# Saturation-tolerant average consensus with controllable rates of convergence [*]

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## Abstract

This paper considers the static average consensus problem for a multi-agent system and proposes a distributed algorithm that enables individual agents to set their own rate of convergence. The algorithm has a two-time scale structure and is constructed using a singular perturbation approach. A fast information processing state uses a Laplacian consensus strategy to calculate the agreement value in a distributed manner. The slow-time dynamic part, termed motion phase, allows each agent to move towards the agreement point at its own desired speed. We provide a complete analysis of the proposed consensus algorithm. This covers the rate of convergence of individual agents, effects of communication delays, robustness to changes in the network topology, implementation in discrete time, and performance guarantees under limited control authority. Our analysis is based on tools from matrix theory, algebraic graph theory and stability analysis. Numerical examples illustrate the benefits of the proposed algorithm.

## 1 Introduction

This paper deals with the static average consensus problem for a network of agents. Given a set of static *input* signals, one per agent, this problem consists of designing distributed strategies that allow agents to obtain the average of the inputs using only their own information and communication with neighboring agents. In recent years, the average consensus problem of networked systems has attracted widespread attention due to its broad usage in a variety of applications. Examples are numerous and we only refer here to multi-vehicle coordination [13], distributed fusion in sensor networks [9], and wireless smart meters where all agents should agree on the network average power demand or consumption [2].

One approach to solve the static average consensus problem is based on reaching agreement regarding the states of $N$ agents with an integrator dynamics of the form $\dot{x}^i(t) = v^i(t)$, $i \in \{1, \ldots, N\}$, where $v^i(t)$ is set to be the weighted sum of the difference between the states of the out-neighbors of an agent and its own state. According to [10], if the network topology satisfies certain conditions, this dynamics converges, with exponential rate, to the average of the initial conditions. The static average consensus thus can be achieved if each agent initializes its dynamics with its static input. Throughout this note, we refer to this algorithm as the *Laplacian consensus* algorithm.

The simple structure of the Laplacian consensus algorithm is very appealing. The literature is by now vast and populated with works that explore a wide variety of aspects of the algorithm, both in continuous and discrete time, see [12, 5, 1] and references therein. Here we only include a few references which are most closely connected to the issues considered in the paper. Several works consider switching topologies and time delays, see e.g., [3, 10, 6, 11, 7]. Increasing the rate of convergence of the Laplacian consensus algorithm using optimal weight design and rewiring of the links of a network to create a so-called small world network are proposed in [15] and [8], respectively. These studies focus on the analysis of the stability and asymptotic convergence properties of the algorithm. In contrast, in this paper, we concentrate on the transient behavior. We make the following observations about the Laplacian consensus algorithm:

- There is no control over the transient behavior of the agents' dynamics. The collective behavior of the agents is governed by the network topology. If the communication topology changes, the transient behavior changes as well;

- The least rate of convergence of the algorithm for all agents is the same. Agents have no control over their own rate of convergence. To accommodate agents with limited control authority, the entire dynamics has to be slowed down;

- Any perturbation in the consensus command (e.g., saturation of $v^i(t)$ at any agent), corrupts the mission of the entire network, i.e., no agent reaches the intended average value.

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This paper addresses the aforementioned 'weaknesses' with an algorithm that is only slightly more complex than the Laplacian consensus algorithm. Our work is motivated by applications where the agreement state corresponds to some physical variable such as position. In such scenarios, agents might have limited control authority. At the same time, robustness to the saturation of control commands, consistent response under different communication topologies, and control over the transient response are highly desirable properties.

The proposed algorithm builds on the observation that the group of agents can use the Laplacian consensus algorithm to quickly obtain the desired average value. Once the desired average value is obtained, the agents can move towards the agreement point at their own desired rate. We call the first stage of this procedure as the *information processing* phase and the corresponding states as the *information* states. We call the second phase as the *motion* phase and its states as the *agreement* states. The innovation here is to combine the information processing and the motion phase in one dynamics using *singularly perturbed* systems, allowing us to eliminate the waiting stage for the information state to converge.

We provide a complete analysis of the proposed consensus algorithm, including the study of the rate of convergence for individual agents, the effect of communication delays, the robustness against changes in the network topology, and the implementation in discrete time. We also study the performance of the proposed algorithm under saturation in the agreement state equation. Our analysis combines matrix theory, algebraic graph theory, and stability analysis.

## 2 Preliminaries

This section gathers basic preliminaries on notation and graph theory and terminology.

**2.1 Notation.** The vector $\mathbf{1}_n$ represents an $n$-dimensional vector with all elements equal to one, and $\boldsymbol{I}_n$ represents the identity matrix with dimension $n \times n$. We denote by $\boldsymbol{A}^\top$ the transpose of matrix $\boldsymbol{A}$. For a square matrix $\boldsymbol{A}$ we define $\mathrm{Sym}(\boldsymbol{A}) = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^\top)$. We use $\mathrm{Diag}(\boldsymbol{A}_1, \cdots, \boldsymbol{A}_N)$ to represent the block-diagonal matrix constructed from matrices $\boldsymbol{A}_1, \ldots, \boldsymbol{A}_N$. The $i^{\text{th}}$ row of a matrix $\boldsymbol{A}$ is indicated by $[\boldsymbol{A}]_i$. For a vector $\boldsymbol{u}$, we use $\|\boldsymbol{u}\|$ to denote the standard Euclidean norm, i.e., $\|\boldsymbol{u}\| = \sqrt{\boldsymbol{u}^\top \boldsymbol{u}}$.

A continuous function $f : [0, a) \rightarrow [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $f(0) = 0$. We let $\delta_1(\epsilon) \in O(\delta_2(\epsilon))$ to denote the fact that there

exist positive constants $c$ and $k$ such that

$$|\delta_1(\epsilon)| \le k|\delta_2(\epsilon)|, \quad \forall |\epsilon| < c.$$

In network related variables, the local variables at each agent are distinguished by a superscript $i$, e.g., $u^i$ is the local static input of agent $i$. If $p^i \in \mathbb{R}$ is a local variable at agent $i$, the aggregated $p^i$'s are represented by $\boldsymbol{p} = (p^1, \ldots, p^N) \in \mathbb{R}^N$.

