Distributed event-triggered communication for dynamic average consensus in networked systems ⋆

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Abstract

This paper presents distributed algorithmic solutions that employ opportunistic inter-agent communication to achieve dynamic average consensus. Our solutions endow individual agents with autonomous criteria that can be checked with the information available to them in order to determine whether to broadcast their state to their neighbors. Our starting point is a distributed coordination strategy that, under continuous-time communication, achieves practical asymptotic tracking of the dynamic average of the time-varying agents' inputs. We propose two different distributed event-triggered communication laws, depending on the directed or undirected nature of the time-varying interactions and under suitable connectivity conditions, that prescribe agent communications at discrete time instants in an opportunistic fashion. In both cases, we establish positive lower bounds on the inter-event times of each agent and characterize their dependence of the algorithm design parameters. This analysis allows us to rule out the presence of Zeno behavior and characterize the asymptotic correctness of the resulting implementations. Several simulations illustrate the results.

Key words: cooperative control, dynamic average consensus, event-triggered communication, weight-balanced directed graphs.

1 Introduction

The dynamic average consensus problem consists of designing a distributed algorithm that allows a group of agents to track the average of individual time-varying inputs, one per agent. This problem has applications in numerous areas that involve distributed sensing and filtering, including distributed tracking [Yang et al., 2007], multi-robot coordination [Yang et al., 2008], sensor fusion [Olfati-Saber, 2007, Olfati-Saber and Shamma, 2005], and distributed estimation [Carron et al., 2013]. Our goal here is to develop algorithmic solutions to the dynamic average consensus problem which rely on agents autonomously deciding when to share information with their neighbors in an opportunistic fashion for greater efficiency and energy savings.

Literature review: The literature of cooperative control has proposed dynamic average consensus algorithms that are executed either in continuous time [Bai et al., 2010, Freeman et al., 2006, Kia et al., 2014b, Olfati-Saber and Shamma, 2005, Spanos et al., 2005] or in discrete time [Kia et al., 2014b, Zhu and Martínez, 2008], with a fixed stepsize. Continuous-time algorithms operate under the assumption of continuous agent-to-agent continuous-time sharing of information. Although discrete-time algorithms are more amenable to practical implementation, they tie the communication and computation stepsizes together, resulting in a conservative stepsize for communication times. This can result in costly implementations, as performing communication usually requires more energy than computation. In addition, the use of fixed communication stepsizes can lead to a wasteful use of the network resources because of the need to select it taking into account worst-case situations. Finally, the assumption of periodic, synchronous communication is unrealistic in many scenarios involving cyber-physical systems, as processors are subject to natural delays and errors which may deviate them from the perfect operational conditions the strategies are designed for. Event-triggered communication offers a way to address these shortcomings by prescribing in an opportunistic way the times for information sharing and allowing individual agents to determine these autonomously. In recent years, an increasing body of work that seeks to trade computation and decision-making for less communication, sensing or actuation effort while guaranteeing a desired level of performance has emerged,

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see e.g., [Heemels et al., 2012, Mazo and Tabuada, 2011, Wang and Lemmon, 2011]. Closest to the problem considered here are the works that study event-triggered communication laws for static average consensus, see e.g., [Dimarogonas et al., 2012, Garcia et al., 2013, Nowzari and Cortés, 2014] and references therein.

Statement of contributions: We propose novel algorithmic solutions to the dynamic average consensus problem that employ opportunistic strategies to determine the communication times among neighboring agents. The basic idea is that agents share their information with neighbors when the uncertainty in the outdated information is such that the monotonic convergent behavior of the overall network can no longer be guaranteed. To realize this concept, we propose and characterize the correctness of two different distributed event-triggered communication laws, depending on whether the interaction topology is described by a time-varying, weight-balanced piecewise constant digraph which is jointly strongly connected over an infinite sequence of contiguous and uniformly bounded time intervals or a time-varying, piecewise continuous undirected connected graph. By establishing positive lower bounds on the inter-event times of each agent, we also show that the proposed distributed event-triggered communication laws are free from Zeno behavior (the undesirable situation where an infinite number of communication rounds are triggered in a finite amount of time). Finally, we analyze the dependence of the inter-event times on the algorithm design parameters. This characterization provides guidelines on the trade-offs between the minimum inter-event times for communication and the performance and energy efficiency of the proposed algorithms. We demonstrate through several comparative simulation studies the advantages of our proposed event-triggered communication strategies over schemes that rely on continuous-time communication as well as discrete-time communication with fixed stepsize.

Organization: Section 2 gathers basic notation and concepts from graph theory. Section 3 presents the network model and the dynamic average consensus problem. Section 4 introduces our continuous-time algorithmic solutions with event-triggered communication. Section 5 presents simulation results and Section 6 gathers our conclusions and ideas for future work.

2 Preliminaries

In this section, we introduce basic notation and concepts from graph theory.

