inter-drogue distances as a function of the initial phase of the wave.
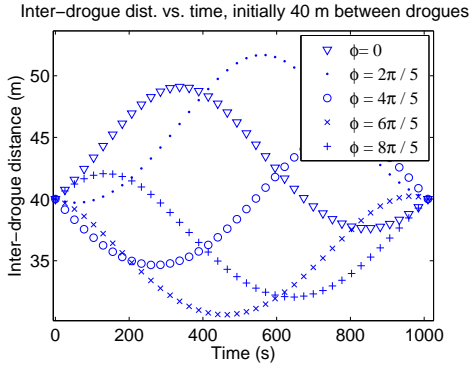
Fig. 2. Inter-drogue distance evolution for drogues initially 40 meters apart, with different phases relative to the wave.

Problem Statement 1: A team of N drogues is deployed in the ocean and their motion is governed by an internal wave. Since the drogues may control their depth, assume all are located at the same depth and each one can measure the relative distance and orientation to the closest M drogues in their own coordinate frame. The objective is to design an algorithm that allows the drogues to collectively determine the parameters $\frac{a}{z_u}$, k, ω, and θ_i defining the internal wave.

IV. NOISE-FREE PARAMETER ESTIMATION

Now we describe a method for determining all the internal wave parameters in the absence of measurement noise, beginning the wave propagation direction. Next, we show that inter-drogue distances are periodic in time and explicitly determine the fundamental period. After that, we devise a method for determining k. Given k, the other two parameters are determined simultaneous. Finally, we gather the methods into one algorithm.

A. Determination of wave propagation direction

We consider the case where the drogues are at the same depth but arbitrarily located. Figure 3 depicts drogue i's local coordinates, inter-drogue distance measurements, and the direction of wave propagation.

Consider the inter-drogue distance to the jth drogue as measured by i, $i \neq j \in \{1, \ldots, N\}$, in its local coordinates

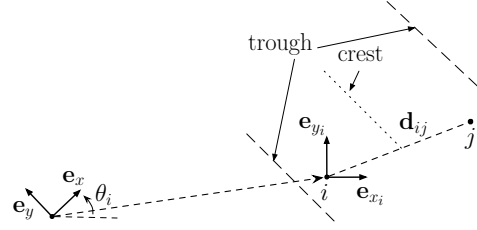$$d_{i,j} = p_j - p_i = (d_{i,j}^{x_i}, d_{i,j}^{y_i}, 0).$$

Fig. 3. Illustration of drogue and wave orientations on relative frame.

For drogues undergoing motion purely caused by an internal wave, inter-drogue distances in their local reference frame can be projected onto the global reference frame $d_{i,j}^g = R_i^g d_{i,j}$ via the transformation matrix R_i^g,

$$R_i^g = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 \\ \sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The global coordinate frame is useful because the inter-drogue distance in the e_y direction is constant, i.e., $\dot{d}_{i,j}^y = 0$. Since θ_i is constant, it can easily be found using the measurements available,

$$\dot{d}_{i,j}^y = \dot{d}_{i,j}^{x_i} \sin\theta_i + \dot{d}_{i,j}^{y_i} \cos\theta_i = 0,$$

$$\theta_i = \arctan \frac{-\dot{d}_{i,j}^{y_i}}{\dot{d}_{i,j}^{x_i}}.$$

B. Fundamental period of inter-drogue distance time-series

In this section, we analytically determine the fundamental period of the inter-drogue distance time-series.

Lemma 4.1 (Drogue trajectories): The solution of (1) starting from $p(0)$ is

$$p^x(t) = c(1 - \beta)t + \psi(t),$$
$$\psi(t) = \frac{2}{k} \arctan\left(\frac{a}{z_u} - \beta \tan\left(\frac{\beta}{2}(\omega t + \gamma_0) \right) \right)$$
$$- \frac{2\pi}{k} \left\lfloor \frac{t}{T} + \frac{\gamma_0}{\omega T} + \frac{1}{2} \right\rfloor + c\beta t - \frac{\phi}{k},$$

where the period is $T = \frac{2\pi}{\omega\beta}$, $\beta = \sqrt{1 - \left(\frac{a}{z_u}\right)^2}$, and

$$\gamma_0 = \frac{2}{\beta} \arctan\left(\frac{1}{\beta} \left(\frac{a}{z_u} - \tan\left(\frac{kp^x(0) + \phi}{2} \right) \right) \right).$$

