Optimal leader allocation in UAV formation pairs ensuring cooperation

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Abstract

This paper considers the problem of optimally allocating the leader task between pairs of selfish unmanned aerial vehicles (UAVs) flying in formation. The UAV that follows the other achieves a fuel benefit. The noncooperative nature of the agents makes it necessary to arbitrate leader-allocation mechanisms that induce collaboration so that the fuel consumption benefits of flying in formation can be realized. Formulated as a nonlinear program, our problem poses two distinct challenges: on the one hand, given a fixed number of leader switches, determine the optimal leader allocation and, on the other, find the optimal number of leader switches. Even though the first problem is nonconvex, we identify a suitable restriction of its feasible set that makes it convex while maintaining the same optimal value. Regarding the second problem, our analysis of the optimal value of the problem as a function of the number of switches allows us to design a search algorithm which is guaranteed to find the solution in logarithmic time. Several simulations illustrate our results.

Key words: point-to-point UAV reconfiguration, noncooperative agents, benefit-driven cooperation, cooperation-enforcing protocols, convex restrictions

1 Introduction

This paper considers pairwise formations between unmanned aerial vehicles (UAVs) where an agent gains a fuel benefit by flying in the wake of another (i.e., a reduction in aerodynamic drag). The objective of each UAV is to travel from source to target locations while consuming the least amount of fuel. UAVs are able to reach their destination alone, however joining in formation can potentially improve their fuel economy. When agents are noncooperative, the potential benefits of flying in formation bring up the issue of distributing fairly the leader task. The goal of this paper is to find optimal cooperation-inducing leader allocations that minimize UAV fuel consumption and provide individual agents with algorithms to compute them. Such benefit-driven cooperation mechanisms are a necessary building block to realize the potential benefits of collaboration in groups of noncooperative agents bargaining over the possibility of teaming up. The results of this paper can be applied to scenarios involving bargaining and auctions, task allocation in teams, and transferable utility games.

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Literature review. In the cooperative control literature, there are numerous instances of the performance benefits obtained by a group of agents collaborating towards the achievement of a common goal, see e.g. [Ren and Beard, 2008, Bullo et al., 2009, Mesbahi and Egerstedt, 2010] and references therein. Several works [Marden et al., 2009, Song et al., 2011, Zhu and Martínez, 2013] have also considered situations where individual agents have objective functions of their own that are aligned with a global objective. Less attention has been paid to scenarios where individual agents can carry out their objectives satisfactorily by themselves, yet performance can improve by collaborating with others. In these cases, a critical issue is how to regulate the collaboration among agents in order to enforce fairness when agents are not altruistic. A particular family of such problems is studied in wireless communications in the context of interference, medium access control, and resource sharing [Stirling and Nokleby, 2009, Nokleby and Aazhang, 2010, Nosratinia et al., 2004]. This paper introduces a different family of such problems that involve a group of UAVs flying in formation towards their goals. The energy savings of flying in formation are apparent in flocks of birds [Weimerskirch et al., 2001, Hummel, 1983]. In theory, the same benefits exist for formations among UAVs [Giulietti et al., 2000, Borrelli et al., 2004]. Moreover, recent improvements in technology make it possible to realize these fuel savings [Vachon et al., 2002, Seanor et al., 2004]. Here, we take inspiration from Ribichini and Frazzoli [2003], who study formation creation in groups of UAVs, and we examine how collaboration can be enforced among (not necessarily cooperative) agents through appropriately designed protocols. In our model for UAV behavior, agents are incentivized to remain in formation, analogously to marginal cost pricing schemes in game theory, see e.g. [Nisan et al., 2007]. Another relevant body of work is that of nonlinear and nonconvex optimization, see e.g. [Floudas and Gounaris, 2009] and references therein. A common approach to solving a nonconvex problem is to convert it into a convex problem, for which efficient solution methods exist [Boyd and Vandenberghe, 2004, Bertsekas et al., 2003], and then establishing the relationship between the solutions of both problems. Such conversions can be performed by way of relaxations of the constraints or restrictions of the feasible set [Açıkmeşe and Blackmore, 2011]. In this paper, we employ the latter approach.

Statement of contributions. Our contributions pertain to the modeling, analysis, and design of UAV formation pairs for optimal point-to-point reconfiguration. Regarding modeling, we introduce the notion of a UAV formation pair as a collection of distances (or leader allocations) in a line along which each one must lead. We also define a cost-to-target function that measures the total fuel consumed along the trajectory. We model the compliance of a UAV to a leader allocation via a parameter, $\epsilon \geq 0$, which quantifies the cost gain that the UAV will forgo before breaking a formation. With these elements, we formulate the problem of finding an optimal leader allocation among those that induce cooperation. This problem is nonconvex in one variable (finding the optimal leader allocation given a fixed number of switches) and combinatorial in the other (finding the optimal number of leader switches). The remaining contributions of the paper concern the analysis of this optimization problem and the design of algorithms that converge to a solution. Our first result builds on a careful characterization of the properties of the objective function and constraints to reformulate the optimization problem as a standard nonlinear program. When switching the lead has no cost, we find the optimal value of the program. This leads us to design the COST REALIZATION ALGORITHM to determine an optimal cooperation-inducing leader allocation. When switching the lead is costly, we restrict the feasible set of leader allocations to mimic those of the solution provided by the COST REALIZATION ALGORITHM. Remarkably, the restriction convexifies the feasible set of the original nonconvex problem while maintaining its optimal value, and its solutions are also feasible. Finally, we establish a quasiconvexity-like property of the optimal value of the problem as a function of the number of leader switches. This property allows us to design the BINARY SEARCH ALGORITHM, which finds the optimal number in logarithmic time. Several simulations throughout the paper illustrate the results.

Organization. Section 2 formulates the optimal leader allocation problem. Section 3 unveils some key properties of the cost-to-target functions and optimal leader allocations. Sections 4 and 5 deal with the no-cost and costly switching cases, respectively. Section 6 gathers our conclusions and ideas for future work.

2 Problem setup

This section describes the problem setup. After introducing the notions of formation, lead distance, and cost-to-target function, we present the optimization problem we seek to solve. Consider a pair of UAVs with unique identifiers (UIDs) i and j evolving in $X \subset \mathbb{R}^3$. Both i and j have synchronized clocks and can communicate with each other. A superscript i (resp. j) denotes a quantity associated with i (resp. j). Agent i has position $x^i(t) \in X$ at time $t \in \mathbb{R}_{\geq 0}$, a target location $\bar{x}^i \in X$, and the objective of flying from origin $x^i(0)$ to target location while consuming the least amount of fuel. The same is valid for agent j. For UAVs flying in close proximity, the inter-agent distance between them is negligible compared to the total distance they must travel to their target. Therefore we make the abstraction that i and j are point masses that may have concurrent position.

2.1 Formations and lead distances

To move from origin to destination efficiently, agents i and j might decide to travel in formation. Here, we formally introduce this notion and examine the associated costs. Without loss of generality (via an appropriate change of coordinate frame), suppose that i and j have rendezvoused at the origin, $x_r = \mathbf{0}$, at time $t = 0$ and are flying in the direction $u = (1, 0, 0)$. Agents i and j are *in formation* at a time t if

(i) $x^i(0) = x^j(0) = x_r$,
(ii) $[x^i(0), x^i(t)] = [x^j(0), x^j(t)] \in \text{ray}(x_r, u)$,
(iii) $d(x^i(\tau), x^j(\tau)) = 0$ for all $\tau \in [0, t]$,

where $\text{ray}(x_r, u)$ is the ray originating at x_r in the direction of u and $d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_{\geq 0}$ is the Euclidean distance between two points. The execution of a formation is completely described by a *vector of lead distances* (VOLD) and the UID of the agent which leads first. Without loss of generality, let i lead the formation first. A VOLD $\ell \in \mathbb{R}^N_{\geq 0}$ is a finite-dimensional vector prescribing which UAV leads the formation when and for how long. For instance, i will initially lead the formation for distance ℓ_1 at which point i and j will switch the lead. Upon completion of the leader switch, j will lead the formation for distance ℓ_2. As such, for n odd (resp. even), ℓ_n is the n^{th} distance led by i (resp. j). We use N to denote the cardinality of a VOLD.

A *leader switch* is a maneuver which takes a distance s to complete, see Figure 1. During a leader switch, the fuel consumption per unit distance is $\Gamma > 1$, and hence, the fuel consumed by both UAVs is $s\Gamma$. We have scaled the quantity Γ relative to the fuel consumption per unit distance of leading the formation (which, by assumption, is 1). Conversely, flying in the wake of another UAV reduces the aerodynamic drag on the following UAV. Thus, the relative fuel cost per unit distance of a UAV following is $\gamma < 1$. Flying solo or leading the formation incur the same fuel consumption per unit distance. Upon completion (or breaking) of the formation, UAVs fly directly to their respective targets. For reasons of presentation, we assume that UAVs are identical in the sense that γ, Γ, and the cost per unit distance of flying solo are the same for all agents. However, the remaining analysis could easily be adapted for agents that are not identical.

