Optimal leader allocation in UAV formation pairs under costly switching

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Abstract— **We study the leader allocation problem in UAV formation pairs when switching the lead incurs a fuel cost. While in formation, UAVs are assumed to adhere to a notion of ϵ-cooperativeness. The problem is formulated as the combination of a non-convex and a discrete optimization problem where the leader allocations are constrained to those that induce cooperation between UAVs. A equivalent formulation of the problem allows us to express the constraint set as a family of equality and inequality constraints. By restricting our search to solutions of a specific form, we replace the non-convex problem with a convex one while preserving the optimal value of the original problem. A necessary and sufficient condition is obtained which is used to verify a solution to the discrete problem. The results are combined to design the OPTIMAL COST ALGORITHM, which efficiently solves the original problem. Our results are verified in simulation.**

I. INTRODUCTION

This paper considers formation pairs between unmanned aerial vehicles (UAVs). Following another UAV in formation benefits a UAV by, among other reasons, reducing aerodynamic drag which improves fuel economy. However, when the agents are selfish neither UAV is willing to lead the formation without assurance that it will be able to follow some time later. To this end, agents must agree on a allocation of lead/follow distances such that both UAVs can be sure that the other will cooperate. If one agent cannot trust that another agent will cooperate, it may be better for that agent to not join in formation at all. A behavioral model for UAVs called ϵ-cooperative ensures the existence of such cooperation-inducing allocations [1]. Roughly speaking, in a network of ϵ-cooperative UAVs each agent is willing to forfeit a little bit of fuel to ensure that the formation occurs. The amount of fuel forfeited is assumed to be small compared to the total benefit of the formation. This paper's aim is to find optimal leader allocations that induce cooperation between ϵ-cooperative agents. By optimal, we mean that there does not exist another leader allocation that further reduces the fuel cost incurred to an agent in the formation.

Motivation. In the case that switching the lead has no cost to UAVs, the problem of finding optimal cooperation-inducing lead/follow distances has been solved [1]. Nonetheless, under no-cost switching, UAVs may switch from leading to following arbitrarily frequently in optimal leader allocations. Clearly, solutions of this type cause the model to break down and the solution may lose its validity. A more appropriate model is to consider a cost associated with switching the lead. However, optimally allocating the leader role in this case has remained unsolved.

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Literature review. The literature most relevant to this work include formation flight, multi-agent cooperation, and optimization. With regards to the former, the self-organization of bird flocks in nature is testament to the energy-saving advantages of flight formation [2], [3]. Indeed, this benefit translates analogously to formations of UAVs [4], [5]. Additional motivations for formation flight of UAVs are discussed in [6]. Moreover, close proximity flight has become a reality due to advances in technology [7],[8].

Recently, some research has been conducted which studies formation creation in networks of UAVs [9]. However, little attention has been paid to the allocation of lead/follow distances in the formation. The aforementioned work [1] is one exception, which assumes a UAV behavioral model analogous to marginal cost pricing [10] in cooperative game theory. The works [11], [12] study a similar model but with the interpretation of trust between agents in a network setting. The contributions of this work are related to [13] which also designs cooperation-inducing mechanisms.

Concerning optimization, a large body of work exists for non-convex problems [14]. Many specific classes of non-convex problems can be solved using a multitude of approaches [15], [16], [17], [18]. A popular strategy in the literature is to construct a convex counterpart to the non-convex problem. This approach is favored because many efficient solvers exist for convex problems, which are better understood in general [19].

Statement of contributions. We model formations of UAV pairs in terms of the distances spent by each UAV while leading, following, and switching between the two. Based on this model, a non-convex optimization problem is provided whose solutions correspond to optimal leader allocations. By understanding important properties of optimal lead allocations, we reformulate the problem in standard optimization form. A restriction on the solution set is proposed which enables us to consider instead a convex problem embedded in a discrete problem. Solutions to the convex problem are optimal leader allocations when the number of leader switches is fixed whereas the discrete problem finds an optimal number of leader switches. Additionally, we develop a necessary and sufficient condition for a number of leader switches to be optimal. Combining all of our results, the OPTIMAL COST ALGORITHM is provided to solve the original optimization problem.

For reasons of space, the proofs of most results are omitted and will appear elsewhere.

Notation. For $a, b \in \mathbb{R}^n$, let $d(a, b) = \|a - b\|$ denote the Euclidean distance between a and b. The closed segment between $a, b \in \mathbb{R}^n$ is $[a, b]$ and the ray starting at a in the direction of $b-a$ is $\mathrm{ray}(a,b)$. The sets of even, odd, and natural numbers are $\mathbb{E}, \mathbb{O}$, and $\mathbb{N}$ respectively. The positive part of $x \in \mathbb{R}$ is given by $x_+ = \max\{0, x\}$.

