# Singularly perturbed algorithms for dynamic average consensus

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*Abstract*— **This paper proposes two continuous-time dynamic average consensus algorithms for networks with strongly connected and weight-balanced interaction topologies. The proposed algorithms, termed $1$st-Order-Input Dynamic Consensus (*FOI-DC*) and $2$nd-Order-Input Dynamic Consensus (*SOI-DC*), respectively, allow agents to track the average of their dynamic inputs within an $O(\epsilon)$-neighborhood with a pre-specified rate. The only requirement on the set of reference inputs is having continuous bounded derivatives, up to second order for *FOI-DC* and up to third order for *SOI-DC*. The correctness analysis of the algorithms relies on singular perturbation theory for non-autonomous dynamical systems. When dynamic inputs are offset from one another by static values, we show that *SOI-DC* converges to the exact dynamic average with no steady-state error. Simulations illustrate our results.**

## I. Introduction

Given a multi-agent system and a set of time-varying input signals, one per agent, the dynamic average consensus problem consists of designing distributed algorithms that allow individual agents to obtain the average of the inputs. This problem has applications in numerous areas, including multi-robot coordination [1], distributed estimation [2], sensor fusion [3], [4], and distributed tracking [5]. In this paper, we employ a singular perturbation approach to design provably correct dynamic consensus algorithms.

*Literature review:* The work [3] generalizes the average static consensus algorithm proposed in [6] to track the average of inputs with uniformly bounded rate which are different from one another by zero-mean white Gaussian noise. The algorithm acts as a low-pass filter which allows agents to track the average of the agents' dynamic inputs with a non-zero steady sate error, which vanishes in the absence of noise. The work [7] proposes a dynamic average consensus algorithm that, from a proper initialization, is able to track with zero steady-state error the average of dynamic inputs whose Laplace transfer function has all its poles in the left half-plane, and has at most one pole at origin. In [8], a proportional dynamic average consensus algorithm can track, with a bounded non-zero steady-state error, the average of reference inputs whose weighted sum with their derivatives is bounded. For static inputs, this algorithm converges with zero-steady-state error if it is initialized properly. This work also proposes a proportional-integral (PI) algorithm which achieves dynamic average consensus, with a non-zero steady-state error, provided signals are slowly varying. The PI algorithm is generalized in [9] to achieve zero-error dynamic average consensus of special class of time-varying inputs with rational Laplace transforms and no poles in the left complex half-plane. The aforementioned algorithms are all designed in continuous time. The work [10] develops instead a discrete-time dynamic average consensus estimator that, with a proper initialization, can track with bounded steady-state error the average of the time-varying inputs whose $n$th-order difference is bounded. The solutions to the dynamic average consensus problem mentioned above each suffer from at least one of the following shortcomings: they require proper initializations that makes them prone to initialization errors and not robust to changes to agents joining and leaving the network; or they relay on knowledge of the dynamics generating the inputs at each agent therefore they are tailored to specified classes of inputs which limits their applicability. In contrast, the algorithms proposed in this paper do not suffer from these shortcomings.

*Statement of contributions:* The starting point for our algorithm design is the following observation: given a hypothetical static average consensus algorithm able to converge 'infinitely' fast, one could solve the dynamic average consensus problem by running it at each time. In practice, however, some time is required for information to flow across the network, and hence the result of the repeated application of any static average consensus algorithm will operate with some error whose size depends on its speed of convergence and how fast inputs change. A follow-up observation is that, in some applications, the task is not just to obtain the average of the dynamic inputs but rather to physically track this value, possibly with limited control authority. In these cases, high rate algorithms might not be implementable. We propose two dynamic average consensus algorithms (termed $1$*st-Order-Input Dynamic Consensus* (*FOI-DC*) and $2$*nd-Order-Input Dynamic Consensus* (*SOI-DC*)) whose design incorporates two time scales, a fast and a slow one. The fast dynamics, which builds on the PI algorithm mentioned above, acts in a similar way to a static average consensus with a high rate of convergence that is able to compute the dynamic input average at each time. The slow dynamics allows agents to track this average at a feasible rate. The novelty here is that these slow and fast dynamics are running simultaneously and, thus, there is no need to wait for convergence of the fast dynamics and then take slow steps towards the input average. Our technical approach uses singular perturbation theory to study the algorithm convergence. We show that, when the derivatives of the agents' inputs are continuous and bounded (up to second order for *FOI-DC* and up to third order for

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*FOI-DC*), the algorithms converge to an $O(\epsilon)$-neighborhood of the dynamic input average. Here, $\epsilon$ is a design parameter. Our algorithms do not require any specific initialization and do not relay on knowledge of the dynamics generating the inputs. We also show how an appropriate variation of our algorithms allows each agent to converge at their own desired rate of convergence. Simulations illustrate our results.

*Organization:* Section II introduces basic notation and concepts from graph theory, static consensus, and singular perturbation theory. Section III presents the problem statement. Section IV motivates the use of singular perturbation theory to solve the dynamic average consensus problem. Section V introduces two novel dynamic average consensus algorithms and analyzes their correctness. Section VI illustrates their performance in simulation.

## II. Preliminaries

This section gathers basic preliminaries on notation, graph theory, static consensus and singularly perturbed dynamical systems.