2.2 Cost-to-target functions

Here we define an agent's cost-to-target. To begin we introduce some auxiliary functions. Given a VOLD $\ell \in \mathbb{R}^N_{\geq 0}$, the distance of the nth switch from the origin is

$$D_n(\ell) = \begin{cases} 0, & n = 0, \\ \sum_{k=1}^{n} \ell_k + (n-1)s, & 1 \leq n \leq N. \end{cases}$$

The total distance of the formation prescribed by ℓ is $D_N(\ell)$. Likewise, given a VOLD ℓ, the number of leader switches that have been initiated when the UAVs have been in formation for distance $D \geq 0$ is

$$\#_{\mathrm{sw}}(\ell, D) = \max\{n \in \{0, 1, \ldots, N\} : D_n(\ell) \leq D\},$$

and the distance from the last switch is $D_{\mathrm{LS}}(\ell, D) = D - D_{\#_{\mathrm{sw}}(\ell,D)}(\ell)$. Given a VOLD and a distance D, a UAV is able to compute the relative fuel consumed on its flight from x_r to its target if it were to break the formation at Du. We refer to it as the UAV's *cost-to-target* function. Formally, for agent i, we have $\mathtt{ct}^i : \mathbb{R}^N_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ given by

$$\begin{aligned} \mathtt{ct}^i(\ell, D) = &\sum_{k \in \mathbb{N} \cap \mathbb{O}}^{\#_{\mathrm{sw}}(\ell,D)} \ell_k + s\Gamma(\#_{\mathrm{sw}}(\ell, D) - 1)_+ \\ &+ \gamma \sum_{k \in \mathbb{N} \cap \mathbb{E}}^{\#_{\mathrm{sw}}(\ell,D)} \ell_k + R^i(\ell, D) + d(Du, \bar{x}^i), \end{aligned}$$

where $\mathbb{N}, \mathbb{O}$ and $\mathbb{E}$ are the set of natural, odd, and even numbers respectively and $(\cdot)_+ := \max\{0, \cdot\}$. The first term is the fuel consumed by leading, the second term is the fuel consumed due to switching the lead, and the third term is the fuel consumed while following. The fourth term is a residual term accounting for the fuel consumed since the last switch. For $\#_{\mathrm{sw}}(\ell, D)$ odd

$$R^i(\ell, D) = \gamma(D_{\mathrm{LS}}(\ell, D) - s)_+ + \Gamma \min\{D_{\mathrm{LS}}(\ell, D), s\},$$

and for $\#_{\mathrm{sw}}(\ell, D)$ even

$$R^i(\ell, D) = (D_{\mathrm{LS}}(\ell, D) - s)_+ + \Gamma \min\{D_{\mathrm{LS}}(\ell, D), s\}.$$

Lastly, the $d(Du, \bar{x}^i)$ is the fuel that i would consume by breaking the formation at Du and flying to its target. Slight variations of a UAV's cost-to-target are the cost-to-target-at-the-kth-switch functions. For $k = 1, \ldots, N$, these are given by $\mathtt{ct}^i_k : \mathbb{R}^k_{\geq 0} \to \mathbb{R}_{>0}$ defined as

$$\begin{aligned} \mathtt{ct}^i_k(\ell_1, \ldots, \ell_k) = &\sum_{n \in \mathbb{N} \cap \mathbb{O}}^{k} \ell_n + \gamma \sum_{n \in \mathbb{N} \cap \mathbb{E}}^{k} \ell_n + s\Gamma(k-1) \\ &+ d\Big(\Big(\sum_{n=1}^{k} \ell_n + s(k-1)\Big)u, \bar{x}^i\Big). \end{aligned}$$

By construction, $\mathtt{ct}^i(\ell, D_k(\ell)) \equiv \mathtt{ct}^i_k(\ell_1, \ldots, \ell_k)$. Analogous $\mathtt{ct}^j$ and $\mathtt{ct}^j_k$ exist for j. In addition to an individual UAV's cost-to-target, it is possible to characterize the combined cost-to-targets of i and j at the end of their formation in terms of the formation breakaway location. To do so, consider any $c^i, c^j \in \mathbb{R}_{>0}$ and suppose there exists an $\ell \in \mathbb{R}^N_{\geq 0}$ such that $c^i = \mathtt{ct}^i_N(\ell)$ and $c^j = \mathtt{ct}^j_N(\ell)$ (i.e., the final cost-to-target at the end of the formation for i and j are c^i and c^j respectively). The UAVs' combined cost-to-targets at the end of the formation is

$$c^j + c^i = \mathtt{ct}^j_N(\ell) + \mathtt{ct}^i_N(\ell).$$

Under a change of variables $L = \sum_{k=1}^{N} \ell_k$ this becomes

$$\begin{aligned} c^j + c^i = \mathtt{ct}^{i+j}_N(L) := &(1+\gamma)L + 2(N-1)s\Gamma \\ &+ d\big((L + s(N-1))u, \bar{x}^j\big) \\ &+ d\big((L + s(N-1))u, \bar{x}^i\big). \end{aligned}$$

We call $\mathtt{ct}^{i+j}_N : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ the *combined cost-to-target* function of a formation with $N-1$ switches. Note that the formation breakaway distance is $L + s(N-1)$.

2.3 Problem statement

Upon arrival at the rendezvous location x_r, the agents need to determine a VOLD to dictate how to execute their formation. Suppose i declares an upper bound C^i on its final cost-to-target. Then j would propose a VOLD which solves the following two-stage optimization problem. First, among VOLDs with a fixed cardinality N, the minimum cost-to-target j can expect is

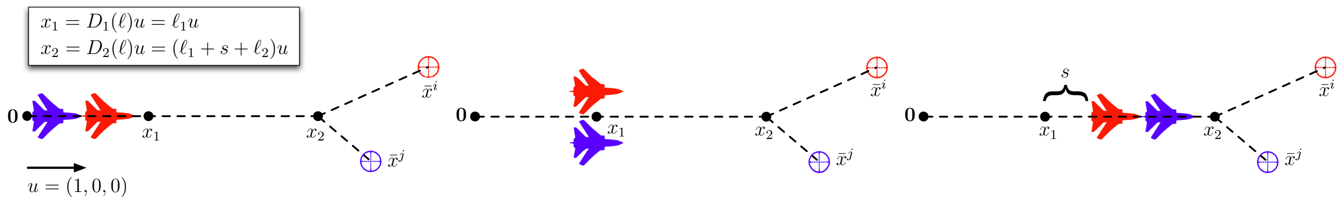
$$\min_{\ell \in \mathbb{R}^N_{\geq 0}} \ \mathtt{ct}^j_N(\ell) \tag{1a}$$

$$\text{s.t.} \quad C^i \geq \mathtt{ct}^i_N(\ell), \tag{1b}$$

$$\mathtt{ct}^i(\ell, D) \leq \mathtt{ct}^i_N(\ell) + \epsilon^i, \forall D \in [0, D_N(\ell)], \tag{1c}$$

$$\mathtt{ct}^j(\ell, D) \leq \mathtt{ct}^j_N(\ell) + \epsilon^j, \forall D \in [0, D_N(\ell)]. \tag{1d}$$

The parameters $\epsilon^i, \epsilon^j \geq 0$, intrinsic to each UAV, model their degree of cooperation (see Remark 2.1). Constraint (1c) ensures that at no point in the formation will



(a) Red begins leading the formation (b) A leader switch is initiated at x_1 (c) Blue leads after the switch

Fig. 1. Example flight behavior of UAVs given a VOLD $\ell = (\ell_1, \ell_2)$. The dashed lines represent the proposed flight paths of the UAVs. During a switch, the UAVs deviate slightly from the formation heading and red (resp. blue) decreases (resp. increases) its speed. After maintaining the new speeds for distance s, the UAVs return to the original heading and speed of the formation. Both UAVs consume $s\Gamma$ amount of fuel in this maneuver. At x_2, the UAVs fly directly to their respective targets.

i's cost-to-target be ϵ^i less than its final cost-to-target. The assumption is that i would break the formation earlier if it could benefit (by more than ϵ^i) in doing so. An analogous reason for j motivates (1d). The optimal value of (1) is denoted $C^j(N)$. Next, among all VOLDs (of any number of leader switches), the minimum cost-to-target that j could expect is

$$\min_{N \in \mathbb{N}_{\geq 2}} \quad C^j(N). \tag{2}$$

If N^* is minimizes (2) and ℓ^* minimizes (1) for fixed N^*, then j would propose ℓ^* to i. For reasons of notation, let $\mathcal{F}(N)$ be the feasible set of (1). In general, $\mathcal{F}(N)$ is non-convex and (2) is combinatorial. We devote much of this paper to transforming (1) into a convex problem (i.e., a convex objective function minimized over a convex set) and developing tools to efficiently solve (2).

From this point on, we assume that the formation heading is not in the same direction as either agents' target,

$$\bar{x}^i \notin \text{ray}(x_r, u) \quad \text{and/or} \quad \bar{x}^j \notin \text{ray}(x_r, u). \tag{3}$$

Without this assumption, inducing cooperation between i and j is trivial: if a UAV cannot "breakaway" from the formation, the other UAV will just follow in the formation until it no longer benefits from doing so.

Remark 2.1 (Selfish vs. fully cooperative UAVs). If $\epsilon^i = \epsilon^j = 0$, constraints (1c)-(1d) imply that i and j only abide by VOLDs for which their cost-to-target at any time in formation is never better than their cost-to-target at the end of the formation. We call such UAVs *selfish*. For selfish UAVs, the solution to problem (1) is trivially $\ell^* = \mathbf{0} \in \mathbb{R}^N$ for any N (agents never fly in formation). This is because neither UAV is willing to be the last to lead the formation compared to flying straight to its target. On the other hand, removing (1c)-(1d) (equivalently, setting $\epsilon^i = \epsilon^j = \infty$) implies that UAVs will abide by any VOLD $\ell \in \mathbb{R}^N_{\geq 0}$. We call such UAVs *fully cooperative*. However, these UAVs could potentially save fuel by breaking the formation earlier. This discussion motivates our problem formulation, which accounts for agents who are selfishly motivated yet willing to forfeit a small amount of fuel to ensure the formation occurs. Figure 2 shows the dependency of C^j_* on ϵ^i, ϵ^j. •

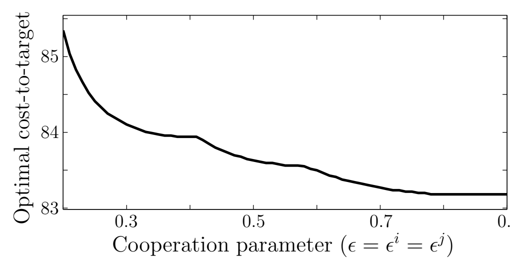


Fig. 2. Optimal value of (2) with respect to $\epsilon = \epsilon^i = \epsilon^j$. As ϵ increases, j's optimal cost-to-target, C^j_* decreases. The nonsmoothness at $\epsilon \approx 0.41$ and $\epsilon \approx 0.58$ is due to decreases in the optimal number of leader switches in the solution to (2). The simulation data are: $C^i = 82$, $s = 0.2$, $\Gamma = 1.7$, $\gamma = 0.5$, $\bar{x}^i = (100, 10)$, $\bar{x}^j = (90, -20)$.

