Distributed, anytime optimization in power-generator networks for economic dispatch

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Abstract— **This paper considers the economic dispatch problem for a group of power generating units communicating over an arbitrary strongly connected, weight-balanced digraph. The goal of the group is to collectively meet a specified load while respecting individual generation bounds and minimizing the total generation cost, which corresponds to the sum of individual arbitrary convex functions. We introduce a distributed coordination algorithm, termed Laplacian-set-valued dynamics, and establish its asymptotic convergence to the solutions of the economic dispatch problem. In addition, we show that the algorithm is anytime, meaning that its executions are feasible solutions at all times and the total cost monotonically decreases as time elapses. The technical approach combines notions and tools from algebraic graph theory, nonsmooth analysis, set-valued dynamical systems, and penalty functions. Several simulations illustrate our results.**

I. Introduction

In future electricity grids, the number of power generating units will increase considerably due to the increasing availability of renewable energy sources, the construction of smart buildings and homes, and a myriad of technological advances. This in turn will make the scale of optimization problems pertaining to power generation and distribution very large and dynamic. In such scenarios, centralized approaches might become impractical. Therefore, there is a need to develop distributed algorithmic solutions that allow units to coordinate with neighboring units to collectively find the solution. Such distributed implementations have the potential to be robust against generation and transmission failures in the grid and can cater to dynamic demands. Motivated by this vision, we study here distributed algorithmic solutions for the economic dispatch (ED) problem. In this problem, a group of power generating units with individual generation costs described by smooth, convex functions seek to determine the generation levels such that the total cost is collectively minimized while satisfying the total load and respecting individual power limits. Our aim is to synthesize distributed algorithms that asymptotically converge to the solutions of the ED problem and are anytime, i.e., its executions are feasible solutions at any time before convergence and they become better and better solutions as time elapses.

Literature review: Traditionally, solution methodologies for the economic dispatch problem have been centralized in nature [1]. Given the expected high density of the future electricity grid [2], the focus has shifted in recent years to distributed algorithmic solutions. Some of these use consensus-based algorithms for quadratic cost functions and communication topologies defined by an undirected [3], [4], [5] or a directed [6] graph. A limitation of these methods is that, in general, they do not generate anytime executions. Center-free algorithms [7], [8] instead, are anytime, and solve an optimal resource allocation problem identical to the ED problem for general convex functions, but they do not consider individual generator constraints. In [9] general convex functions and general local constraints for the units are considered but the trajectories of the proposed distributed algorithm only converge to suboptimal points by solving a regularized version of the problem. Our work has also connections with the emerging body of research on distributed optimization, see e.g., [10], [11], [12], [13] and references therein, where network agents communicate and update their individual estimates of the complete solution vector. In contrast, in our setting, each unit communicates and optimizes its own local variable, and all the local variables are connected via a global constraint. We have also studied the ED problem in the journal version [14] of this work. However, the Laplacian set-valued dynamics introduced here as well as the analysis of its asymptotic correctness are novel and have not been presented elsewhere.

Statement of contributions: Our starting point is the formulation of the ED problem for a group of power-generator units that communicate over an arbitrary weight-balanced, strongly connected digraph. Our main contribution is the design of the Laplacian-set-valued dynamics and the analysis of its asymptotic convergence to the set of solutions of the ED problem. We establish the anytime nature of the algorithm, i.e., the load constraint and capacity bounds are satisfied at all times and the aggregate cost decreases monotonically along the evolutions. The algorithm design is based on an alternative, exact-penalty formulation of the ED problem that, unlike the original problem, has no bounds on the capacity of individual generators. In comparison with the dynamics we presented in [14], the advantage of the dynamics proposed here is that convergence can be guaranteed under less stringent conditions, independent of the network topology, on the parameter associated to the exact penalty formulation. Our technical approach relies on tools from nonsmooth analysis, constrained optimization, and graph theory. Simulations illustrate our results.

Organization: Section II contains basic preliminaries. Section III presents the ED problem. Section IV introduces the Laplacian-set-valued dynamics and establishes its convergence properties. Section V presents simulations. Section VI summarizes our conclusions and ideas for future work.