### A. Notation

The vector $\mathbf{1}_n$ represents a $n$-dimensional vector with all elements equal to one, and $\boldsymbol{I}_n$ represents the identity matrix with dimension $n\times n$. We denote by $\boldsymbol{A}^\top$ the transpose of matrix $\boldsymbol{A}$. We use $\mathrm{Diag}(\boldsymbol{A}_1, \cdots, \boldsymbol{A}_N)$ to represent the block-diagonal matrix constructed from matrices $\boldsymbol{A}_1, \dots, \boldsymbol{A}_N$. We let $\delta_1(\epsilon) \in O(\delta_2(\epsilon))$ denote the fact that there exist positive constants $c$ and $k$ such that

$$|\delta_1(\epsilon)| \leq k|\delta_2(\epsilon)|, \quad \forall\, |\epsilon| < c.$$

For network related variables, the local variables at each agent are distinguished by a superscript $i$, e.g., $u^i$ is the local input of agent $i$. We denote the aggregate vector of local variables $p^i$'s by $\boldsymbol{p} = (p^1, \dots, p^N) \in \mathbb{R}^N$.

### B. Graph theory

Here, we briefly review some basic concepts from graph theory and linear algebra following [11]. A *directed graph*, or simply a *digraph*, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ is the *node set* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the *edge set*. We make the convention that an edge from $i$ to $j$, denoted by $(i, j)$, models the fact that agent $j$ can send information to $i$. For an edge $(i, j) \in \mathcal{E}$, $i$ is called an *in-neighbor* of $j$ and $j$ is called an *out-neighbor* of $i$. A *directed path* is an ordered sequence of vertices such that any ordered pair of vertices appearing consecutively is an edge of the digraph. A digraph is called *strongly connected* if for every pair of vertices there is a directed path between them.

A *weighted digraph* is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \boldsymbol{\mathcal{A}})$, where $(\mathcal{V}, \mathcal{E})$ is a digraph and $\boldsymbol{\mathcal{A}} \in \mathbb{R}^{N\times N}$ is a weighted *adjacency* matrix with the property that $a_{ij} > 0$ if $(i, j) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. A weighted digraph is *undirected* if $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{V}$. The *weighted out-degree* and *weighted in-degree* of a node $i$, are respectively, $\mathrm{d}^{\mathrm{in}}(i) = \sum_{j=1}^N a_{ji}$ and $\mathrm{d}^{\mathrm{out}}(i) = \sum_{j=1}^N a_{ij}$. A digraph is *weight-balanced* if at each node $i \in \mathcal{V}$, the weighted out-degree and weighted in-degree coincide (although they might be different at different nodes).

The out-degree matrix $\boldsymbol{D}^{\mathrm{out}}$ is the diagonal matrix whose $\boldsymbol{D}^{\mathrm{out}}_{ii} = \mathrm{d}^{\mathrm{out}}(i)$, for $i \in \mathcal{V}$. The *(out-) Laplacian* matrix is $\boldsymbol{L} = \boldsymbol{D}^{\mathrm{out}} - \boldsymbol{\mathcal{A}}$. Based on the structure of $\boldsymbol{L}$, at least one of the eigenvalues of $\boldsymbol{L}$ is zero and the rest of them have nonnegative real parts. Also, $\boldsymbol{L}\mathbf{1}_N = \mathbf{0}$. For a strongly connected digraph, zero is a simple eigenvalue of $\boldsymbol{L}$. A digraph $\mathcal{G}$ is weight-balanced if and only if $\mathbf{1}_N^T\boldsymbol{L} = 0$.

### C. Static consensus

Here, we briefly review the solution given in [8] to find in a distributed way the average of a set of static inputs. Let $\mathcal{G}$ be a strongly connected and weight-balanced digraph. Assume each node $i \in \{1, \dots, N\}$ has access to a static input $u^i \in \mathbb{R}$. Consider the dynamics

$$\begin{aligned} \dot{x}^i &= -(x^i - u^i) - \textstyle\sum_{j=1}^N \boldsymbol{L}_{ij}x^j - \sum_{j=1}^N \boldsymbol{L}_{ij}\nu^j, \\ \dot{\nu}^i &= \textstyle\sum_{j=1}^N \boldsymbol{L}_{ji}x^j. \end{aligned} \tag{1}$$

The paper [8] establishes that, starting from any initial condition $x^i(0), \nu^i(0) \in \mathbb{R}$, the variable $x^i$ converges to $\frac{1}{N}\sum_{j=1}^N u^j$ exponentially fast for all $i \in \{1, \dots, N\}$. Note that the distributed implementation of this algorithm requires each agent to know the weights of its in-neighbors. If the graph is undirected, this requirement is trivially satisfied.

### D. Singularly perturbed dynamical systems

Here we give a short account of the terminology employed in singularly perturbed dynamical systems following [12, Chapter 11]. Let

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{z}, \epsilon), \qquad \boldsymbol{x}(t_0) = \boldsymbol{\eta}(\epsilon), \tag{2a}$$

$$\epsilon\, \dot{\boldsymbol{z}} = \boldsymbol{g}(t, \boldsymbol{x}, \boldsymbol{z}, \epsilon), \qquad \boldsymbol{z}(t_0) = \boldsymbol{\zeta}(\epsilon). \tag{2b}$$

