Distributed coordination for economic dispatch with varying load and generator commitment

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Abstract— **This paper considers the economic dispatch problem for a group of power generating units. The collective aim is to meet a power demand while respecting individual generator constraints and minimizing the total generation cost. Assuming that the units communicate over a strongly connected, weight-balanced digraph, we propose a distributed coordination algorithm that provably converges to the solution of the dispatch problem starting from any initial power allocation. Additionally, we establish that the proposed strategy is robust against mismatch between load and total generation (and thus able to handle time-varying loads), and against intermittent generation commitment, a plausible scenario due to the integration of renewable energy sources into the grid. Our technical approach uses notions and tools from algebraic graph theory, nonsmooth analysis, set-valued dynamical systems, and dynamic average consensus. Several simulations illustrate our results.**

I. Introduction

Power generation and distribution in electricity grids is becoming increasingly decentralized with the recent advances in renewable energy technologies and the attempts of integrating them into the grid. As a consequence, grid optimization problems are becoming large-scale and dynamic in nature, in turn, making traditional centralized, top-down solution approaches impractical. This motivates the design of distributed algorithms that are efficient in handling dynamic loads, robust against transmission and generation failures, allow for plug-and-play, and adequately preserve the privacy of the entities involved. In this paper, we consider the design of distributed algorithmic solutions for the economic dispatch (ED) problem, where a group of power generators aims to meet a power demand while minimizing the total generation cost (the summation of individual costs) and respecting the individual generators' capacity constraints. Our objective is to synthesize solution strategies that find the solution to the ED problem starting from any initial power allocation. Further, we want these algorithms to handle time-varying loads and be robust against intermittent power generation by the units.

Literature review: Traditionally, solution algorithms for the ED problem have been centralized in nature, see e.g. [1] and references therein. As we move towards a smarter electricity grid [2], distributed solution strategies are taking the center stage when it comes to optimizing the power grid. Along this transition, various distributed algorithmic solutions have emerged in the literature for the ED problem. A majority of them leverage upon the specific form of the solutions of the optimization problem and design consensus-based algorithms. The predominant approach is to consider convex, quadratic cost functions for the power generators and perform consensus over their incremental costs under undirected [3], [4] or directed [5], [6] communication topologies. Alternatively, some works consider general convex cost functions as we do here, but they either assume the algorithm to be initialized with a feasible power allocation [7], [8], need feedback on the power mismatch from the shift in steady-state frequency due to primary control [9], or do not consider capacity constraints on the generators [10]. In addition to load and capacity constraints, [6], [11] include transmission losses in their formulation. [12] additionally considers valve-point loading effects and prohibited operating zones. These constraints make the problem nonconvex and prevent these works from theoretically guaranteeing global convergence of the algorithms. In [13], the authors propose best-response dynamics for a potential-game formulation of the nonconvex ED problem, but the implementation requires all-to-all communication among the generators. In [14], [15] distributed methods are proposed to solve a resource allocation problem that is similar to the ED problem but without any individual agent constraints. While these constraints are incorporated in the formulation of [16], the proposed algorithm only arrives at suboptimal solutions of the optimization problem. Our algorithm design builds on our previous work [7], which requires a proper algorithm initialization, and employs tools from dynamic average consensus [17], [18] to synthesize a coordination strategy that converges from any initial condition.

Statement of contributions: Our starting point is the formal definition of the ED problem for a group of power generating units that communicate over a strongly connected, weight-balanced digraph. This optimization problem is convex as the individual cost functions are smooth and convex, the load satisfaction is a linear constraint, and the capacity bounds of the generators are convex inequality constraints. Our first contribution is the design of a centralized scheme, termed "load mismatch + Laplacian-nonsmooth-gradient" dynamics, that finds the solution of the ED problem starting from any initial power allocation. This algorithm has two components. The first component optimizes the total generation cost of the network while keeping the total generation constant. The second component uses the feedback on the error between the desired load and total network generation and drives the power allocations of the generators to load satisfaction at

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an exponential rate starting from any initial point. These observations set the basis for our second contribution, which is the synthesis of a distributed coordination algorithm, termed "dynamic average consensus + Laplacian-nonsmooth-gradient" dynamics, with the same convergence guarantees. Our design consists of two coupled dynamical systems: a dynamic average consensus algorithm to estimate the mismatch between generation and desired load in a distributed fashion and a distributed Laplacian-nonsmooth-gradient dynamics that employs these estimates to dynamically allocate the unit generation levels. Our final contribution is the formal characterization of the robustness properties of the distributed algorithm. Using the fact that the dynamics of mismatch between network generation and total load is exponentially convergent and input-to-state stable, we establish the algorithm's ability to track time-varying loads and its robustness in scenarios with intermittent power generation. For reasons of space, all proofs are omitted and will appear elsewhere.

Organization: Section II gathers notation and basic concepts. Section III defines formally the problem statement. Section IV proposes a motivating centralized solution strategy. Section V presents the distributed solution strategy along with its convergence analysis and robustness properties. Simulation examples are provided in Section VI and Section VII summarizes our conclusions and ideas for future work.

II. PRELIMINARIES

This section introduces basic concepts and preliminaries. We begin with some notational conventions. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{Z}_{\geq 1}$ denote the real, nonnegative real, positive real, and positive integer numbers, resp. For $r \in \mathbb{R}$ we denote $\mathcal{H}_r = \{x \in \mathbb{R}^n \mid \mathbf{1}_n^\top x = r\}$. The 2- and ∞-norms on $\mathbb{R}^n$ and their respective induced norms on $\mathbb{R}^{n\times n}$ are denoted with $\|\cdot\|$ and $\|\cdot\|_\infty$, resp. We let $B(x,\delta) = \{y \in \mathbb{R}^n \mid \|y-x\| < \delta\}$. For $x \in \mathbb{R}^n$, $x_i \in \mathbb{R}$ denotes its i-th component. Given vectors $x, y \in \mathbb{R}^n$, $x \leq y$ if and only if $x_i \leq y_i$ for all $i \in \{1,\dots,n\}$. We denote $\mathbf{1}_n = (1,\dots,1) \in \mathbb{R}^n$.A set-valued map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ associates to each point in $\mathbb{R}^n$ a set in $\mathbb{R}^m$. For a symmetric matrix $A \in \mathbb{R}^{n\times n}$, $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the minimum and maximum eigenvalues of A. Finally, we let $[u]^+ = \max\{0,u\}$ for $u \in \mathbb{R}$.