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II. Preliminaries

This section introduces basic concepts from graph theory, nonsmooth analysis, discontinuous dynamics, and constrained optimization. We begin with some notational conventions. Let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{R}_{>0}$, $\mathbb{Z}_{\geq 1}$ denote the set of real, nonnegative real, positive real, and positive integer numbers, respectively. We denote the 2-norm and ∞-norm on $\mathbb{R}^n$ by $\|\cdot\|_2$ and $\|\cdot\|_\infty$, respectively. We let $B(x,\delta) = \{y \in \mathbb{R}^n \mid \|y-x\|_2 < \delta\}$ denote the open ball centered at $x \in \mathbb{R}^n$ with radius $\delta > 0$. The projection of a point $x \in \mathbb{R}^n$ onto a closed and convex set D is denoted by $\pi_D(x)$, where $\pi_D(x)$ satisfies $\|x - \pi_D(x)\|_2 = \min_{y \in D} \|x-y\|_2$. We use the shorthand notation $\mathbf{0}_n = (0,\dots,0) \in \mathbb{R}^n$, $\mathbf{1}_n = (1,\dots,1) \in \mathbb{R}^n$, and $I_n \in \mathbb{R}^{n\times n}$ for the identity matrix. Given $x \in \mathbb{R}^n$, x_i denotes the i-th component of x. For $x, y \in \mathbb{R}^n$, $x \leq y$ denotes $x_i \leq y_i$ for $i \in \{1,\dots,n\}$. A set-valued map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ associates to each point in $\mathbb{R}^n$ a set in $\mathbb{R}^m$. Given a set-valued map $f : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and a matrix $A \in \mathbb{R}^{p\times m}$, their composition $h = Af : \mathbb{R}^n \rightrightarrows \mathbb{R}^p$ is the set-valued map defined by $h(x) = \{z \in \mathbb{R}^p \mid z = Ay \text{ with } y \in f(x)\}$. We let $[u]^+ = \max\{0,u\}$ for $u \in \mathbb{R}$.

A. Graph theory

We present some basic notions from algebraic graph theory following [15]. A *directed graph*, or simply *digraph*, is a pair $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where $\mathcal{V}$ is a finite set called the vertex set and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the edge set. A path is an ordered sequence of vertices such that any ordered pair of vertices appearing consecutively is an edge of the digraph. A digraph is *strongly connected* if there is a path between any pair of distinct vertices. For a digraph, $N_{\text{out}}(v_i)$ and $N_{\text{in}}(v_i)$ are the sets of out- and in-neighbors of v_i, respectively, i.e., $N_{\text{out}}(v_i) = \{v_j \in \mathcal{V} \mid (v_i,v_j) \in \mathcal{E}\}$ and $N_{\text{in}}(v_i) = \{v_j \in \mathcal{V} \mid (v_j,v_i) \in \mathcal{E}\}$. A *weighted digraph* is a triplet $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathsf{A})$, where $(\mathcal{V},\mathcal{E})$ is a digraph and $\mathsf{A} \in \mathbb{R}^{n\times n}_{\geq 0}$ is the *adjacency matrix* of $\mathcal{G}$, with the property that $a_{ij} > 0$ if $(v_i,v_j) \in \mathcal{E}$ and $a_{ij} = 0$, otherwise. The weighted out-degree and in-degree of v_i, $i \in \{1,\dots,n\}$, are respectively, $d_{\text{out}}(v_i) = \sum_{j=1}^n a_{ij}$ and $d_{\text{in}}(v_i) = \sum_{j=1}^n a_{ji}$. The *weighted out-degree matrix* D_{out} is the diagonal matrix defined by $(\mathsf{D}_{\text{out}})_{ii} = d_{\text{out}}(i)$, for all $i \in \{1,\dots,n\}$. The *Laplacian* matrix is $\mathsf{L} = \mathsf{D}_{\text{out}} - \mathsf{A}$. Note that $\mathsf{L}\mathbf{1}_n = 0$. If $\mathcal{G}$ is strongly connected, then zero is a simple eigenvalue of L. $\mathcal{G}$ is undirected if $\mathsf{L} = \mathsf{L}^\top$ and *weight-balanced* if $d_{\text{out}}(v) = d_{\text{in}}(v)$, for all $v \in \mathcal{V}$. Equivalently, $\mathcal{G}$ is weight-balanced if and only if $\mathbf{1}_n^\top\mathsf{L} = 0$ if and only if $\mathsf{L}+\mathsf{L}^\top$ is positive semidefinite.

B. Nonsmooth analysis

Here, we introduce some key notions on nonsmooth analysis following [16]. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *locally Lipschitz* at $x \in \mathbb{R}^n$ if there exist $L_x, \epsilon \in (0,\infty)$ such that

$$\|f(y) - f(y')\|_2 \leq L_x \|y - y'\|_2,$$

for all $y, y' \in B(x,\epsilon)$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^n$ if, for all $v \in \mathbb{R}^n$, the right directional derivative of f at x in the direction of v exists, and coincides with the generalized directional derivative of f at x in the direction of v, see [17] for definitions of these notions. A function that is continuously differentiable at x is regular at x. Also, a convex function is regular.

A set-valued map $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is *upper semicontinuous* at $x \in \mathbb{R}^n$ if, for all $\epsilon \in (0,\infty)$, there exists $\delta \in (0,\infty)$ such that $\mathcal{H}(y) \subset \mathcal{H}(x) + B(0,\epsilon)$ for all $y \in B(x,\delta)$. Also, $\mathcal{H}$ is *locally bounded* at $x \in \mathbb{R}^n$ if there exist $\epsilon, \delta \in (0,\infty)$ such that $\|z\|_2 \leq \epsilon$ for all $z \in \mathcal{H}(y)$, and all $y \in B(x,\delta)$.

Given a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, let Ω_f be the set (of measure zero) of points where f is not differentiable. The *generalized gradient* $\partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of f is defined by

$$\partial f(x) = \operatorname{co}\{\lim_{i\to\infty} \nabla f(x_i) \mid x_i \to x, x_i \notin S \cup \Omega_f\},$$

where co is the convex hull and $S \subset \mathbb{R}^n$ is any set of measure zero. The set-valued map ∂f is locally bounded, upper semicontinuous, and takes non-empty, compact, and convex values. A *critical point* $x \in \mathbb{R}^n$ of f satisfies $0 \in \partial f(x)$.