The use of a small constant $\epsilon > 0$ creates two-time scales in the system, resulting into a fast and a slow dynamics. Singular perturbation theory establishes precise conditions under which the behavior of the system follows that of the limiting system when $\epsilon$ goes to 0. We assume that $\boldsymbol{f}$ and $\boldsymbol{g}$ are continuously differentiable in their arguments for $(t, \boldsymbol{x}, \boldsymbol{z}, \epsilon) \in [0, \infty) \times D_x \times D_z \times [0, \epsilon_0]$, where $D_x \subset \mathbb{R}^n$ and $D_z \subset \mathbb{R}^m$ are open connected sets. When we set $\epsilon = 0$ in (2), the dimension of the state equation reduces from $n+m$ to $n$ because the differential equation (2b) degenerates into the algebraic equation

$$\mathbf{0} = \boldsymbol{g}(t, \boldsymbol{x}, \boldsymbol{z}, 0). \tag{3}$$

We say that the model (2) is in standard form if (3) has $k \geq 1$ isolated real roots

$$\boldsymbol{z}_i = \boldsymbol{h}_i(t, \boldsymbol{x}), \quad i \in \{1, \dots, k\}, \tag{4}$$

for each $(t, \boldsymbol{x}) \in [0, \infty) \times D_x$. This assumption assures that a well-defined $n-$dimensional *reduced model* (slow dynamics) will correspond to each root of (3). To obtain the $i$th reduced model, we substitute (4) into (2a), at $\epsilon = 0$, to obtain

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}, \boldsymbol{h}(t, \boldsymbol{x}), 0), \tag{5}$$

where we have dropped the subscript $i$ from $\boldsymbol{h}$. The *boundary-layer* system (fast dynamics) is

$$\frac{d\boldsymbol{z}}{d\tau} = \boldsymbol{g}(t, \boldsymbol{x}, \boldsymbol{z}, 0), \qquad \tau = \frac{t}{\epsilon}, \tag{6}$$

where $\boldsymbol{x}$ and $t$ are treated as fixed parameters. The stability and convergence properties of these dynamical systems can be established via [12, Theorem 11.2].

## III. Problem Statement

We consider a network of $N$ agents with single-integrator dynamics given by

$$\dot{x}^i = c^i, \quad i \in \{1, \ldots, N\}, \tag{7}$$

where $x^i \in \mathbb{R}$ is the *agreement state* and $c^i \in \mathbb{R}$ is the *driving command* of agent $i$. The network interaction topology is modeled by a weighted digraph $\mathcal{G}$. Agent $i \in \{1, \ldots, N\}$ has access to a time-varying input signal $u^i : [0, \infty) \to \mathbb{R}$. The problem we seek to solve is stated next.

*Problem 1: (Dynamic average consensus with pre-specified least rate of convergence):* Let $\mathcal{G}$ be strongly connected and weight-balanced. Design a distributed algorithm such that each $x^i$ in (7) *tracks* the average $\frac{1}{N}\sum_{j=1}^N u^j(t)$ of the inputs with a convergence rate less than or equal to $\beta > 0$, i.e., $\exists \kappa > 0$ such that $|x^i(t) - \frac{1}{N}\sum_{j=1}^N u^j(t)| \leq \kappa |x^i(0) - \frac{1}{N}\sum_{j=1}^N u^j(0)| \mathrm{e}^{-\beta t}$ for all $t \geq 0$.

For vector-valued inputs, one can apply the solution of Problems 1 in each dimension independently. Note that the algorithm (1) provides a solution to Problem 1 for static inputs and no pre-specified rate of convergence. It is worth noticing the fact that $\beta$ in the problem statement is an upper bound of the convergence rate, not a lower bound. This is motivated by scenarios where agents have limited control authority and cannot implement arbitrary driving commands dictated by the consensus algorithm. This is normally the case when (7) corresponds to a model of a physical process.

## IV. Motivation to Use Singular Perturbation Theory for the Design of Consensus Algorithms

In this section, we explain the rationale behind the design of our algorithmic solutions to solve Problem 1. The simplest dynamics that achieves for each agent $x^i(t) \to \frac{1}{N}\sum_{j=1}^N u^j(t)$, as $t \to \infty$, exponentially fast with rate $\beta$, is the following

$$\dot{x}^i = -\beta\Big(x^i - \frac{1}{N}\sum_{j=1}^N u^j(t)\Big) + \frac{1}{N}\sum_{j=1}^N \dot{u}^i(t).$$

To decentralize this dynamics, we can make use of a mechanism that generates the average of the inputs and also the average of the derivative of inputs, *rapidly*, in each agent in a distributed fashion. Then, the distributed dynamic consensus algorithm becomes a two-time scale operation, a fast dynamics to generate each average and a slow dynamics to track the input average. Such dynamics could be realized by means of the following mixed discrete/continuous-time algorithm running synchronously at each node $i \in \{1, \ldots, N\}$.