II. Problem formulation

Consider a pair of UAVs, each with unique identifiers (UIDs) i and j evolving in $X \subset \mathbb{R}^3$. We assume that i and j have synchronized clocks and can communicate with each other. The subscript i denotes a quantity associated with agent i. Thus, the position of UAV i at time t is denoted by $x_i(t)$. Each agent has an origin, a target location, and an objective which is to arrive at its target location while consuming the least amount of fuel. We use $\bar{x}_i$ (resp. $\bar{x}_j$) to denote the target location of i (resp. j). A UAV's fuel consumption can be reduced by flying in formation in the wake of another UAV. The fuel consumption per unit distance of following in formation is $\gamma \in \mathbb{R}_{>0}$, whereas flying solo or leading in a formation incurs a fuel consumption of $\Gamma \in \mathbb{R}_{>0}$ per unit distance. By assumption, $\Gamma > \gamma$.

A. Formations and lead distances

A *formation* is a tuple $(\{i,j\}, x_r, u_h) \in \mathbb{N}^2 \times X^2$, where i and j are the UIDs of the UAVs involved in the formation, x_r is a rendezvous location and the vector u_h is the heading direction of the formation. The remainder of this paper assumes that $x_r = \mathbf{0}$ and $u_h = e_1$, where (e_1, e_2, e_3) are the standard basis in $\mathbb{R}^3$ (this is done without loss of generality, because a simple transformation of the coordinate frame ensures that it is satisfied). When we say that i and j have been *in formation* for distance D, we mean that there exist $T_2 > T_1 > 0$ such that (i) $x_i(T_1) = x_j(T_1) = x_r$, (ii) $[x_i(T_1), x_i(T_2)] = [x_j(T_1), x_j(T_2)] = [x_r, x_r + Du_h]$, and (iii) $d(x_i(t), x_j(t)) < c$ for $t \in [T_1, T_2]$. Here $c > 0$ is assumed small compared to the total flight distances. Thus, we use the abstraction that $d(x_i(t), x_j(t)) = 0$ while i and j are in formation.

The execution of a formation is completely described by a *vector of lead distances* (VOLD) and the UID of the UAV which leads the formation first. Without loss of generality, UAV i always leads the formation first. A VOLD is a finite-dimensional vector $\Delta = (\delta_1, \dots, \delta_N) \in \mathbb{R}^N_{>0}$. For n odd, δ_n denotes the n^{th} distance led by i before i and j swap the lead. For n even, δ_n denotes the n^{th} distance led by j before i and j swap the lead. The distance required for i and j to switch from leading to following (resp. following to leading) is given by $s \geq 0$. During a switch, both UAVs consume $\Gamma_s \geq \Gamma$ fuel per unit distance. As such, the cost of switching the lead is $s\Gamma_s$ to both i and j. Note that for $\Delta \in \mathbb{R}^N_{>0}$ there are $N-1$ leader switches prescribed by Δ. The distance from x_r of the n^{th} switch is represented by,

$$D_n(\Delta) = \begin{cases} 0, & n = 0, \\ \sum_{k=1}^{n} \delta_k + (n-1)s, & 1 \leq n \leq N. \end{cases}$$

The total distance of the formation is $D_N(\Delta)$. Figure 1 shows the execution of a formation given a VOLD as well as the process of UAVs transitioning from following to leading (resp. leading to following).

We conclude this section with the definition of two functions that will be used subsequently. Given a VOLD $\Delta \in \mathbb{R}^N_{>0}$ and $D \geq 0$, let $\#_s : \mathbb{R}^N_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{N}_{[0,N]}$ return the number of leader switches that have been initiated when the UAVs have been in formation for D distance. Precisely,

$$\#_s(\Delta, D) = \max\{n \in \mathbb{N}_{[0,N]} : D_n(\Delta) \leq D\}.$$

Consequently, the distance that i and j have been in formation since the initiation of the last switch is given by,

$$D_{\mathrm{LS}}(\Delta, D) = D - D_{\#_s(\Delta,D)}(\Delta).$$

B. Objective function

Given a VOLD and a distance D, a UAV can compute the fuel consumed in the flight from the rendezvous location to its target if the UAV were to leave (or break) the formation at De_1. We represent this cost-to-go as $J_i : \mathbb{R}^N_{>0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$. An analogous function J_j exists for j. The function J_i can be expressed as,

$$J_i(\Delta, D) = \Gamma \sum_{\substack{n=1 \\ n \text{ odd}}}^{\#_s(\Delta,D)} \delta_n + \gamma \sum_{\substack{n=2 \\ n \text{ even}}}^{\#_s(\Delta,D)} \delta_n \qquad (1)$$
$$+ s\Gamma_s(\#_s(\Delta,D) - 1)_+ + R_i(\Delta, D) + \Gamma d(De_1, \bar{x}_i),$$