From Lemma 4.1, the solution of (1) is the sum of a linear function in t and a periodic function ψ with fundamental period T. Since the linear function does not depend on the initial condition, the inter-drogue distance evolution is periodic with period T,

$$d_{i,j}(t) = \frac{2}{k} \arctan\left(\frac{a}{z_u} - \beta \tan\left(\frac{\beta}{2}(\omega t + \gamma_{0,j}) \right) \right)$$
$$- \frac{2}{k} \arctan\left(\frac{a}{z_u} - \beta \tan\left(\frac{\beta}{2}(\omega t + \gamma_{0,i}) \right) \right)$$
$$- \frac{2\pi}{k} \left\lfloor \frac{t}{T} + \frac{\gamma_{0,j}}{\omega T} + \frac{1}{2} \right\rfloor + \frac{2\pi}{k} \left\lfloor \frac{t}{T} + \frac{\gamma_{0,i}}{\omega T} + \frac{1}{2} \right\rfloor.$$

C. Zero-derivative method and algorithm description

Now we detail a method for determining the wave parameters k, ω, and $\frac{a}{z_u}$, beginning with k. After deriving situations when k can be determined, we leverage this knowledge to determine the other two parameters ω and $\frac{a}{z_u}$ simultaneously.

1) Inter-drogue motion: Once the wave propagation direction is known, the drogues can write their motion in the global coordinate frame. Since the only motion is in the wave propagation direction, we can drop the superindex on distances and positions, considering only distances in the x-direction. According to (1), for N drogues, for any $i \in \{1, \ldots, N\}$, the following dynamics completely describe the drogues' motion in the e_x direction:

$$\dot{d}_{i,j} = 2\frac{\omega a}{kz_u}\sin(\frac{kd_{i,j}}{2})\cos(\frac{kd_{i,j}}{2} + v_i), \quad \forall i \neq j \quad (2a)$$

$$\dot{v}_i = \omega(\frac{a}{z_u}\sin(v_i) - 1), \quad (2b)$$

with $v_i = kp_i^x - \omega t + \phi$.

2) Determination of k: We note that each inter-drogue distance equation in (2a) contains 3 unknowns: v_i, $\frac{\omega a}{kz_u}$, and k. Since k is one of the parameters of interest, our aim in this section is to first create an equation which is only a function of measurement data and k, and then solve for the value of k that fits the measurement data. We first begin by noting that the ratio of two of these (2a) equations eliminates $\frac{\omega a}{kz_u}$.

Secondly, we can write v_i explicitly at any time instance in terms of any two inter-drogue distances, say $d_{i,j}$ and $d_{i,l}$, their derivatives $\dot{d}_{i,j}$ and $\dot{d}_{i,l}$, and k:

$$v_i(k, d_{i,j}, d_{i,l}, \dot{d}_{i,j}, \dot{d}_{i,l}) =$$
$$\text{atan}\left(\frac{\dot{d}_{i,j}\sin(\frac{kd_{i,l}}{2})\cos(\frac{kd_{i,l}}{2}) - \dot{d}_{i,l}\sin(\frac{kd_{i,j}}{2})\cos(\frac{kd_{i,j}}{2})}{\dot{d}_{i,j}\sin^2(\frac{kd_{i,l}}{2}) - \dot{d}_{i,l}\sin^2(\frac{kd_{i,j}}{2})}\right). \quad (3)$$

Thus, by substituting this expression for v_i into a ratio of inter-drogue distance equations other than i, j or i, l, say i, m and i, n, we've created an equation with only one unknown: k. We call this function the distance rate quotient:

$$\text{drq}(k, X) = \frac{\sin(\frac{kd_{i,m}}{2})\cos(\frac{kd_{i,m}}{2} + v_i(k, d_{i,j}, d_{i,l}, \dot{d}_{i,j}, \dot{d}_{i,l}))}{\sin(\frac{kd_{i,n}}{2})\cos(\frac{kd_{i,n}}{2} + v_i(k, d_{i,j}, d_{i,l}, \dot{d}_{i,j}, \dot{d}_{i,l}))}$$

where $X = (d_{i,j}, d_{i,l}, d_{i,m}, d_{i,n}, \dot{d}_{i,j}, \dot{d}_{i,l}, \dot{d}_{i,m}, \dot{d}_{i,n})$ is the collection of all 4 inter-drogue distances and their derivatives.