2.1 Notation

We let $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$ denote the set of real, nonnegative real, nonnegative integer and natural numbers, respectively. We use $\Re(\cdot)$ to denote the real part of a complex number. The transpose of a matrix $\mathbf{A}$ is $\mathbf{A}^\top$. We let $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) denote the vector of n ones (resp. n zeros), and denote by $\mathbf{I}_n$ the $n \times n$ identity matrix. We let $\mathbf{\Pi}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$. When clear from the context, we do not specify the matrix dimensions. For $\mathbf{u} \in \mathbb{R}^d$, $\|\mathbf{u}\| = \sqrt{\mathbf{u}^\top\mathbf{u}}$ denotes the standard Euclidean norm. For $u \in \mathbb{R}$, we denote by $|u|$ its absolute value. For a time-varying measurable locally essentially bounded signal $\mathbf{u}: \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, we denote by $\|\mathbf{u}\|_{\text{ess}}$ the essential supremum norm. For a scalar signal u, we use $|u|_{\text{ess}}$ instead. For vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$, we let $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_m)$ represent the aggregated vector. In a networked system, we distinguish the local variables at each agent by a superscript, e.g., $\mathbf{x}^i$ is the local state of agent i. If $\mathbf{p}^i \in \mathbb{R}^d$ is a variable of agent i, the aggregate of a network with N agents is $\mathbf{p} = (\mathbf{p}^1, \dots, \mathbf{p}^N) \in (\mathbb{R}^d)^N$. To deal with the dynamics that follow, it is convenient to introduce the matrices $\mathbf{T} \in \mathbb{R}^{N\times N}$ and $\mathbf{R} \in \mathbb{R}^{N\times N-1}$, and the vector $\mathbf{r} \in \mathbb{R}^N$ as follows,

$$\mathbf{T} = [\mathbf{r},\ \mathbf{R}],\quad \mathbf{r} = \frac{1}{\sqrt{N}}\mathbf{1}_N,\quad \mathbf{r}^\top\mathbf{R} = \mathbf{0},\quad \mathbf{R}^\top\mathbf{R} = \mathbf{I}_{N-1}.$$

Notice that $\mathbf{T}^\top\mathbf{T} = \mathbf{I}_N$.

2.2 Graph theory

Here, we briefly review some basic concepts from graph theory and linear algebra following [Bullo et al., 2009]. A *directed graph*, or simply a *digraph*, is a pair $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, \dots, N\}$ is the *node set* and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the *edge set*. For an edge $(i,j) \in \mathcal{E}$, i is called an *in-neighbor* of j and j is called an *out-neighbor* of i. We let $\mathcal{N}^i$ denote the set of out-neighbors of $i \in \{1, \dots, N\}$. A graph is *undirected* if $(i,j) \in \mathcal{E}$ when $(j,i) \in \mathcal{E}$. A *directed path* is a sequence of nodes connected by edges. A digraph is called *strongly connected* if for every pair of vertices there is a directed path connecting them. Given digraphs $\mathcal{G}_i = (\mathcal{V}, \mathcal{E}_i)$, $i \in \{1, \dots, m\}$, their *union* is the graph $\cup_{i=1}^n \mathcal{G}_i = (\mathcal{V}, \mathcal{E}_1 \cup \mathcal{E}_2 \cup \dots \cup \mathcal{E}_m)$.

A *weighted digraph* is a triplet $(\mathcal{V}, \mathcal{E}, \mathsf{A})$, where $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a digraph and $\mathsf{A} \in \mathbb{R}^{N\times N}$ is a weighted *adjacency* matrix with the property that $\mathsf{a}_{ij} > 0$ if $(i,j) \in \mathcal{E}$ and $\mathsf{a}_{ij} = 0$, otherwise. A weighted digraph is *undirected* if $\mathsf{a}_{ij} = \mathsf{a}_{ji}$ for all $i, j \in \mathcal{V}$. We refer to a strongly connected and undirected graph as *connected*. The *weighted out-* and *in-degrees* of a node i are, respectively, $\mathsf{d}^i_{\text{out}} = \sum_{j=1}^N \mathsf{a}_{ij}$ and $\mathsf{d}^i_{\text{in}} = \sum_{j=1}^N \mathsf{a}_{ji}$. A digraph is *weight-balanced* if at each node $i \in \mathcal{V}$, the weighted out- and in-degrees coincide (although they might be different across different nodes). The *(out-) Laplacian* matrix is $\mathbf{L} = \mathsf{D}^{\text{out}} - \mathsf{A}$, where $\mathsf{D}^{\text{out}} = \text{Diag}(\mathsf{d}^1_{\text{out}}, \cdots, \mathsf{d}^N_{\text{out}}) \in \mathbb{R}^{N\times N}$. Note that $\mathbf{L}\mathbf{1}_N = \mathbf{0}$. A digraph is weight-balanced if and only if $\mathbf{1}_N^T\mathbf{L} = \mathbf{0}$ if and only if $\text{Sym}(\mathbf{L}) = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T)$ is positive semi-definite. Based on the structure of $\mathbf{L}$, at least one of the eigenvalues of $\mathbf{L}$, denoted by $\lambda_1, \dots, \lambda_N$, is zero and the rest of them have nonnegative real parts. We let $\lambda_1 = 0$ and $\Re(\lambda_i) \leq \Re(\lambda_j)$, for $i < j$. We denote the eigenvalues of $\mathrm{Sym}(\mathbf{L})$ by $\hat{\lambda}_1, \ldots, \hat{\lambda}_N$. For a strongly connected and weight-balanced digraph, zero is a simple eigenvalue of both $\mathbf{L}$ and $\mathrm{Sym}(\mathbf{L})$. In this case, we order the eigenvalues of $\mathrm{Sym}(\mathbf{L})$ as $0 = \hat{\lambda}_1 < \hat{\lambda}_2 \leq \hat{\lambda}_3 \leq \ldots \leq \hat{\lambda}_N$.