1: (Initialization) at $k = 0$ initialize $x^i(0) \in \mathbb{R}^n$
2: **while** data exists **do**
3: Obtain inputs $u^i(k)$ and $\dot{u}^i(k)$
4: Initialize $z^i(0), \nu^i(0) \in \mathbb{R}$
5: Solve the following dynamical equation

$$\begin{cases} \dot{z}^i(t) = & -(z^i(t) + \beta u^i(k) + \dot{u}^i(k)) \\ & -\sum_{i=j}^N \boldsymbol{L}_{ij}(z^j(t) + \nu^j(t)), \\ \dot{\nu}^i(t) = & \sum_{j=1}^N \boldsymbol{L}_{ji} z^j(t), \end{cases} \tag{8}$$

6: Let $z^i$ converge to equilibrium $\bar{z}^i$
7: Define:

$$x^i(k+1) = x^i(k) - \Delta t \beta \left(x^i(k) + \bar{\boldsymbol{z}}^i(k)\right) \tag{9}$$

8: $k \leftarrow k + 1$
9: **end while**

In the above algorithm $\Delta t$ is the stepsize. Building on the discussion in Section II-C, at each timestep $k$, the dynamical system (8) acts as a static consensus algorithm with static input $\beta u^i(k) + \dot{u}^i(k)$. This algorithm converges exponentially to $\bar{z}^i(k) = -\frac{1}{N}\sum_{j=1}^N(\beta u^i(k) + \dot{u}^i(k))$. Therefore, at any timestep $k$, for all $i \in \{1, \ldots, N\}$, (9) becomes

$$x^i(k+1) = x^i(k) - \Delta t \beta\Big(x^i(k) - \frac{1}{N}\sum_{j=1}^N (u^i(k) + \dot{u}^i(k))\Big).$$

For small $\Delta t$, the stability and convergence of the above difference equation can be studied using the following continuous-time model

$$\dot{y}^i = -\beta\, y^i, \quad i \in \{1, \ldots, N\}. \tag{10}$$

where

$$y^i = x^i - \frac{1}{N}\sum_{j=1}^N u^j, \quad i \in \{1, \ldots, N\}. \tag{11}$$

The dynamical system (10) is a stable linear system with eigenvalue $-\beta$. Therefore, it converges to zero exponentially fast with rate $\beta$. As a result, $x^i$ in (9) converges to $\frac{1}{N}\sum_{j=1}^N u^j(t)$ exponentially, for all $i \in \{1, \ldots, N\}$.

The aforementioned algorithm solves Problem 1, provided (8) converges to its equilibrium in a time interval with length $\Delta t$. Hence, this algorithm is only conceptual: the cost of solving (8) at each timestep makes it un-implementable. Inspired by the multi-time scale structure observed above, we use singular perturbation theory to weave together steps 5–7 and devise a continuous-time dynamic average consensus algorithm. By doing so, we avoid solving the fast dynamics at each iteration, i.e., the slow dynamics does not need to wait for the fast dynamics to converge.

## V. Dynamic Continuous-time Consensus algorithms via Singularly Perturbed Dynamics

In this section, we present novel continuous-time dynamic average consensus algorithms whose design is based on singular perturbation theory. For $\beta > 0$ and $i \in \{1, \dots, N\}$, consider the following dynamical systems

- 1*st-Order-Input Dynamic Consensus* (*FOI-DC*):

$$\begin{cases} \epsilon\,\dot{z}^i = -(z^i + \beta\,u^i + \dot{u}^i) - \sum_{i=j}^{N} \boldsymbol{L}_{ij}(z^j + \nu^j), \\ \epsilon\,\dot{\nu}^i = \sum_{j=1}^{N} \boldsymbol{L}_{ji} z^j, \end{cases} \tag{12a}$$

$$\dot{x}^i = -\beta\,x^i - z^i, \tag{12b}$$

- 2*nd-Order-Input Dynamic Consensus* (*SOI-DC*):

$$\begin{cases} \epsilon\,\dot{z}^i = -(z^i + \beta\,u^i + \dot{u}^i) - \sum_{j=1}^{N} \boldsymbol{L}_{ij}(z^j + \nu^j) \\ \qquad - \epsilon(\beta\,\dot{u}^i + \ddot{u}^i), \\ \epsilon\,\dot{\nu}^i = \sum_{j=1}^{N} \boldsymbol{L}_{ji} z^j, \end{cases} \tag{13a}$$

$$\dot{x}^i = -\beta\,x^i - z^i, \tag{13b}$$

The following result establishes in what sense 1*st-Order-Input Dynamic Consensus* solves Problem 1.

*Theorem 5.1 (Convergence of FOI-DC):* Let $\mathcal{G}$ be a strongly connected and weight-balanced digraph. Assume that the first and the second derivatives of the input signal $u^i$ at each agent $i \in \{1, \dots, N\}$ are continuous and bounded for $t \geq 0$. Then, there exists $\epsilon^\star > 0$ such that, for all $\epsilon \in (0, \epsilon^\star]$, starting from any initial conditions $\boldsymbol{x}(0), \boldsymbol{z}(0), \boldsymbol{\nu}(0) \in \mathbb{R}^N$, the state $x^i$, $i \in \{1, \dots, N\}$, of the algorithm (12) converges exponentially fast with rate $\beta$ to an $O(\epsilon)$-neighborhood of $\frac{1}{N}\sum_{j=1}^{N} u^j(t)$.