3 Unveiling the structure of optimal VOLDs

This section describes properties of the cost-to-target functions and of the solutions to (1). Using them, we provide a more explicit description of the feasible set, allowing us to express (1) in standard form. Without loss of generality, many of the results only refer to i.

3.1 Properties of the cost-to-target functions

A cost-to-target function is continuous and piecewise differentiable with respect to distance D. In particular, $\texttt{ct}^i$ is not differentiable at distances where leader switches are initiated or completed. That is, $\partial_D \texttt{ct}^i$ exists at (ℓ, D) iff $D_{\text{LS}}(\ell, D) \notin \{0, s\}$. The following reveals a useful convexity-like property of the cost-to-target functions.

Lemma 3.1 (Leading, following, and switching become more costly as the formation progresses). *Let ℓ be a VOLD and $D_1 < D_2$. Suppose that, under ℓ, UAV i is leading (or following, or switching) at both $D_1 u$ and $D_2 u$. Then $\partial_D \texttt{ct}^i(\ell, D_1) < \partial_D \texttt{ct}^i(\ell, D_2)$. Moreover, if i is leading or switching at $D_1 u$, then $\partial_D \texttt{ct}^i(\ell, D_1) > 0$.*

Proof. The derivative of $\mathtt{ct}^i$ with respect to D is

$$\partial_D \mathtt{ct}^i(\ell, D) = \partial_D d(Du, \bar{x}^i) + \begin{cases} \gamma, & \text{if } i \text{ follows at } Du, \\ 1, & \text{if } i \text{ leads at } Du, \\ \Gamma, & \text{otherwise.} \end{cases}$$

The function $D \mapsto d(Du, \bar{x}^i)$ is strictly convex under (3). Thus, $\partial_D d(Du, \bar{x}^i)$ is strictly increasing. Suppose that i is leading at both $D_1 u$ and $D_2 u$. Then

$$\begin{aligned} \partial_D \mathtt{ct}^i(\ell, D_1) &= \partial_D d(D_1 u, \bar{x}^i) + 1, \\ &< \partial_D d(D_2 u, \bar{x}^i) + 1 = \partial_D \mathtt{ct}^i(\ell, D_2). \end{aligned}$$

Similar analysis holds for when i is following (or switching) at both $D_1 u$ and $D_2 u$. To show that $\partial_D \mathtt{ct}^i(\ell, D_1) > 0$ when i is leading at $D_1 u$, let $a > 0$ be sufficiently small such that i is also leading at $(D_1 - a)u$. Then

$$\begin{aligned} \mathtt{ct}^i(\ell, D_1) = \mathtt{ct}^i(\ell, D_1 - a) - d((D_1 - a)u, \bar{x}^i) \\ + a + d(D_1 u, \bar{x}^i) > \mathtt{ct}^i(\ell, D_1 - a), \end{aligned}$$

where we have used the triangle inequality. Since a can be taken arbitrarily small, $\partial_D \mathtt{ct}^i(\ell, D_1) > 0$ follows. If i is switching at $D_1 u$, the above argument with Γa instead of a, together with $\Gamma > 1$, yields the same conclusion. □

Roughly speaking, Lemma 3.1 states that it is more costly to lead (or follow or switch) in the formation as it progresses. The last statement in Lemma 3.1 simply states that leading or switching is always costly. This is to be distinguished from following which, as we show later, decreases the cost-to-target function in an optimal VOLD. Using a similar argument as in the proof of Lemma 3.1, the following states some properties of the cost-to-target-at-the-kth-switch functions.

Lemma 3.2 (Properties of the cost-to-target-at-the-kth-switch functions). *For $\ell \in \mathbb{R}^N_{\geq 0}$*

(P1) $\partial_{\ell_1} \mathtt{ct}^i_k(\ell_1, \ldots, \ell_k) > 0$, *for* $k \geq 1$,
(P2) $\partial_{\ell_2} \mathtt{ct}^j_k(\ell_1, \ldots, \ell_k) > 0$, *for* $k \geq 2$,
(P3) $\partial_{\ell_n} \mathtt{ct}^m_k(\ell_1, \ldots, \ell_k) = \partial_{\ell_{n+2}} \mathtt{ct}^m_k(\ell_1, \ldots, \ell_k)$, *for* $k \geq n + 2$ *and* $m = i, j$,
(P4) $\partial_{\ell_n} \mathtt{ct}^m_k(\ell_1, \ldots, \ell_k) < \partial_{\ell_n} \mathtt{ct}^m_{k+2}(\ell_1, \ldots, \ell_k)$, *for* $k \geq n$ *and* $m = i, j$.

3.2 Properties of the optimal VOLDs

This section explores an important property of the breakaway distance prescribed by a solution to (1). For some $\ell \in \mathbb{R}_{\geq 0}$ let $c^j = \mathtt{ct}^j_N(\ell)$ and $c^i = \mathtt{ct}^i_N(\ell)$. Recalling the discussion on the combined cost-to-target function (cf. Section 2.2), the possible breakaway locations of the formation can be described by all L satisfying

$$c^j + c^i = \mathtt{ct}^{i+j}_N(L).$$

Since $\mathtt{ct}^{i+j}_N$ is strictly convex, there exist two solutions L_1, L_2 to this equation (note that L_1, L_2 may not be distinct). Letting $L^*_N = \operatorname{argmin}_L \mathtt{ct}^{i+j}_N(L)$, we assume without loss of generality that $L_1 \leq L^*_N \leq L_2$. The following result states that $L_1 + s(N-1)$, and not $L_2 + s(N-1)$, is the breakaway location for a solution to (1).

Proposition 3.3 (UAVs breakaway as soon as possible). *For $N \in \mathbb{N}$, let ℓ^* be a solution to* (1)*. Then,*

$$\ell^* \in \mathbb{L}_N := \left\{ \ell \in \mathbb{R}^N_{\geq 0} : \sum_{k=1}^N \ell_k \leq L^*_N \right\}.$$

Proof. Consider the case when N is even so that j leads the last segment. Proving the result by contradiction, suppose ℓ solves (1) but $\sum_{k=1}^N \ell_k = \hat{L} > L^*_N$. Since $\mathtt{ct}^{i+j}_N$ is strictly convex, we know $\partial_L \mathtt{ct}^{i+j}_N(\hat{L}) > 0$. That is

$$\begin{aligned} \gamma + \partial_L d((\hat{L} + s(N-1))u, \bar{x}^i) > \\ -1 - \partial_L d((\hat{L} + s(N-1))u, \bar{x}^j). \end{aligned} \tag{4}$$

In other words, $\partial_{\ell_N} \mathtt{ct}^i_N(\ell) > -\partial_{\ell_N} \mathtt{ct}^j_N(\ell)$. A rearrangement of (4) reveals that $\partial_{\ell_{N-1}} \mathtt{ct}^j_N(\ell) > -\partial_{\ell_{N-1}} \mathtt{ct}^i_N(\ell)$ also. Consider the alternative VOLD $\ell' \in \mathbb{R}^N_{\geq 0}$ where, for some $a, b \geq 0$ to be designed, $\ell'_{N-1} = \ell_{N-1} - a$, $\ell'_N = \ell_N - b$, and $\ell'_k = \ell_k$ otherwise. To reach a contradiction of ℓ being a solution of (1), we want to design a, b such that $\mathtt{ct}^i_N(\ell') \leq \mathtt{ct}^i_N(\ell)$, $\mathtt{ct}^j_N(\ell') < \mathtt{ct}^j_N(\ell)$ and ℓ' satisfies (1c)-(1d). First, let a, b be sufficiently small such that the following linear approximations are valid. We desire

$$a \partial_{\ell_{N-1}} \mathtt{ct}^j_N(\ell) + b \partial_{\ell_N} \mathtt{ct}^j_N(\ell) > 0, \tag{5a}$$

$$a \partial_{\ell_{N-1}} \mathtt{ct}^i_N(\ell) + b \partial_{\ell_N} \mathtt{ct}^i_N(\ell) \geq 0. \tag{5b}$$

There are four cases: (i) $\partial_{\ell_N} \mathtt{ct}^i_N(\ell), \partial_{\ell_{N-1}} \mathtt{ct}^j_N(\ell) > 0$ (ii) $\partial_{\ell_N} \mathtt{ct}^i_N(\ell) < 0 < \partial_{\ell_{N-1}} \mathtt{ct}^j_N(\ell)$ (iii) $\partial_{\ell_{N-1}} \mathtt{ct}^j_N(\ell) < 0 < \partial_{\ell_N} \mathtt{ct}^i_N(\ell)$ (iv) $\partial_{\ell_N} \mathtt{ct}^i_N(\ell), \partial_{\ell_{N-1}} \mathtt{ct}^j_N(\ell) < 0$. For the sake of space, consider only case (iv) which we claim is the most complex. Combining (5) yields

$$-a \frac{\partial_{\ell_{N-1}} \mathtt{ct}^i_N(\ell)}{\partial_{\ell_N} \mathtt{ct}^i_N(\ell)} < b < -a \frac{\partial_{\ell_{N-1}} \mathtt{ct}^j_N(\ell)}{\partial_{\ell_N} \mathtt{ct}^j_N(\ell)}.$$