Throughout the paper, we deal with time-varying digraphs with fixed vertex set. A time-varying digraph $\mathbb{R}_{\geq 0} \ni t \mapsto \mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t), \mathsf{A}(t))$ is *piecewise continuous (respectively, piecewise constant)* if the map $t \mapsto \mathsf{A}(t)$ is piecewise continuous (respectively, piecewise constant) from the right. In such case, we denote by $\{s_k\}_{k\in\mathbb{Z}_{\geq 0}}$ the time instants at which this map is discontinuous and refer to them as *switching times*. By convention, $s_0 = 0$. A time-varying digraph $\mathbb{R}_{\geq 0} \ni t \mapsto \mathcal{G}(t)$ has *uniformly bounded weights* if, for all $t \in \mathbb{R}_{\geq 0}$, $\mathsf{a}_{ij}(t) \in [\underline{\mathsf{a}}, \bar{\mathsf{a}}]$, with $0 < \underline{\mathsf{a}} \leq \bar{\mathsf{a}}$, if $(j, i) \in \mathcal{E}(t)$, and $\mathsf{a}_{ij} = 0$ otherwise. A time-varying digraph $\mathbb{R}_{\geq 0} \ni t \mapsto \mathcal{G}(t)$ is *strongly connected* if each $\mathcal{G}(t)$ is strongly connected, and is *jointly strongly connected* over $[t_1, t_2)$ if $\cup_{t\in[t_1,t_2)}\mathcal{G}(t)$ is strongly connected. A piecewise constant time-varying digraph $\mathbb{R}_{\geq 0} \ni t \mapsto \mathcal{G}(t)$ is *recurrently jointly strongly connected* if the sequence of inter-switching times $\{s_{k+1} - s_k\}_{k\in\mathbb{Z}_{\geq 0}}$ is uniformly lower bounded and there exists an infinite sequence of contiguous uniformly bounded intervals $\{[s_{k_j}, s_{k_{j+1}})\}_{j\in\mathbb{Z}_{\geq 0}}$, with $s_{k_0} = s_0$, such that $\mathcal{G}(t)$ is jointly strongly connected over $[s_{k_j}, s_{k_{j+1}})$, for all $j \in \mathbb{Z}_{\geq 0}$. Finally, a time-varying digraph $\mathbb{R}_{\geq 0} \ni t \mapsto \mathcal{G}(t)$ is *weight-balanced* if each $\mathcal{G}(t)$ is weight-balanced. For this type of digraphs, we define

$$\begin{aligned}
\overline{\|\mathbf{L}\|} &= \sup\{\|\mathbf{L}_t\| \mid t \in \mathbb{R}_{\geq 0}\}, \\
\underline{\hat{\lambda}}_2 &= \inf\{\hat{\lambda}_2(\mathbf{L}_t) \mid t \in \mathbb{R}_{\geq 0}\}, \\
\bar{\mathsf{d}}^i_{\mathrm{out}} &= \sup\{(\mathsf{d}^i_{\mathrm{out}})_t \mid t \in \mathbb{R}_{\geq 0}\},\ i \in \{1, \ldots, N\},
\end{aligned}$$

where $\mathbf{L}_t$ is the Laplacian of $\mathcal{G}(t)$. Note that if $\mathcal{G}$ has uniformly bounded weights and is weight-balanced and strongly connected, then $\underline{\hat{\lambda}}_2 > 0$. If $\mathcal{G}$ is a time-varying connected graph, we use the notation $\underline{\lambda}_2$ instead of $\underline{\hat{\lambda}}_2$. The following result, taken from [Kia et al., 2014b, Lemma 4.5], is useful when dealing with recurrently jointly strongly connected digraphs. For a weight-balanced and recurrently jointly strongly connected time-varying digraph with uniformly bounded weights, there exist $\hat{\lambda}_\sigma > 0$ and $\rho > 0$ such that

$$\left\| \mathrm{e}^{-\beta \mathbf{R}^\top \mathbf{L}_t \mathbf{R}(t-t_0)} \right\| \leq \rho\, \mathrm{e}^{-\beta\hat{\lambda}_\sigma(t-t_0)}, \quad \forall t \geq t_0 \geq 0, \quad (1)$$

for any $\beta > 0$. If the digraph is additionally strongly connected, then (1) is satisfied with $\rho = 1$ and $\hat{\lambda}_\sigma = \underline{\hat{\lambda}}_2$.