A. Graph theory

We present basic notions from algebraic graph theory following [19]. A *directed graph* (or *digraph*) is a pair $\mathcal{G} = (\mathcal{V},\mathcal{E})$, with $\mathcal{V} = \{1,\dots,n\}$ the vertex set and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ the edge set. A path is a sequence of vertices connected by edges. A digraph is *strongly connected* if there is a path between any pair of vertices. The sets of out- and in-neighbors of v are, resp., $\mathcal{N}^{\text{out}}(v) = \{w \in \mathcal{V} \mid (v,w) \in \mathcal{E}\}$ and $\mathcal{N}^{\text{in}}(v) = \{w \in \mathcal{V} \mid (w,v) \in \mathcal{E}\}$. A *weighted digraph* $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathsf{A})$ is composed of a digraph $(\mathcal{V},\mathcal{E})$ and an *adjacency matrix* $\mathsf{A} \in \mathbb{R}_{\geq 0}^{n\times n}$ with $a_{ij} > 0$ iff $(i,j) \in \mathcal{E}$. The weighted out- and in-degree of i are, resp., $d_{\text{out}}(i) = \sum_{j=1}^n a_{ij}$ and $d_{\text{in}}(i) = \sum_{j=1}^n a_{ji}$. The *Laplacian* matrix is $\mathsf{L} = \mathsf{D}_{\text{out}} - \mathsf{A}$, where D_{out} is the diagonal matrix with $(\mathsf{D}_{\text{out}})_{ii} = d_{\text{out}}(i)$, for all $i \in \{1,\dots,n\}$. Note that $\mathsf{L}\mathbf{1}_n = 0$. If $\mathcal{G}$ is strongly connected, then 0 is a simple eigenvalue of L. $\mathcal{G}$ is undirected if $\mathsf{L} = \mathsf{L}^\top$. $\mathcal{G}$ is *weight-balanced* if $d_{\text{out}}(v) = d_{\text{in}}(v)$, for all $v \in \mathcal{V}$ iff $\mathbf{1}_n^\top\mathsf{L} = 0$ iff $\mathsf{L}+\mathsf{L}^\top \geq 0$.

B. Dynamic average consensus

Here, we introduce notions on dynamic average consensus following [18]. Consider $n \in \mathbb{Z}_{\geq 1}$ agents communicating over a strongly connected, weight-balanced digraph $\mathcal{G}$ whose Laplacian is denoted as L. Each agent is associated with a state $x_i \in \mathbb{R}$ and an input signal $t \mapsto u_i(t) \subset \mathbb{R}$ that is measurable and locally essentially bounded. The aim is to provide a distributed dynamics such that the state of each agent $x_i(t)$ tracks the average signal $\frac{1}{n}\sum_{i=1}^n u_i(t)$ asymptotically. This can be achieved via the dynamics $X_{dac} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$,

$$\begin{aligned}\dot{x} &= -\alpha x - \beta\mathsf{L}x - v + \nu u,\\ \dot{v} &= \alpha\beta\mathsf{L}x,\end{aligned}$$

where $\alpha, \beta, \nu > 0$ are design parameters and $v \in \mathbb{R}^n$ is an auxiliary state. If the initial condition satisfies $\mathbf{1}_n^\top v(0) = 0$ and the time-derivatives of the input signals are bounded, then one can show, cf. [18, Corollary 4.1], that the error signal $t \mapsto \left|x_i(t) - \frac{1}{n}\sum_{i=1}^n u_i(t)\right|$ is ultimately bounded for each $i \in \{1,\dots,n\}$. Moreover, this error vanishes if the input signal converges to a constant value.

C. Nonsmooth analysis and differential inclusions

We review here some notions from nonsmooth analysis and differential inclusions following [20]. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *locally Lipschitz* at $x \in \mathbb{R}^n$ if there exist $L_x, \epsilon \in (0,\infty)$ such that $\|f(y) - f(y')\| \leq L_x\|y-y'\|$, for all $y, y' \in B(x,\epsilon)$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^n$ if, for all $v \in \mathbb{R}^n$, the right and generalized directional derivatives of f at x in the direction of v coincide, see [20] for definitions of these notions. A function that is continuously differentiable at x is regular at x. Also, a convex function is regular. A set-valued map $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *upper semicontinuous* at $x \in \mathbb{R}^n$ if, for all $\epsilon \in (0,\infty)$, there exists $\delta \in (0,\infty)$ such that $\mathcal{H}(y) \subset \mathcal{H}(x) + B(0,\epsilon)$ for all $y \in B(x,\delta)$. Also, $\mathcal{H}$ is *locally bounded* at $x \in \mathbb{R}^n$ if there exist $\epsilon, \delta \in (0,\infty)$ such that $\|z\| \leq \epsilon$ for all $z \in \mathcal{H}(y)$ and $y \in B(x,\delta)$.

Given a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, let Ω_f be the set (of measure zero) of points where f is not differentiable. The *generalized gradient* $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is

$$\partial f(x) = \operatorname{co}\{\lim_{i\to\infty} \nabla f(x_i) \mid x_i \to x, x_i \notin S \cup \Omega_f\},$$

where co denotes convex hull and $S \subset \mathbb{R}^n$ is any set of measure zero. The map ∂f is locally bounded, upper semicontinuous, and takes non-empty, compact, and convex values. A *critical point* x of f satisfies $0 \in \partial f(x)$.