Given $a > 0$, $\exists b > 0$ since $\partial_{\ell_{N-1}} \mathtt{ct}^j_N(\ell) > -\partial_{\ell_{N-1}} \mathtt{ct}^i_N(\ell)$ and $\partial_{\ell_N} \mathtt{ct}^i_N(\ell) > -\partial_{\ell_N} \mathtt{ct}^j_N(\ell)$. It remains to show that ℓ' satisfies (1c)-(1d). Consider first the interval $D \in [0, D_{N-1}(\ell')]$ where, by construction of ℓ', $\mathtt{ct}^i(\ell', D) = \mathtt{ct}^i(\ell, D)$. Since ℓ satisfies (1c)-(1d) on this interval

$$\mathtt{ct}^i(\ell', D) \geq \mathtt{ct}^i_N(\ell) - \epsilon^i > \mathtt{ct}^i_N(\ell') - \epsilon^i.$$

Similar analysis holds for j (i.e., cooperation is induced up until $D_{N-1}(\ell')u$). Beyond $D_{N-1}(\ell')u$, UAV j is switching and leading. So, $\forall D \in (D_{N-1}(\ell'), D_N(\ell')]$

$$\mathtt{ct}^j(\ell', D) > \mathtt{ct}^j_{N-1}(\ell') > \mathtt{ct}^j_N(\ell') - \epsilon^j,$$

due to Lemma 3.1. Thus, j will cooperate under ℓ'. Still under case (iv), note that $\partial_{\ell_N} \mathtt{ct}^i_N(\ell) < 0 \Rightarrow \partial^-_D \mathtt{ct}^i(\ell, D_N(\ell)) < 0 \Rightarrow \partial^-_D \mathtt{ct}^i(\ell', D_N(\ell')) < 0$ which, due to Lemma 3.1, further implies that following always decreases i's cost-to-target function in the formation. Agent i only switches and follows beyond $D_{N-1}(\ell')u$. So, $\forall D \in (D_{N-1}(\ell'), D_{N-1}(\ell') + s]$

$$\mathtt{ct}^i(\ell', D) > \mathtt{ct}^i(\ell', D_{N-1}(\ell')) > \mathtt{ct}^i(\ell', D_N(\ell')) - \epsilon^i,$$

and $\forall D \in [D_{N-1}(\ell') + s, D_N(\ell'))$

$$\mathtt{ct}^i(\ell', D) > \mathtt{ct}^i(\ell', D_N(\ell')) > \mathtt{ct}^i(\ell', D_N(\ell')) - \epsilon^i.$$

Thus, i will also cooperate (i.e., ℓ' satisfies (1c)-(1d)). Similar arguments hold for cases (i)-(iii). In summary, ℓ' decreases (1a) while satisfying (1d) which contradicts ℓ solving (1). N odd is dealt with analogously. □

Proposition 3.3 gives an upper bound on the breakaway distance of an optimal VOLD. Thus, we can restrict the feasible set of (1) to $\ell \in \mathcal{F}(N) \cap \mathbb{L}_N$. Following this result, for $\ell \in \mathbb{L}_N$, one has the additional property that a UAV's cost-to-target strictly decreases while following.

Corollary 3.4 (Following is beneficial). *If $\ell \in \mathbb{L}_N$ has i following at $\hat{D}u$, $\partial_D \mathtt{ct}^i(\ell, \hat{D}) < -\partial_D \mathtt{ct}^j(\ell, \hat{D}) < 0$.*

Proof. Following the proof of Proposition 3.3, if $\ell \in \mathbb{L}_N$ then we arrive at (4) with the inequality reversed. However, the LHS of (4) is $\partial_D \mathtt{ct}^i(\ell, \hat{D})$ where $\hat{D} = \hat{L}$. The result follows from applying Lemma 3.1 □

Corollary 3.4 also allows us to identify additional properties of the cost-to-target-at-the-k^{th}-switch functions.

Lemma 3.5 (Properties of the cost-to-target-at-the-kth-switch functions - continued). *For $\ell \in \mathbb{L}_N$,*

(P5) $\partial_{\ell_1} \mathtt{ct}^j_k(\ell_1, \dots, \ell_k) \le -\partial_{\ell_1} \mathtt{ct}^i_k(\ell_1, \dots, \ell_k) < 0,\ k \ge 2,$

(P6) $\partial_{\ell_2} \mathtt{ct}^i_k(\ell_1, \dots, \ell_k) \le -\partial_{\ell_2} \mathtt{ct}^j_k(\ell_1, \dots, \ell_k) < 0,\ k \ge 2.$

The results thus far are now used to state a fact about the final cost-to-target for UAV i given a solution to (1).

Lemma 3.6 (i receives its bound on final cost-to-target). *If ℓ is a solution to (1) then $C^i = \mathtt{ct}^i_N(\ell)$.*

Proof. The proof is by contradiction, so let ℓ solve (1) and assume $C^i < \mathtt{ct}^i_N(\ell)$. Decrease ℓ_2 by some amount $a > 0$, thus increasing i's cost-to-target and decreasing j's cost-to-target (cf. Lemma 3.5(P6)). For a sufficiently small, (1b) is still satisfied and (1a) decreases. Also (1c)-(1d) are satisfied, as shown by repeated application of Lemma 3.2(P4). Thus, we have reached a contradiction (i.e., if $C^i < \mathtt{ct}^i_N(\ell)$ then ℓ does not solve (1)). □

3.3 Equivalent formulation

Here, we combine the results established above to reduce the feasibility set of (1) to only those VOLDs exhibiting properties of optimal VOLDs. In particular, Lemma 3.1 and Corollary 3.4 reveal that, for $\ell \in \mathbb{L}_N$, the local minima of $\mathtt{ct}^i$ and $\mathtt{ct}^j$ occur at the distances where an agent initiates a switch from following to leading. So, if cooperation is induced at those points, then cooperation is induced for the entire formation. Additionally, we can now fix i's final cost-to-target at C^i. To summarize, we reformulate (1) in terms of the cost-to-target-at-the-kth-switch functions as follows. For fixed $N \in \mathbb{N}$

$$\min_{\ell \in \mathbb{L}_N} \quad \mathtt{ct}^j_N(\ell) \tag{6a}$$

$$\text{s.t. } c^j = \mathtt{ct}^j_N(\ell), \quad C^i = \mathtt{ct}^i_N(\ell), \tag{6b}$$

$$c^j \le \mathtt{ct}^j_k(\ell_1, \dots, \ell_k) + \epsilon^j, \quad k \in \mathbb{O}_{[1,N-1]}, \tag{6c}$$

$$C^i \le \mathtt{ct}^i_k(\ell_1, \dots, \ell_k) + \epsilon^i, \quad k \in \mathbb{E}_{[2,N-1]}. \tag{6d}$$

Given the above discussion, the set of solutions to (6) is the set of solutions to (1). The equality constraints in (6) are not affine and substituting them into the inequality constraints yields nonconvex inequality constraints.

Remark 3.7 (Total lead distance functions). For $\ell \in \mathbb{L}_N$ satisfying (6b)-(6d), one can show that, without knowing the specific elements of ℓ, there exists a unique distance that i must lead the formation. This is

$$\mathtt{tl}^i_N(c^j) := \big[L + d((L + (N-1)s)u, \bar{x}^j) + (N-1)s\Gamma - c^j\big]/(1-\gamma) \equiv \textstyle\sum_{k \in \mathbb{N} \cap \mathbb{O}}^N \ell_k,$$

where L satisfies $c^j + C^i = \mathtt{ct}^{i+j}_N(L)$ ($\mathtt{tl}^i_N$ is well-defined since L is unique). Also, $\mathtt{tl}^j_N(c^j) := L - \mathtt{tl}^i_N(c^j)$. •

4 Optimal VOLDs under no-cost switching

This section solves problem (2) when switching the lead does not incur a cost to UAVs (i.e., $s = s\Gamma = 0$). We start by characterizing the optimal value C^j_* and then design the COST REALIZATION ALGORITHM to generate a VOLD that realizes the optimal fuel consumption of UAV j in the formation. Note that, under no-cost switching

$$\mathtt{ct}^{i+j}_{N'}(L) = \mathtt{ct}^{i+j}_{N''}(L), \quad \forall N', N'' \in \mathbb{N}.$$

This can be interpreted as follows. Given a fixed break-away location (in this case, L because $s = 0$) and i's cost-to-target, the final cost-to-target for j is independent of the number of leader switches in the VOLD. Based on this observation, we are able to prove the following.