3 Network model and problem statement

This section formalizes the problem of interest. Consider a network of N agents with single-integrator dynamics,

$$\dot{x}^i = g^i, \quad i \in \{1, \ldots, N\},$$

where $x^i \in \mathbb{R}$ is the *agreement state* and $g^i \in \mathbb{R}$ is the *driving command* of agent i. Our consideration of simple dynamics is motivated by the fact that the state of the agents does not necessarily correspond to some physical quantity, but instead to some logical variable on which agents perform computation and processing. Each agent $i \in \{1, \ldots, N\}$ has access to a time-varying input signal $\mathsf{u}^i : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Agents transmit information to other agents through wireless communication and their interaction topology is modeled by a time-varying weighted digraph $\mathcal{G}$. An edge (i, j) from i to j at time t means that agent j can send information to agent i at t. Given that communication occurs at discrete instants of time, we let $\hat{x}^i$ denote the last known state of agent $i \in \{1, \ldots, N\}$ transmitted to its in-neighbors. We let $\{t^i_k\} \subset \mathbb{R}_{\geq 0}$ denote the sequence of times at which agent i communicates with its in-neighbors, so that $\hat{x}^i(t) = x^i(t^i_k)$ for $t \in [t^i_k, t^i_{k+1})$. The variable $\tilde{x}^i(t) = \hat{x}^i(t) - x^i(t)$ denotes the mismatch between the last transmitted state and the state of agent i at time t.

Under the network model described above, our goal is to design a distributed algorithm that allows each agent to asymptotically track the average of the inputs $\frac{1}{N}\sum_{j=1}^N \mathsf{u}^j(t)$ across the group. The algorithm design amounts to specifying, for each agent $i \in \{1, \ldots, N\}$, a suitable distributed driving command $g^i : \mathbb{R}^{\mathcal{N}^i} \to \mathbb{R}$ together with a mechanism for triggering communication with its in-neighbors in an opportunistic fashion. By *distributed*, we mean that each agent only needs to receive information from its out-neighbors to evaluate g^i and the communication triggering law. By *opportunistic*, we mean that the transmission of information to its in-neighbors should happen at times when it is needed to preserve the stability and convergence of the coordination algorithm. A key requirement on the communication triggering mechanism is that the resulting network evolution is free from Zeno behavior, i.e., does not exhibit an infinite amount of communication rounds in any finite amount of time.

4 Continuous-time computation with distributed discrete-time communication

Here, we present our solution to the problem stated in Section 3. Our starting point is the continuous-time algorithm for dynamic average consensus proposed in our previous work [Kia et al., 2014b],

$$\dot{v}^i = \alpha\beta \sum_{j=1}^{N} \mathsf{a}_{ij}(x^i - x^j), \tag{2a}$$

$$\dot{x}^i = \dot{\mathsf{u}}^i - \alpha(x^i - \mathsf{u}^i) - \beta \sum_{j=1}^{N} \mathsf{a}_{ij}(x^i - x^j) - v^i, \tag{2b}$$

for each $i \in \{1, \ldots, N\}$. In (2), $\alpha, \beta \in \mathbb{R}_{>0}$ are design parameters. Note that the execution of this algorithm requires agents to continuously interchange information about their respective variable x with their neighbors. The following result summarizes for reference the asymptotic correctness guarantees of (2).

Theorem 4.1 (Convergence of (2) over weight-balanced and recurrently jointly strongly connected digraphs [Kia et al., 2014b]): *Assume the agent inputs satisfy* $\|\mathbf{\Pi}_N \dot{\mathbf{u}}\|_{ess} = \gamma < \infty$. *Let the communication topology be a weight-balanced and recurrently jointly strongly connected time-varying digraph* $\mathcal{G}$ *with uniformly bounded weights. Then, for any* $\alpha, \beta \in \mathbb{R}_{>0}$, *the evolution of the algorithm* (2) *over* $\mathcal{G}$ *initialized at* $z^i(0), v^i(0) \in \mathbb{R}$ *with* $\sum_{i=1}^N v^i(0) = 0$ *is bounded and satisfies, for* $i \in \{1, \ldots, N\}$,

$$\limsup_{t\to\infty} \Big| x^i(t) - \frac{1}{N}\sum_{j=1}^{N} \mathsf{u}^j(t) \Big| \le \rho \frac{\gamma}{\beta \hat{\lambda}_\sigma}, \tag{3}$$

with $\hat{\lambda}_\sigma$ *and* ρ *satisfying* (1).