*Proof:* We show the algorithm satisfies the conditions of [12, Theorem 11.2] globally (for the terminology used here we refer to Section II-D). The boundary-layer (fast) dynamics of the algorithm (12) is, for $i \in \{1, \dots, N\}$,

$$\begin{aligned} \tfrac{d\,z^i}{d\tau} &= -(z^i + \beta\,u^i(t) + \dot{u}^i(t)) - \textstyle\sum_{i=1}^{N} \boldsymbol{L}_{ij}(z^j + \nu^j), \\ \tfrac{d\,\nu^i}{d\tau} &= \textstyle\sum_{i=1}^{N} \boldsymbol{L}_{ji} z^j. \end{aligned}$$

Invoking the discussion in Section II-C, this fast dynamics globally exponentially converges to

$$z^i = -\frac{1}{N}\sum_{j=1}^{N}(\beta\,u^i + \dot{u}^i), \quad i \in \{1, \dots, N\}. \tag{14}$$

Substituting (14) into (12b), and using the change of variables (11), we obtain (10) as the reduced system (slow dynamics) model. For $\beta > 0$, (10) is a stable linear system with system matrix eigenvalue equal to $-\beta$. Thus, for all $i \in \{1, \dots, N\}$, $y^i(t)$ converges globally exponentially fast to zero with a rate of $\beta$, which $\forall t \geq 0$ is equivalent to

$$|x^i(t) - \frac{1}{N}\sum_{j=1}^{N} u^j(t)| \leq |x^i(0) - \frac{1}{N}\sum_{j=1}^{N} u^j(0)|\mathrm{e}^{-\beta t}. \tag{15}$$

Based on the required conditions for input signals, the algorithm *FOI-DC* satisfies the differentiability and Lipschitz conditions of [12, Theorem 11.2] on any compact set of $(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\nu})$. Thus, all the conditions of [12, Theorem 11.2] are satisfied globally. As a result, for all $i \in \{1, \dots, N\}$, $|x^i(t,\epsilon) - x^i(t)| \leq O(\epsilon)$ where $x^i(t,\epsilon)$ is the solution of the singularly perturbed system (12) and $x^i(t)$ is the solution of the slow dynamics. Recall (15), then for all $i \in \{1, \dots, N\}$ and all $t \geq 0$ we have

$$|x^i(t,\epsilon) - \frac{1}{N}\sum_{j=1}^{N} u^j(t)| < O(\epsilon) + |x^i(0) - \frac{1}{N}\sum_{j=1}^{N} u^j(0)|\mathrm{e}^{-\beta t},$$

which concludes our proof. ■

The following result establishes in what sense 2*nd-Order-Input Dynamic Consensus* solves Problem 1.

*Theorem 5.2 (Convergence of SOI-DC):* Let $\mathcal{G}$ be a strongly connected and weight-balanced digraph. Assume that the first, second, and third derivatives of the input signal $u^i$ at each agent $i \in \{1, \dots, N\}$ are continuous and bounded for $t \geq 0$. Then, there exists $\epsilon^\star > 0$ such that, for all $\epsilon \in (0, \epsilon^\star]$, starting from any initial conditions $\boldsymbol{x}(0), \boldsymbol{z}(0), \boldsymbol{\nu}(0) \in \mathbb{R}^N$, the state $x^i$, $i \in \{1, \dots, N\}$, of the algorithm (13) converges exponentially fast with rate $\beta$ to an $O(\epsilon)$-neighborhood of $\frac{1}{N}\sum_{j=1}^{N} u^j(t)$.

*Proof:* The proof of this result is very similar to the proof of Theorem 5.1 so we only provide a brief sketch. Notice that the parallelism between the slow and fast dynamics of *SOI-DC* and *FOI-DC*. As shown in the proof of Theorem 5.1, these dynamics are both globally exponentially stable. Based on the required conditions for the input signals, the algorithm *SOI-DC* also satisfies the differentiability and Lipschitz conditions of [12, Theorem 11.2] on any compact set of $(\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{\nu})$. Thus, all the conditions of [12, Theorem 11.2] are satisfied globally. ■

It is worth noticing that, with respect to the algorithms available in the literature, both (12) and (13) perform tracking with a pre-specified rate of convergence, can handle arbitrary initial conditions (and are therefore robust to initialization errors) and do not require any knowledge of the dynamics generating the inputs. In the following, we show that the algorithm *SOI-DC* has some advantages over *FOI-DC* at the expense of the extra condition on the input signals.

*Lemma 5.1: (SOI-DC for inputs offset by a static value):* Let $\mathcal{G}$ be a strongly connected and weight-balanced digraph. Assume that the difference in the input signals is a static offset, i.e., $u^i(t) = u_c(t) + \bar{u}^i$, where $\bar{u}^i$ is a constant scalar for each $i \in \{1, \dots, N\}$. Then, starting from any initial conditions $\boldsymbol{x}(0), \boldsymbol{z}(0), \boldsymbol{\nu}(0) \in \mathbb{R}^N$, for any $\epsilon > 0$ and $\beta > 0$, the algorithm *SOI-DC* converges exponentially fast to the exact input average, i.e., $x^i(t) \to \frac{1}{N}\sum_{j=1}^{N} u^j(t)$ as $t \to \infty$ for all $i \in \{1, \dots, N\}$.

*Proof:* Consider the following change of variables:

$$\boldsymbol{p} = \boldsymbol{z} + (\beta\,u_c + \dot{u}_c)\mathbf{1}_N,\ \boldsymbol{q} = \boldsymbol{x} - u_c\mathbf{1}_N,\ \boldsymbol{\eta} = \begin{bmatrix}\boldsymbol{r} & \boldsymbol{R}\end{bmatrix}^\top \boldsymbol{\nu},$$

where $\boldsymbol{r} = \frac{1}{\sqrt{N}}\mathbf{1}$ and $\boldsymbol{R} \in \mathbb{R}^{N\times N-1}$ satisfies $\boldsymbol{R}^\top \boldsymbol{R} = \boldsymbol{I}_{N-1}$ and $\boldsymbol{r}^\top \boldsymbol{R} = \mathbf{0}$. We let $\boldsymbol{\eta} = (\eta_1, \boldsymbol{\eta}_{2:N})$ where $\eta_1 \in \mathbb{R}$ and $\boldsymbol{\eta}_{2:N} \in \mathbb{R}^{N-1}$. Then, we can re-write (13) as follows