Introducing the function kfind as

$$\text{kfind}(k, X) =$$
$$\text{drq}(k, v_i(k, d_{i,j}, d_{i,l}, \dot{d}_{i,j}, \dot{d}_{i,l}), d_{i,m}, d_{i,n}) - \frac{\dot{d}_{i,m}}{\dot{d}_{i,n}} \quad (4)$$

we can see that by definition for $k' = k$,

$$\text{kfind}(k', X) = 0. \quad (5)$$

The goal is now to determine a condition for when only the true k satisfies (5). To aid in this, we investigate the structure of v_i. We note that drq is a complicated trigonometric function of k due to the structure of v_i, so we wish to

determine a time when v_i has a simpler form; it is when $\dot{d}_{i,j} = 0$. From the interpretation of Lemma 4.1, we note that $d_{i,j}$ is bounded and periodic. We also know that for any $t \in \mathbb{R}_{>0}$, $\dot{v}_i(t) < -\omega(1 - \frac{a}{z_u}) < 0$ because it is physically impossible for $\frac{a}{z_u} \geq 1$. Given these facts it is clear from (2a) that there exist times within one period T, one of which is t_{crit} when $\dot{d}_{i,j}(t_{\text{crit}}) = 0$. We assume that $0 < |kd_{i,j}| < 2\pi$, so $v_i(t_{\text{crit}}) = \pm\frac{\pi}{2} - \frac{k}{2}d_{i,j}(t_{\text{crit}})$.

Now, using this fact about v_i, we are almost ready to state the next lemma which establishes when the true k is the unique value that satisfies (5) and can be found. This is established for a neighborhood in $\frac{d_{i,j}}{d_{i,l}}$ around 0. However, before doing so, we introduce some useful notation,

$$\epsilon_{\max}(x, C_1, C_2) = \quad (6)$$
$$= \max_{\gamma \in (0, \Gamma(x))} \begin{cases} R(x, \gamma, C_1, C_2) & \frac{\pi}{4} > x - \gamma > 0, \\ \frac{x - \frac{1}{2}}{C_2} & 2\pi > x - \gamma > \frac{\pi}{4}, \end{cases}$$

where $\Gamma(x) = x - \sin(x)\cos(x)$ and

$$R(x, \gamma, C_1, C_2) = \min\{\frac{\frac{1}{2}\arcsin(2(x - \gamma)) - x}{C_1}, \frac{\gamma}{C_2}\}.$$

Lemma 4.2 (Neighborhood around zero-derivative): For noiseless distance and its derivative measurements, where

$$k \in [0, \frac{2\pi}{d_{i,n}}), \; 0 < d_{i,j} < d_{i,l} < d_{i,m} < d_{i,n}, \; |\dot{d}_{i,j}/\dot{d}_{i,l}| < \delta,$$

$$\delta = \min\left\{\frac{\sin(\frac{kd_{i,j}}{2})\sin(\frac{k}{2}(d_{i,n} - d_{i,j}))}{\sin(\frac{kd_{i,l}}{2})\sin(\frac{k}{2}(d_{i,n} - d_{i,l}))}, \right.$$
$$\left. \epsilon_{\max}(\frac{k(d_{i,m} - d_{i,j})}{2}, L_1, L_2)\right\},$$

and L_1 and L_2 are Lipschitz constants with respect to $\frac{\dot{d}_{i,j}}{\dot{d}_{i,l}}$ for v and $\partial_k v$, respectively, the true value of k can be uniquely determined.

We generalize this result to determine an open interval around t_{crit} such that the true k can be uniquely determined.

Corollary 4.3 (Neighborhood around t_{crit}): Given noiseless distance and derivative measurements X at t_{crit} with

$$k \in [0, \frac{2\pi}{d_{i,n}}), \; 0 < d_{i,j} < d_{i,l} < d_{i,m} < d_{i,n}, \; \dot{d}_{i,j}(t_{\text{crit}}) = 0.$$

Then, the true value of k can be uniquely determined using measurements $X(t)$, for any $t \in (t_{\text{crit}} - \Delta_{\max}, t_{\text{crit}} + \Delta_{\max})$ where $\Delta_{\max} = \frac{L_4}{L_3}\frac{\delta}{1+\delta}$, δ is defined in Lemma 4.2, $L_3 = 4(\frac{\omega a}{kz_u})^2 k + 2\frac{\omega a}{kz_u}\omega(1 + \frac{a}{z_u})$, and $0 < L_4 \leq |\dot{d}_{i,l}(t_{\text{crit}})|$.
The following remark describes the minimum sampling rate for drogues to find k.