**2.2 Graph Theory.** Here, we briefly review some basic concepts from graph theory and linear algebra, see e.g. [1]. A *directed graph*, or simply a *digraph*, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \ldots, N\}$ is the *node set* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the *edge set*. We make the convention that an edge from $i$ to $j$, denoted by $(i, j)$, models the fact that agent $j$ can send information to $i$. For an edge $(i, j) \in \mathcal{E}$, $i$ is called an *in-neighbor* of $j$ and $j$ is called an *out-neighbor* of $i$. A *directed path* is an ordered sequence of vertices such that any ordered pair of vertices appearing consecutively is an edge of the digraph. A digraph is called *strongly connected* if for every pair of vertices there is a directed path between them.

A *weighted digraph* is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}})$, where $(\mathcal{V}, \mathcal{E})$ is a digraph and $\boldsymbol{\mathcal{A}} \in \mathbb{R}^{N \times N}$ is a weighted *adjacency* matrix with the property that $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. A weighted digraph is *undirected* if $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{V}$. The *weighted out-degree* and *weighted in-degree* of a node $i$, are respectively, $\mathrm{d}^{\text{in}}(i) = \sum_{j=1}^N a_{ji}$ and $\mathrm{d}^{\text{out}}(i) = \sum_{j=1}^N a_{ij}$. We let $\mathrm{d}^{\text{out}}_{\max} = \max_{i \in \{1, \ldots, N\}} \mathrm{d}^{\text{out}}(i)$ denote the maximum weighted out-degree. A digraph is *weight-balanced* if at each node $i \in \mathcal{V}$, the weighted out-degree and weighted in-degree coincide (although they might be different across different nodes).

The out-degree matrix $\boldsymbol{D}^{\text{out}}$ is the diagonal matrix whose $\boldsymbol{D}^{\text{out}}_{ii} = \mathrm{d}^{\text{out}}(i)$, for $i \in \mathcal{V}$. The *(out-) Laplacian* matrix is $\boldsymbol{L} = \boldsymbol{D}^{\text{out}} - \boldsymbol{\mathcal{A}}$. Based on the structure of $\boldsymbol{L}$, at least one of the eigenvalues of $\boldsymbol{L}$ is zero and the rest of them have nonnegative real parts. Also, $\boldsymbol{L}\mathbf{1}_N = \boldsymbol{0}$. For an undirected graph, $\boldsymbol{L}$ is a symmetric positive semidefinite matrix. For a strongly connected digraph, zero is a simple eigenvalue of $\boldsymbol{L}$. A weighted digraph $\mathcal{G}$ is weight-balanced if and only if $\mathbf{1}_N^T \boldsymbol{L} = 0$. We denote the eigenvalues of the Laplacian matrix by $\lambda_i$, $i \in \mathcal{V}$, where $\lambda_1 = 0$ and $\Re(\lambda_i) \le \Re(\lambda_j)$, for $i < j$. We denote the eigenvalues of $\mathrm{Sym}(\boldsymbol{L})$ by $\hat{\lambda}_i$, $i \in \mathcal{V}$. For a strongly connected and weight-balanced digraph, zero is a simple eigenvalue of $\mathrm{Sym}(\boldsymbol{L})$. For such a digraph, we order the eigenvalues of $\mathrm{Sym}(\boldsymbol{L})$ as $\hat{\lambda}_1 = 0 < \hat{\lambda}_2 \le \hat{\lambda}_3 \le \cdots \le \hat{\lambda}_N$.

## 3   Problem Definition

Consider a network of $N$ agents with single-integrator dynamics given by

$$(3.1) \qquad \dot{x}^i = v^i, \quad i \in \{1, \ldots, N\},$$

where $x^i \in \mathbb{R}$ is the *agreement state* and $v^i \in \mathbb{R}$ is the *driving command* of agent $i$. The network interaction topology is modeled by a weighted digraph $\mathcal{G}$. Each agent $i \in \{1, \ldots, N\}$ has a static input $u^i \in \mathbb{R}$. When $\mathcal{G}$ is a strongly connected and weight-balanced digraph, [10] showed that the first-order integrator system (3.1) with
(3.2)

$$v^i = -\sum_{j=1}^N a_{ij}\,(x^i - x^j), \quad x^i(0) = u^i, \; i \in \{1, \ldots, N\},$$

satisfies the following:

- It converges exponentially to the average of initial conditions of the agents, i.e., $\boldsymbol{x}(t) \to (\frac{1}{N}\sum_{j=1}^N u^j)\mathbf{1}_N$ as $t \to \infty$;

- The least rate of convergence of all agents are the same and it is governed by the smallest non-zero eigenvalue of $\mathrm{Sym}(\boldsymbol{L})$, i.e., $\hat{\lambda}_2$;

- For time-varying networks that remain strongly connected and weight-balanced, it converges exponentially fast to the agreement $(\frac{1}{N}\sum_{j=1}^N u^j)\mathbf{1}_N$ with the least rate of the minimum of $\hat{\lambda}_2$ of all the graph topologies;

- The discrete version of the algorithm with stepsize $\delta$, i.e., $\boldsymbol{x}(k+1) = \boldsymbol{x}(k) - \delta\boldsymbol{L}\boldsymbol{x}(k)$, converges to the agreement value $(\frac{1}{N}\sum_{j=1}^N u^j)\mathbf{1}_N$, as long as $\delta \in (0, \mathrm{d}_{\max}^{\mathrm{out}})$;

- For networks with undirected and connected graph topologies where the communication time delay $\tau$ is the same across all links, if $\tau \in (0, \pi/2\lambda_N)$ or the Nyquist plot of $\Phi(s) = \mathrm{e}^{-\tau s}/s$ has a zero encirclement around $-\lambda_i{}^{-1}$, for $i \in \{2, \ldots, N\}$, then $\boldsymbol{x}(t) \to (\frac{1}{N}\sum_{j=1}^N u^j)\mathbf{1}_N$ asymptotically as $t \to \infty$.