Theorem 4.1 (Optimal value under no-cost switching). *For $s = 0$, $\max\{\epsilon^i, \epsilon^j\} > 0$, and any $N \geq 2$, C^j_* is the optimal value of the convex problem*

$$\min_L \{\mathtt{ct}_N^{i+j}(L) - C^i\}. \tag{7}$$

Proof. The proof is constructive. For $s = 0$, let $c^j = \min_L \mathtt{ct}_N^{i+j}(L) - C^i$ for any $N \geq 2$ and for brevity, let $\ell^i_L = \mathtt{tl}^i_N(c^j)$ and $\ell^j_L = \mathtt{tl}^j_N(c^j)$ (cf. Remark 3.7). Begin with the VOLD $\ell = (\ell^i_L, \ell^j_L)$. If ℓ is not feasible then it must be that $\mathtt{ct}^j_1(\ell^i_L) < c^j - \epsilon^j$. By assumption, $\mathtt{ct}^j_1(0) > c^j - \epsilon^j$. Therefore, by the intermediate value theorem, there exists a $\ell_1 \in (0, \ell^i_L)$ such that $\mathtt{ct}^j_1(\ell_1) = c^j - \epsilon^j$. With a slight abuse of notation, let $\ell = (\ell_1, \ell^j_L, \ell^i_L - \ell_1)$. Then, because the break-away distance has been preserved, i and j still realize their costs of C^i and c^j, respectively. Again, if ℓ is not feasible, then it must be that $\mathtt{ct}^i_2(\ell_1, \ell^j_L) < C^i - \epsilon^i$ and, by the intermediate value theorem, there exists a $\ell_2 \in (0, \ell^j_L)$ such that $\mathtt{ct}^i_2(\ell_1, \ell_2) = C^i - \epsilon^i$. Then, update $\ell = (\ell_1, \ell_2, \ell^i_L - \ell_1, \ell^j_L - \ell_2)$. This process may be repeated as long as ℓ is not feasible. If it never happens that ℓ is feasible, this implies that $\ell_k \to 0$ (we view $\{\ell_k\}$ as a sequence which is bounded and monotonic). This further implies that there exists a $L := \sum_{k=1}^{\infty} \ell_k$ such that $c^j - \epsilon^j + C^i - \epsilon^i = \mathtt{ct}_N^{i+j}(L)$. However, if $\max\{\epsilon^i, \epsilon^j\} > 0$, this contradicts c^j being the optimal value of (7). Therefore, it must be that for some finite number of steps, ℓ becomes feasible under the proposed procedure. □

The above result establishes that the optimal value of (2) under no-cost switching can be found as the optimal value of a simple convex problem. This result is useful for j as it is able to know a priori what final cost-to-target it can expect from a formation with i. However, i and j still do not know how to realize these cost-to-targets. The COST REALIZATION ALGORITHM provided in Figure 3 resolves this issue. Its design is inspired by the constructive proof of Theorem 4.1. Agents i and j implement this algorithm on-the-fly while in formation and only require knowledge of the optimal value to (2).

Corollary 4.2 (Inducing optimal solutions: no-cost switching). *For $s\Gamma = 0$ and input C^j_*, the* COST REALIZATION ALGORITHM *induces a VOLD that solves* (2).

Figure 4 reports the cost-to-targets in a simulation of two UAVs flying from origin to target locations while implementing the COST REALIZATION ALGORITHM. A leader switch is indicated when an agents' cost-to-target transitions from increasing to decreasing (or vice versa). As one can see, the COST REALIZATION ALGORITHM schedules a leader switch whenever one of the agents' cost-to-target reaches ϵ below the projected final cost-to-target.

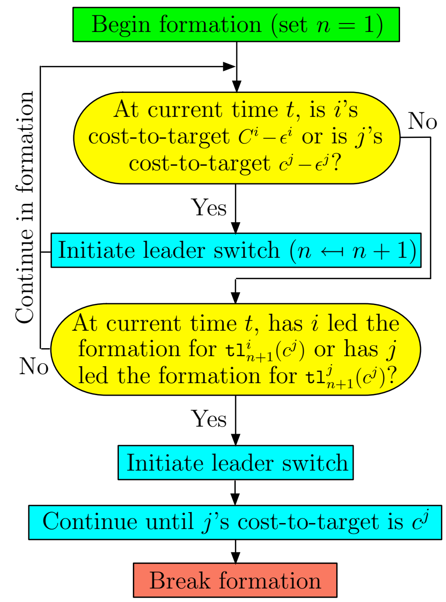
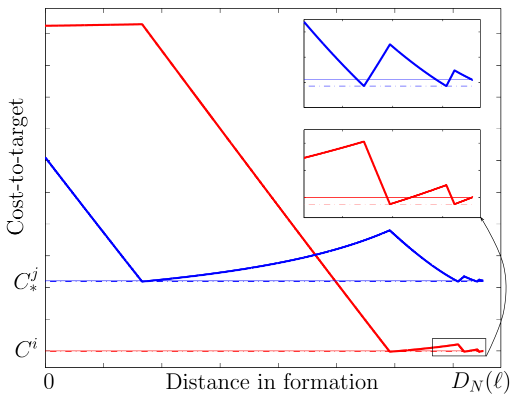


Fig. 3. The COST REALIZATION ALGORITHM, with input c^j, over ϵ-cooperative agents i and j. Upon breaking the formation, UAVs fly directly to their respective targets. UAVs have knowledge of the following parameters when implementing the algorithm: C^i, s, Γ, γ, ϵ^i, ϵ^j, $\bar{x}^i$, $\bar{x}^j$.

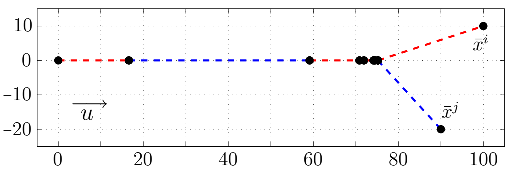
Remark 4.3 (Robustness of the COST REALIZATION ALGORITHM**).** Small measurement, modeling (i.e., unmodeled wind effects), and computational uncertainties result in small perturbations to an agent's final cost-to-target resulting from the COST REALIZATION ALGORITHM. Thus, for ϵ^i (resp. ϵ^j) sufficiently large, i (resp. j) is willing to remain in formation despite these perturbations to its expected final cost-to-target. In this sense, the parameters ϵ^i and ϵ^j ensure that the COST REALIZATION ALGORITHM is robust to small uncertainties. •

5 Optimal VOLDs under costly switching

This section solves problem (2) when switching is costly. The key difference with respect to Section 4 is that, under no-cost switching, whenever an inequality constraint in (6) becomes active, agents can initiate a leader switch to ensure cooperation is maintained without affecting their final cost-to-targets. However, under costly switching, the same logic does not hold because adding a leader switch increases the final cost-to-target of both agents.



(a) Cost-to-targets under no-cost switching



(b) Switch locations under no-cost switching.

Fig. 4. The COST REALIZATION ALGORITHM implemented under no-cost switching with input $C^j_* = 84$. (a) shows the cost-to-targets for i (red) and j (blue) resulting from the induced VOLD. Horizontal lines are final cost-to-targets and dash-dot lines are $\epsilon^i = \epsilon^j$ below that. (b) shows the actual flight paths: a red (resp. blue) dotted line is a segment on which i (resp. j) leads. Simulation data are $C^i = 80$,$\gamma = 0.5$, $\epsilon^i = \epsilon^j = 0.05$, $\bar{x}^i = (100, 10)$, $\bar{x}^j = (90, -20)$. The cost-benefits of the formation are $d(x_r, \bar{x}^i) - C^i = 20$ (20%) and $d(x_r, \bar{x}^j) - C^j_* = 8$ (8.7%) for i and j, resp.

5.1 *Convex restriction*

We start by restricting the feasible set of (6) to VOLDs that exhibit the same structure as in the no-cost case (i.e., equality constraints for $k = 1, \ldots, N-2$). That is,

$$\min_{c^j} \quad c^j \tag{8a}$$

$$\text{s.t. } c^j = \mathtt{ct}^j_k(\ell_1, \ldots, \ell_k) + \epsilon^j, \quad k \in \mathbb{O}_{[1,N-2]}, \tag{8b}$$

$$C^i = \mathtt{ct}^i_k(\ell_1, \ldots, \ell_k) + \epsilon^i, \quad k \in \mathbb{E}_{[2,N-2]}, \tag{8c}$$

$$c^j = \mathtt{ct}^j_N(\ell_1, \ldots, \ell_N), \tag{8d}$$

$$C^i = \mathtt{ct}^i_N(\ell_1, \ldots, \ell_N), \tag{8e}$$

$$\ell \in \mathbb{L}_N, \tag{8f}$$

$$c^j \leq \mathtt{ct}^j_{N-1}(\ell_1, \ldots, \ell_{N-1}) + \epsilon^j,\ N \in \mathbb{E}, \tag{8g}$$

$$C^i \leq \mathtt{ct}^i_{N-1}(\ell_1, \ldots, \ell_{N-1}) + \epsilon^i,\ N \in \mathbb{O}. \tag{8h}$$

Given c^j, (8b)-(8e) define a unique ℓ. Thus, the variable of optimization is now c^j. Constraints (8g)-(8h) ensure that the entire VOLD induces cooperation. Denote the set of feasible c^j in the above problem by $\mathcal{F}_r(N)$ and the optimal value by $C^j_r(N)$. In general, for any given $N \in \mathbb{N}$ and optimal value $C^j(N)$, one has $C^j_r(N) \geq C^j(N)$. However, for some N, we have $C^j_r(N) = C^j_*$.