Remark 4.4: (Minimum sampling rate) Using Corollary 4.3 and a lowerbound on inter-drogue distances, one can find a sampling rate $f_{s,\min}$ guaranteeing that the drogues are able to determine k. ●

3) Method for finding $\frac{a}{z_u}$ and ω: With k known, $\frac{a}{z_u}$ and ω can be found with the following method. We can calculate v_i from (3) now. From these measurements of v_i, we can construct $\dot{v}_i$ using the method outlined in Section II-A.

Lemma 4.5 (Determination of $\frac{a}{z_u}$ and ω): For any t_1 and t_2 such that $t_2 - t_1 < T$ and $\sin(v_i(t_1)) \neq \sin(v_i(t_2))$, then ω and $\frac{a}{z_u}$ can be determined from

$$\omega = -\beta_2, \quad \frac{a}{z_u} = \frac{-\beta_1}{\beta_2}, \quad \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \sin(v_i(t_1)) & 1 \\ \sin(v_i(t_2)) & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot{v}_i(t_1) \\ \dot{v}_i(t_2) \end{bmatrix}$$

4) Algorithm and its correctness: Finally, we gather the methods above into Algorithm 1.

Algorithm 1: PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD.

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1 Calculate wave propagation direction, $\theta_i = \arctan \frac{-d_{i,j}^{y_i}}{d_{i,j}^{x_i}}$

2 if $\left| \frac{\dot{d}_{i,j}(t_\kappa)}{\dot{d}_{i,j_2}(t_\kappa)} \right| < \delta$ **then**

3 $\quad$ Solve kfind$(k, X) = 0$

4 $\quad$ For any t_1, t_2 with $t_2 - t_1 < T$ and $\sin(v_i(t_1)) \neq \sin(v_i(t_2))$, calculate

5 $\quad \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \sin(v_i(t_1)) & 1 \\ \sin(v_i(t_2)) & 1 \end{bmatrix}^{-1} \begin{bmatrix} \dot{v}_i(t_1) \\ \dot{v}_i(t_2) \end{bmatrix}$

6 $\quad \omega = -\beta_2, \quad \frac{a}{z_u} = \frac{-\beta_1}{\beta_2}$

7 end

The next result establishes the correctness of the PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD. It is a straightforward application of Corollary 4.3 and Lemma 4.5.

Proposition 4.6 (Algorithm correctness): Using noiseless distance and distance derivative measurements, drogue i can exactly determine the parameters θ_i, $\frac{a}{z_u}$, ω, and k using the PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD.

V. ROBUSTNESS ANALYSIS

In this section we bound the error in our algorithm's k estimation in terms of the errors in the measurements needed and then define methods of aggregating k estimates.

A. Error bound

With noiseless measurements X, one finds the unique k satisfying (5). As described in Section III, we assume an additive Gaussian error model for the measured inter-drogue distances. Using trigonometric identities, we can describe the noisy drq in terms of the noiseless drq.

Lemma 5.1 (Relating noisy drq to noiseless drq): The function drq evaluated at noisy measurements can be explicitly described as drq evaluated at the true values times a function of k, the true values, and the noise:

$$\mathrm{drq}(k, y', z', v') = \mathrm{drq}(k, y, z, v) G(k, y, z, v, \epsilon_y, \epsilon_z, \epsilon_v)$$

where

$$G(k, y, z, v, \epsilon_y, \epsilon_z, \epsilon_v) = \frac{\cos(\epsilon_y) + \cot(\frac{ky}{2})\sin(\epsilon_y)}{\cos(\epsilon_z) + \cot(\frac{kz}{2})\sin(\epsilon_z)}$$
$$\cdot \frac{\cos(\epsilon_y + \epsilon_v) + \tan(\frac{ky}{2})\sin(\epsilon_y + \epsilon_v)}{\cos(\epsilon_z + \epsilon_v) + \tan(\frac{kz}{2})\sin(\epsilon_z + \epsilon_v)}$$

We use the previous lemma to construct a bound on the difference between the true k and our estimated one.