Given a set-valued map $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, a differential inclusion on $\mathbb{R}^n$ is

$$\dot{x} \in \mathcal{H}(x). \tag{2}$$

A solution of (2) on $[0, T] \subset \mathbb{R}$ is an absolutely continuous map $x : [0, T] \to \mathbb{R}^n$ that satisfies (2) for almost all $t \in [0, T]$. If $\mathcal{H}$ is locally bounded, upper semicontinuous, and takes non-empty, compact, and convex values, then existence of solutions is guaranteed. The set of equilibria of (2) is $\mathrm{Eq}(\mathcal{H}) = \{x \in \mathbb{R}^n \mid 0 \in \mathcal{H}(x)\}$.

III. PROBLEM STATEMENT

Consider $n \in \mathbb{Z}_{\geq 1}$ power generators communicating over a strongly connected and weight-balanced digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathsf{A})$. Each generator corresponds to a vertex in the digraph and an edge (i, j) represents the ability of generator j to send information to generator i. The cost of power generation for unit i is measured by $f_i : \mathbb{R} \to \mathbb{R}_{\geq 0}$, assumed to be convex and continuously differentiable. Representing the power generated by unit i by $P_i \in \mathbb{R}$, the total cost incurred by the network with the power allocation $P = (P_1, \ldots, P_n) \in \mathbb{R}^n$ is measured by $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ as

$$f(P) = \sum_{i=1}^{n} f_i(P_i).$$

Note that f is convex and continuously differentiable. The generators aim to minimize the total cost $f(P)$ while meeting the total power load $P_l \in \mathbb{R}_{>0}$, i.e., $\sum_{i=1}^{n} P_i = P_l$. Each generator has an upper and a lower limit on the power it can produce, $P_i^m \leq P_i \leq P_i^M$ for $i \in \{1, \ldots, n\}$. Formally, the economic dispatch (ED) problem is

$$\begin{aligned} &\text{minimize} && f(P), && \text{(3a)} \\ &\text{subject to} && \mathbf{1}_n^\top P = P_l, && \text{(3b)} \\ & && P^m \leq P \leq P^M. && \text{(3c)} \end{aligned}$$

The constraint (3b) is the *load condition* and (3c) are the *box constraints*. We denote the feasibility set of (3) as $\mathcal{F}_{\mathrm{ED}} = \{P \in \mathbb{R}^n \mid P^m \leq P \leq P^M \text{ and } \mathbf{1}_n^\top P = P_l\}$ and the set of solutions as $\mathcal{F}_{\mathrm{ED}}^*$. Since $\mathcal{F}_{\mathrm{ED}}$ is compact, $\mathcal{F}_{\mathrm{ED}}^*$ is compact. Note that $P^M \in \mathcal{F}_{\mathrm{ED}}$ implies $\mathcal{F}_{\mathrm{ED}} = \{P^M\}$. Similarly $P^m \in \mathcal{F}_{\mathrm{ED}}$ implies $\mathcal{F}_{\mathrm{ED}} = \{P^m\}$. Therefore, we assume P^M and P^m are not feasible.

Our objective is to design a distributed coordination algorithm that allows the team of generators to solve the ED problem (3) starting from any initial condition, can handle time-varying loads, and is robust to intermittent power generation.

Remark 3.1: (Additional practical constraints): We do not consider here, for simplicity, other constraints on the ED problem such as transmission losses, transmission line capacities, valve-point loading effects, ramp rate limits, and prohibited operating zones. As our forthcoming treatment will show, the design and analysis of algorithmic solutions to the ED problem without these additional constraints is already quite challenging given our performance requirements. Nevertheless, Remark 5.3 later comments on how to adapt our algorithm to deal with more general scenarios. •

Our design strategy relies on the following reformulation of the ED problem without inequality constraints. Consider the modified ED problem

$$\begin{aligned} &\text{minimize} && f^\epsilon(P), && \text{(4a)} \\ &\text{subject to} && \mathbf{1}_n^\top P = P_l, && \text{(4b)} \end{aligned}$$

where the objective function is

$$f^\epsilon(P) = \sum_{i=1}^{n} f_i(P_i) + \frac{1}{\epsilon}\Big(\sum_{i=1}^{n}([P_i - P_i^M]^+ + [P_i^m - P_i]^+)\Big).$$

This corresponds to each generator $i \in \{1, \ldots, n\}$ having the modified local cost

$$f_i^\epsilon(P_i) = f_i(P_i) + \frac{1}{\epsilon}([P_i - P_i^M]^+ + [P_i^m - P_i]^+).$$

Note that f_i^ϵ is convex, locally Lipschitz, and continuously differentiable on $\mathbb{R}$ except at $P_i = P_i^m$ and $P_i = P_i^M$. Moreover, the total cost f^ϵ is convex, locally Lipschitz, and regular. According to our previous work [7, Proposition 5.2], the solutions to the original (3) and the modified (4) ED problems coincide for $\epsilon \in \mathbb{R}_{>0}$ such that

$$\epsilon < \frac{1}{2 \max_{P \in \mathcal{F}_{\mathrm{ED}}} \|\nabla f(P)\|_\infty}. \tag{5}$$

Throughout the paper, we assume the parameter ϵ satisfies this condition. A useful fact is that $P^* \in \mathbb{R}^n$ is a solution of (4) if and only if there exists $\mu \in \mathbb{R}$ such that

$$\mu \mathbf{1}_n \in \partial f^\epsilon(P^*) \quad \text{and} \quad \mathbf{1}_n^\top P^* = P_l. \tag{6}$$

IV. ROBUST CENTRALIZED ALGORITHMIC SOLUTION

This section presents a robust strategy to make the network power allocation converge to the solution set of the ED problem starting from any initial condition. Even though this algorithm is centralized, its design provides enough insight to tackle later the design of a distributed algorithmic solution. Consider the "load mismatch + Laplacian-nonsmooth-gradient" (abbreviated $\mathtt{lm}$+$\mathsf{L}\partial$) dynamics, represented by the set-valued map $X_{\mathtt{lm}+\mathsf{L}\partial} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$,

$$\dot{P} \in -\mathsf{L}\partial f^\epsilon(P) + \frac{1}{n}(P_l - \mathbf{1}_n^\top P)\mathbf{1}_n, \tag{7}$$

where L is the Laplacian associated to the strongly connected and weight-balanced communication digraph $\mathcal{G}$. For each generator, the first term seeks to minimize the total cost while leaving unchanged the total generated power. The second term is a feedback element that seeks to drive the units towards the satisfaction of the load. The first term is computable using information from its neighbors but the second term requires them to know the aggregated state of the whole network, which makes it not directly implementable in a distributed manner. The next result states the convergence properties of (7).