Theorem 5.1 (Restriction is exact). *Let $C^j_* < \infty$ be the optimal value of* (2)*. Then there exists an $N \geq 2$ such that $C^j_r(N) = C^j_*$.*

Proof. Let N be a minimizer of (2) and suppose ℓ is a solution of (6) for fixed N. Our method is to build a new VOLD, ℓ' satisfying (8b)-(8e), from ℓ. Initially, set $\ell' = \ell$ and let $k_0 \in [1, N-2]$ be the smallest k such that one of the constraints (6c)-(6d) is not active when evaluated at ℓ'. Assume k_0 is odd, so $C^j_* - \epsilon^j < \mathtt{ct}^j_{k_0}(\ell'_1, \ldots, \ell'_{k_0})$. Increase ℓ'_{k_0} and decrease ℓ'_{k_0+2} at the same rate until $C^j_* - \epsilon^j = \mathtt{ct}^j_{k_0}(\ell'_1, \ldots, \ell'_{k_0})$ (this is possible due to (P5)). After performing this procedure, the k_0 constraint is active, the RHS of the $k_0 + 1$ constraint has increased in value (see (P1) and (P3)), and all other constraint functions have maintained their original value (thus, the final cost-to-go for i and j have also remained at C^j_* and C^i respectively). Thus, ℓ' still solves (6). Next, we focus on the $k_0 + 1$ constraint whose RHS has increased in value and thus $C^i - \epsilon^i < \mathtt{ct}^i_{k_0+1}(\ell'_1, \ldots, \ell'_{k_0+1})$. Again, increase ℓ'_{k_0+1} and decrease ℓ'_{k_0+3} at the same rate until the $k_0 + 1$ constraint becomes active. We are able to repeat this procedure until the $N-1$ constraint is reached. It is not possible to make this constraint active using the same procedure because there is no ℓ'_{N+1} component to decrease. Therefore, once the $N-1$ constraint is reached, ℓ' satisfies (8b)-(8e). One point of concern in the proposed procedure occurs if when decreasing (say) ℓ'_{k_0+2} it happens that $\ell'_{k_0+2} \leq 0$. However, in this event, one can show that N is not a minimizer of (2): a new VOLD with one less leader switch can be constructed that decreases the objective function and satisfies the constraints. □

Recall that, given input C^j_*, the COST REALIZATION ALGORITHM generates a VOLD satisfying (8b)-(8h) for some N. Thus, the following is a result of Theorem 5.1.

Corollary 5.2 (Constructing an optimal solution: costly switching). *Under costly switching, the* COST REALIZATION ALGORITHM *with input C^j_* induces a VOLD which solves* (2)*.*

Corollary 5.2 generalizes Corollary 4.2. Next, we state an analogous result to Theorem 4.1, allowing us to find the optimal value of (2) under costly switching.

Theorem 5.3 (Restriction is convex). *The problem* (8) *is convex.*

Proof. It suffices to show that $\mathcal{F}_r(N)$ is convex. Suppose that N is even, begin with $c^j \in \text{int}(\mathcal{F}_r(N))$, and let $a > 0$ and $\mathbf{b} \in \mathbb{R}^N$ be sufficiently small such that the following analysis holds. Let $\ell + \mathbf{b}$ satisfy (8b)-(8e) for $c^j + a$. Towards characterizing $\mathbf{b}$, notice that $c^j + a - \epsilon^j =$

$\mathtt{ct}_1^j(\ell_1+b_1)$. Hence, $a = b_1\partial_{\ell_1}\mathtt{ct}_1^j(\ell_1)$, implying $b_1 < 0$. Next, note that $C^i - \epsilon^i = \mathtt{ct}_2^i(\ell_1+b_1, \ell_2+b_2)$. Therefore

$$0 = b_1\partial_{\ell_1}\mathtt{ct}_2^i(\ell_1,\ell_2) + b_2\partial_{\ell_2}\mathtt{ct}_2^i(\ell_1,\ell_2),$$

from which we see that $b_2 < 0$. Repeating this argument while invoking Lemmas 3.2 and 3.5, we see that $b_k < 0$ for $k = 1, \ldots, N-2$ and b_{N-1}, b_N need to satisfy

$$b_{N-1}\partial_{\ell_2}\mathtt{ct}_N^j(\ell) + b_N\partial_{\ell_1}\mathtt{ct}_N^j(\ell) > 0,$$
$$b_{N-1}\partial_{\ell_2}\mathtt{ct}_N^i(\ell) + b_N\partial_{\ell_1}\mathtt{ct}_N^i(\ell) > 0.$$

Evoking (P5)-(P6), we deduce $b_{N-1}, b_N < 0$ as well, and hence $\mathbf{b} < 0$. Next, we study how (8g) changes as we increase slightly c^j. In particular, $\mathbf{b} < 0$ satisfies the equation $c^j + a = \mathtt{ct}_N^j(\ell+\mathbf{b})$. Or, in other words

$$\begin{aligned} a &= \partial_{\ell_1}\mathtt{ct}_N^j(\ell)\textstyle\sum_{k\in\mathbb{N}\cap\mathbb{O}}^{N-1} b_k + \partial_{\ell_2}\mathtt{ct}_N^j(\ell)\textstyle\sum_{k\in\mathbb{N}\cap\mathbb{E}}^{N} b_k, \\ a &< \partial_{\ell_1}\mathtt{ct}_{N-1}^j(\ell_1,\ldots,\ell_{N-1})\textstyle\sum_{k\in\mathbb{N}\cap\mathbb{O}}^{N-1} b_k \\ &\quad + \partial_{\ell_2}\mathtt{ct}_{N-1}^j(\ell_1,\ldots,\ell_{N-1})\textstyle\sum_{k\in\mathbb{N}\cap\mathbb{E}}^{N-2} b_k, \end{aligned} \tag{9}$$

where (P4) has been used. The LHS (resp. RHS) of (9) represents the increase in the LHS (resp. RHS) of (8g). Thus, (8g) remains satisfied by increasing c^j. Therefore, by increasing c^j the only constraint one may violate is $\ell \in \mathbb{R}_{\geq 0}^N \supset \mathbb{L}_N$. However, increasing c^j more further decreases each ℓ_i. Thus, $\mathcal{F}_r(N)$ must be convex. □

By Theorem 5.3, given $N \in \mathbb{N}$, the optimal value of (8) can be efficiently found under costly switching. Note that the restriction to the feasible set does not limit the type of real-world scenarios that we can solve. Moreover, the solution of (8) maximizes the distance between switches. From an implementation point of view, this is a desirable and robust switching protocol because UAVs are not required to perform switching maneuvers arbitrarily fast. To find the optimal value of (2), we next study how to determine the optimal number of leader switches.

5.2 Optimal number of leader switches

Here, we identify a criterion that allows us to determine an optimal N and helps us search for it. The following result provides such a criterion via a quasiconvexity-like property of $C_r^j(N)$. Figure 5 illustrates Theorem 5.4.

Theorem 5.4 (Certificate for optimal number of switches). *For $N \in \mathbb{N}$, the following statements hold*

(i) if adding two switches increases (8)

$$C_r^j(N) < C_r^j(N+2),$$

then adding any more multiple of two switches also increases it

$$C_r^j(N+2k) \leq C_r^j(N+2(k+1)), \quad \forall k \in \mathbb{N}.$$

The inequality is strict iff (8) *is feasible for $N+2k$.*

(ii) if removing two switches increases (8)

$$C_r^j(N) < C_r^j(N-2),$$

then removing any more multiple of two switches also increases it

$$C_r^j(N-2k) \leq C_r^j(N-2(k+1)), \quad \forall k \leq N/2-2.$$

The inequality is strict iff (8) *is feasible for $N-2k$.*

Proof. The result follows from the combination of:

(S1) If $C_r^j(N) < C_r^j(N+2)$ for some $N \in \mathbb{N}$, then $C_r^j(N) \leq C_r^j(N+2k)$ for all $k \in \mathbb{N}$. The inequality is strict iff (8) is feasible for $N+2$.

(S2) If $C_r^j(N) < C_r^j(N-2)$ for some $N \in \mathbb{N}$, then $C_r^j(N) \leq C_r^j(N-2k)$ for all $k \in \mathbb{N}$ such that $k \leq N/2-2$. The inequality is strict iff (8) is feasible for $N-2$.

Suppose N is even. Consider first (S1). If $C_r^j(N) < C_r^j(N+2)$ then $C_r^j(N) \notin \mathcal{F}_r(N+2)$. That is, for $N+2$ and $c^j = C_r^j(N)$, either (S1.1) constraints (8b)-(8f) are violated or (S1.2) the constraint (8g) is violated.

Consider (S1.1). Let $c_{\min}^j(N)$ be the minimum c^j such that (8b)-(8f) are satisfied for N. As per the analysis in the proof of Theorem 5.3, $c_{\min}^j(N) \equiv \min_L \mathtt{ct}_N^{i+j}(L) - C^i$. By the properties of $\mathtt{ct}_N^{i+j}$, it follows that $c_{\min}^j(N) < c_{\min}^j(N+2k)$ for any $k \in \mathbb{N}$. Therefore, if (S1.1) is true, this means that $C_r^j(N) \notin \mathcal{F}_r(N+2k) \Rightarrow C_r^j(N) < C_r^j(N+2k)$ and this would prove (S1).