Lemma 5.2 (Bound on error in k): The error between the estimated k and the true k can be bounded by the following function of the noise:

$$|k' - k| \leq \frac{\epsilon_{\frac{\dot{y}}{z}} + \frac{\dot{y}}{z}(1 - G(k, y, z, v, \epsilon_y, \epsilon_z, \epsilon_v))}{L_{\min}}$$

and $L_{\min} = \min_k \partial_k \mathrm{drq}(k, y', z', v')$.

B. Estimate aggregation

Given that measurements of inter-drogue distances and their derivatives are corrupted by additive Gaussian noise, we first verify that PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD is able to find a k estimate from noisy measurements. We do this by showing that there exists an analytic implicit function $\mathbf{k}$ which solves (5). Given that $\mathbf{k}$ is implicitly defined and the nonlinearity of (4), we construct approximations for $\mathbf{k}$. Finally, we use these approximations to define methods for aggregating estimates of k. However, for interests of space, we jump right to the result which determines the approximations for $\mathbf{k}$. Proving the existence of the analytic implicit function simply amounts to finding the domain where both kfind is analytic and its partial derivative with respect to k is non-zero.

Lemma 5.3 (Approximations of $\mathbf{k}$): The 1-st and 2-nd order approximation of $\mathbf{k}$ are

$$\mathrm{k}^{\mathrm{frst}}(X', X) = k + J_X(X' - X),$$
$$\mathrm{k}^{\mathrm{second}}(X', X) = k + J_X(X' - X) + \frac{1}{2}(X' - X)H_X(X' - X),$$

respectively, where

$$J_X = \frac{-D_X}{B_X}, \quad H_X = \frac{E_X}{B_X} - \frac{D_X C_X^T + C_X D_X^T}{B_X^2} + \frac{A_X D_X D_X^T}{B_X^3}$$

and $A_X = \partial_k^2 \mathrm{kfind}(k, X)$, $B_X = \partial_k \mathrm{kfind}(k, X)$, $C_X = \partial_k \partial_X \mathrm{kfind}(k, X)$, $D_X = \partial_X \mathrm{kfind}(k, X)$, and $E_X = \partial_X \partial_X \mathrm{kfind}(k, X)$.

Next, we introduce an aggregation scheme that we employ for aggregating k measurements. Given independent random variables x_1 and x_2 with mean $\mathrm{E}[x_1] = \mathrm{E}[x_2] = \mu$ and variances $\mathrm{Var}[x_1] = \sigma_1^2$, $\mathrm{Var}[x_2] = \sigma_2^2$, define the optimal aggregating function OptAgg by

$$\mathrm{OptAgg}(x_1, \sigma_1^2, x_2, \sigma_2^2) = \left(\frac{\sigma_2^2 x_1 + \sigma_1^2 x_2}{\sigma_1^2 + \sigma_2^2}, \frac{\sigma_1^2 \sigma_2^2}{\sigma_1^2 + \sigma_2^2} \right).$$

Note that $\frac{\sigma_2^2 x_1 + \sigma_1^2 x_2}{\sigma_1^2 + \sigma_2^2}$ is the convex combination of x_1 and x_2 with smallest variance. $\{(\hat{k}_i, \hat{X}_i), i \in \mathbb{N}\}$ is a sequence of random variables with $\hat{k}_i$ as the k estimate of our algorithm for the noisy data $\hat{X}_i$. We call the following iterative aggregation process FIRST-ORDER K FUSION:

$$(\mathrm{k}^{\mathrm{first}}_{\mathrm{agg}\ i+1}, \mathrm{Var}[\mathrm{k}^{\mathrm{first}}_{\mathrm{agg}\ i+1}]) =$$
$$\mathrm{OptAgg}(\mathrm{k}^{\mathrm{first}}_{\mathrm{agg}\ i}, \mathrm{Var}[\mathrm{k}^{\mathrm{first}}_{\mathrm{agg}\ i}], \hat{k}_{i+1}, \mathrm{Var}[\mathrm{k}^{\mathrm{frst}}(\hat{X}_{i+1})])$$

and this one SECOND-ORDER K FUSION:

$$(\mathrm{k}^{\mathrm{scnd}}_{\mathrm{agg}\ i+1}, \mathrm{Var}[\mathrm{k}^{\mathrm{scnd}}_{\mathrm{agg}\ i+1}]) =$$
$$\mathrm{OptAgg}(\mathrm{k}^{\mathrm{scnd}}_{\mathrm{agg}\ i}, \mathrm{Var}[\mathrm{k}^{\mathrm{scnd}}_{\mathrm{agg}\ i}],$$
$$\hat{k}_{i+1} - \mathrm{E}[\mathrm{k}^{\mathrm{second}}(\hat{X}_{i+1})], \mathrm{Var}[\mathrm{k}^{\mathrm{second}}(\hat{X}_{i+1})]).$$