Given the network model of Section 3, where the transmission of information is limited to discrete instants of time, we propose here the following implementation of (2) with discrete-time communication,

$$\dot{v}^i = \alpha\beta \sum_{j=1}^{N} \mathsf{a}_{ij}(\hat{x}^i - \hat{x}^j), \tag{4a}$$

$$\dot{x}^i = \dot{\mathsf{u}}^i - \alpha(x^i - \mathsf{u}^i) - \beta \sum_{j=1}^{N} \mathsf{a}_{ij}(\hat{x}^i - \hat{x}^j) - v^i, \tag{4b}$$

for each $i \in \{1, \ldots, N\}$. Our task is to provide individual agents with triggers that allow them to determine in an opportunistic fashion when to transmit information to their in-neighbors. The design of such triggers is challenging because of the following requirements: triggers need to be distributed, so that agents can check them with the information available to them from their out-neighbors, they must guarantee the absence of Zeno behavior, and they have to ensure the network achieves dynamic average consensus even though agents operate with outdated information while inputs are changing with time.

4.1 Compact-form algorithm representations

Here we present two equivalent compact-form representations of the algorithm (4) for analysis purposes. For the first representation, let $\bar{\mathbf{u}} = \frac{1}{N}\sum_{j=1}^N \mathsf{u}^j \mathbf{1}_N$, and consider the change of variables

$$\mathbf{y} = \mathbf{x} - \bar{\mathbf{u}}, \tag{5a}$$
$$\mathbf{w} = \mathbf{v} - \alpha \mathbf{\Pi}_N \mathbf{u}. \tag{5b}$$

In these new variables, the dynamics looks like

$$\dot{\mathbf{y}} = -\alpha \mathbf{y} - \beta \mathbf{L}_t \mathbf{y} - \beta \mathbf{L}_t \tilde{\mathbf{x}} + \mathbf{\Pi}_N \dot{\mathbf{u}} - \mathbf{w}, \tag{6a}$$
$$\dot{\mathbf{w}} = \alpha\beta \mathbf{L}_t \mathbf{y} + \alpha\beta \mathbf{L}_t \tilde{\mathbf{x}} - \alpha \mathbf{\Pi}_N \dot{\mathbf{u}}, \tag{6b}$$

where we have used $\mathbf{L}_t \hat{\mathbf{x}} = \mathbf{L}_t(\mathbf{x} + \tilde{\mathbf{x}}) = \mathbf{L}_t \mathbf{y} + \mathbf{L}_t \tilde{\mathbf{x}}$. For the second representation, consider the following change of variables,

$$q_1 = \mathbf{r}^\top \mathbf{w}, \quad \mathbf{q}_{2:N} = \alpha \mathbf{R}^\top \mathbf{y} + \mathbf{R}^\top \mathbf{w}, \quad \mathbf{z} = \mathbf{T}^\top \mathbf{y}. \tag{7}$$

We partition the new variable $\mathbf{z}$ as $(z_1, \mathbf{z}_{2:N})$, where $z_1 \in \mathbb{R}$. Then, if the network interaction topology is weight-balanced, the algorithm (6) can be written as,

$$\dot{q}_1 = 0, \tag{8a}$$
$$\dot{\mathbf{q}}_{2:N} = -\alpha \mathbf{q}_{2:N}, \tag{8b}$$
$$\dot{z}_1 = -\alpha z_1 - q_1, \tag{8c}$$
$$\dot{\mathbf{z}}_{2:N} = -\beta \mathbf{R}^\top \mathbf{L}_t \mathbf{R} \mathbf{z}_{2:N} - \beta \mathbf{R}^\top \mathbf{L}_t \tilde{\mathbf{x}} + \mathbf{R}^\top \dot{\mathbf{u}} - \mathbf{q}_{2:N}. \tag{8d}$$

We close this section by describing the relationship between the initial conditions of the variables for each representation. Note that $\mathbf{q}_{2:N} = \mathbf{R}^\top(\alpha \mathbf{y} + \mathbf{w}) = \mathbf{R}^\top(\alpha(\mathbf{x} - \mathbf{u}) + \mathbf{v})$. Then, given $\mathbf{x}(0), \mathbf{v}(0) \in \mathbb{R}^N$ with $\sum_{i=1}^N v^i(0) = 0$, and using $\mathbf{r}^\top \mathbf{\Pi}_N = \mathbf{0}$ and $\mathbf{R}\mathbf{R}^\top = \mathbf{\Pi}_N = \mathbf{\Pi}_N^2$,

$$q_1(0) = \mathbf{r}^\top \mathbf{w}(0) = \mathbf{r}^\top \mathbf{v}(0) = 0, \tag{9a}$$
$$\|\mathbf{q}_{2:N}(0)\| = \|\alpha \mathbf{\Pi}_N(\mathbf{x}(0) - \mathbf{u}(0)) + \mathbf{v}(0)\|, \tag{9b}$$
$$z_1(0) = \mathbf{r}^\top \mathbf{y}(0) = \mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0)), \tag{9c}$$
$$\|\mathbf{z}_{2:N}(0)\| = \|\mathbf{\Pi}_N(\mathbf{x}(0) - \bar{\mathbf{u}}(0))\|. \tag{9d}$$

4.2 Weight-balanced and recurrently jointly strongly connected digraphs

In this section, we consider networks whose interaction is modeled by a time-varying digraph and introduce a distributed event-triggered mechanism that agents can employ to determine their sequence of communication times. For each agent, the execution of this mechanism relies on local variables and the triggered states received from its out-neighbors. This naturally results in asynchronous schedules of communication, which poses additional analysis challenges. Nevertheless, we are able to overcome them in the following result which states that the closed-loop network execution is free from Zeno behavior and guaranteed to achieve practical dynamic average consensus.