Consider (S1.2). Let ℓ^N (resp. ℓ^{N+2}) (resp. ℓ^{N+4}) satisfy (8g) for $C_r^j(N)$ and N (resp. $N+2$) (resp. $N+4$). We know $\ell_k^{N+2} = \ell_k^N$ for $k = 1, \ldots, N-1$. Denote $\ell_N^{N+2} = \ell_N^N + a$. Note that

$$\mathtt{ct}_N^i(\ell^N) = \mathtt{ct}_N^i(\ell_1^N, \ldots, \ell_N^N + a) + \epsilon^i, \tag{10}$$

so $a > 0$. Also, because (8g) is violated

$$C_r^j(N) > \mathtt{ct}_{N+1}^j(\ell_1^N, \ldots, \ell_{N-1}^N, \ell_N^N + a, \ell_{N+1}^{N+2}) + \epsilon^j. \tag{11}$$

For now let $\ell_k^{N+4} = \ell_k^{N+2}$ for $k = 1, \ldots, N+1$ and denote $\ell_{N+2}^{N+4} = \ell_{N+2}^{N+2} + b$. Define $\ell_{N+3}^{N+4}, \ell_{N+4}^{N+4}$ implicitly by

$$\begin{aligned} C^i &= \mathtt{ct}_{N+2}^i(\ell_1^{N+2}, \ldots, \ell_{N+2}^{N+2} + b) + \epsilon^i, \\ c^j &= \mathtt{ct}_{N+4}^j(\ell_1^{N+4}, \ldots, \ell_{N+4}^{N+4}), \\ C^i &= \mathtt{ct}_{N+4}^i(\ell_1^{N+4}, \ldots, \ell_{N+4}^{N+4}). \end{aligned}$$

Likewise, $b > 0$ since

$$\mathtt{ct}_{N+2}^i(\ell^{N+2}) = \mathtt{ct}_{N+2}^i(\ell_1^{N+2}, \ldots, \ell_{N+2}^{N+2} + b) + \epsilon^i.$$

Comparing the above and (10), we see that $a < b$. Thus $\mathtt{ct}^j_N(\ell^N_1, \ldots, \ell^N_N + a) < \mathtt{ct}^j_{N+2}(\ell^{N+2}_1, \ldots, \ell^{N+2}_{N+2} + b)$. Therefore, the conditions of Lemma A.1 are satisfied and $\ell^{N+4}_{N+4} > \ell^{N+2}_{N+2}$. Since j leads last, this means that $\mathtt{ct}^j_{N+3}(\ell^{N+4}_1, \ldots, \ell^{N+4}_{N+3}) < \mathtt{ct}^j_{N+1}(\ell^{N+2}_1, \ldots, \ell^{N+2}_{N+1})$. Recalling (11), the above means that (8d) is violated. As a final step to creating ℓ^{N+4}, employ the strategy as in the proof of Theorem 5.1: decrease ℓ^{N+4}_{N+1} and increase ℓ^{N+4}_{N+3} by the same amount until (8b) is satisfied. However, increasing ℓ^{N+4}_{N+3} further violates (8d). Therefore, $C^j_r(N) < C^j_r(N+4)$. We can repeat this process for $N+6$ and so on to attain the desired result. So long as (8b)-(8f) are satisfied for $C^j_r(N)$ and $N + 2k$, the above construction is valid. However, if $C^j_r(N + 2) = \infty$, recall (S1.1). Then $C^j_r(N + 2k) = \infty$ for all k. This completes the proof of (S1).

Next, we prove (S2). Let ℓ^N (resp. ℓ^{N-2}) (resp. ℓ^{N-4}) satisfy (8g) for $C^j_r(N)$ and N (resp. $N-2$) (resp. $N-4$). To reach a contradiction, suppose

$$C^j_r(N-4) \le C^j_r(N) < C^j_r(N-2). \tag{12}$$

First, let $a > b > 0$ be such that

$$\begin{aligned} C^i &= \mathtt{ct}^i_{N-2}(\ell^N_1, \ldots, \ell^N_{N-3}, \ell^N_{N-2} - a), \\ &= \mathtt{ct}^i_{N-4}(\ell^N_1, \ldots, \ell^N_{N-5}, \ell^N_{N-4} - b). \end{aligned}$$

Let $L = \sum_{k=1}^{N-2} \ell^N_{N-2} - a$. If $c^j \ge \mathtt{ct}^j_{N-2}(\ell^N_1, \ldots, \ell^N_{N-2} - a)$ then $c^j + C^i > \mathtt{ct}^{i+j}_{N-2}(L)$ (i.e., the formation length is too long). Since we know $\ell^N_k = \ell^{N-2}_k$ for $k = 1, \ldots, N-4$, decreasing the formation distance must be accomplished by decreasing $\ell^N_{N-3} + \ell^N_{N-2} - a$. Since i's cost-to-target must be maintained at C^i, both ℓ^N_{N-3} and $\ell^N_{N-2} - a$ must decrease (i.e., $\ell^{N-2}_{N-3} \le \ell^N_{N-3}$). But since (8g) is violated for ℓ^{N-2}, it must be that $\ell^{N-2}_{N-3} > \ell^N_{N-3}$ which is a contradiction. Thus, it must be that $c^j < \mathtt{ct}^j_{N-2}(\ell^N_1, \ldots, \ell^N_{N-2} - a)$. Under the assumption of (12), a similar argument can be made to show that $c^j \ge \mathtt{ct}^j_{N-4}(\ell^N_1, \ldots, \ell^N_{N-4} - b)$. Let us now reverse the change of a (resp. b) in ℓ_{N-2} (resp. ℓ_{N-4}). Then we see that $\mathtt{ct}^j_{N-4}(\ell^N_1, \ldots, \ell^N_{N-4}) < \mathtt{ct}^j_{N-2}(\ell^N_1, \ldots, \ell^N_{N-2})$. But, by Corollary A.2, this would mean that $C^j_r(N-2) < C^j_r(N)$, contradicting (12). The claim can be extended analogously for cases where more switches are removed. Thus, $C^j_r(N) < C^j_r(N-4)$. So long as (8b)-(8f) are satisfied for $C^j_r(N)$ and $N - 2k$, the above construction is valid. Thus, $C^j_r(N-2) = \infty \Rightarrow C^j_r(N-4) = \infty$, and so on. This proves (S2). □

Next, we design a method to find the optimal N. Define

$$\Delta_N := C^j_r(N) - C^j_r(N+2).$$

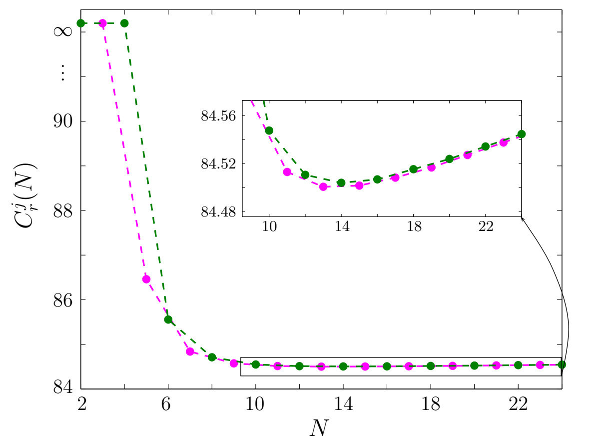


Fig. 5. An example of the optimal value of (6) with respect to N. The magenta (resp. dark green) dots represent $C^j_r(N)$ for odd (resp. even) N.

If $N^* \in \mathbb{E}$ is optimal, then Theorem 5.4 implies $\Delta_N > 0$ for all $N \in \mathbb{N}_{[2,N^*)} \cap \mathbb{E}$ and $\Delta_N < 0$ for all $N \in \mathbb{N}_{(N^*,\infty)} \cap \mathbb{E}$ (so long as Δ_N is finite). Also, $0 \in [\Delta_{N^*}, \Delta_{N^*-2}]$. Thus, the problem of finding an optimal N is well-suited for a binary search (see [Cormen et al., 2009]), which is presented in Algorithm 1 adapted to our problem.

Algorithm 1 The BINARY SEARCH ALGORITHM

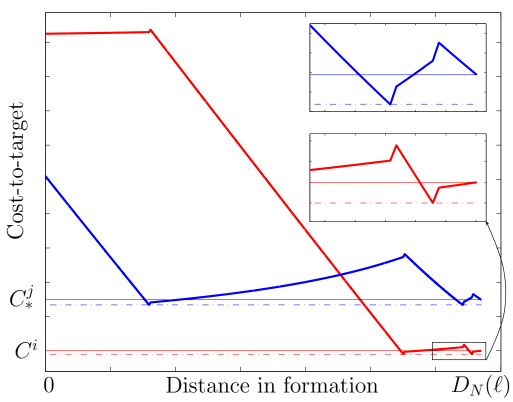
Input: N with Δ_N or Δ_{N+2} finite

if $\Delta_N \ge 0$ **then**
  $N_l := N$ and $N_u := N + 2$
  while $\Delta_{N_u} \ge 0$ **do**
    $N_l := N_u$
    $N_u \hookleftarrow 2N_u$
  end while
else $N_l := 2 + (N \mod 2)$ and $N_u := N$
end if
while $N_u - N_l \ge 4$ and $\Delta_N \ne 0$ **do**
  $N := (N_u - N_l)/2$
  if $\Delta_N \ge 0$
    $N_l := N$
  else $N_u := N$ **end if**
end while
return N if $\Delta_N \le 0$, $N + 1$ otherwise

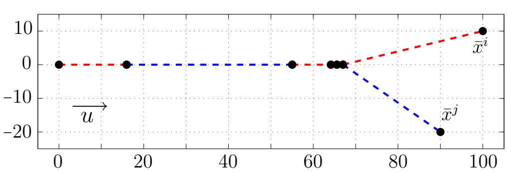
The method is implemented for odd and even inputs, the optimal N being the lesser of the two outputs. The computational intensity of the BINARY SEARCH ALGORITHM stems from the evaluation of Δ_N, which solves (8).