where the first term is the fuel consumed while i leads the formation, the second term is the fuel consumed while i follows in the formation, and the third term is the fuel consumed while i has been switching from leader to follower (and vice versa). In addition R_i is a residual term that accounts for any fuel consumed between the initiation of the last switch and De_1. For $\#_s(\Delta, D)$ odd, we have,

$$R_i(\Delta, D) = \gamma(D_{\mathrm{LS}}(\Delta, D) - s)_+ + \min\{D_{\mathrm{LS}}(\Delta, D), s\}\Gamma_s,$$

and for $\#_s(\Delta, D)$ even,

$$R_i(\Delta, D) = \Gamma(D_{\mathrm{LS}}(\Delta, D) - s)_+ + \min\{D_{\mathrm{LS}}(\Delta, D), s\}\Gamma_s.$$

Finally, the last term is the fuel required to fly directly to the target $\bar{x}_i$ should i decide to break the formation at De_1.

C. Behavior of UAVs in formation

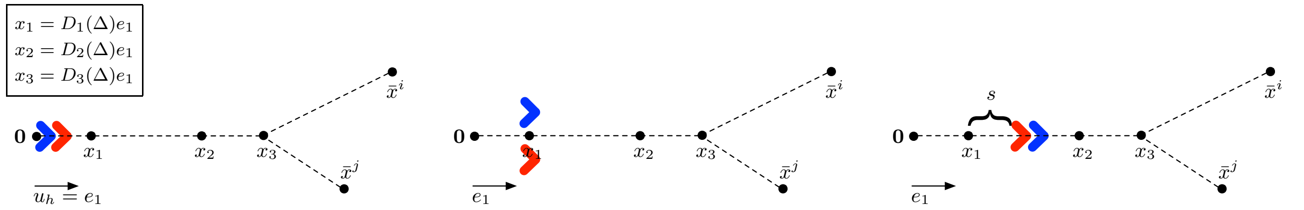
We use the following model for the behavior of i and j while in formation. Agent i is *ϵ-cooperative* if it will abide by any VOLD $\Delta \in \mathbb{R}^N_{>0}$ of any length $N \in \mathbb{N}$ iff Δ satisfies,

$$J_i(\Delta, D_N(\Delta)) \leq J_i(\Delta, D) + \epsilon, \quad \forall D \in [0, D_N(\Delta)],$$

for some $\epsilon \geq 0$. In other words, an ϵ-cooperative UAV will abide by a VOLD so long as there is no point in the formation where its cost-to-go is ϵ less than the final cost-to-go upon completion of the formation. A thorough justification for this model is found in [1]. Note in particular that selfish and fully cooperative UAVs correspond to 0- and ∞-cooperative UAVs, respectively.

We refer to the set of VOLDs that both UAVs will abide by as *cooperation-inducing*. Formally, for $N \in \mathbb{N}$,

$$\mathscr{D}^N_c(\epsilon_i, \epsilon_j) = \{\Delta \in \mathbb{R}^N_{>0} : J_i(\Delta, D_N(\Delta)) \leq J_i(\Delta, D) + \epsilon_i$$
$$\text{and } J_j(\Delta, D_N(\Delta)) \leq J_j(\Delta, D) + \epsilon_j$$
$$\text{for all } D \in [0, D_N(\Delta)]\}.$$



(a) Red begins leading the formation (b) A leader switch is initiated at x_1 (c) Blue leads the formation after the switch

Fig. 1. Example flight behavior of UAVs given a VOLD $\Delta = (\delta_1, \delta_2, \delta_3)$. The dashed lines represent the proposed flight paths of the UAVs. During a switch, the red UAV will decrease its speed while the blue UAV increases its speed. The new speeds are maintained for a distance s, after which the UAVs return to the original heading and speed of the formation (the fuel consumed by both the red and blue UAV in this maneuver is $s\Gamma_s$). At x_3, the UAVs fly directly to their respective targets.

In words, if i (resp. j) is ϵ_i-cooperative (resp. ϵ_j-cooperative), then both i and j will abide by $\Delta \in \mathscr{D}_c^N(\epsilon_i, \epsilon_j)$. When it is clear from the context, we use the shorthand notation $\mathscr{D}_c^N$.