Theorem 4.1: (Convergence of the trajectories of $X_{\mathtt{lm}+\mathsf{L}\partial}$

to the solutions of ED *problem):* The trajectories of (7) starting from any point in $\mathbb{R}^n$ converge to the set of solutions of (3).

Interestingly, by computing the evolution of $V_1(P) = (P_l - \mathbf{1}_n^\top P)^2$ along (7) one can deduce that the feedback term in (7) drives the mismatch between generation and load to zero at an exponential rate, no matter what the initial power allocation. This is a good indication of its robustness properties: time-varying loads or scenarios with generators going down and coming back online can be handled as long as the rate of these changes is lower than the exponential rate of convergence associated to the load satisfaction. We provide a formal characterization of these properties for the distributed implementation of this strategy in the next section.

V. Robust distributed algorithmic solution

This section presents a distributed strategy to solve the ED problem starting from any initial power allocation. We build on the centralized design presented in Section IV. We also formally characterize the robustness properties against addition and deletion of generators and time-varying loads.

Given the discussion on the centralized nature of the dynamics (7), the core idea of our design is to employ a dynamic average consensus algorithm that allows each unit in the network to estimate the mismatch in load satisfaction. To this end, we assume the total load P_l is only known to one generator $r \in \{1, \dots, n\}$ (its specific identity is arbitrary). Following Section II-B, consider the dynamics,

$$\begin{aligned}\dot{z} &= -\alpha z - \beta \mathsf{L} z - v + \nu_2 (P_l e_r - P),\\ \dot{v} &= \alpha\beta \mathsf{L} z,\end{aligned}$$

where $e_r \in \mathbb{R}^n$ is the unit vector along the r-th direction and $\alpha, \beta, \nu_2 > 0$ are design parameters. Note that this dynamics is distributed over the communication graph $\mathcal{G}$. For each $i \in \{1, \dots, n\}$, z_i plays the role of an estimator associated to i which aims to track the average signal $t \mapsto \frac{1}{n}(P_l - \mathbf{1}_n^\top P(t))$. This observation justifies substituting the feedback term in (7) by $z \in \mathbb{R}^n$, giving rise to the "dynamic average consensus + Laplacian-nonsmooth-gradient" dynamics, abbreviated $\mathtt{dac}$+$\mathsf{L}\partial$ for convenience, mathematically represented by the set-valued map $X_{\mathtt{dac}+\mathsf{L}\partial} : \mathbb{R}^{3n} \rightrightarrows \mathbb{R}^{3n}$,

$$\dot{P} \in -\mathsf{L}\partial f^{\epsilon}(P) + \nu_1 z, \tag{9a}$$

$$\dot{z} = -\alpha z - \beta \mathsf{L} z - v + \nu_2 (P_l e_r - P), \tag{9b}$$

$$\dot{v} = \alpha\beta \mathsf{L} z, \tag{9c}$$

where $\nu_1 > 0$ is a design parameter. Unlike (7), this dynamics is distributed, as each agent only needs to interact with its neighbors to implement it. The next result formalizes the convergence properties of the $\mathtt{dac}$+$\mathsf{L}\partial$ dynamics to the set of solutions of the ED problem.

Theorem 5.1: (Convergence of the $\mathtt{dac}$+$\mathsf{L}\partial$ *dynamics to the solutions of* ED *problem):* For $\alpha, \beta, \nu_1, \nu_2 > 0$ with

$$\frac{\nu_1}{\beta \nu_2 \lambda_2(\mathsf{L} + \mathsf{L}^\top)} + \frac{\nu_2^2 \lambda_{\max}(\mathsf{L}^\top \mathsf{L})}{2\alpha} < \lambda_2(\mathsf{L} + \mathsf{L}^\top), \tag{10}$$

the trajectories of (9) starting from any point in $\mathbb{R}^n \times \mathbb{R}^n \times \mathcal{H}_0$ converge to the set $\mathcal{F}^*_{\text{aug}} = \{(P, z, v) \in \mathcal{F}^*_{\text{ED}} \times \{0\} \times \mathbb{R}^n \mid v = \nu_2(P_l e_r - P)\}$.

Note that as a consequence of the above result, the $\mathtt{dac}$+$\mathsf{L}\partial$ dynamics does not require any specific pre-processing for the initialization of the power allocations. Each generator can select any generation level, independent of the other units, and the algorithm guarantees convergence to the solutions of the ED problem.