C. Stability of differential inclusions

We gather here some useful tools to analyze the stability properties of differential inclusions [16],

$$\dot{x} \in \mathcal{H}(x), \tag{1}$$

where $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map. A solution of (1) on $[0,T] \subset \mathbb{R}$ is an absolutely continuous map $x : [0,T] \to \mathbb{R}^n$ that satisfies (1) for almost all $t \in [0,T]$. If the set-valued map $\mathcal{H}$ is locally bounded, upper semicontinuous, and takes non-empty, compact, and convex values, then the existence of solutions is guaranteed. The set of equilibria of (1) is denoted by $\operatorname{Eq}(\mathcal{H}) = \{x \in \mathbb{R}^n \mid 0 \in \mathcal{H}(x)\}$. A set $S \subset \mathbb{R}^n$ is *strongly positively invariant* under (1) if, for each $x \in S$, all solutions starting from x are entirely contained in S.

Given a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$, the *set-valued Lie derivative* $\mathcal{L}_{\mathcal{H}} f : \mathbb{R}^n \rightrightarrows \mathbb{R}$ of f with respect to (1) at x is defined as

$$\mathcal{L}_{\mathcal{H}} f = \{a \in \mathbb{R} \mid \text{ there exists } v \in \mathcal{H}(x) \text{ such that } \zeta^\top v = a \text{ for all } \zeta \in \partial f(x)\}.$$

The next result, see e.g., [16, Theorem 2], provides a way to establish the asymptotic convergence of (1).

Theorem 2.1: (LaSalle Invariance Principle for differential inclusions): Let $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be locally bounded, upper semicontinuous, with non-empty, compact, and convex values. Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and regular. If $S \subset \mathbb{R}^n$ is compact and strongly invariant under (1) and $\max \mathcal{L}_{\mathcal{H}} f(x) \leq 0$ for all $x \in S$, then the solutions of (1) starting at S converge to the largest weakly invariant set M contained in $S \cap \{x \in \mathbb{R}^n \mid 0 \in \mathcal{L}_{\mathcal{H}} f(x)\}$. Moreover, if the set M consists of a finite number of points, then the limit of each solution starting in S exists and is an element of M.

D. Constrained optimization and exact penalty functions

Here, we introduce some notions on constrained optimization problems and exact penalty functions following [18],

[19]. Consider the constrained optimization problem,

$$\begin{aligned} &\text{minimize} \quad f(x), && \text{(2a)}\\ &\text{subject to} \quad g(x) \le \mathbf{0}_m, \quad h(x) = \mathbf{0}_p, && \text{(2b)} \end{aligned}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, and $h : \mathbb{R}^n \to \mathbb{R}^p$, with $p \le n$, are continuously differentiable. The *refined Slater condition* is satisfied by (2) if there exists $x \in \mathbb{R}^n$ such that $h(x) = \mathbf{0}_p$, $g(x) \le \mathbf{0}_m$, and $g_i(x) < 0$ for all nonaffine functions g_i. The optimization (2) is convex if f and g are convex and h affine. For convex optimization problems, the refined Slater condition implies that strong duality holds.

A point $x \in \mathbb{R}^n$ is a Karush-Kuhn-Tucker (KKT) point of (2) if there exist Lagrange multipliers $\lambda \in \mathbb{R}^m_{\ge 0}$ and $\nu \in \mathbb{R}^p$ such that

$$\begin{aligned} &g(x) \le \mathbf{0}_m, \quad h(x) = \mathbf{0}_p, \quad \lambda^\top g(x) = 0,\\ &\nabla f(x) + \sum_{i=1}^m \lambda_i \nabla g_i(x) + \sum_{i=1}^p \nu_i \nabla h_i(x) = 0. \end{aligned}$$

If the optimization (2) is convex and strong duality holds, then a point is a solution of (2) if and only if it is a KKT point.

In the presence of inequality constraints in (2), we are interested in using exact penalty function methods to eliminate them while keeping the equality constraints intact. To this end, we follow the exposition in [19] to construct a nonsmooth exact penalty function $f^\epsilon : \mathbb{R}^n \to \mathbb{R}$,

$$f^\epsilon(x) = f(x) + \frac{1}{\epsilon}\sum_{i=1}^m [g_i(x)]^+$$

with $\epsilon > 0$, and define the minimization problem

$$\begin{aligned} &\text{minimize} \quad f^\epsilon(x), && \text{(3a)}\\ &\text{subject to} \quad h(x) = \mathbf{0}_p. && \text{(3b)} \end{aligned}$$

Note that, if f is convex, then f^ϵ is convex because the function $t \mapsto \frac{1}{\epsilon}[t]^+$ is convex. Therefore, if the problem (2) is convex, then the problem (3) is convex as well. The following result, see e.g. [19, Proposition 1], identifies conditions under which the solutions of the problems (2) and (3) coincide.