The algorithm (3.2) is a static average consensus algorithm, which we refer to as *Laplacian consensus algorithm*. As our brief overview above shows, this algorithm only addresses the collective behavior of the agents and is designed under the assumption of unlimited control authority. In this paper, motivated by the applications where (3.1) is a model of a physical process, we solve the following two problems. The first problem below states the desire of agents to converge to the agreement value with their own rate of convergence.

PROBLEM 1. *Let $\mathcal{G}$ be a strongly connected and weight-balanced digraph. Design a distributed average consensus algorithm such that, for each $i \in \{1, \ldots, N\}$, the agreement state $x^i$ arrives at $\frac{1}{N}\sum_{j=1}^N u^j$ with its own desired rate of convergence $\beta^i$.*

The next problem addresses the static average consensus problem for systems with limited control authority.

PROBLEM 2. *Let $\mathcal{G}$ be a strongly connected and weight-balanced digraph. Assume the driving command of every agent $i \in \{1, \ldots, N\}$ is limited by some value $\bar{v}^i \in \mathbb{R}$, i.e., $|v^i(t)| < \bar{v}^i$ for all $t \geq 0$. Design a distributed average consensus algorithm such that the agreement state $x^i$ arrives at $\frac{1}{N}\sum_{j=1}^N u^j$.*

## 4   Static Average Consensus Algorithm with Controllable Rates of Convergence

In this section, we solve Problem 1. The simplest dynamics that generates $x^i \to \frac{1}{N}\sum_{j=1}^N u^j$, exponentially with rate $\beta^i > 0$ for each agent $i$, is

$$\dot{x}^i = -\beta^i(x^i - \frac{1}{N}\sum_{j=1}^N u^j).$$

To decentralize this dynamics, each agent needs a mechanism that generates the average of the inputs in a distributed manner. Once the agents know the average, they can move towards this point with their desired rate $\beta^i$. This procedure can be realized as follows:

- Information processing phase: wait for

$$\dot{\boldsymbol{z}} = -\boldsymbol{L}\boldsymbol{z}, \quad z^i(0) = u^i,$$

  to converge to its equilibrium $\bar{z}^i = \frac{1}{N}\sum_{j=1}^N u^j$;

- Motion phase: use the resulting $\bar{z}^i$ in

$$\dot{x}^i = -\beta^i(x^i - \bar{z}^i), \quad x^i(0) \in \mathbb{R}.$$

The problem with this setup is that, it takes infinite time for $z^i(t)$ to converge to its exact equilibrium point. The aforementioned procedure can be interpreted as a two-time scales operation, a fast dynamics to generate the average and a slow dynamics to move towards the input average with the desired rate. Note that both the fast and slow dynamics are linear and exponentially stable. The framework of singularly perturbed dynamical systems offers the possibility of combining the slow and fast dynamics to avoid the wait for the fast dynamics to converge.

Consider therefore the dynamics

$$(4.3a) \quad \epsilon \, \dot{z}^i = -\sum_{j=1}^{N} a_{ij} \left( z^i - z^j \right), \quad z^i(0) = u^i \in \mathbb{R},$$

$$(4.3b) \quad \dot{x}^i = -\beta^i (x^i - z^i), \quad x^i(0) \in \mathbb{R},$$

According to the singular perturbation theorem on infinite intervals, cf. [4, Theorem 11.2], there is an $\epsilon^\star > 0$ such that for any $0 < \epsilon \leq \epsilon^\star$, the solution for each $i \in \{1, \ldots, N\}$ of the above dynamics converges to a $O(\epsilon)$-neighborhood of the solution of the slow dynamics, i.e., $|x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j|, |z^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j| < O(\epsilon)$ as $t \to \infty$. In the following, we show that, in fact, convergence is exact and exponential for any $\epsilon > 0$ and $\beta^i > 0$, $i \in \{1, \ldots, N\}$.

THEOREM 4.1. *Let $\mathcal{G}$ be strongly connected and weight-balanced digraph. Following the algorithm (4.3), for any $\epsilon > 0$ and $\beta^i > 0$, $i \in \{1, \ldots, N\}$, we have $x^i(t), z^i(t) \to \frac{1}{N}\sum_{j=1}^{N} u^j$ as $t \to \infty$, exponentially.*

*Proof.* We can rewrite (4.3) in the following compact form:

$$(4.4) \quad \begin{bmatrix} \dot{\boldsymbol{z}} \\ \dot{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} -\epsilon^{-1}\boldsymbol{L} & \boldsymbol{0} \\ \boldsymbol{B} & -\boldsymbol{B} \end{bmatrix} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{x} \end{bmatrix} = \tilde{\boldsymbol{L}} \begin{bmatrix} \boldsymbol{z} \\ \boldsymbol{x} \end{bmatrix}.$$

Here, $\boldsymbol{B} \in \mathbb{R}^{N \times N}$ is the diagonal matrix whose diagonal element $\boldsymbol{B}_{ii}$ is equal to $\beta^i$. The eigenvalues of $\tilde{\boldsymbol{L}}$ are equal to the eigenvalues of $-\epsilon^{-1}\boldsymbol{L}$ and $-\boldsymbol{B}$. i.e.,

$$(4.5) \quad 0, -\epsilon^{-1}\lambda_2, \ldots, -\epsilon^{-1}\lambda_N, -\beta^1, \ldots, -\beta^N.$$