D. Problem statement

Suppose that at time t, UAVs i and j are about to join in formation with each other (that is, $x_i(t) = x_j(t) = x_r$) but have not yet agreed upon a VOLD to determine the execution of the formation. Also, i has provided to j an upper bound, $\hat{C}_i \in \mathbb{R}_{>0}$, on the amount of fuel it wishes to consume on its flight to $\bar{x}_i$. Then, to determine an appropriate VOLD for the formation, j would like to solve the following problem,

$$\min_{N \in \mathbb{N}} \quad \min_{\Delta \in \mathscr{D}_c^N} \quad J_j(\Delta, D_N(\Delta)) \tag{2a}$$

$$\text{subject to} \qquad \hat{C}_i \geq J_i(\Delta, D_N(\Delta)). \tag{2b}$$

Denote the inner (resp. outer) problem by $\mathcal{I}$ (resp. $\mathcal{O}$) with solution set $\omega(N)$ (resp. Ω). A solution to $\mathcal{I}$ is an optimal cooperation-inducing VOLD of fixed dimension N from the perspective of j. A VOLD that j would propose to i is a solution to $\mathcal{O}$.

In previous work [1], we have solved the problem above for the case when switching the lead is instantaneous with no cost (i.e., $s = s\Gamma_s = 0$). Solving (2) with costs associated with switching the lead is instead more complex. Without stating it from now on, we make the assumption that $\bar{x}_i, \bar{x}_j \notin \mathrm{ray}(x_r, u_h)$. That is, breaking away from a formation is always an option. When this assumption does not hold, then the optimal solution to (2) is trivial: the optimal VOLD for j is to follow i as long as it is benefits from doing so, and i has no choice but to "cooperate".

III. Equivalent formulation of the inner optimization problem

In this section, we develop an equivalent formulation of the inner optimization problem $\mathcal{I}$. This result is a stepping stone towards our ultimate goal of efficiently solving the original optimization problem (2). Section III-A and III-B characterize, respectively, properties of the objective function and the optimal solutions of the inner optimization problem. These results are the basis for the equivalent formulation of the inner optimization problem in Section III-C. Without loss of generality, we note that most of the results that follow make reference only to UAV i.

A. Properties of the objective function

The objective functions J_i and J_j, as defined in (1), are piecewise differentiable with respect to the distance D. Specifically, J_i and J_j are not differentiable (but continuous) at distances where leader switches are initiated or completed, i.e., $\partial_D J_i$ and $\partial_D J_j$ exist at (Δ, D) iff $D_{\mathrm{LS}}(\Delta, D) \notin \{0, s\}$. The next result states a useful property of these functions.

Lemma III.1 (Convexity-like property of the objective function). *Let $D_1 < D_2$. Suppose that i is leading (or following, or switching) at both $D_1 e_1$ and $D_2 e_1$. Then $\partial_D J_i(\Delta, D_1) < \partial_D J_i(\Delta, D_2)$. Moreover, if i is leading or switching at $D_1 e_1$ then $\partial_D J_i(\Delta, D_1) > 0$.*

Proof: The derivative of J_i with respect to D is,

$$\partial_D J_i(\Delta, D) = \Gamma \partial_D d(De_1, \bar{x}_i) + \begin{cases} \gamma, & \text{if } i \text{ follows at } De_1, \\ \Gamma, & \text{if } i \text{ leads at } De_1, \\ \Gamma_s, & \text{otherwise.} \end{cases}$$

The function $D \mapsto d(De_1, \bar{x}_i)$ is strictly convex when $\bar{x}_i \notin \mathrm{ray}(\mathbf{0}, e_1)$. Thus, $\partial_D d(De_1, \bar{x}_i)$ is strictly increasing. Suppose that i is leading at both $D_1 e_1$ and $D_2 e_1$. Then,

$$\begin{aligned} \partial_D J_i(\Delta, D_1) &= \Gamma \partial_D d(D_1 e_1, \bar{x}_i) + \Gamma, \\ &< \Gamma \partial_D d(D_2 e_1, \bar{x}_i) + \Gamma = \partial_D J_i(\Delta, D_2). \end{aligned}$$

Similar analysis holds for when i is following (or switching) at both $D_1 e_1$ and $D_2 e_1$.

To show that $\partial_D J_i(\Delta, D_1) > 0$ when i is leading at $D_1 e_1$, let $a > 0$ be sufficiently small such that i is also leading at $(D_1 - a)e_1$. Then,

$$\begin{aligned} J_i(\Delta, D_1) = J_i(\Delta, D_1 - a) - \Gamma d((D_1 - a)e_1, \bar{x}_i) \\ + \Gamma a + \Gamma d(D_1 e_1, \bar{x}_i) > J_i(\Delta, D_1 - a), \end{aligned}$$

where we have used the triangle inequality. Since a can be taken arbitrarily small, $\partial_D J_i(\Delta, D_1) > 0$ follows. If i is switching at $D_1 e_1$, the above argument with $\Gamma_s a$ instead of Γa, together with $\Gamma_s a \geq \Gamma a$, yields the same conclusion. ■

Roughly speaking, Lemma III.1 states that it is more costly to lead (or follow or switch) in the formation as the formation progresses. The last statement in Lemma III.1 simply states that leading or switching is always costly. This is to be distinguished from following which, as we show later, decreases the cost-to-go function in an optimal VOLD.