Proposition 2.2: (Equivalence of problems (2) *and* (3)*):* Assume that the optimization problem (2) is convex, has nonempty and compact solution set, and satisfies the refined Slater condition. Then, the problems (2) and (3) have exactly the same solutions if

$$\frac{1}{\epsilon} > \|\lambda\|_\infty,$$

for some Lagrange multiplier $\lambda \in \mathbb{R}^m_{\ge 0}$ of the problem (2).

Note that a Lagrange multiplier for (2) exists because, the refined Slater condition is satisfied, and hence every solution is a KKT point.

III. Problem Statement

Consider a network of $n \in \mathbb{Z}_{\ge 1}$ power generators whose communication topology is represented by a strongly connected and weight-balanced digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathsf{A})$. Each generator corresponds to a vertex of the digraph and an edge of the form (i, j) represents the capability of generator j to transmit information to generator i.

The power generated by the vertex i is denoted by $P_i \in \mathbb{R}$. Each generator $i \in \{1, \dots, n\}$ has a cost function $f_i : \mathbb{R} \to \mathbb{R}_{\ge 0}$, which is assumed to be convex and continuously differentiable. The cost incurred by vertex i to generate power P_i is then $f_i(P_i)$. The total cost incurred by the network with the power allocation $P = (P_1, \dots, P_n) \in \mathbb{R}^n$ is given by $f : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ as

$$f(P) = \sum_{i=1}^n f_i(P_i).$$

Note that the function f is also convex and continuously differentiable. The group of generators are given a total power load $P_l \in \mathbb{R}_{>0}$ that must be met, i.e., $\sum_{i=1}^n P_i = P_l$, while at the same time minimizing the total incurred cost $f(P)$. For each generator there exist an upper and a lower limit on the power it can produce, i.e., $P_i^m \le P_i \le P_i^M$ for each $i \in \{1, \dots, n\}$. Formally, the economic dispatch (ED) problem is defined by

$$\begin{aligned} &\text{minimize} \quad f(P), && \text{(4a)}\\ &\text{subject to} \quad \mathbf{1}_n^\top P = P_l, && \text{(4b)}\\ &\qquad\qquad P^m \le P \le P^M. && \text{(4c)} \end{aligned}$$

We refer to (4b) as the load condition and to (4c) as the box constraints. We let $\mathcal{F}_{\text{ED}} = \{P \in \mathbb{R}^n \mid P^m \le P \le P^M \text{ and } \mathbf{1}_n^\top P = P_l\}$ denote the feasibility set of (4). Since the set $\mathcal{F}_{\text{ED}}$ is compact, the set of solutions of (4) is compact. Moreover, since the constraints (4b) and (4c) are affine, feasibility of the ED problem implies that the refined Slater condition is satisfied and strong duality holds for the problem (4). Note that if either $P^M \in \mathcal{F}_{\text{ED}}$ or $P^m \in \mathcal{F}_{\text{ED}}$, then $\mathcal{F}_{\text{ED}}$ is a singleton. For this reason, and without loss of generality, we assume that P^M and P^m are not feasible points for the ED problem. Our objective is to design a distributed procedure that allows the network of power-generators to solve the ED problem.

IV. Distributed Algorithmic Solution to the Economic Dispatch Problem

This section presents our main contribution: the design and analysis of the Laplacian-set-valued dynamics as a distributed algorithmic solution to the ED problem. The synthesis of this strategy is based on an alternative formulation of the ED problem proposed in [14] using an exact penalty function approach. This formulation, which we review next, sets the basis for the design of our Laplacian-based consensus dynamics using the generalized gradients of the modified local cost functions.

A. Exact penalty function formulation

In this section, we employ the exact penalty function approach described in Section II-D to provide an equivalent formulation of the ED problem without the box constraints.

Consider the nonsmooth objective function

$$f^\epsilon(P) = \sum_{i=1}^n f_i(P_i) + \frac{1}{\epsilon}(\sum_{i=1}^n ([P_i - P_i^M]^+ + [P_i^m - P_i]^+)).$$

Note that this corresponds to a scenario where generator $i \in \{1, \dots, n\}$ has local cost given by

$$f_i^\epsilon(P_i) = f_i(P_i) + \frac{1}{\epsilon}([P_i - P_i^M]^+ + [P_i^m - P_i]^+). \quad (5)$$

This function is convex, locally Lipschitz, and continuously differentiable in $\mathbb{R}$ except at $P_i = P_i^m$ and $P_i = P_i^M$. Its generalized gradient $\partial f_i^\epsilon : \mathbb{R} \rightrightarrows \mathbb{R}$ is given by

$$\partial f_i^\epsilon(P_i) = \begin{cases} \{\nabla f_i(P_i) - \frac{1}{\epsilon}\} & P_i < P_i^m, \\ [\nabla f_i(P_i) - \frac{1}{\epsilon}, \nabla f_i(P_i)] & P_i = P_i^m, \\ \{\nabla f_i(P_i)\} & P_i^m < P_i < P_i^M, \\ [\nabla f_i(P_i), \nabla f_i(P_i) + \frac{1}{\epsilon}] & P_i = P_i^M, \\ \{\nabla f_i(P_i) + \frac{1}{\epsilon}\} & P_i > P_i^M. \end{cases}$$