Theorem 4.2 (Convergence of (4) over recurrently jointly strongly connected and weight-balanced digraph with asynchronous distributed event-triggered communication): *Assume that the input of each agent $i \in \{1,\dots,N\}$ satisfies $|\dot{\mathrm{u}}^i|_{ess} = \kappa^i < \infty$, and the input differences satisfy $\|\mathbf{\Pi}_N \dot{\mathbf{u}}\|_{ess} = \gamma < \infty$. Let the communication topology be a weight-balanced and recurrently jointly strongly connected time-varying digraph $\mathcal{G}$ with uniformly bounded weights. For $\boldsymbol{\epsilon} \in \mathbb{R}^N_{>0}$, consider an implementation of the algorithm (4) over $\mathcal{G}$ where each agent $i \in \{1,\dots,N\}$ communicates with its neighbors at times $\{t^i_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$, starting at $t^i_1 = 0$, determined by*

$$t^i_{k+1} = \operatorname{argmax}\{t \in [t^i_k, \infty) \,|\, |x^i(t^i_k) - x^i(t)| \leq \epsilon_i\}. \quad (10)$$

Then, for any $\alpha, \beta \in \mathbb{R}_{>0}$, the evolution starting from $x^i(0) \in \mathbb{R}$ and $v^i(0) \in \mathbb{R}$ with $\sum_{i=1}^N v^i(0) = 0$ satisfies

$$\limsup_{t\to\infty} \Big| x^i(t) - \frac{1}{N}\sum_{j=1}^N \mathrm{u}^j(t) \Big| \leq \frac{(\gamma + \beta\overline{\|\mathbf{L}\|}\,\|\boldsymbol{\epsilon}\|)}{\beta\hat{\lambda}_\sigma}\rho, \quad (11)$$

for $i \in \{1,\dots,N\}$ with an exponential rate of convergence of $\min\{\alpha, \beta\hat{\lambda}_\sigma\}$. Here, $\hat{\lambda}_\sigma$ and ρ satisfy (1). Furthermore, the inter-execution times of agent $i \in \{1,\dots,N\}$ are lower bounded by

$$\tau^i = \frac{1}{\alpha}\ln\Big(1 + \frac{\alpha\epsilon_i}{c^i}\Big), \quad (12)$$

where

$$c^i = \kappa^i + (\alpha + 2\beta\bar{\mathrm{d}}^i_{\mathrm{out}})\sqrt{\eta^2 + |\mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0))|^2} + \|\mathbf{\Pi}_N(\alpha(\mathbf{x}(0) - \mathbf{u}(0)) + \mathbf{v}(0))\| + \alpha\eta, \quad (13)$$

and

$$\eta = \rho\frac{(\gamma + \beta\overline{\|\mathbf{L}\|}\,\|\boldsymbol{\epsilon}\|^2)}{\beta\hat{\lambda}_\sigma} + \|\mathbf{\Pi}_N(\mathbf{x}(0) - \bar{\mathbf{u}}(0))\| + \rho\|\mathbf{q}_{2:N}(0)\| \times \begin{cases} \frac{1}{\alpha - \beta\hat{\lambda}_\sigma}\big((\frac{\beta\hat{\lambda}_\sigma}{\alpha})^{\frac{\beta\hat{\lambda}_\sigma}{\alpha-\beta\hat{\lambda}_\sigma}} - (\frac{\beta\hat{\lambda}_\sigma}{\alpha})^{\frac{\alpha}{\alpha-\beta\hat{\lambda}_\sigma}}\big), & \text{if } \beta\hat{\lambda}_\sigma \neq \alpha, \\ \frac{1}{\beta\hat{\lambda}_\sigma e}, & \text{if } \beta\hat{\lambda}_\sigma = \alpha. \end{cases}$$

PROOF. Given an initial condition, let $[0,T)$ be the maximal interval on which there is no accumulation point in the set of event times $\{t_k\}_{k\in\mathbb{N}} = \cup_{i=1}^N \cup_{k\in\mathbb{N}} t^i_k$. Note that $T > 0$, since the number of agents is finite and, for each $i \in \{1,\dots,N\}$, $\epsilon_i > 0$ and $\tilde{x}^i(0) = \hat{x}^i(0) - x^i(0) = \mathbf{0}$. The dynamics (4), under the event-triggered communication scheme (10), has a unique solution in the time interval $[0,T)$. Our first step is to show that the trajectory stays bounded in $[0,T)$. Consider the compact-form representation (8) of the algorithm. Given $\sum_{i=1}^N v^i(0) = 0$ and (9), for $t \in \mathbb{R}_{\geq 0}$,