For a strongly connected and weight-balanced digraph, $\tilde{\boldsymbol{L}}$ has a simple zero eigenvalue and the rest of the eigenvalues have negative real parts, provided $\epsilon > 0$ and $\beta^i > 0$. Therefore, (4.4) is a stable linear system. Note that $\tilde{\boldsymbol{L}}\boldsymbol{1}_{2N} = \boldsymbol{0}$, therefore, $\boldsymbol{1}_{2N}$ is the right eigenvector of $\tilde{\boldsymbol{L}}$ corresponding to its zero eigenvalue. As zero is the simple eigenvalue of $\tilde{\boldsymbol{L}}$, with corresponding eigenvector $\boldsymbol{1}_{2N}$, then the equilibrium of (4.4) is $\alpha\boldsymbol{1}_{2N}$ where $\alpha \in \mathbb{R}$. For a weight-balanced network topology, multiplying the collective $\boldsymbol{z}$ state equation from left by $\boldsymbol{1}^\top$, we obtain $\sum_{j=1}^{N} \dot{z}^j = 0$. Consequently, $\sum_{j=1}^{N} z^j = \sum_{j=1}^{N} z^j(0) = \sum_{j=1}^{N} u^j$, for $t \geq 0$. Additionally, at $t \to \infty$, we have $N\alpha = \sum_{j=1}^{N} u^j$. As a result, $\alpha = \frac{1}{N}\sum_{j=1}^{N} u^j$. This completes the proof. $\square$

Next, we show under what conditions on $\epsilon$, we can solve Problem 1.

LEMMA 4.1. *Let $\mathcal{G}$ be strongly connected and weight-balanced digraph. For given $\epsilon > 0$ and $\beta^i > 0$ such that $\beta^i \neq \epsilon^{-1}\hat{\lambda}_2$, the worst rate of convergence of the agreement state $x^i$, $i \in \{1, \ldots, N\}$, following algorithm (4.3) is $\min\{\beta^i, \epsilon^{-1}\hat{\lambda}_2\}$.*

*Proof.* We write the algorithm (4.3) in the following equivalent form:

$$(4.6a) \quad \dot{\boldsymbol{z}} = -\epsilon^{-1}\boldsymbol{L}\boldsymbol{z}, \quad z^i(0) = u^i \in \mathbb{R},$$

$$(4.6b) \quad \dot{x}^i = -\beta^i(x^i - [\boldsymbol{I}_N]_i\boldsymbol{z}), \quad x^i(0) \in \mathbb{R}.$$

We can look at $\boldsymbol{z}$ as a dynamical input to (4.6b). Therefore, for a given initial condition $x^i(0)$, the solution of (4.6b) is

$$(4.7) \quad x^i(t) = x^i(0)e^{-\beta^i t} + \beta^i \int_0^t e^{-\beta^i(t-\tau)}[\boldsymbol{I}_N]_i\boldsymbol{z}(\tau)d\tau.$$

We can write this solution as

$$x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j = (x^i(0) - \frac{1}{N}\sum_{j=1}^{N} u^j)e^{-\beta^i t} +$$

$$\beta^i \int_0^t e^{-\beta^i(t-\tau)}[\boldsymbol{I}_N]_i(\boldsymbol{z}(\tau) - \frac{1}{N}\sum_{j=1}^{N} u^j\boldsymbol{1}_N)d\tau.$$

We define $\kappa_x = |x^i(0) - \frac{1}{N}\sum_{j=1}^{N} u^j|$. Using [10, Theorem 8] for strongly connected and weight-balanced digraphs, we have the following convergence bound for (4.6a):

$$(4.8) \quad \|\boldsymbol{z}(t) - \frac{1}{N}\sum_{j=1}^{N} u^j\boldsymbol{1}_N\| \leq \kappa_z \, e^{-\epsilon^{-1}\hat{\lambda}_2 t},$$

where $\kappa_z = \|\boldsymbol{z}(0) - \frac{1}{N}\sum_{j=1}^{N} u^j\boldsymbol{1}_N\|$. Recall that $\hat{\lambda}_2$ is the smallest non-zero eigenvalue of $\text{Sym}(\boldsymbol{L})$. Thus,

$$|x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j| \leq \kappa_x e^{-\beta^i t} + \beta^i \kappa_z e^{-\beta^i t}\int_0^t e^{-(\epsilon^{-1}\hat{\lambda}_2-\beta^i)\tau}d\tau.$$

- For $\beta^i = \epsilon^{-1}\hat{\lambda}_2$:

$$|x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j| \leq \kappa_x e^{-\beta^i t} + t\,\beta^i \kappa_z e^{-\beta^i t};$$

- For $\beta^i \neq \epsilon^{-1}\hat{\lambda}_2$:

$$|x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j| \leq \kappa_x e^{-\beta^i t} +$$

$$\frac{\beta^i \kappa_z}{\beta^i - \epsilon^{-1}\hat{\lambda}_2}(e^{-\epsilon^{-1}\hat{\lambda}_2 t} - e^{-\beta^i t}).$$

Hence, for $\beta^i \neq \epsilon^{-1}\hat{\lambda}_2$, as $t \to \infty$, $x^i(t)$ goes to $\frac{1}{N}\sum_{j=1}^{N} u^j$, with an exponential rate of $\min\{\beta^i, \epsilon^{-1}\hat{\lambda}_2\}$. $\square$

COROLLARY 4.1. *A solution to Problem 1 is to follow algorithm (4.3) with $\epsilon > 0$ satisfying the following condition*

$$(4.9) \quad \epsilon < \bar{\beta}^{-1}\hat{\lambda}_2, \quad \bar{\beta} = \max\{\beta^1, \ldots, \beta^N\}.$$

**4.1 Dynamically Changing Interaction Topologies.** Here, we study the convergence of the algorithm (4.3) over time-varying interaction topologies. We consider strongly connected and weight-balanced digraphs $(\mathcal{V}, \mathcal{E})$ whose adjacency matrices have nonzero entries that are both uniformly lower and upper bounded, i.e, $\boldsymbol{\mathcal{A}} \in \mathcal{S}_{\mathcal{A}}(\mathcal{E}) = \{\boldsymbol{\mathcal{A}}|\ 0 < \underline{a} \leq a_{ij} \leq \bar{a}$ if $(i,j) \in \mathcal{E}$ otherwise $a_{ij} = 0\}$. We represent the set of all such weighted digraphs by

$$\Gamma = \{\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}}) \mid \boldsymbol{\mathcal{A}} \in \mathcal{S}_{\mathcal{A}}(\mathcal{E}), \mathbf{1}_N^\top \boldsymbol{L}(\mathcal{G}) = \boldsymbol{0},$$
$$(4.10) \quad \boldsymbol{L}(\mathcal{G})\mathbf{1}_N = \boldsymbol{0}, \text{rank}(\boldsymbol{L}(\mathcal{G})) = N - 1\}.$$