As a result, the total cost f^ϵ is convex, locally Lipschitz, and regular. Its generalized gradient at $P = (P_1, \dots, P_n) \in \mathbb{R}^n$ is given by $\partial f^\epsilon(P) = \partial f_1^\epsilon(P_1) \times \dots \times \partial f_n^\epsilon(P_n)$. Next, consider the modified ED problem as

$$\text{minimize} \quad f^\epsilon(P), \quad (6a)$$
$$\text{subject to} \quad \mathbf{1}_n^\top P = P_l. \quad (6b)$$

The following result from [14] establishes the equivalence of this optimization problem with the ED problem. The proof follows using Proposition 2.2.

Proposition 4.1: (Equivalence between (4) *and* (6) *[14]):* The solutions of (4) and (6) coincide for $\epsilon \in \mathbb{R}_{>0}$ such that

$$\epsilon < \frac{1}{2 \max_{P \in \mathcal{F}_{\mathrm{ED}}} \|\nabla f(P)\|_\infty}. \quad (7)$$

B. Laplacian-set-valued dynamics

Here, we propose a continuous-time, distributed set-valued dynamics to solve the ED problem. Our algorithm design is based on the alternative formulation (6) of the problem, cf. Proposition 4.1. Consider the Laplacian-set-valued dynamics

$$\dot{P} \in -\mathsf{L}\mathcal{H}(P), \quad (8)$$

where $\mathcal{H}(P) = \mathcal{H}_1(P_1) \times \dots \times \mathcal{H}_n(P_n)$, and,

$$\mathcal{H}_i(P_i) = \begin{cases} \{-\frac{1}{\epsilon}\} & P_i < P_i^m, \\ [-\frac{1}{\epsilon}, \nabla f_i(P_i)] & P_i = P_i^m, \\ \{\nabla f_i(P_i)\} & P_i^m < P_i < P_i^M, \\ [\nabla f_i(P_i), \frac{1}{\epsilon}] & P_i = P_i^M, \\ \{\frac{1}{\epsilon}\} & P_i > P_i^M, \end{cases} \quad (9)$$

with ϵ satisfying (7). Note that $\mathcal{H}$ does not exactly correspond to the generalized gradient of f^ϵ. Specifically, $\mathcal{H}$ and ∂f^ϵ only coincide when evaluated at points in the interior of $\mathcal{F}_{\mathrm{ED}}$, and are otherwise different. As we show below in Lemma 4.2, the use of $\mathcal{H}$ makes the feasibility set $\mathcal{F}_{\mathrm{ED}}$ strongly invariant under the dynamics (8) without imposing further conditions on the choice of parameter ϵ (in contrast with the case when ∂f^ϵ is used, which requires additional constraints that depend on the network topology [14]).

When convenient, we use $X_{\text{L-sv}} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ to refer to the Laplacian-set-valued dynamics (8). Note that for ϵ satisfying (7), the set-valued map $\mathcal{H}$ and hence, $X_{\text{L-sv}}$, is nonempty, locally bounded, upper semicontinuous, and takes compact, convex values. Therefore, a solution is guaranteed to exist starting from any point for the Laplacian-set-valued dynamics (8). Moreover, this dynamics is distributed in the sense that, to implement it, each generator i selects an element from the set $\mathcal{H}_i(P_i)$ and communicates it to its in-neighbors and at the same time it receives similar information from its out-neighbors.

The following result establishes that the feasibility set $\mathcal{F}_{\mathrm{ED}}$ is strongly positively invariant for the dynamics (8).

Lemma 4.2: (Invariance of the constraint set under the Laplacian-set-valued dynamics): The feasibility set $\mathcal{F}_{\mathrm{ED}}$ is strongly positively invariant under the Laplacian-set-valued dynamics (8).

Proof: We start by defining two sets

$$\mathcal{F}_B = \{P \in \mathbb{R}^n \mid P^m \le P \le P^M\},$$
$$\mathcal{F}_L = \{P \in \mathbb{R}^n \mid \mathbf{1}_n^\top P = P_l\}.$$

Note that $\mathcal{F}_{\mathrm{ED}} = \mathcal{F}_L \cap \mathcal{F}_B$. Therefore, to prove our claim, it is sufficient to show that each set is strongly positively invariant under $X_{\text{L-sv}}$. First, consider the set-valued Lie derivative of the total power generated by the network along $X_{\text{L-sv}}$

$$\mathcal{L}_{X_{\text{L-sv}}}(\mathbf{1}_n^\top P) = \{-\mathbf{1}_n^\top \mathsf{L}\zeta \mid \zeta \in \mathcal{H}(P)\}.$$