Remark 5.2: (Distributed selection of algorithm design parameters): The convergence of the $\mathtt{dac}$+$\mathsf{L}\partial$ dynamics relies on a selection of the parameters α, β, ν_1 and $\nu_2 \in \mathbb{R}_{>0}$ that satisfy (10). Checking this inequality requires knowledge of the spectrum of matrices related to the Laplacian matrix, and hence the entire network structure. Here, we provide an alternative condition that implies (10) and can be checked by the units in a distributed way. Let n_{max} be an upper bound on the number of units, $d_{\text{out,max}}$ be an upper bound on the out-degree of all units, and a_{min} be a lower bound on the edge weights,

$$n \le n_{\text{max}}, \ \max_{i \in \mathcal{V}} d_{\text{out}}(i) \le d_{\text{out,max}}, \ \min_{(i,j) \in \mathcal{E}} a_{ij} \ge a_{\text{min}}. \tag{11}$$

A straightforward generalization of [21, Theorem 4.2] for weighted graphs gives rise to the following lower bound on $\lambda_2(\mathsf{L} + \mathsf{L}^\top)$,

$$\frac{4 a_{\text{min}}}{n_{\text{max}}^2} \le \lambda_2(\mathsf{L} + \mathsf{L}^\top). \tag{12}$$

On the other hand, using properties of matrix norms [22, Chapter 9], one can deduce

$$\begin{aligned}\lambda_{\max}(\mathsf{L}^\top \mathsf{L}) = \|\mathsf{L}\|^2 &\le (\sqrt{n}\|\mathsf{L}\|_\infty)^2\\ &\le (2\sqrt{n} d_{\text{out,max}})^2 \le 4 n_{\text{max}} (d_{\text{out,max}})^2.\end{aligned} \tag{13}$$

Using (12)-(13), the left-hand side of (10) can be upper bounded by

$$\begin{aligned}&\frac{\nu_1}{\beta \nu_2 \lambda_2(\mathsf{L} + \mathsf{L}^\top)} + \frac{\nu_2^2 \lambda_{\max}(L^\top L)}{2\alpha}\\ &\qquad \le \frac{\nu_1 n_{\text{max}}^2}{4 a_{\text{min}} \beta \nu_2} + \frac{2\nu_2^2 n_{\text{max}} (d_{\text{out,max}})^2}{\alpha}.\end{aligned}$$

Further, the right-hand side of (10) can be lower bounded using (12). Putting the two together, we obtain the new condition

$$\frac{\nu_1 n_{\text{max}}^2}{4 a_{\text{min}} \beta \nu_2} + \frac{2\nu_2^2 n_{\text{max}} (d_{\text{out,max}})^2}{\alpha} < \frac{4 a_{\text{min}}}{n_{\text{max}}^2}, \tag{14}$$

which implies (10). The network can ensure that this condition is met in various ways. For instance, if the bounds n_{max}, $d_{\text{out,max}}$, and a_{min} are not available, the network can implement distributed algorithms for max- and min-consensus [23] to compute them in finite time. Once known, any generator can select α, β, ν_1 and ν_2 satisfying (14) and broadcast its choice. Alternatively, the computation of the design parameters can be implemented concurrently with the determination of the bounds via consensus by providing a specific formula to select them that is guaranteed to satisfy (14). Note that the units necessarily need to agree on the parameters, otherwise if each unit selects a different set of parameters, the dynamic average consensus would not track the average input signal. •

Remark 5.3: (Distributed loads and transmission losses): Here we expand on our observations in Remark 3.1 regarding the inclusion of additional constraints on the ED problem. Our algorithmic solution can be easily modified to deal with the alternative scenarios studied in [24], [4], [6], [11], where each generator has the knowledge of the load at the corresponding bus that it is connected to and the total load is the aggregate of these individual loads. Mathematically, denoting the load demanded at generator bus i by $P_i^L \in \mathbb{R}$, the total load is given by $P_l = \sum_{i=1}^n P_i^L$. For this case, replacing the vector $P_l e_r$ by P^L in the dac+L∂ dynamics (9b) gives an algorithm that solves the ED problem for the load P_l. Our solution strategy can also handle transmission losses as modeled in [6], where it is assumed that each generator i can estimate the power loss in the transmission lines adjacent to it. With those values available, the generator could add them to the quantity P_i^L, which would make the network find a power allocation that takes care of the transmission losses. •

A. Robustness analysis

In this section, we study the robustness properties of the dac+L∂ dynamics in the presence of time-varying loads and intermittent power generation. Our analysis relies on the exponential stability of the mismatch dynamics between total generation and load, a fact that is established next. Define $x_1(t) = \mathbf{1}_n^\top P(t) - P_l$ and $x_2(t) = \dot{x}_1(t)$. Then, the dynamics of x under (9) can be written as a first-order system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\nu_1\nu_2 & -\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{15}$$

Evaluating the Lie derivative of the positive definite, radially unbounded function $V_2(x_1, x_2) = \nu_1\nu_2 x_1^2 + x_2^2$ along the above dynamics and applying the LaSalle Invariance Principle [25], we deduce that $x_1(t) \to 0$ and $x_2(t) \to 0$ as $t \to \infty$, that is, $\mathbf{1}_n^\top P(t) \to P_l$ and $\mathbf{1}_n^\top z(t) \to 0$. Since the system (15) is linear, the convergence is exponential. This implies that (15) is input-to-state stable (ISS) [25, Lemma 4.6], and consequently robust against arbitrary bounded perturbations. The following result provides an explicit, exponentially decaying, bound for the evolution of any trajectory of (15).

Lemma 5.4: (Convergence rate of the mismatch dynamics (15)): Let $R \in \mathbb{R}^{2\times 2}$ be defined by

$$R = \frac{1}{2\alpha\nu_1\nu_2} \begin{bmatrix} \alpha^2 + \nu_1\nu_2 + (\nu_1\nu_2)^2 & \alpha \\ \alpha & 1 + \nu_1\nu_2 \end{bmatrix}.$$

Then $R \succ 0$ and any trajectory $t \mapsto x(t)$ of the dynamics (15) satisfies $\|x(t)\| \leq c_1 e^{-c_2 t}\|x(0)\|$, where $c_1 = \sqrt{\lambda_{\max}(R)/\lambda_{\min}(R)}$ and $c_2 = 1/2\lambda_{\max}(R)$.

In the above result, it is interesting to note that the convergence rate is independent of the specific communication digraph (as long as it is weight-balanced). We use next the exponentially decaying bound obtained above to illustrate the extent to which the network can collectively track a dynamic load (which corresponds to a time-varying perturbation in the mismatch dynamics) and is robust to intermittent power generation (which corresponds to perturbations in the state of the mismatch dynamics).