Notice that the index set associated with the elements of $\Gamma$, represented by $\mathcal{I}_\Gamma$, can have infinite cardinality. The consensus algorithm (4.3) on a network whose topology at each time belongs to $\Gamma$ becomes a linear switching system below characterized by the switching signal $\sigma : \mathbb{R} \to \mathcal{I}_\Gamma$

$$(4.11a) \quad \epsilon \dot{\boldsymbol{z}} = -\epsilon^{-1}\boldsymbol{L}(\mathcal{G}_{\sigma(t)}), \quad \boldsymbol{z}(0) = \boldsymbol{u} \in \mathbb{R}^N,$$

$$(4.11b) \quad \dot{\boldsymbol{x}} = -\boldsymbol{B}(\boldsymbol{x} - \boldsymbol{z}), \quad \boldsymbol{x}(0) \in \mathbb{R}^N,$$

where at each time $\mathcal{G}_{\sigma(t)} \in \Gamma$. Recall $\boldsymbol{B} \in \mathbb{R}^{N \times N}$ is the diagonal matrix whose diagonal element $\boldsymbol{B}_{ii}$ is equal to $\beta^i$. The following result studies its convergence.

LEMMA 4.2. *Let $\mathcal{G}_\sigma \in \Gamma$, where $\Gamma$ is given in (4.10) and $\sigma : \mathbb{R} \to \mathcal{I}_\Gamma$ is any arbitrary switching signal. Then, for any $\epsilon > 0$ and $\beta^i > 0$ such that $\beta^i \neq \epsilon^{-1} \min_{\mathcal{G}_k \in \Gamma} \{\hat{\lambda}_2(\boldsymbol{L}(\mathcal{G}_k))\}$, both $x^i(t)$ and $z^i(t)$ in (4.11) converge exponentially fast to $\frac{1}{N}\sum_{j=1}^N z^j(0) = \frac{1}{N}\sum_{j=1}^N u^j$, as $t \to \infty$, $i \in \{1, \ldots, N\}$. Furthermore, the least rate of convergence for $x^i$ is $\min\{\beta^i, \epsilon^{-1} \min_{\mathcal{G}_k \in \Gamma} \{\hat{\lambda}_2(\boldsymbol{L}(\mathcal{G}_k))\}\}$, $i \in \{1, \ldots, N\}$.*

*Proof.* Given the conditions in the statement of this lemma, using the results on the Laplacian consensus algorithm, for $\epsilon > 0$, we have $z^i(t) \to \frac{1}{N}\sum_{j=1}^N u^j$, exponentially fast, as $t \to \infty$, $i \in \{1, \ldots, N\}$, with a least rate of convergence of $\epsilon^{-1} \min_{\mathcal{G}_k \in \Gamma} \{\hat{\lambda}_2(\boldsymbol{L}(\mathcal{G}_k)\}$. Using the similar treatment in the proof of Lemma 4.1 (substitute $\min_{\mathcal{G}_k \in \Gamma}(\hat{\lambda}_2(\boldsymbol{L}(\mathcal{G}_k))$ for $\hat{\lambda}_2$ in (4.8)), we can show that $x^i(t) \to \frac{1}{N}\sum_{j=1}^N u^j$ exponentially fast as $t \to \infty$ with a worst rate of convergence of $\min\{\beta^i, \epsilon^{-1} \min_{\mathcal{G}_k \in \Gamma} \{\hat{\lambda}_2(\boldsymbol{L}(\mathcal{G}_k))\}\}$, $i \in \{1, \ldots, N\}$. $\square$

**4.2 Time Delay.** In this section, we assume that the network topology is an undirected and connected static graph. We assume that the information state of node $i$, i.e. $z^i$, passes through a communication channel $(i, j)$

with a time-delay $\tau_{ij} > 0$ before getting to node $j$. The average consensus algorithm (4.3) in this case is

$$(4.12a) \quad \epsilon\dot{z}^i(t) = -\sum_{j=1}^N a_{ij}\left(z^i(t - \tau_{ij}) - z^j(t - \tau_{ji})\right),$$

$$(4.12b) \quad \dot{x}^i(t) = -\beta^i(x^i(t) - z^i(t)),$$

where the initial conditions are $z^i(0) = u^i \in \mathbb{R}$ and $x^i(0) \in \mathbb{R}$, $i \in \{1, \ldots, N\}$. Here, we focus on the simplest possible case where the time-delays in all channels are equal to $\tau > 0$. Our main result is as follows:

LEMMA 4.3. *Consider the algorithm (4.12) with equal communication time-delay $\tau > 0$ in all links. Assume the network topology is a static, undirected, and connected graph. Then, for any $\epsilon > 0$ and $\beta^i > 0$, (4.12) globally asymptotically converges to the consensus value $\frac{1}{N}\sum_{j=1}^N u^j$ for both information and agreement states $z^i$ and $x^i$, $i \in \{1, \ldots, N\}$, if and only if either of the following equivalent conditions are satisfied:*

*(i) $\tau \in (0, \epsilon\pi/2\lambda_N)$;*

*(ii) The Nyquist plot of $\Phi(s) = e^{-\tau s}/s$ has a zero encirclement around $-\epsilon\lambda_i^{-1}$, for $i \in \{2, \ldots, N\}$.*