Since $\mathcal{G}$ is weight-balanced, $\mathcal{L}_{X_{\text{L-sv}}}(\mathbf{1}_n^\top P) = \{0\}$. Therefore, the total power generated by the network is conserved along all evolutions of $X_{\text{L-sv}}$ and hence, $\mathcal{F}_L$ is strongly positively invariant. To establish the strong invariance of $\mathcal{F}_B$, using [20, Proposition 1, pp 234], it is sufficient to show that for $P \notin \mathcal{F}_B$, $(P - \pi_{\mathcal{F}_B}(P))^\top v \le 0$ for all $v \in -\mathsf{L}\mathcal{H}(P)$. Let $P \notin \mathcal{F}_B$, and define the sets of indices $I_m(P) = \{i \in \{1, \dots, n\} \mid P_i < P_i^m\}$, $I_I(P) = \{i \in \{1, \dots, n\} \mid P_i^m \le P_i \le P_i^M\}$, and $I_M(P) = \{i \in \{1, \dots, n\} \mid P_i > P_i^M\}$. The projection of P on the set $\mathcal{F}_B$, $\pi_{\mathcal{F}_B}(P)$ is

$$(\pi_{\mathcal{F}_B}(P))_i = \begin{cases} P_i^m & i \in I_m(P), \\ P_i & i \in I_I(P), \\ P_i^M & i \in I_M(P), \end{cases}$$

and hence $P - \pi_{\mathcal{F}_B}(P)$ is

$$(P - \pi_{\mathcal{F}_B}(P))_i = \begin{cases} P_i - P_i^m & i \in I_m(P), \\ 0 & i \in I_I(P), \\ P_i - P_i^M & i \in I_M(P). \end{cases}$$

Now consider an element $v \in -\mathsf{L}\mathcal{H}(P)$. Let $\xi \in \mathcal{H}(P)$ be such that $v = -\mathsf{L}\xi$. Then, by definition of the Laplacian matrix, $v_i = -\sum_{j \in N_{\mathrm{out}}(i)} a_{ij}(\xi_i - \xi_j)$. Moreover, from (9), we observe that for $i \in I_m(P)$, $\xi_i = -\frac{1}{\epsilon}$ and $\xi_j \ge -\frac{1}{\epsilon}$ for all $j \in N_{\mathrm{out}}(i)$. Thus, we obtain $v_i \ge 0$ for $i \in I_m(P)$. Similarly, $v_i \le 0$ for $i \in I_M(P)$. With this property of v and the form of $P - \pi_{\mathcal{F}_B}(P)$, we deduce that for all $P \notin \mathcal{F}_B$, $(P - \pi_{\mathcal{F}_B}(P))^\top v \le 0$ for all $v \in -\mathsf{L}\mathcal{H}(P)$, as claimed. ■

Lemma 4.2 implies that all evolutions of (8) from a point $P_0 \in \mathcal{F}_{\text{ED}}$ are contained in $\mathcal{F}_{\text{ED}}$. The next result establishes some important properties of the evolution of the function f^ϵ along the Laplacian-set-valued dynamics.

Lemma 4.3: (Nonsmooth objective function is nonincreasing along the Laplacian-set-valued dynamics): For $P \in \mathcal{F}_{\text{ED}}$,

(i) If $a \in \mathcal{L}_{X_{\text{L-sv}}} f^\epsilon(P)$, then $a = -\xi^\top \mathsf{L}\xi$ for a $\xi \in \mathcal{H}(P)$.

(ii) $\max \mathcal{L}_{X_{\text{L-sv}}} f^\epsilon(P) \leq 0$ (f^ϵ is nonincreasing along $X_{\text{L-sv}}$).

(iii) If $0 \in \mathcal{L}_{X_{\text{L-sv}}} f^\epsilon(P)$ then there exists $\mu \in \mathbb{R}$ such that $\mu \mathbf{1}_n \in \mathcal{H}(P) \cap \partial f^\epsilon(P)$.

Proof: For convenience we define the following sets of indices for $P \in \mathcal{F}_{\text{ED}}$: $I_0(P) = \{i \in \{1, \dots, n\} \mid P_i^m < P < P_i^M\}$, $I_+(P) = \{i \in \{1, \dots, n\} \mid P_i = P_i^M\}$, and $I_-(P) = \{i \in \{1, \dots, n\} \mid P_i = P_i^m\}$. Re (i), for $P \in \mathcal{F}_{\text{ED}}$, let $a \in \mathcal{L}_{X_{\text{L-sv}}} f^\epsilon(P)$. For a, let $\xi \in \mathcal{H}(P)$ and $X_{\text{L-sv}}(P) \ni v = -\mathsf{L}\xi$ be such that $\zeta^\top v = a$ for all $\zeta \in \partial f^\epsilon(P)$. Since $\zeta^\top v = a$ for all $\zeta \in \partial f^\epsilon(P)$ and $\partial f_i^\epsilon(P_i)$ is not a singleton set for $i \in (I_+(P) \cup I_-(P))$, we deduce that $v_i = 0$ for all $i \in (I_+(P) \cup I_-(P))$. Next, from the definition of sets $\partial f^\epsilon(P)$ and $\mathcal{H}(P)$, $\zeta_i - \xi_i = 0$ for all $i \in I_0(P)$ and for all $\zeta \in \partial f^\epsilon(P)$. With these properties of v and $(\zeta - \xi)$ we obtain that $(\zeta - \xi)^\top v = 0$ for all $\zeta \in \partial f^\epsilon$. Which implies that $a = \zeta^\top v = \xi^\top v = -\xi^\top \mathsf{L}\xi$. Fact (ii) follows from (i) and the fact that $\mathcal{G}$ is weight-balanced. Re (iii), from (i), if $0 \in \mathcal{L}_{X_{\text{L-sv}}} f^\epsilon(P)$ then $-\xi^\top \mathsf{L}\xi = 0$ for some $\xi \in \mathcal{H}(P)$. Since $\mathcal{G}$ is weight-balanced, $\xi = \mu \mathbf{1}_n$ for some $\mu \in \mathbb{R}$ and hence $\mu \mathbf{1}_n \in \mathcal{H}(P)$. To prove $\mu \mathbf{1}_n \in \partial f^\epsilon(P)$ we first show that $|\mu| \leq \|\nabla f(P)\|_\infty$. From (9) we have