Next, consider the cost-to-go for agent i at the k^{th} switch which, for brevity, is given by functions $J_i^k : \mathbb{R}_{>0}^k \to \mathbb{R}_{>0}$ defined for $k = 1, \ldots, N$ as,

$$J_i^k(\delta_1, \ldots, \delta_k) = \Gamma \sum_{\substack{n=1 \\ n \text{ odd}}}^{k} \delta_n + \gamma \sum_{\substack{n=2 \\ n \text{ even}}}^{k} \delta_n + s\Gamma_s(k-1) + \Gamma d\Bigg(\bigg(\sum_{n=1}^{k} \delta_n + s(k-1)\bigg)e_1, \bar{x}_i\Bigg).$$

By construction, $J_i(\Delta, D_k(\Delta)) \equiv J_i^k(\delta_1, \ldots, \delta_k)$. Analogous functions J_j^k exist such that $J_j(\Delta, D_k(\Delta)) \equiv J_j^k(\delta_1, \ldots, \delta_k)$. The following state some properties of these functions which one may prove using a similar argument as in the proof of Lemma III.1.

Lemma III.2 *For $\Delta \in \mathbb{R}_{>0}^N$,*

(P1) $\partial_{\delta_1} J_i^k(\delta_1, \ldots, \delta_k) > 0$,
(P2) $\partial_{\delta_2} J_j^k(\delta_1, \ldots, \delta_k) > 0$, *for* $k \geq 2$,
(P3) $\partial_{\delta_n} J_m^k(\delta_1, \ldots, \delta_k) = \partial_{\delta_{n+2}} J_m^k(\delta_1, \ldots, \delta_k)$, *for* $k \geq n+2$ *and* $m = i, j$,
(P4) $\partial_{\delta_n} J_m^k(\delta_1, \ldots, \delta_k) < \partial_{\delta_n} J_m^{k+2}(\delta_1, \ldots, \delta_k)$, *for* $k \geq n$ *and* $m = i, j$.

B. Properties of optimal VOLDs

This section explores an important property of the breakaway location prescribed by a solution to $\mathcal{I}$. To begin, consider any $C_i, C_j \in \mathbb{R}_{>0}$ and suppose there exists a VOLD $\Delta \in \mathbb{R}_{>0}^N$ such that $C_i = J_i^N(\Delta)$ and $C_j = J_j^N(\Delta)$ (i.e., the final cost-to-go for i and j are C_i and C_j respectively). Towards a characterization of the possible breakaway locations, consider the combined cost-to-go for i and j,

$$C_j + C_i = J_j^N(\Delta) + J_i^N(\Delta).$$

Under the change of variables $\psi = \sum_{k=1}^{N} \delta_k$ this becomes,

$$\begin{aligned} C_j + C_i = J_{i+j}^N(\psi) := (\Gamma + \gamma)\psi + 2(N-1)s\Gamma_s \\ + \Gamma d\big((\psi + s(N-1))e_1, \bar{x}_j\big) \\ + \Gamma d\big((\psi + s(N-1))e_1, \bar{x}_i\big). \end{aligned}$$

Note that ψ is related to the breakaway location of the formation by $D_N(\Delta) = \psi + s(N-1)$. In words, $J_{i+j}^N(\psi)$ is the combined cost-to-go of i and j for a formation with $N-1$ leader switches which breaks at $\psi + s(N-1)$. Therefore, without knowledge of the specific elements of Δ but given C_j, C_i, and N we can describe the possible breakaway locations of the formation implicitly in terms of all ψ which satisfy $C_j + C_i = J_{i+j}^N(\psi)$. Since J_{i+j}^N is strictly convex, there exist only two ψ_1, ψ_2 which satisfy,

$$C_j + C_i = J_{i+j}^N(\psi_1) = J_{i+j}^N(\psi_2),$$

for given C_i, C_j, N (note that ψ_1, ψ_2 may not be unique but, because we assumed that such a VOLD exists, are guaranteed to be real). We further characterize ψ_1, ψ_2 by $\psi_1 \leq \psi_{\max}^N \leq \psi_2$ where $\psi_{\max}^N = \operatorname{argmin}_\psi J_{i+j}^N(\psi)$. The following result makes known which of $\psi_1 + s(N-1)$ or $\psi_2 + s(N-1)$ is a possible breakaway location for a solution to $\mathcal{I}$. First, we introduce some notation. For $N \in \mathbb{N}$, let Q^N be the N-dimensional simplex bounded by $\psi_{\max}^N$. Formally,

$$Q^N = \left\{\Delta \in \mathbb{R}_{>0}^N : \sum_{k=1}^{N} \delta_k \leq \psi_{\max}^N\right\}.$$

If $\Delta \in Q^N$, then it must be that the breakaway location is $\psi_1 + s(N-1)$. We see next that this is indeed the case.