*Proof.* Consider the following change of variables:

$$(4.13) \quad p^i = \beta^i(x^i - z^i), \quad i \in \{1, \ldots, N\}.$$

Then, we can write (4.12b) in the equivalent form of

$$(4.14) \quad \dot{p}^i = -\beta^i p^i - \beta^i \dot{z}^i, \quad i \in \{1, \ldots, N\},$$

a dynamical system with input $\dot{z}^i(t)$. For $\beta^i > 0$, the unforced system (when $\dot{z}^i(t) \equiv 0$) of (4.14) is exponentially stable, with equilibrium at $p^i = 0$. Note that (4.14) is globally Lipschitz in $(p^i, \dot{z}^i)$. Invoking Lemma 4.6 of [4], then (4.14) is globally ISS, $i \in \{1, \ldots, N\}$. Consider (4.12a); given the conditions (i) and (ii) in the lemma, using the results on the Laplacian consensus algorithm, for $\epsilon > 0$, we have $z^i(t) \to \frac{1}{N}\sum_{j=1}^N u^j$ asymptotically, as $t \to \infty$, $i \in \{1, \ldots, N\}$. As a result, $\dot{z}^i(t)$ goes to zero as $t \to \infty$, $i \in \{1, \ldots, N\}$. For ISS systems when the input signal converges to zero as $t \to \infty$, so does the states of the system. Therefore, $p^i(t) \to 0$ as $t \to \infty$ in (4.14), i.e., $x^i(t) \to z^i(t)$ as $t \to \infty$. As a result, in (4.12), $x^i(t)$ and $z^i(t)$ both asymptotically tend to $\frac{1}{N}\sum_{j=1}^N z^j(0) = \frac{1}{N}\sum_{j=1}^N u^j$ as $t \to \infty$. $\square$

**4.3 Discrete-time Implementation.** An iterative form of (4.3), using first-order Euler discretization, can be stated as follows:

$$(4.15\text{a}) \quad z^i(k+1) = z^i(k) - \delta\epsilon^{-1}\sum_{j=1}^{N} a_{ij}\left(z^i(k) - z^j(k)\right),$$

$$(4.15\text{b}) \quad x^i(k+1) = x^i(k) - \delta(\beta^i(x^i(k) - z^i(k))),$$

where $\delta > 0$ is the stepsize. The following result, studies the convergence of this algorithm and characterizes the range of admissible stepsizes.

LEMMA 4.4. *Let $\mathcal{G}$ be strongly connected and weight-balanced digraph topology. For a given $\epsilon > 0$ and $\beta^i > 0$, $i \in \{1, \ldots, N\}$, for any $x^i(0) \in \mathbb{R}$ and $z^i(0) = u^i \in \mathbb{R}$, following (4.15) with a $\delta \in (0, \min\{\epsilon d_{\max}^{out}{}^{-1}, \bar{\beta}^{-1}\})$, where $\bar{\beta} = \max\{\beta^1, \cdots, \beta^N\}$, we have $x^i(k) \to \frac{1}{N}\sum_{j=1}^{N} u^j$ and $z^i(k) \to \frac{1}{N}\sum_{j=1}^{N} u^j$ asymptotically as $k \to \infty$, for $i \in \{1, \ldots, N\}$.*

*Proof.* The collective form of (4.15) can be written as

$$\begin{bmatrix} \boldsymbol{z}(k+1) \\ \boldsymbol{x}(k+1) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_N - \epsilon^{-1}\delta\boldsymbol{L} & \boldsymbol{0} \\ \delta\boldsymbol{B} & \boldsymbol{I}_N - \delta\boldsymbol{B} \end{bmatrix}\begin{bmatrix} \boldsymbol{z}(k) \\ \boldsymbol{x}(k) \end{bmatrix} = \boldsymbol{P}_\delta\begin{bmatrix} \boldsymbol{z}(k) \\ \boldsymbol{x}(k) \end{bmatrix}.$$

For $0 < \delta < \epsilon d_{\max}^{out}{}^{-1}$, using the Gersgorin disk theorem (see, e.g., [1]), we can show the eigenvalues of $\boldsymbol{I}_N - \epsilon^{-1}\delta\boldsymbol{L}$ are inside the unit circle in the complex plane. For a strongly connected digraph this matrix has a simple eigenvalue at 1. Because $\boldsymbol{B} > 0$, for $0 < \delta < \bar{\beta}^{-1}$, the eigenvalues of $\boldsymbol{I}_N - \delta\boldsymbol{B}$ are all located strictly inside the unit circle. As a result, for $\delta \in (0, \min\{\epsilon d_{\max}^{in}{}^{-1}, \bar{\beta}^{-1}\})$, $\boldsymbol{P}_\delta$ has an eigenvalue equal to 1 and the rest of the eigenvalues are located inside the unit circle. Therefore, $\boldsymbol{P}_\delta$ is a semi-convergent matrix, i.e., $\lim_{k\to\infty} \boldsymbol{P}_\delta^k = \bar{\boldsymbol{P}}$ where $\bar{\boldsymbol{P}}$ is a constant matrix. Therefore,

$$\lim_{k\to\infty}\left(\begin{bmatrix} \boldsymbol{z}(k+1) \\ \boldsymbol{x}(k+1) \end{bmatrix} - \begin{bmatrix} \boldsymbol{z}(k) \\ \boldsymbol{x}(k) \end{bmatrix}\right) = \lim_{k\to\infty}\left(\delta\tilde{\boldsymbol{L}}\begin{bmatrix} \boldsymbol{z}(k) \\ \boldsymbol{x}(k) \end{bmatrix}\right) = \boldsymbol{0},$$

where $\tilde{\boldsymbol{L}}$ is defined in (4.4). Recall that the nullspace of $\tilde{\boldsymbol{L}}$ is spanned by $\boldsymbol{1}_{2N}$, therefore,

$$(4.16) \quad \lim_{k\to\infty}\left(\begin{bmatrix} \boldsymbol{z}(k) \\ \boldsymbol{x}(k) \end{bmatrix}\right) = \alpha\boldsymbol{1}_{2N}, \quad \alpha \in \mathbb{R}.$$