$$\begin{cases} -\frac{1}{\epsilon} \leq \mu \leq \nabla f_i(P_i), & i \in I_-(P), \\ \mu = \nabla f_i(P_i), & i \in I_0(P), \\ \nabla f_i(P_i) \leq \mu \leq \frac{1}{\epsilon}, & i \in I_+(P). \end{cases} \quad (10)$$

If $I_0(P)$ is non-empty, then from (10), $|\mu| \leq \|\nabla f(P)\|_\infty$. If $I_0(P)$ is empty, (10) yields, $\nabla f_j(P_j) \leq \mu \leq \nabla f_k(P_k)$ for all $j \in I_+(P)$ and $k \in I_-(P)$. In this case, $I_-(P)$ and $I_+(P)$ are non-empty because otherwise either $P^m \in \mathcal{F}_{\text{ED}}$ or $P^M \in \mathcal{F}_{\text{ED}}$ which we assume not to be true. Therefore, we get $|\mu| \leq \|\nabla f(P)\|_\infty$. This establishes that if $\mu \mathbf{1}_n \in \mathcal{H}(P)$ then $|\mu| \leq \|\nabla f(P)\|_\infty$. By choice, ϵ satisfies (7), hence $\frac{1}{\epsilon} \geq 2\|\nabla f(P)\|_\infty$. This, along with the derived bound on $|\mu|$ gives us $\nabla f_i(P_i) - \frac{1}{\epsilon} \leq \mu \leq \nabla f_i(P_i) + \frac{1}{\epsilon}$ for all $i \in \{1, \dots, n\}$. Finally, this condition and (10) together imply that $\mu \mathbf{1}_n \in \partial f^\epsilon(P)$. ■

Next, we show the convergence of the Laplacian-set-valued dynamics to the solutions of the ED problem.

Theorem 4.4: (Convergence of the Laplacian-set-valued dynamics to the solutions of ED *problem):* The evolution of the Laplacian-set-valued dynamics $X_{\text{L-sv}}$ from any point $P_0 \in \mathcal{F}_{\text{ED}}$ converges to the solutions of the ED problem (4).

Proof: Recall that, with the choice of ϵ satisfying (7), $X_{\text{L-sv}}$ is locally bounded, upper semicontinuous and takes non-empty, compact and convex values. To establish convergence, we study the behavior of the function f^ϵ along $X_{\text{L-sv}}$. Recall that f^ϵ is locally Lipschitz and regular. From Lemma 4.2, $\mathcal{F}_{\text{ED}}$ is strongly positively invariant under the dynamics $X_{\text{L-sv}}$ and from Lemma 4.3, $\max \mathcal{L}_{X_{\text{L-sv}}} f^\epsilon(P) \leq 0$ for all $P \in \mathcal{F}_{\text{ED}}$. Therefore, from LaSalle Invariance Principle, cf. Theorem 2.1, all evolutions of (8) starting at $P_0 \in \mathcal{F}_{\text{ED}}$ converge to the largest weakly invariant set M contained in $\mathcal{F}_{\text{ED}} \cap \{P \in \mathbb{R}^n | 0 \in \mathcal{L}_{X_{\text{L-sv}}} f^\epsilon(P)\}$. For $P \in \mathcal{F}_{\text{ED}}$ with $0 \in \mathcal{L}_{X_{\text{L-sv}}} f^\epsilon(P)$, Lemma 4.3 implies that there exists $\mu \in \mathbb{R}$ such that $\mu \mathbf{1}_n \in \partial f^\epsilon(P)$. This fact, together with $\mathbf{1}_n^\top P = P_l$, implies that P is a solution of (6). Therefore, M corresponds to the set of solutions of (6). Finally, since ϵ satisfies (7), from Proposition 4.1, the set M is also the set of solutions of (4). ■

Since, $\mathcal{F}_{\text{ED}}$ is strongly positively invariant under $X_{\text{L-sv}}$, and f^ϵ is nonincreasing along $X_{\text{L-sv}}$ (Lemma 4.3), we deduce that f is nonincreasing along $X_{\text{L-sv}}$. This follows from the fact that $f^\epsilon(P) = f(P)$ for $P \in \mathcal{F}_{\text{ED}}$. From these, we conclude that the Laplacian-set-valued dynamics is an anytime algorithm, i.e., starting from $\mathcal{F}_{\text{ED}}$ its trajectories are feasible solutions at any time before convergence, and they become better and better solutions as time elapses. Also, as the total power generated by the units is conserved along the trajectories, any initial error in load satisfaction does not grow while the system evolves.