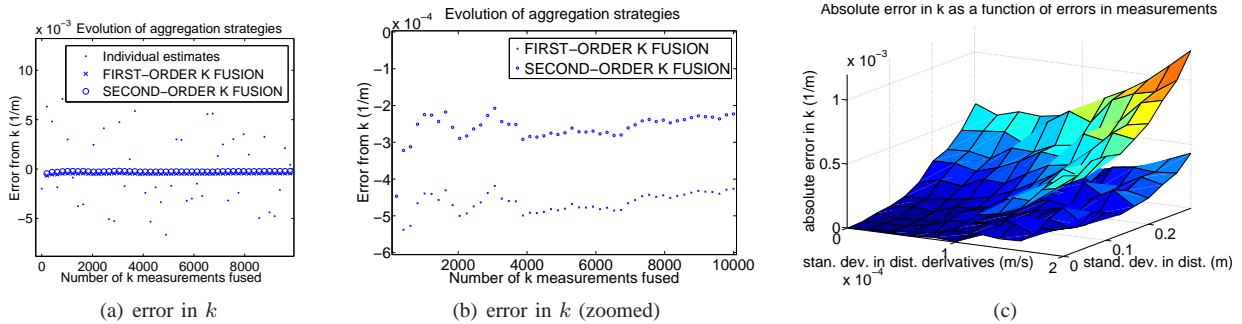| (a) error in k | (b) error in k (zoomed) | (c) |

Fig. 4. (a) shows how both FIRST-ORDER and SECOND-ORDER K FUSION, each using k estimates from PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD, converge to an error smaller than that of the individual measurements. (b) shows that SECOND-ORDER K FUSION converges to a smaller error than FIRST-ORDER K FUSION. (c) shows the absolute error of FIRST-ORDER K FUSION and SECOND-ORDER K FUSION as a function of the standard deviations in inter-drogue distances and their derivatives, highlighting how SECOND-ORDER K FUSION outperforms FIRST-ORDER K FUSION.

Now we illustrate the performance of the aggregation schemes through simulation. Figure 4(a) and (b) shows the evolution of the FIRST-ORDER and SECOND-ORDER K FUSION methods, which aggregate k estimates from PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD. Both evolutions converge very quickly to errors smaller than individual estimates, but the second-order method has a smaller error. Figure 4(c) shows that absolute error of FIRST-ORDER and SECOND-ORDER K FUSION as a function of the standard deviation in inter-drogue distance measurements and in their derivatives. SECOND-ORDER K FUSION has smaller errors.

VI. CONCLUSIONS

We have considered the task of estimating the physical parameters which capture the dynamics of the ocean's linear internal waves. We have designed the PARAMETER ESTIMATION BY ZERO-DERIVATIVE METHOD to be run on a group of drogues using only inter-drogue distance measurements. With noiseless measurements, we have determined tight conditions on the minimal sampling rate under which our algorithm precisely estimates the internal wave parameters. When measurements are corrupted by additive Gaussian noise, we have precisely bounded the error in the horizontal wavenumber estimate and derived a second-order approximation for it. We built on this model to create two aggregating schemes that fuse information from different time instants: FIRST-ORDER and SECOND-ORDER K FUSION. We have shown in simulations that these schemes converge to values with smaller errors than the individual k estimates. Future work will be devoted to formalizing the performance guarantees of the FIRST-ORDER and SECOND-ORDER K FUSION as well as extending our analysis to scenarios with multiple linear internal waves, as well as more general models of internal waves, such as weakly nonlinear internal waves.

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APPENDIX

Lemma A.1: For any $x \in (0, 2\pi)$, for any $C_1, C_2 \in \mathbb{R}_{>0}$, for all $\epsilon \in (-\epsilon_{\max}(x, C_1, C_2), \epsilon_{\max}(x, C_1, C_2))$,

$$\frac{1}{2} \sin(2(x + C_1 \epsilon)) - x + C_2 \epsilon < 0, \qquad (7)$$

where $\epsilon_{\max}(x, C_1, C_2)$ is defined by (6).