For a weight-balanced network topology, multiplying the collective $\boldsymbol{z}$ state equation from left by $\boldsymbol{1}^\top$, we obtain $\sum_{j=1}^{N} z^j(k+1) = \sum_{i=1}^{N} z^j(k)$. Consequently, $\sum_{j=1}^{N} z^j(k) = \sum_{j=1}^{N} z^j(0) = \sum_{j=1}^{N} u^j$, for all $k$. Invoking (4.16), then at $k = \infty$ we have $N\alpha = \sum_{j=1}^{N} u^j$. As a result, $\alpha = \frac{1}{N}\sum_{j=1}^{N} u^j$. This completes the proof. $\square$

REMARK 4.1. *According to Corollary 4.1, when the desired rate of convergence $\beta^i$ of any agent $i$ is greater than $\hat{\lambda}_2$, then we are forced to use $\epsilon < \bar{\beta}^{-1}\hat{\lambda}_2 < 1$ to accommodate it. Compared to the Laplacian consensus algorithm, this results in a smaller stepsize in the corresponding discrete-time implementation. However, if the desired rates of convergence of the agents are smaller than $\hat{\lambda}_2$, one can use an $\epsilon > 1$, and use a larger stepsize in the discrete-time implementation.*

## 5  Consensus in the Presence of Saturation

In this section, we show that the static average consensus algorithm (4.3) also solves Problem 2. We start by studying the stability of the consensus algorithm (4.3) if the driving command is saturated. In this case the consensus algorithm is:

$$(5.17\text{a}) \qquad \epsilon\dot{z}^i = -\sum_{j=1}^{N} a_{ij}\left(z^i - z^j\right), \quad z^i = u^i \in \mathbb{R},$$

$$(5.17\text{b}) \qquad \dot{x}^i = -\text{sat}^i(\beta^i(x^i - z^i)), \quad x^i \in \mathbb{R},$$

where for $p \in \mathbb{R}$ and a given bound $\bar{v}^i > 0$ at agent $i$:

$$\text{sat}^i(p) = \begin{cases} p & |p| \le \bar{v}^i, \\ \text{sign}(p)\bar{v}^i & |p| > \bar{v}^i. \end{cases}$$

LEMMA 5.1. *Let $\mathcal{G}$ be strongly connected and weight-balanced digraph topology. Starting from any initial conditions, for any $\epsilon > 0$ and $\beta^i > 0$, the algorithm (5.17) makes $x^i(t)$ and $z^i(t)$ converge to $\frac{1}{N}\sum_{j=1}^{N} u^j$ as $t \to \infty$.*

*Proof.* Consider the following change of variables:

$$p^i = \beta^i(x^i - \frac{1}{N}\sum_{j=1}^{N} u^j), \quad q^i = -\beta^i(z^i - \frac{1}{N}\sum_{j=1}^{N} u^j).$$

Then, we can write (5.17b) in the following equivalent form:

$$(5.18) \qquad \dot{p}^i = -\beta^i\,\text{sat}^i(p^i + q^i), \quad i \in \{1, \ldots, N\}.$$

Following the approach in [14] and using the ISS Lyapunov function

$$V^i = \frac{1}{3\beta^i\bar{v}^i}(|p^i|)^3 + \frac{1}{2\beta^i}(p^i)^2, \quad i \in \{1, \ldots, N\},$$

one can show that (5.18) is globally ISS. Using the results on the Laplacian consensus algorithm for strongly connected and weight-balanced digraphs, in (5.17a), for $\epsilon > 0$, $z^i(t)$ tends to $\frac{1}{N}\sum_{j=1}^{N} z^j(0) = \frac{1}{N}\sum_{j=1}^{N} u^j$ as $t \to \infty$. As a result $q^i$ in (5.18) is a bounded and vanishing input signal. For ISS systems when the input
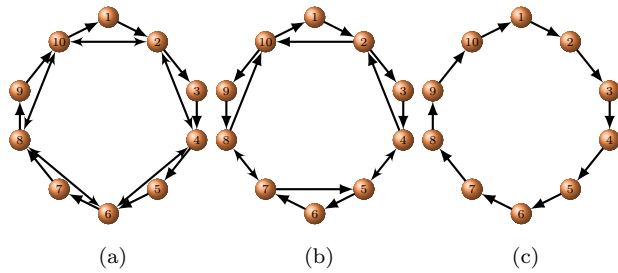
Figure 1: Network topologies

signal converges to zero as $t \to \infty$, so do the states of the system. Therefore, in (5.18), $p^i(t) \to 0$ as $t \to \infty$ i.e., $x^i(t) \to \frac{1}{N}\sum_{j=1}^{N} u^j$ as $t \to \infty$. As a result, in the algorithm (5.17), $x^i(t)$ and $z^i(t)$ both converge to $\frac{1}{N}\sum_{j=1}^{N} z^j(0) = \frac{1}{N}\sum_{j=1}^{N} u^j$ as $t \to \infty$. □

Lemma 5.1 can be extended to deal with the case of networks with dynamic interaction topologies as discussed in Section 4.1 and the case of time delays as discussed in in Section 4.2.



(a) Corresponds to Fig.1(a)

(b) Corresponds to Fig.1(b)

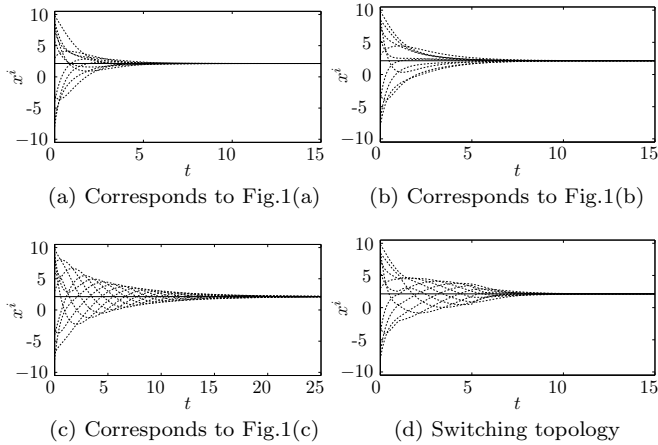(c) Corresponds to Fig.1(c)

(d) Switching topology

Figure 2: Simulation results using the Laplacian consensus algorithm.