Proposition III.3 (UAVs breakaway as soon as possible). $\omega(N) \subset Q^N$.

The result above provides an upper bound on the breakaway distance prescribed by a VOLD which is a solution to $\mathcal{I}$. Therefore, we can restrict the feasible set of VOLDs in $\mathcal{I}$ to $\Delta \in \mathscr{D}_c^N \cap Q^N$. For $\Delta \in Q^N$, we have the additional property that a UAVs cost-to-go strictly decreases while following, as stated next.

Corollary III.4 (Following is beneficial) *If $\Delta \in Q^N$ has i following at $\hat{D}e_1$ then $\partial_D J_i(\Delta, \hat{D}) < -\partial_D J_j(\Delta, \hat{D}) < 0$.*

The above analysis also allows us to identify additional properties of the cost-to-go functions at the k^{th} switch.

Lemma III.5 *If $\Delta \in Q^N$ then,*

(P5) $\partial_{\delta_1} J_j^k(\delta_1, \ldots, \delta_k) \leq -\partial_{\delta_1} J_i^k(\delta_1, \ldots, \delta_k) < 0$,
(P6) $\partial_{\delta_2} J_i^k(\delta_1, \ldots, \delta_k) \leq -\partial_{\delta_2} J_j^k(\delta_1, \ldots, \delta_k) < 0, k \geq 2$

The results thus far are now used to state a fact about the final cost-to-go for UAV i given a solution to $\mathcal{I}$.

Lemma III.6 (**i receives its proposed cost-to-go).** *If $\Delta \in \omega(N)$ then $\hat{C}^i = J_i^N(\Delta)$.*

C. Equivalent formulation

Here, we combine the results established above to reduce the feasibility set to only those VOLDs exhibiting properties of optimal VOLDs. In particular, Lemma III.1 and Corollary III.4 reveal that, for $\Delta \in Q^N$, the local minima of J_i and J_j occur at the distances where an agent initiates a switch from following to leading. Therefore, $\Delta \in \mathscr{D}_c^N \cap Q^N$ iff,

$$J_j(\Delta, D_N(\Delta)) \leq J_j(\Delta, D_k(\Delta)) + \epsilon_j, \quad k \in [1, N-1] \cap \mathbb{O},$$
$$J_i(\Delta, D_N(\Delta)) \leq J_i(\Delta, D_k(\Delta)) + \epsilon_i, \quad k \in [2, N-1] \cap \mathbb{E}.$$

Thus we reformulate $\mathcal{I}$ as follows. For fixed $N \in \mathbb{N}$,

$$\underset{\Delta \in Q^N}{\text{minimize}} \quad C_j \tag{3a}$$

subject to

$$C_j \leq J_j^k(\delta_1, \ldots, \delta_k) + \epsilon_j, \quad k \in [1, N-1] \cap \mathbb{O}, \tag{3b}$$
$$\hat{C}_i \leq J_i^k(\delta_1, \ldots, \delta_k) + \epsilon_i, \quad k \in [2, N-1] \cap \mathbb{E}, \tag{3c}$$
$$C_j = J_j^N(\Delta), \tag{3d}$$
$$\hat{C}_i = J_i^N(\Delta), \tag{3e}$$

where we use $J_i(\Delta, D_k(\Delta)) \equiv J_i^k(\delta_1, \ldots, \delta_k)$. The next result states the equivalence of (3) and $\mathcal{I}$.

Corollary III.7 *For $N \in \mathbb{N}$, the set of solutions to* (3) *is $\omega(N)$.*

Note that the equality constraints (3d)-(3e) are not affine and substituting (3d) in (3a) and (3b) yields non-convex inequality constraints. Moreover, since $\hat{C}_i$ is a fixed quantity, (3e) cannot be combined with (3c). We deal with these issues in the next section by further restricting the feasibility set.

IV. Convex restriction of the inner optimization problem

This section proposes a method to further restrict the set of feasible VOLDs in (3) to those where (3b)-(3c) are active for $k = 1, \dots, N-2$. We show that if $\mathcal{O}$ is feasible, then there exists at least one solution with this desired structure and that, for fixed N, the restriction of the feasible set to VOLDs of this desired structure is convex.