1) Tracking dynamic loads: Here we consider a time-varying total load given by a twice continuously differentiable trajectory $\mathbb{R}_{\geq 0} \ni t \mapsto P_l(t)$ and show how the total generation of the network under the dac+L∂ dynamics tracks it. We assume the signal is known to an arbitrary unit $r \in \{1, \dots, n\}$. In this case, the dynamics (15) takes the following form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\nu_1\nu_2 & -\alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\alpha\dot{P}_l - \ddot{P}_l \end{bmatrix}.$$

Using Lemma 5.4, one can compute the following bound on any trajectory of the above system

$$\|x(t)\| \leq c_1 e^{-c_2 t}\|x(0)\| + \frac{c_1}{c_2} \sup_{s\in[0,t]} \left|\alpha\dot{P}_l(s) + \ddot{P}_l(s)\right|.$$

In particular, for a signal with bounded $\dot{P}_l$ and $\ddot{P}_l$, the mismatch between generation and load, i.e., $x_1(t)$ is bounded. Also, the mismatch has an ultimate bound as $t \to \infty$. The following result summarizes this notion formally. The proof is straightforward application of Lemma 5.4 following the exposition of input-to-state stability in [25].

Proposition 5.5: (Power mismatch is ultimately bounded for dynamic load under dac+L∂ dynamics): Let $\mathbb{R}_{\geq 0} \ni t \mapsto P_l(t)$ be twice continuously differentiable such that

$$\sup_{t\geq 0} \left|\dot{P}_l(t)\right| \leq d_1, \quad \sup_{t\geq 0} \left|\ddot{P}_l(t)\right| \leq d_2,$$

for some $d_1, d_2 > 0$. Then, the mismatch $\mathbf{1}_n^\top P(t) - P_l(t)$ between load and generation is bounded along the trajectories of (9) and has ultimate bound $\frac{c_1}{c_2}(\alpha d_1 + d_2)$, with c_1, c_2 given in Lemma 5.4. Moreover, if $\dot{P}_l(t) \to 0$ and $\ddot{P}_l(t) \to 0$ as $t \to \infty$, then $\mathbf{1}_n^\top P(t) \to P_l(t)$ as $t \to \infty$.

2) Robustness to intermittent power generation: Here, we characterize the algorithm robustness against unit addition and deletion to capture scenarios with intermittent power generation. Addition and deletion events are modeled via a time-varying communication digraph, which we assume remains strongly connected and weight-balanced at all times. When a unit stops generating power (deletion event), the corresponding vertex and its adjacent edges are removed. When a unit starts providing power (addition event), the corresponding node is added to the digraph along with a set of edges. Given the intricacies of the convergence analysis for the $\mathtt{dac}+\mathsf{L}\partial$ dynamics, cf. Theorem 5.1, it is important to make sure that the state v remains in the set $\mathcal{H}_0$, irrespectively of the discontinuities caused by the events. The following routine makes sure that this is the case.

> TRAJECTORY INVARIANCE: When a unit i joins the network at time t, it starts with $v_i(t) = 0$. When a unit i leaves the network at time t, it passes a token with value $v_i(t)$ to one of its in-neighbors $j \in \mathcal{N}^{\text{in}}(i)$, who resets its value to $v_j(t) + v_i(t)$.

The TRAJECTORY INVARIANCE routine ensures that the dynamics (15) is the appropriate description for the evolution of the load satisfaction mismatch. This, together with the ISS property established in Lemma 5.4, implies that the mismatch effect in power generation caused by addition/deletion events vanishes exponentially fast. In particular, if the number of addition/deletion events is finite, then the set of generators converge to the solution of the ED problem. We formalize this next.

Proposition 5.6: (Convergence of $\mathtt{dac}+\mathsf{L}\partial$ dynamics under intermittent power generation): Let $n_{\max}$ be the maximum number of generators that can contribute to the power generation at any time. Let $\Sigma_{n_{\max}}$ be the set of digraphs that are strongly connected and weight-balanced and whose vertex set is included in $\{1, \dots, n_{\max}\}$. Let $\sigma : [0, \infty) \to \Sigma_{n_{\max}}$ be a piecewise constant, right-continuous switching signal described by the set of switching times $\{t_1, t_2, \dots\} \subset \mathbb{R}_{\geq 0}$, with $t_k \leq t_{k+1}$, each corresponding to either an addition or a deletion event. Denote by $X^{\sigma}_{\mathtt{dac}+\mathsf{L}\partial}$ the switching $\mathtt{dac}+\mathsf{L}\partial$ dynamics corresponding to σ, defined by (9) with L replaced by $\mathsf{L}(\sigma(t))$ for all $t \geq 0$, and assume agents execute the TRAJECTORY INVARIANCE routine when they leave or join the network. Then,

(i) at any time $t \in \{0\} \cup \{t_1, t_2, \dots\}$, if the variables $(P(t), z(t))$ for the generators in $\sigma(t)$ satisfy $\left|\mathbf{1}_n^\top P(t) - P_l\right| \leq M_1$ and $\left|\mathbf{1}_n^\top z(t)\right| \leq M_2$ for some $M_1, M_2 > 0$, then the magnitude of the mismatch between generation and load becomes less than or equal to $\rho > 0$ in time
$$t_\rho = \frac{1}{c_2} \ln\Big(\frac{c_1(M_1 + \nu_1 M_2)}{\rho}\Big),$$
provided no event occurs in the interval $(t, t + t_\rho)$;

(ii) if the number of events is finite, say N, then the trajectories of $X^{\sigma}_{\mathtt{dac}+\mathsf{L}\partial}$ converge to the set of solutions of the ED problem for the group of generators in $\sigma(t_N)$ provided (10) is met for $\sigma(t_N)$.

Note that the generators can ensure that the condition (10), required for the convergence of the $\mathtt{dac}+\mathsf{L}\partial$ dynamics, holds at all times even under addition and deletion events, if they rely on verifying that (14) holds and the bounds (11) are valid for all the topologies in $\Sigma_{n_{\max}}$.