## 6 Numerical Example

Consider the networked system with three possible communication topologies depicted in Fig. 1. The inputs in the agents are $\boldsymbol{u} = \begin{bmatrix} 9.8 & 7.5 & 3 & -4 & 9 & -8 & -3 & -7 & 8.4 & 5.7 \end{bmatrix}^\top$. Agents use edge weights 0 and 1. For the average consensus task at hand, the network can run on a static topology, selected from one of the topologies in Fig. 1, or can switch among them. The digraphs are all strongly connected and weight-balanced. The $\hat{\lambda}_2$ of

each of the digraphs shown in Fig. 1 are as follows: for Fig. 1(a) $(\hat{\lambda}_2)_a = 0.627$, for Fig. 1(b) $(\hat{\lambda}_2)_b = 0.4258$ and for Fig. 1(c) $(\hat{\lambda}_2)_c = 0.1910$.

Figure 2(a)-(c) shows the results of simulations when the Laplacian consensus algorithm (with driving command is given by (3.2)) is run over each of the networks shown in Fig. 1(a)-(c) (static case). Figure 2(d) shows the simulation results when this algorithm is run over a dynamic network with switching scenario as follows: for $t < 1$ the communication topology is the one depicted in Fig. 1(c); for $1 \le t < 5$, it is Fig. 1(b); for $t > 5$ it is Fig. 1(a). In the plots, the solid horizontal line is the average and the agreement states are represented by dashed lines. These plots show that the Laplacian consensus algorithm converges to the average of the inputs for both static and dynamic communication topologies. However, the transient response is different for each communication topology.

Figure 3 shows the simulation results for the static and dynamic topology cases explained above using the average consensus algorithm (4.3) when $\boldsymbol{B} = \text{Diag}(0.2, 0.3., \ldots, 1.1)$ and $\epsilon = 0.1$. We use the same initial conditions for the $x^i(0)$'s in all four scenarios considered – these are generated randomly in $[-20, 20]$– and we depict the simulation results in one plot. Because $\max\{\beta^1, \cdots, \beta^N\} < \epsilon^{-1}\min\{(\hat{\lambda}_2)_a, (\hat{\lambda}_2)_b, (\hat{\lambda}_2)_c\}$, according to results of Corollary 4.1 and Lemma 4.2, the dominant rate of convergence at each agent $i$, for all four communication topologies, is set by $\beta^i$ of that agent. Therefore, as shown in Fig. 3, the agreement states of each agent for all four scenarios considered have almost the same transient behavior and as expected $x^i(t) \to \frac{1}{N}\sum_{j=1}^{N} z^j(0) = \frac{1}{N}\sum_{j=1}^{N} u^j$ as $t \to \infty$ with an exponential rate of $\beta^i$, $\forall i \in \{1, \ldots, N\}$.

Next, we evaluate the performance of the algorithms when the driving command $v^i$ is bounded, i.e., $\dot{x}^i = -\text{sat}^i(v^i(t))$. The saturation bound is set to 1 for all the agents. We use $\epsilon = 1$ and $\beta^i = 1$ in our proposed consensus algorithm and initial conditions for the $x^i(0)$'s are generated randomly in $[-20, 20]$. The simulation results are shown in Fig. 4. As this figure shows, our algorithm (Fig. 4(b)) converges to the right agreement value despite the saturation, as guaranteed by Lemma 5.1. However, this is not the case for the Laplacian consensus algorithm (Fig. 4(a)).

## 7 Conclusions

We have proposed a consensus algorithm that enables individual agents to agree on the average of their static signals and set their own rate of convergence. The proposed algorithm builds on the theory of singular perturbed systems and is robust to switching topologies,
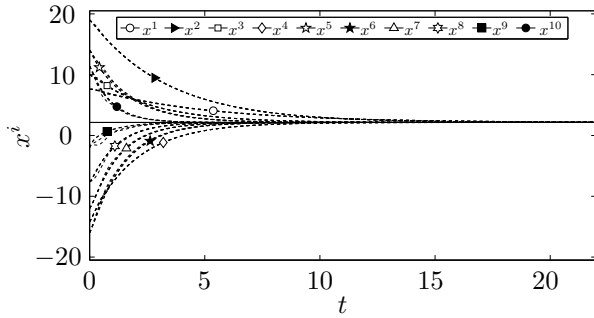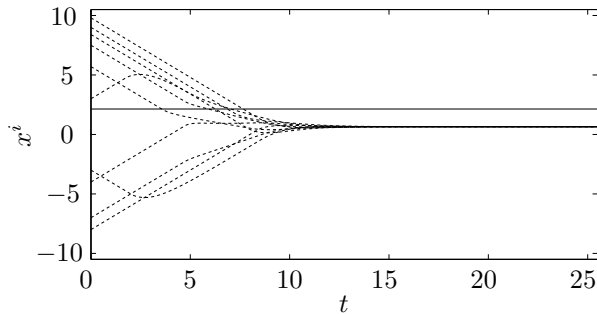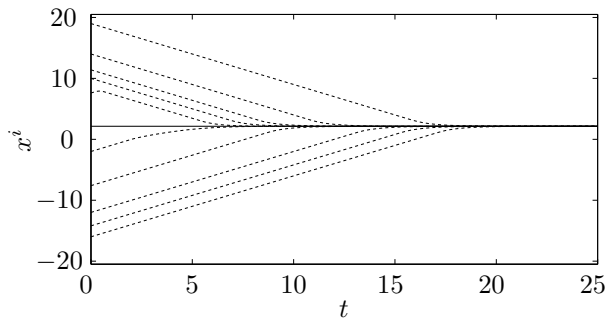
Figure 3: Simulation results using the average consensus algorithm (4.3).



(a) Laplacian consensus algorithm



(b) Consensus algorithm (4.3)

Figure 4: Simulation results when the driving command saturates.

communication time delays, and saturation. We have also studied the discrete-time implementation of the algorithm and derived bounds on the stepsize that guarantee asymptotic convergence. Future work will explore the extension of the results to dynamic signals.

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