$$q_1(t) = 0,\ \mathbf{q}_{2:N}(t) = \mathbf{q}_{2:N}(0)\mathrm{e}^{-\alpha t}, \\ z_1(t) = z_1(0)\mathrm{e}^{-\alpha t}. \quad (14)$$

and hence these functions are bounded. To bound $t \mapsto \mathbf{z}_{2:N}(t)$, we look into the solution of (8d) by substituting $\mathbf{q}_{2:N}(t) = \mathbf{q}_{2:N}(0)\mathrm{e}^{-\alpha t}$ and considering $(\tilde{\mathbf{x}}, \mathbf{R}\dot{\mathbf{u}})$ as exogenous inputs. In the time interval $t \in [0,T)$ that this solution exists, one has

$$\mathbf{z}_{2:N}(t) = \mathbf{\Phi}(t,0)\mathbf{z}_{2:N}(0) + \int_0^t \mathbf{\Phi}(t,\tau)\,\mathrm{e}^{-\alpha\tau}\,\mathbf{q}_{2:N}(0)\mathrm{d}\tau - \int_0^t \mathbf{\Phi}(t,\tau)(\beta\mathbf{R}^\top\mathbf{L}_\tau\tilde{\mathbf{x}}(\tau) - \mathbf{R}^\top\dot{\mathbf{u}}(\tau))\mathrm{d}\tau$$

where $\mathbf{\Phi}(t,\tau) = \mathrm{e}^{-\beta\mathbf{R}^\top\mathbf{L}_t\mathbf{R}(t-\tau)}$. Recall that under the event-triggered communication law (10), we have $\|\tilde{\mathbf{x}}\| \leq \|\boldsymbol{\epsilon}\|$. Then, for $t \in [0,T)$ we have

$$\|\mathbf{z}_{2:N}(t)\| \leq \rho\mathrm{e}^{-\beta\hat{\lambda}_\sigma t}\|\mathbf{z}_{2:N}(0)\| + \rho\frac{\gamma + \beta\overline{\|\mathbf{L}\|}\,\|\boldsymbol{\epsilon}\|}{\beta\hat{\lambda}_\sigma}(1 - \mathrm{e}^{-\beta\hat{\lambda}_\sigma t}) + \rho\|\mathbf{q}_{2:N}(0)\| \times \begin{cases} \frac{1}{\alpha-\beta\hat{\lambda}_\sigma}(\mathrm{e}^{-\beta\hat{\lambda}_\sigma t} - \mathrm{e}^{-\alpha t}), & \text{if } \beta\hat{\lambda}_\sigma \neq \alpha, \\ t\mathrm{e}^{-\beta\hat{\lambda}_\sigma t}, & \text{if } \beta\hat{\lambda}_\sigma = \alpha. \end{cases} \quad (15)$$

Here, we also used $\|\mathbf{R}^\top\dot{\mathbf{u}}\| = \|\mathbf{\Pi}\dot{\mathbf{u}}\| \leq \gamma$ and (1). Taking the maximum of each term in the righthand side of (15) and using (9), we have

$$\|\mathbf{z}_{2:N}(t)\| \leq \eta, \quad t \in [0,T), \quad (16)$$

were the constant η is given in the statement.

Next, we establish a lower bound on the inter-execution times of any agent. To do this, we determine a lower bound on the amount of time it takes for agent $i \in \{1,\dots,N\}$ to have $|\hat{x}^i - x^i|$ evolve from 0 to ϵ_i. Note that

$$\|\mathbf{y}(t)\| = \|\mathbf{z}(t)\| \leq \sqrt{\eta^2 + |\mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0))|^2}, \quad (17)$$

for all $t \in [0,T)$, where we have used (9), (14) and (16). On the other hand, since $\mathbf{r}^\top\mathbf{w}(t) = 0$ for all $t \in \mathbb{R}_{\geq 0}$ by (14), we have $\mathbf{\Pi}_N\mathbf{w}(t) = \mathbf{w}(t)$. Multiplying the second equation in (7) by $\mathbf{R}$ and using $\mathbf{R}\mathbf{R}^\top = \mathbf{\Pi}_N$, we obtain $\mathbf{w}(t) = \mathbf{R}\mathbf{q}_{2:N}(t) - \alpha\mathbf{R}\mathbf{z}_{2:N}(t)$. Using now $\mathbf{R}^\top\mathbf{R} = \mathbf{I}_{N-1}$, we deduce

$$\|\mathbf{w}(t)\| \leq \|\mathbf{q}_{2:N}(t)\| + \alpha\|\mathbf{z}_{2:N}(t)\| \\ \leq \|\alpha\mathbf{\Pi}_N(\mathbf{x}(0) - \mathbf{u}(0)) + \mathbf{v}(0)\| + \alpha\eta,$$

for all $t \in [0,T)$, where we have again used (9), (14) and (16). The application of Lemma A.1 now yields

$$\frac{d}{dt}|\hat{x}^i - x^i| \le \alpha|\hat{x}^i - x^i| + c^i, \quad (18)$$

where c^i is given in (13). Using the Comparison Lemma and the fact that $\hat{x}^i = x^i(t_k^i)$, we deduce

$$|\hat{x}^i - x^i(t)| \le \frac{c^i}{\alpha}(\mathrm{e}^{\alpha(t-t_k^i)} - 1), \quad t \ge t_k^i.$$