$$\epsilon\, \dot{\boldsymbol{p}} = -(\boldsymbol{p} + \beta\, \bar{\boldsymbol{u}}) - \boldsymbol{L}\boldsymbol{p} - \boldsymbol{L}\boldsymbol{R}\boldsymbol{\eta}_{2:N}, \tag{16a}$$

$$\epsilon\, \dot{\boldsymbol{\eta}}_{2:N} = \boldsymbol{R}^\top \boldsymbol{L}^\top \boldsymbol{p}, \tag{16b}$$

$$\epsilon \dot{\eta}_1 = 0, \tag{16c}$$

$$\dot{\boldsymbol{q}} = -\beta\, \boldsymbol{q} - \boldsymbol{p}. \tag{16d}$$

We can show that the equilibrium point of this system is ($\bar{\boldsymbol{p}} = -(\frac{\beta}{N}\sum_{j=1}^N \bar{u}^j)\mathbf{1}_N,\ \bar{\boldsymbol{\eta}}_{2:N} = -\beta(\boldsymbol{R}^\top \boldsymbol{L}\boldsymbol{R})^{-1}\boldsymbol{R}^\top \bar{\boldsymbol{u}},\ \bar{\eta}_1 = \frac{1}{N}\sum_{j=1}^N \nu^j(0),\ \bar{\boldsymbol{q}} = (\frac{1}{N}\sum_{j=1}^N \bar{u}^j)\mathbf{1}_N$). Consider the following Lyapunov function where $\tilde{\boldsymbol{q}} = \boldsymbol{q} - \bar{\boldsymbol{q}}$, $\tilde{\boldsymbol{p}} = \boldsymbol{p} - \bar{\boldsymbol{p}}$ and $\tilde{\boldsymbol{\eta}}_{2:N} = \boldsymbol{\eta}_{2:N} - \bar{\boldsymbol{\eta}}_{2:N}$.

$$V = \frac{\beta}{2}\tilde{\boldsymbol{q}}^\top \tilde{\boldsymbol{q}} + \frac{\epsilon}{8}\tilde{\boldsymbol{p}}^\top \tilde{\boldsymbol{p}} + \frac{\epsilon}{8}\tilde{\boldsymbol{\eta}}_{2:N}^\top \tilde{\boldsymbol{\eta}}_{2:N}.$$

The derivative of this Lyapunov function along the trajectories of (16a), (16b) and (16d) is

$$\dot{V} = -\frac{1}{8}\tilde{\boldsymbol{p}}^\top (\boldsymbol{L} + \boldsymbol{L}^\top)\tilde{\boldsymbol{p}} - (\frac{1}{2}\tilde{\boldsymbol{p}} + \beta\, \tilde{\boldsymbol{q}})^\top (\frac{1}{2}\tilde{\boldsymbol{p}} + \beta\, \tilde{\boldsymbol{q}}),$$

which for strongly connected digraph, it is negative semi-definite. For a strongly connected and weight-balanced digraph, we have $\mathcal{S} = \{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}} \in \mathbb{R}^N, \tilde{\boldsymbol{\eta}}_{2:N} \in \mathbb{R}^{N-1} \,|\, \dot{V} = 0\} = \{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{q}} \in \mathbb{R}^N, \tilde{\boldsymbol{\eta}}_{2:N} \in \mathbb{R}^{N-1} \,|\, \tilde{\boldsymbol{p}} = \alpha\mathbf{1}_N,\ \tilde{\boldsymbol{p}} = -2\beta\, \tilde{\boldsymbol{q}},\ \alpha \in \mathbb{R}\}$. Next, we show that no solution of (16a), (16b) and (16d) can stay in $\mathcal{S}$ except $\{\tilde{\boldsymbol{p}} = \mathbf{0},\ \tilde{\boldsymbol{q}} = \mathbf{0},\ \tilde{\boldsymbol{\eta}}_{2:N} = \mathbf{0}\}$. A trajectory $t \mapsto (\boldsymbol{p}(t), \boldsymbol{q}(t), \boldsymbol{\eta}_{2:N}(t))$ belonging to $\mathcal{S}$ must satisfy $\tilde{\boldsymbol{p}}(t) \equiv \alpha(t)\mathbf{1}_N$ and $\tilde{\boldsymbol{p}}(t) \equiv -2\beta\, \tilde{\boldsymbol{q}}(t)$. Then, (16a), (16b) and (16d) become, respectively,

$$\epsilon\dot{\alpha}\mathbf{1}_N = -\alpha\mathbf{1}_N - \boldsymbol{L}\boldsymbol{R}\tilde{\boldsymbol{\eta}}_{2:N}, \tag{17a}$$