Corollary 5.5 (Correctness and complexity). *Suppose* (2) *is feasible and N^* is its minimizer. Let $N^{\mathrm{o}}_0 \in \mathbb{O}$ (resp. $N^{\mathrm{e}}_0 \in \mathbb{E}$) be a valid input to the* BINARY SEARCH ALGORITHM *with output N^{o} (resp. N^{e}). Then $N^* \in \{N^{\mathrm{o}}, N^{\mathrm{e}}\}$. Moreover, to determine N^* the problem* (8) *is solved at most $4\lceil \log_2 N^* \rceil$ times.*

The correctness result in Corollary 5.5 follows from Theorem 5.4 and the complexity result is inherited from binary search algorithms [Cormen et al., 2009]. Figure 6 presents simulation results verifying the correctness of the BINARY SEARCH ALGORITHM. The main differences when compared to the simulations presented in Figure 4 pertain to the cost of switching ($s\Gamma$) being positive and the degree of cooperation between agents (ϵ^i, ϵ^j). The agents in Figure 4 are able to induce cooperation even when ϵ^i, ϵ^j are small because adding a switch does not increase their final cost-to-target. On the other hand, when there is a cost associated with switching, the agents are not able to switch arbitrarily fast without increasing their final cost-to-targets. In fact, the problem (6) is infeasible for small ϵ^i, ϵ^j under costly switching. In terms of real-world implementation, the BINARY SEARCH ALGORITHM is run prior to agents beginning their formation. For this reason, the implementation time of the BINARY SEARCH ALGORITHM is not reflected in the simulation of Figure 6. However, on-board processors must be able to run the BINARY SEARCH ALGORITHM within the time required for UAVs to fly from their current location to the formation rendezvous location. If executed online, the complexity bound for the binary search (cf. Corollary 5.5) provides guidance as to how fast a processor should be with respect to UAV motion.



(a) Cost-to-targets under an optimal VOLD



(b) Switch locations under the above VOLD

Fig. 6. N^* is computed using the BINARY SEARCH ALGORITHM, $C_r^j(N^*) = 85$ is fed to the COST REALIZATION ALGORITHM, and an optimal VOLD is attained. Note the effect of costly switching on the cost-to-target function. The data for the simulation are: $C^i = 82$, $s = 0.2$, $\Gamma = 1.7$, $\gamma = 0.5$, $\epsilon^i = 0.2$, $\epsilon^j = 0.3$, $\bar{x}^i = (100, 10)$, $\bar{x}^j = (90, -20)$. The cost-benefits of the formation are 18 (18%) and 7 (7.6%) for i and j, resp. (slightly less for each UAV than the no-cost of switching case).

6 Conclusions

We have considered the problem of optimally allocating the leader task between pairs of selfish UAVs flying in formation. Formulated as a nonlinear program, our problem poses two distinct challenges: given a fixed number of leader switches, determining the optimal leader allocation and finding the optimal number of leader switches. We showed that, when switching the lead has no cost, the optimal value can be obtained via a convex program and designed the COST REALIZATION ALGORITHM to determine an optimal cooperation-inducing leader allocation. In the costly switching case, we restricted the feasible set of allocations to mimic the structure of the solutions provided by this policy. The resulting restriction has the same optimal value and, for a fixed number of leader switches, is convex. We also unveiled a quasiconvexity-like property of the optimal value as a function of the number of switches and designed the BINARY SEARCH ALGORITHM to find the optimal number in logarithmic time. Future work will include extensions to formations of more than two UAVs, scenarios with obstacle avoidance/no-fly zones, and problems where UAVs bargain over the possibility of joining in formation.

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A Appendix

Here, we state two results used to prove Theorem 5.4.

Lemma A.1 (Property of the last two lead/follow distances). *For $k \in \mathbb{N}$, let $\ell^N \in \mathbb{L}_N$ and $\ell^{N+2k} \in \mathbb{L}_{N+2k}$ such that*

(i) $\sum_{k=1}^{N-2} \ell_k^N \le \sum_{k=1}^{N+2(k-1)} \ell_k^{N+2k}$,

(ii) $\mathtt{ct}_{N-2}^j(\ell_1^N, \dots, \ell_{N-2}^N) \le \mathtt{ct}_{N+2(k-1)}^j(\ell_1^{N+2k}, \dots, \ell_{N+2(k-1)}^{N+2k})$,

(iii) $\mathtt{ct}_{N-2}^i(\ell_1^N, \dots, \ell_{N-2}^N) = \mathtt{ct}_{N+2(k-1)}^i(\ell_1^{N+2k}, \dots, \ell_{N+2(k-1)}^{N+2k})$,

(iv) $\mathtt{ct}_N^j(\ell^N) = \mathtt{ct}_{N+2k}^j(\ell^{N+2k})$, $\mathtt{ct}_N^i(\ell^N) = \mathtt{ct}_{N+2k}^i(\ell^{N+2k})$.

Then, $\ell_{N-1}^N < \ell_{N+2k-1}^{N+2k}$ and $\ell_N^N < \ell_{N+2k}^{N+2k}$.

Proof. Suppose that $k = 1$ and N is even. Recall (cf. Section 3.2) that $\mathtt{ct}_N^{i+j}$ is strictly decreasing and convex on $\mathbb{L}_N$. Also, $\mathtt{ct}_N^{i+j}$ has the convexity-like property

$$0 < \mathtt{ct}_N^{i+j}(L) - \mathtt{ct}_{N-2}^{i+j}(L) < \mathtt{ct}_{N+2}^{i+j}(L) - \mathtt{ct}_N^{i+j}(L).$$

Therefore, if for some $L_1 \le L_2$ and $a, b > 0$,

$$\begin{aligned} 0 &> \mathtt{ct}_N^{i+j}(L_1 + a) - \mathtt{ct}_{N-2}^{i+j}(L_1) \\ &> \mathtt{ct}_{N+2}^{i+j}(L_2 + b) - \mathtt{ct}_N^{i+j}(L_2), \end{aligned} \tag{A.1}$$

then $a < b$. Take $L_1 = \sum_{k=1}^{N-2} \ell_k^N$, $L_2 = \sum_{k=1}^{N+2k-2} \ell_k^{N+2k}$, $a = \ell_{N-1}^N + \ell_N^N$, and $b = \ell_{N+1}^{N+2} + \ell_{N+2}^{N+2}$. Now note that, expressing $(ii)-(iv)$ in terms of the combined fuel functions, shows that the condition (A.1) is satisfied for this choice of values. Thus $a = \ell_{N-1}^N + \ell_N^N < b = \ell_{N+1}^{N+2} + \ell_{N+2}^{N+2}$. Next, we show by contradiction that $\ell_{N-1}^N < \ell_{N+1}^{N+2}$ and $\ell_N^N < \ell_{N+2}^{N+2}$. Suppose $\ell_{N-1}^N \ge \ell_{N+1}^{N+2}$. Thus

$$\begin{aligned} &\mathtt{ct}_{N-2}^j(\ell_1^N, \dots, \ell_{N-2}^N) - \mathtt{ct}_{N-1}^j(\ell_1^N, \dots, \ell_{N-1}^N) \\ &> \mathtt{ct}_N^j(\ell_1^{N+2}, \dots, \ell_N^{N+2}) - \mathtt{ct}_{N+1}^j(\ell_1^{N+2}, \dots, \ell_{N+1}^{N+2}). \end{aligned}$$

Since (ii) is true and following is more beneficial to j earlier in the formation, the above implies that $\mathtt{ct}_{N-1}^j(\ell_1^N, \dots, \ell_{N-1}^N) < \mathtt{ct}_{N+1}^j(\ell_1^{N+2}, \dots, \ell_{N+1}^{N+2})$. To satisfy (iv), this would mean

$$\begin{aligned} &\mathtt{ct}_N^j(\ell^N) - \mathtt{ct}_{N-1}^j(\ell_1^N, \dots, \ell_{N-1}^N) \\ &\quad > \mathtt{ct}_{N+2}^j(\ell^{N+2}) - \mathtt{ct}_{N+1}^j(\ell_1^{N+2}, \dots, \ell_{N+1}^{N+2}). \end{aligned}$$

Since leading is more costly further in the formation, the above can only be satisfied if $\ell_N^N > \ell_{N+2}^{N+2}$. However, this would contradict $\ell_{N-1}^N + \ell_N^N < \ell_{N+1}^{N+2} + \ell_{N+2}^{N+2}$. Reasoning instead with i's cost-to-target and starting with $\ell_N^N > \ell_{N+2}^{N+2}$, a similar contradiction can be reached. N odd and $k \ge 2$ can be handled similarly. □

Corollary A.2 (Sufficient condition to benefit from switch removal). *Let ℓ solve* (6) *for $N \in \mathbb{N}$. If*

$$\mathtt{ct}_{N-2(k+1)}^j(\ell_1, \dots, \ell_{N-2(k+1)}) \le \mathtt{ct}_{N-2}^j(\ell_1, \dots, \ell_{N-2}),$$

for some $k \le N/2 - 2$ then $C_r^j(N-2k) < C_r^j(N)$.

Proof. Suppose that N is even and let ℓ^{N-2k} satisfy (8b)-(8f) for $N-2k$ and $C_r^j(N)$. Since the assumptions of Lemma A.1 are satisfied, $\ell_{N-2k}^{N-2k} < \ell_N^N$. Thus

$$\begin{aligned} &\mathtt{ct}_{N-2k}^j(\ell^{N-2k}) - \mathtt{ct}_{N-2k-1}^j(\ell_1^{N-2k}, \dots, \ell_{N-2k-1}^{N-2k}) \\ &\quad < \mathtt{ct}_N^j(\ell^N) - \mathtt{ct}_{N-1}^j(\ell_1^N, \dots, \ell_{N-1}^N), \\ \Rightarrow\ &\mathtt{ct}_{N-2k-1}^j(\ell_1^{N-2k}, \dots, \ell_{N-2k-1}^{N-2k}) \\ &\quad > \mathtt{ct}_{N-1}^j(\ell_1^N, \dots, \ell_{N-1}^N) \ge C_r^j(N) - \epsilon^j. \end{aligned}$$

In other words, (8g) is satisfied for ℓ^{N-2k}, and thus $C_r^j(N-2k) < C_r^j(N)$ (this relation is strict because (8g) is not active and thus there exists a feasible $c^j < C_r^j(N)$). N odd can be dealt with analogously. □