Unit	a_i	b_i	c_i	P_i^m	P_i^M
1	671	10.1	0.000299	150	455
2	574	10.2	0.000183	150	455
3	374	8.8	0.001126	20	130
4	374	8.8	0.001126	20	130
5	461	10.4	0.000205	150	470
6	630	10.1	0.000301	135	460
7	548	9.8	0.000364	135	465
8	227	11.2	0.000338	60	300
9	173	11.2	0.000807	25	162
10	175	10.7	0.001203	25	160
11	186	10.2	0.003586	20	80
12	230	9.9	0.005513	20	80
13	225	13.1	0.000371	25	85
14	309	12.1	0.001929	15	55
15	323	12.4	0.004447	15	55

TABLE I

COEFFICIENTS OF THE QUADRATIC COST FUNCTION $f_i(P_i) = a_i + b_i P_i + c_i P_i^2$ AND LOWER P_i^m AND UPPER P_i^M GENERATION LIMITS FOR EACH UNIT i.

VI. SIMULATIONS

Here, we illustrate the convergence of the $\mathtt{dac}+\mathsf{L}\partial$ dynamics to the solutions of the ED problem (3) starting from any initial power allocation. We consider a 15 bus system [26]. Table I gives the cost function of each generator and its capacity bounds. For all the scenarios considered, we select the initial condition for the dynamics to be $(P(0), z(0), v(0)) = (0.5 * (P^m + P^M), 0, 0)$ and the design parameters to be $\nu_1 = 1, \nu_2 = 2, \alpha = 5, \beta = 20$, and $\epsilon = 0.0253$, which satisfy the conditions (5) and (10).

For the first case, the communication topology is $\mathcal{G}$, as described in Table II. The total load is 2630 for the first 300 seconds, and 2550 for the next 300 seconds, and is known to unit 3. Figure 1(a)-(c) shows the evolution of the power allocation, total cost, and the mismatch between the total generation and load under the $\mathtt{dac}+\mathsf{L}\partial$ dynamics. The generators initially converge to an optimal allocation that meets the load 2630. Later, with the decrease in desired load to 2550, the network decreases the total generation while minimizing the total cost.

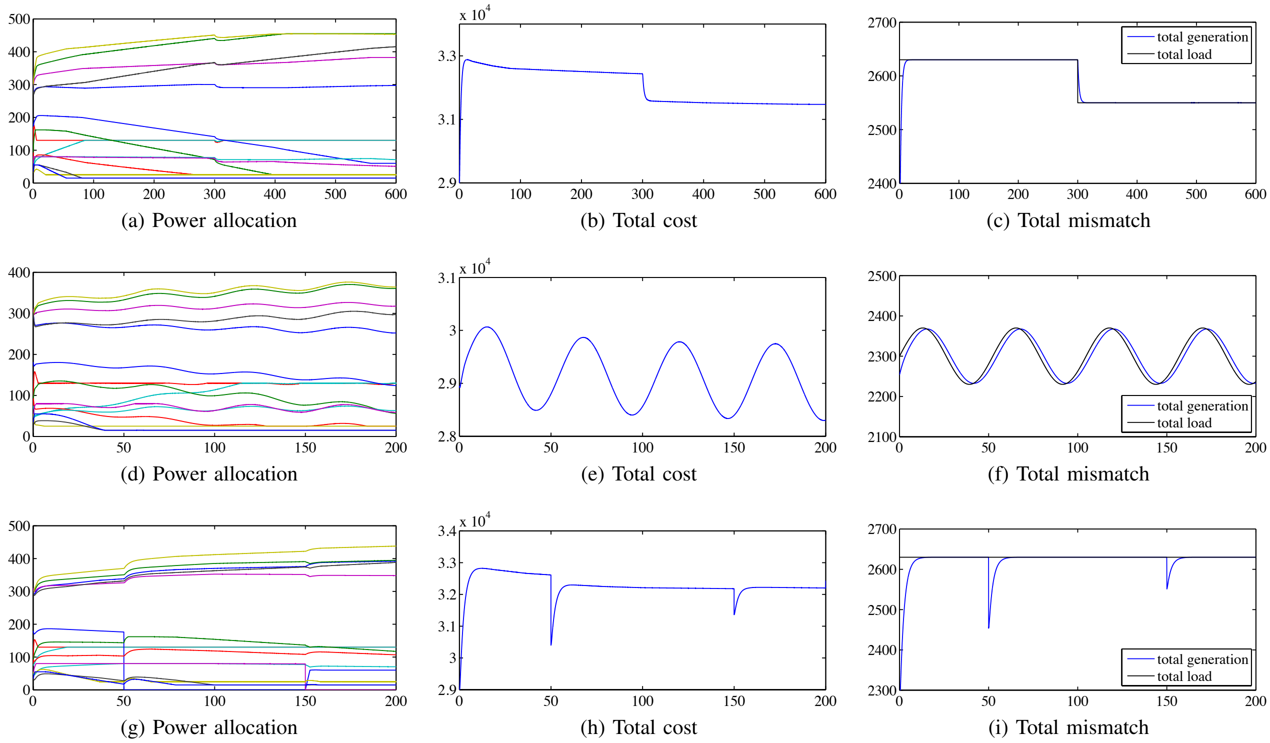
Next, we consider a time-varying total load given by a constant plus a sinusoid, $P_l(t) = 2300 + 70\sin(0.05t)$. With the same communication topology $\mathcal{G}$ among the units, Figure 1 (d)-(f) depicts the evolutiong of the network under the $\mathtt{dac}+\mathsf{L}\partial$ dynamics. As established in Proposition 5.5, the total generation tracks the time-varying load signal and the mismatch between these values is ultimately bounded. Additionally, to illustrate how that the mismatch vanishes if the load becomes constant, we show in Figure 2 a load signal that consists of short bursts of sinusoidal variation that decay exponentially. As the load tends towards a constant signal, the mismatch between generation and load becomes smaller and smaller.