Remark 4.5: (Initialization of the Laplacian-set-valued dynamics): The asymptotic convergence result of Theorem 4.4 requires a initial network configuration that is feasible, i.e., that satisfies the load and the box constraints. The linear-iterative algorithms suggested in [21] can be used for initialization purposes, but they only guarantee convergence to a feasible point $P_0 \in \mathcal{F}_{\text{ED}}$ asymptotically. We have proposed in [14] the DETERMINE FEASIBLE ALLOCATION algorithm, that can find a network configuration in $\mathcal{F}_{\text{ED}}$ in finite time and works over undirected communication topologies. •

V. SIMULATIONS

Here, we provide simulation results illustrating the application of Laplacian-set-valued dynamics (8) for solving the economic dispatch problem. As an example, we consider the dispatch problem for a standard system of 6 generating units [22]. The cost function and generation limits for each unit is given in Table I. The cost for each generator is a quadratic function of the power it generates. The total load on the system of generators is 1263 MW, i.e., $P_l = 1263$. To employ the set-valued dynamics, we consider the units to have a communication topology of a strongly connected weight-balanced digraph with adjacency matrix as

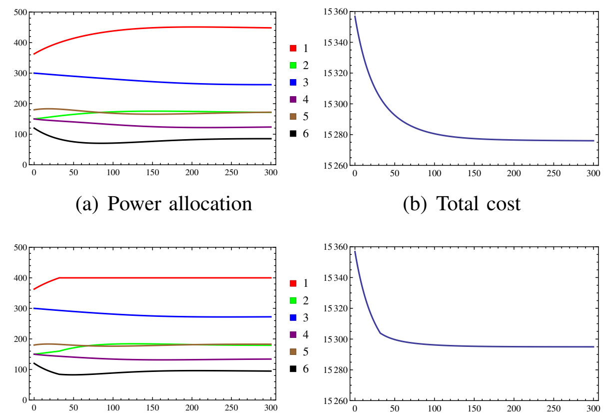
$$\mathsf{A} = \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (11)$$

We choose $\epsilon = \frac{1}{30}$ that satisfies (7) for the current system. The simulation results for this system are presented in Figure 1(a)-(b). The power allocation of the system satisfies the load condition and the box constraints at all time instances and converges to the optimal solution. To illustrate the invariance property of the Laplacian-set-valued dynamics, we repeated the simulation with the generation limits for unit 1 modified from $[100, 500]$ to $[100, 400]$. This is depicted in Figure 1(c)-(d). In this case, unit 1 reaches its upper limit 400 MW and stays there (cf. red line in Figure 1(c)) while the other units converge. As expected, the more restrictive box constraints on generator 1 result in a higher final cost.

Unit	a($)	b($/MW)	c($/MW.MW)	P_i^m(MW)	P_i^M(MW)
1	240	7.0	0.0070	100	500
2	200	10.0	0.0095	50	200
3	220	8.5	0.0090	80	300
4	200	11.0	0.0090	50	150
5	220	10.5	0.0080	50	200
6	190	12.0	0.0075	50	120

TABLE I

COST FUNCTION COEFFICIENTS (a, b, c) AND GENERATION LIMITS P_i^M, P_i^m. THE COST FUNCTION FOR EACH UNIT IS $f_i(P_i) = a + bP_i + cP_i^2$.



(a) Power allocation (b) Total cost

(c) Power allocation (d) Total cost

Fig. 1. Simulation results depicting the application of the Laplacian-set-valued dynamics to solve the economic dispatch problem for a system of 6 generation units. The cost function and the generation limits are given in Table I. The communication topology is defined by (11), $\epsilon = \frac{1}{30}$, and $P_l = 1263$. (a) and (b) show the evolution of power allocations and total cost for the system starting from the feasible power allocation $(363, 150, 300, 150, 180, 120)$. The system converges to the optimal solution $(448, 172, 262, 124, 172, 85)$ and total cost is \$15276. (c) and (d) show the results for the same example with the constraints on unit 1 changed from $[100, 500]$ to $[100, 400]$. The system now converges to $(400, 179, 272, 134, 183, 95)$, with total cost \$15295.

VI. CONCLUSIONS

We have proposed the Laplacian-set-valued dynamics to solve the economic dispatch problem over a group of power-generators with arbitrary convex cost functions and capacity bounds. We have shown that, when the communication topology among the generators is described by a weight-balanced, strongly connected digraph, this distributed dynamics provably converges to the solutions of the economic dispatch problem. Our analysis, based on tools from algebraic graph theory, nonsmooth analysis, and optimization, has also established that the algorithm is anytime. Future work will study the characterization of the rate of convergence of the Laplacian-set-valued dynamics and the analysis of dynamic communication topologies, losses over the transmission lines, and generator dynamics with ramp constraints.

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