A. Reduction of the set of solutions

We desire a solution to (3) of a specific form which are characterized by elements of the set,

$$\begin{aligned}\omega_0(N) = \{\Delta \in \omega(N) : C_j = J_j^N(\Delta),\\ C_j = J_j^k(\delta_1, \dots, \delta_k) + \epsilon_j, \quad k \in [1, N-2] \cap \mathbb{O},\\ \hat{C}_i = J_i^k(\delta_1, \dots, \delta_k) + \epsilon_i, \quad k \in [2, N-2] \cap \mathbb{E}\}.\end{aligned}$$

From an implementation standpoint, solutions in $\omega_0(N)$ have the desirable property that agents lead/follow as long as possible before initiating a leader switch. Specifically, UAV i initially leads the formation until j attains a cost-to-go of $C_j - \epsilon_j$. At that point, a switch is initiated and j leads until i attains a cost-to-go of $\hat{C}_i - \epsilon_i$. This procedure is repeated until the last switch, whose location is determined by the amount of residual leading/following necessary for i and j to simultaneously attain $\hat{C}_i$ and C_j, respectively (this corresponds to satisfying constraints (3d)-(3e)). Note that, in general, for any given $N \in \mathbb{N}$, the set $\omega_0(N)$ might be empty. However, the following result states that a solution in $\omega_0(N)$ exists for the optimization problem (2).

Proposition IV.1 (Most constraints can be active) *If $\Omega \neq \emptyset$, then there exists $N \in \mathbb{N}$ such that $\emptyset \neq \omega_0(N) \subseteq \Omega$.*

Inspired by the above, let us consider the restricted problem

$$\underset{\Delta \in Q^N}{\text{minimize}} \quad C_j \tag{4a}$$

subject to

$$C_j = J_j^k(\delta_1, \dots, \delta_k) + \epsilon_j, \quad k \in [1, N-2] \cap \mathbb{O}, \tag{4b}$$

$$\hat{C}_i = J_i^k(\delta_1, \dots, \delta_k) + \epsilon_i, \quad k \in [2, N-2] \cap \mathbb{E}, \tag{4c}$$

$$\hat{C}_i \leq J_i^{N-1}(\delta_1, \dots, \delta_{N-1}) + \epsilon_i, \tag{4d}$$

$$\hat{C}_j \leq J_j^{N-1}(\delta_1, \dots, \delta_{N-1}) + \epsilon_j, \tag{4e}$$

$$C_j = J_j^N(\Delta), \quad \hat{C}_i = J_i^N(\Delta). \tag{4f}$$

Constraints (4b)-(4c) impose the desired structure on a VOLD, whereas (4d)-(4e) ensure that cooperation is induced.

Algorithm 1: OPTIMAL COST ALGORITHM

parameters: $\Gamma, \Gamma_s, \gamma, s, \bar{x}_i, \bar{x}_j, \epsilon_i, \epsilon_j$
input : $\hat{C}_i$
output : C_j^* (optimal cost-to-go for j)

1 $k := 0$
2 **foreach** $N_0 \in \{1, 2\}$ **do**
3   **repeat**
4     $N := N_0 + 2k$
5   For N, solve convex problem (4)
6   Let $C_j^{N_0}(N)$ be the optimal value from line 5
7   **if** $(k \neq 0) \wedge (C_j^{N_0}(N) > C_j^{N_0}(N-2))$ **then**
8       $C_j^{N_0,*} := C_j^{N_0}(N-2)$ and goto line 11
9   **endif**
10     $k := k + 1$
11   **end**
12 **end**
13 $C_j^* := \min\{C_j^{1,*}, C_j^{2,*}\}$

For $N \in \mathbb{N}$ fixed, let $\sigma(N)$ denote the solution to (4) (it can be shown that $\sigma(N)$ is a singleton). Because the feasibility set of (4) is contained in the feasibility set of (3), in general $\sigma(N) \not\subset \omega(N)$ (and thus $\sigma(N) \not\subset \omega_0(N)$). Nevertheless, the following result follows from Proposition IV.1 and the fact that if $\omega_0(N) \neq \emptyset$ for some N then $\sigma(N) = \omega_0(N)$.

Corollary IV.2 *If $\Omega \neq \emptyset$, then there exists $N \in \mathbb{N}$ such that $\emptyset \neq \sigma(N) \subset \Omega$.*

The following result, combined with Corollary IV.2, shows that (4) achieves the goal of replacing the non-convex problem $\mathcal{I}$ with a convex counterpart.

Theorem IV.3 *The problem* (4) *is convex.*

This result concludes our study of the inner optimization problem: since $\mathcal{I}$ can be replaced with (4) and Theorem IV.3 states that the latter is a convex optimization problem, existing efficient techniques can be used to find an optimal solution for a given $N \in \mathbb{N}$. From the optimal value of (4), an explicit construction of the corresponding VOLD can be achieved using, for instance, the PARTITION REFINEMENT ALGORITHM [1]. This algorithm efficiently computes the optimal VOLDs as solutions to N algebraic equations.