Our final scenario considers addition and deletion of generators. The initial communication topology is the undirected graph $\hat{\mathcal{G}}$ described in Table II. The total load is 2630 and is the same at all times. For the first 50 seconds, the power

$\mathcal{G}$	digraph over 15 vertices consisting of a directed cycle through vertices $1, \ldots, 15$ and bi-directional edges $\{(i, \mathrm{id}_{15}(i+3)), (i, \mathrm{id}_{15}(i+6))\}$ for each $i \in \{1, \ldots, 15\}$, where $\mathrm{id}_{15}(x) = x$ if $x \in \{1, \ldots, 15\}$ and $x - 15$ otherwise. All edge weights are 0.1.
$\hat{\mathcal{G}}$	obtained from $\mathcal{G}$ by replacing the directed cycle with an undirected one keeping the edge weights same
$\hat{\mathcal{G}}_{\setminus\{8\}}$	obtained from $\hat{\mathcal{G}}$ by removing the vertex $\{8\}$ and the edges adjacent to it
$\hat{\mathcal{G}}_{\setminus\{12\}}$	obtained from $\hat{\mathcal{G}}$ by removing the vertex $\{12\}$ and the edges adjacent to it

TABLE II

DEFINITION OF THE DIGRAPHS $\mathcal{G}$, $\hat{\mathcal{G}}$, $\hat{\mathcal{G}}_{\setminus\{8\}}$, AND $\hat{\mathcal{G}}_{\setminus\{12\}}$.



(a) Power allocation (b) Total cost (c) Total mismatch

(d) Power allocation (e) Total cost (f) Total mismatch

(g) Power allocation (h) Total cost (i) Total mismatch

Fig. 1. Evolution of the power allocation, the total cost, and the total mismatch between generation and load under the dac+L∂ dynamics for the 15 bus example in different scenarios. The design parameters remain the same for all the cases and are set as $\nu_1 = 1$, $\nu_2 = 2$, $\alpha = 5$, $\beta = 20$, and $\epsilon = 0.0253$. In the first case (a)-(c), the communication topology is $\mathcal{G}$. The load is initially 2630 and later 2550. In the second scenario (d)-(f), the digraph remains the same but the load is time-varying, $P_l(t) = 2300 + 70 \sin(0.05t)$. In the last case (g)-(i), the communication graph is initially the graph $\hat{\mathcal{G}}$. At $t = 50s$, unit 8 leaves the network, resulting in the communication topology $\hat{\mathcal{G}}_{\setminus\{8\}}$, and the remaining agents run the TRAJECTORY INVARIANCE routine. Later, at $t = 150s$, unit 8 joins the network while unit 12 leaves it, resulting in the communication topology $\hat{\mathcal{G}}_{\setminus\{12\}}$. After implementing the TRAJECTORY INVARIANCE routine, the dac+L∂ dynamics eventually converges to an optimizer of the ED problem for the network $\hat{\mathcal{G}}_{\setminus\{12\}}$.

allocations converge to a neighborhood of a solution of the ED problem for the set of generators in $\hat{\mathcal{G}}$. At time $t = 50s$, the units 8 stops generating power and leaves the network. We select this generator because of its substantial impact in the total power generation. After this event, the resulting communication graph is $\hat{\mathcal{G}}_{\setminus\{8\}}$, cf. Table II. The generators implement the TRAJECTORY INVARIANCE routine, after which the dac+L∂ dynamics drives the mismatch to zero and minimizes the total cost. At $t_2 = 150s$, another event occurs, the unit 8 gets added back to the network while the unit 12 leaves. The resulting communication topology is $\hat{\mathcal{G}}_{\setminus\{12\}}$, cf. Table II. After executing the TRAJECTORY INVARIANCE routine, the dynamics converges eventually to the optimizers of the ED problem for the set of generators in $\hat{\mathcal{G}}_{\setminus\{12\}}$, as shown in Figure 1(g)-(i). This example illustrates the robustness of the dac+L∂ dynamics against intermittent generation by the units, as formally established in Proposition 5.6. In addition to the presented examples, we also successfully simulated scenarios of the kind described in Remark 5.3, where the total load is not known to a single generator and is instead the aggregate of the local loads connected to each of the generator buses, but we do not report here for space reasons.

VII. CONCLUSIONS

We have designed a novel provably-correct distributed strategy that allows a group of generators to solve the economic dispatch problem starting from any initial power allocation. Our algorithm design combines elements from average consensus to dynamically estimate the mismatch between generation and desired load and ideas from distributed optimization to dynamically allocate the unit generation

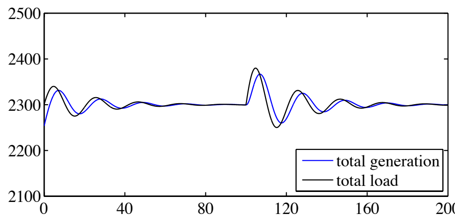


Fig. 2. Evolution of the total power generation for the 15 bus example under the dac+L∂ dynamics for the communication digraph $\mathcal{G}$, design parameters $\nu_1 = 1$, $\nu_2 = 2$, $\alpha = 5$, $\beta = 20$ and $\epsilon = 0.0253$, and time-varying total load. The plot depicts the input-to-state stability of the mismatch dynamics.

levels. Our analysis has shown that the mismatch dynamics between total generation and load is input-to-state stable and, as a consequence, the coordination algorithm is robust to initialization errors, dynamic load signals, and intermittent power generation. Future work will explore the study of the preservation of the generator box constraints under the proposed coordination strategy, the extension to scenarios that involve additional constraints, such as transmission losses, transmission line capacity constraints, ramp rate limits, prohibited operating zones, and valve-point loading effects, and the study of the stability and convergence properties of algorithm designs that combine our approach here with traditional primary and secondary generator controllers.

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