$$\epsilon\dot{\tilde{\boldsymbol{\eta}}}_{2:N} = \mathbf{0}, \tag{17b}$$

$$\dot{\alpha}\mathbf{1}_N = \beta\alpha\mathbf{1}_N. \tag{17c}$$

From (17b), $t \mapsto \boldsymbol{\eta}_{2:N}(t)$ must be constant. Recall that for strongly connected and weight-balanced digraphs $\boldsymbol{R}^\top \boldsymbol{L}\boldsymbol{R}$ is invertible. Therefore, multiplying (17a) by $\boldsymbol{R}^\top$ from the left, we conclude that $\tilde{\boldsymbol{\eta}}_{2:N} = \mathbf{0}$. As a result, from (17a) and (17c) we deduce that $t \mapsto \alpha(t) = 0$. In other words, $\{\tilde{\boldsymbol{p}} = \mathbf{0},\ \tilde{\boldsymbol{q}} = \mathbf{0},\ \tilde{\boldsymbol{\eta}}_{2:N} = \mathbf{0}\}$ is the only solution of (16) that identically belongs to $\mathcal{S}$. Invoking the LaSalle invariant principle [12, Theorem 4.4 and Corollary 4.2]), we conclude that $\tilde{\boldsymbol{q}} \to \mathbf{0}$ and as a result $x^i \to \frac{1}{N}\sum_{j=1}^N u^j(t)$, for all $i \in \{1,\dots,N\}$, globally asymptotically as $t \to \infty$. The systems (16a), (16b) and (16d) are linear time-invariant, therefore the rate of convergence is exponential. ■

*Remark 5.1: (Relationship between the size of $\beta$ and $\epsilon$):* As stated in Theorems 5.1, and 5.2, $\beta$ is the convergence rate of $x^i$ to an $O(\epsilon)$-neighborhood of $\frac{1}{N}\sum_{j=1}^N u^j(t)$. One can increase the rate of convergence by choosing a large $\beta$. However, to keep the two-time scale structure of the algorithms, one would then be forced to use a smaller $\epsilon$. Quantifying this trade-off, and specifically the range of admissible values of $\epsilon$ for a given $\beta$, is left as future work. □

We conclude this section by describing a variation of the algorithms *FOI-DC* and *SOI-DC* that does not require all agents to use the same parameter $\beta$ to guarantee convergence. Consider the dynamical system

$$\begin{cases} \epsilon\, \dot{z}^i = -(z^i + u^i) - \sum_{i=j}^N \boldsymbol{L}_{ij}(z^j + \nu^j), \\ \epsilon\, \dot{\nu}^i = \sum_{j=1}^N \boldsymbol{L}_{ji} z^j, \end{cases} \tag{18a}$$

$$\begin{cases} \epsilon\, \dot{y}^i = -(y^i + \dot{u}^i) - \sum_{i=j}^N \boldsymbol{L}_{ij}(y^j + \mu^j), \\ \epsilon\, \dot{\mu}^i = \sum_{j=1}^N \boldsymbol{L}_{ji} y^j, \end{cases} \tag{18b}$$

$$\dot{x}^i = -\beta^i\, x^i - \beta^i\, z^i - y^i, \tag{18c}$$

where $\beta^i > 0$'s for all $i \in \{1,\dots,N\}$. Note that this algorithm has the benefit of each agent using its own local parameter. The drawback is the need for additional distributed processing and communication. The following result characterizes the convergence properties of this algorithm. Its proof is along the same lines of the proof of Theorem 5.1 and omitted for brevity.

*Theorem 5.3 (Convergence of* (18)*):* Let $\mathcal{G}$ be a strongly connected and weight-balanced digraph. Assume that the first and the second derivatives of the input signal $u^i$ at each agent $i \in \{1,\dots,N\}$ are continuous and bounded for $t \geq 0$. Then, there exists $\epsilon^\star > 0$ such that, for all $\epsilon \in (0, \epsilon^\star]$ starting from any initial conditions $\boldsymbol{x}(0), \boldsymbol{z}(0), \boldsymbol{\nu}(0) \in \mathbb{R}^N$, the state $x^i$, $i \in \{1,\dots,N\}$, of the algorithm (18) converges exponentially fast with rate $\beta^i$ to an $O(\epsilon)$-neighborhood of $\frac{1}{N}\sum_{j=1}^N u^j(t)$.

## VI. Numerical Examples

Here, we present three numerical examples to demonstrate the performance of the algorithms *FOI-DC*, *SOI-DC* and (18). First, we consider a randomly generated undirected network (using Matlab BGL package [13]) consisting of $N = 100$ agents. The local input signals are

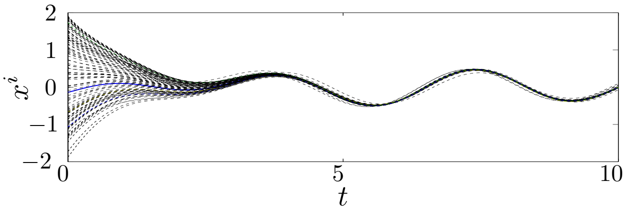
$$u^i(t) = a^i\, \sin(b^i\, t + c^i), \quad i \in \{1,\dots,N\}, \tag{19}$$

where the input coefficients are generated randomly uniformly in the following ranges: $a^i \sim \mathcal{U}[-5,5]$, $b^i \sim \mathcal{U}[1,2]$, $c^i \sim \mathcal{U}[0,\pi/2]$. Figure 1 shows the time histories of the local internal states $x^i$ generated by the algorithm *FOI-DC* for different values of $\epsilon$ and $\beta$.