V. Optimal number of leader switches

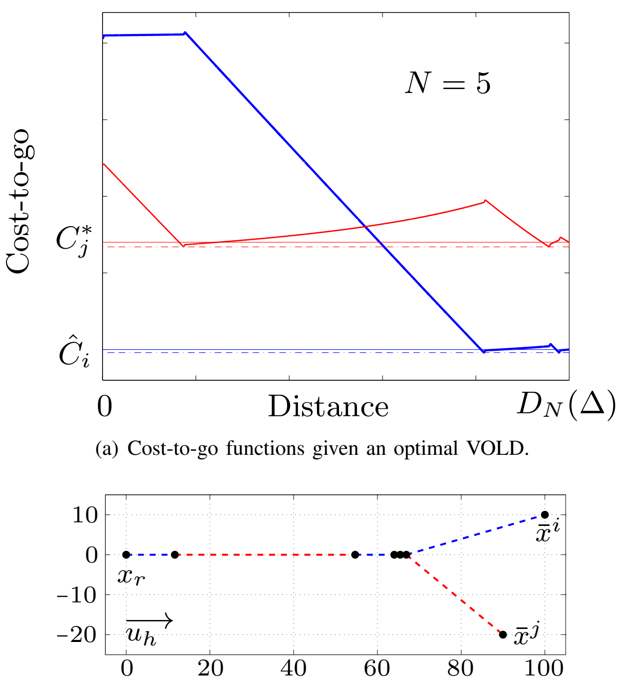
Having described how to efficiently solve the inner optimization problem $\mathcal{I}$, we turn our attention to the outer optimization problem $\mathcal{O}$. Specifically, we are interested in identifying a criterion that allows us to identify an optimal N and helps us search for it. The following result provides such a criterion. Before stating it, we note that the optimal value of (4) for a given $N \in \mathbb{N}$ can be expressed as $J_j^N(\sigma(N))$.

Theorem V.1 *If $J_j^N(\sigma(N)) < J_j^{N+2}(\sigma(N+2))$ for some $N \in \mathbb{N}$, then $J_j^N(\sigma(N)) < J_j^{N+2k}(\sigma(N+2k))$ for $k \in \mathbb{N}$.*

By Theorem V.1, if adding two switches yields a larger optimal value of (4), then it is not necessary to check the optimal value of (4) for any additional multiples of two more switches. Together with Corollary IV.2 and Theorem IV.3, this leads to the design of the OPTIMAL COST ALGORITHM.

Corollary V.2 *Let $\Delta \in \Omega$ with $N = |\Delta|$. If $\hat{C}_i$ is the input to the* OPTIMAL COST ALGORITHM *and C_j^* is the output, then $C_j^* = J_j^N(\Delta)$.*

In Figure 2, the OPTIMAL COST ALGORITHM determines the optimal cost-to-go for j and then the PARTITION REFINEMENT ALGORITHM constructs an optimal VOLD.



(a) Cost-to-go functions given an optimal VOLD.

(b) Resulting flight paths of UAVs under an optimal VOLD.

Fig. 2. An optimal VOLD which solves (2). Here, $\Gamma = 1, \Gamma_s = 1.7, \gamma = 0.5, s = 0.2, \bar{x}_i = (100, 10), \bar{x}_j = (90, -20), \epsilon_i = 0.2, \epsilon_j = 0.3$ and the input to the OPTIMAL COST ALGORITHM is $\hat{C}_i = 80$. The cost-to-go for UAV i (resp. j) is plotted in blue (resp. thinner red) and the optimal cost to j is $C_j^* = 87$. The solid horizontal lines are the final cost-to-gos while the dashed lines are $\epsilon_{i,j}$ below. The goal of a cooperation-inducing VOLD is to keep both agents' cost-to-go above the dashed line.

VI. CONCLUSIONS AND FUTURE WORK

We have studied the problem of finding optimal leader allocations for ϵ-cooperative UAVs that travel in formation towards their destination goals under costly leader switching. We have formulated this problem as a constrained optimization problem and established its lack of convexity. Our analysis of the properties of the objective function and the feasible set has allowed us to restrict the latter and turn the original problem into a combination of a convex optimization problem (that determines the optimal leader allocation given a fixed number of switches) and a discrete problem (that seeks to determine an optimal numbers of switches). Moreover, we have developed a criterion for the discrete problem that allows us to determine an optimal number of switches and helps us search for them. Future work will be devoted to (i) developing tight bounds on the numbers of optimal switches to help us better initialize the OPTIMAL COST ALGORITHM, (ii) considering formations of more than two UAVs, and (iii) incorporating the results obtained here in matching processes, where agents make and receive proposals about the possibility of joining in formation.

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