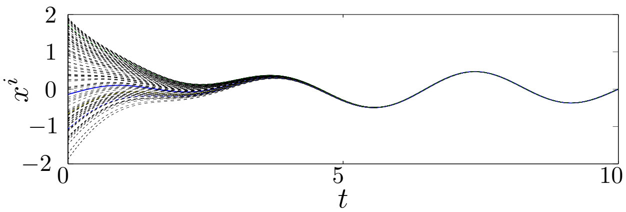
Figure 3 demonstrates the performance of the algorithms *FOI-DC* and *SOI-DC* when the difference in the input signals is a static offset. Figure 2 shows the network and inputs employed (we set $\theta = 0$ in all the input signals). Figure 4 demonstrates the performance of the algorithm (18) when agents use different $\beta$'s. Figure 2 shows the network and inputs employed (here, we set $\theta = 1$ in all the input signals).

## VII. Conclusions

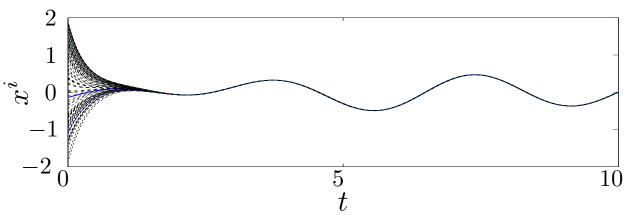
We have proposed two continuous-time dynamic average consensus algorithms for networks with strongly connected and weight-balanced interaction topologies. The proposed strategies have a two-time scale structure and do not require model information on the dynamic inputs. Using singular




(a) $\epsilon = 0.01$ and $\beta = 1$

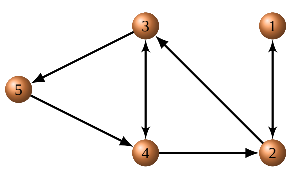



(b) $\epsilon = 0.001$ and $\beta = 1$



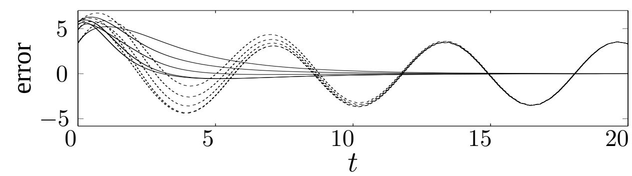

(c) $\epsilon = 0.001$ and $\beta = 3$

Fig. 1: Performance evaluation of the algorithm *FOI-DC* with respect to different choices for $\epsilon$ and $\beta$ using a random network of $N = 100$ agents and inputs given in (19): Smaller $\epsilon$ results in smaller error and larger $\beta$ results in faster convergence. The solid blue line is the average of the inputs and the dashed lines are the agreement states of agents.

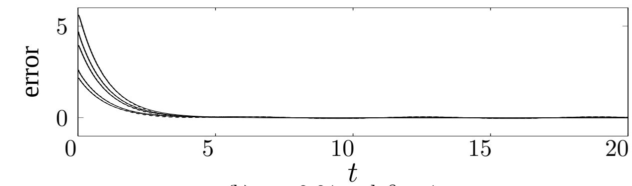



$$u^1(t) = 5\sin(t) + \theta\,\tfrac{10}{t+2} + 1,$$
$$u^2(t) = 5\sin(t) + \theta\, 0.5t - 1,$$
$$u^3(t) = 5\sin(t) + \theta\, \cos(0.5t) + 4,$$
$$u^4(t) = 5\sin(t) + \theta \log(t+1) + 5,$$
$$u^5(t) = 5\sin(t) + \theta\, \mathrm{atan}(t) + 10.$$

Fig. 2: A digraph and the corresponding input at each agent.




(a) $\epsilon = 1$ and $\beta = 1$




(b) $\epsilon = 0.01$ and $\beta = 1$

Fig. 3: Performance evaluation of the algorithms *FOI-DC* and *SOI-DC* with respect to input signals which are different from one another by static values. Figure 2 shows the network and inputs employed (here, with $\theta = 0$). Dashed (resp. solid) lines represent the error between the agreement states of the algorithm *FOI-DC* (resp. *SOI-DC*) and the input average. As guaranteed by Lemma 5.1, the algorithm *SOI-DC* converges with zero steady-state error for arbitrary values of $\epsilon$.

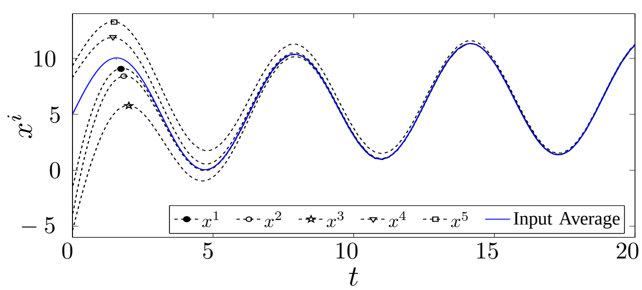



Fig. 4: Execution of the algorithm (18) over the networked system of Figure 2, with $\theta = 1$, $\epsilon = 0.01$ and $\beta^1 = 1.2$, $\beta^2 = 1$, $\beta^3 = 0.5$, $\beta^4 = 0.4$, $\beta^5 = 0.2$. Agent $i$ has rate of convergence $\beta^i$, hence, agent 5 has the slowest rate of convergence.

perturbation analysis, we have shown that the algorithms reach an $O(\epsilon)$-neighborhood of the dynamic input average with an exponential rate irrespective of the initial conditions. Future work will be devoted to rigorously characterizing the $O(\epsilon)$-convergence neighborhood and extending the results to networks with switching topology.

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