Therefore, the time it takes $|\hat{x}^i - x^i|$ to reach ϵ_i is lower bounded by $\tau^i > 0$ as given in (12). This fact also implies that $T = \infty$. To see this, we reason by contradiction, i.e., suppose $T < \infty$. Then, the sequence of events $\{t_k\}_{k\in\mathbb{N}}$ has an accumulation point at T. Because there is a finite number of agents, this implies that there is an agent $i \in \{1,\dots,N\}$ for which $\{t_k^i\}_{k\in\mathbb{N}}$ has an accumulation point at T. This implies that i transmits infinitely often in the time interval $[T-\Delta, T)$ for any $\Delta \in (0,T]$, which contradicts the fact that the inter-event times are lower bounded by $\tau^i > 0$ on $[0,T)$. Finally, note that

$$|x^i - \frac{1}{N}\sum_{j=1}^{N}\mathsf{u}^j| \le \|\mathbf{x} - \bar{\mathbf{u}}\| = \|\mathbf{y}\| = \|\mathbf{z}\|, \quad (19)$$

for $i \in \{1,\dots,N\}$, so the bound (17) holds for all $t \in \mathbb{R}_{\ge 0}$. From (14), $\lim_{t\to\infty} z_1(t) = 0$ with an exponential convergence rate α. From (15), $\limsup_{t\to\infty}\|z_{2:N}(t)\| \le (\gamma + \beta\overline{\|\mathbf{L}\|}\|\boldsymbol{\epsilon}\|)/(\beta\hat{\underline{\lambda}}_2)$ with an exponential rate of convergence $\min\{\alpha, \beta\hat{\underline{\lambda}}_2\}$. Combining these facts with (19) yields (11), and this concludes the proof. □

Not surprisingly, the ultimate convergence error bound (11) obtained under event-triggered discrete-time communication is worse than the bound (3) obtained when agents communicate continuously. The trigger (10) does not use the full state of the agent and hence can be interpreted as an output feedback event-triggered controller, see e.g., [Donkers and Heemels, 2012], for which guaranteeing the existence of lower-bounded inter-execution times is in general difficult.

Remark 4.3 (Inter-event times as a function of the design parameters): The lower bound τ^i in (12) on the inter-event times allows a designer to compute bounds on the maximum number of communication rounds (and associated energy spent) by each agent $i \in \{1,\dots,N\}$ (and hence the network) during any given time interval. It is interesting to analyze how this lower bound depends on the various problem ingredients: τ^i is an increasing function of ϵ_i and a decreasing function of α and c^i. Through the latter variable, the bound also depends on the graph topology and the design parameter β. Given the definition of c^i, one can deduce that the faster an input of an agent is changing (larger κ^i) or the farther the agent initially starts from the average of the inputs, the more often that agent would need to trigger communication. The connection between the network performance and the communication overhead can also be observed here. Increasing β or decreasing ϵ_i to improve the ultimate tracking error bound (11) results in smaller inter-event times. Given that the rate of convergence of (4) under (10) is $\min\{\alpha, \beta\hat{\lambda}_\sigma\}$, decreasing α to increase the inter-event times slows down the convergence. •

4.3 *Time-varying connected undirected graphs*

Here, we design an alternative distributed event-triggered communication law for the algorithm (4) over networks with time-varying connected undirected graph interaction topologies. While the results of the previous section are of course valid for these topologies, here we show that the structural properties of the Laplacian matrix in the undirected case allows the alternative event-triggered law to have longer inter-event times with similar dynamic average tracking performance.

Proposition 4.4 (Convergence of (4) over time-varying connected graphs with asynchronous distributed event-triggered communication): *Assume that the input of each agent $i \in \{1,\dots,N\}$ satisfies $|\dot{\mathsf{u}}^i|_{ess} = \kappa^i < \infty$, and the input differences satisfy $\|\mathbf{\Pi}_N\dot{\mathbf{u}}\|_{ess} = \gamma < \infty$. Let the communication topology be a connected, piecewise continuous time-varying graph $\mathcal{G}$ with uniformly bounded weights. For $\boldsymbol{\epsilon} \in \mathbb{R}_{>0}$, consider an implementation of the algorithm (4) over $\mathcal{G}$, where agent $i \in \{1,\dots,N\}$ communicates with its neighbors at times $\{t_k^i\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\ge 0}$, starting at $t_1^i = 0$, determined by*

$$t_{k+1}^i = \operatorname{argmax}\{t \in [t_k^i, \infty) \mid |\hat{x}^i(t) - x^i(t)|^2 \le \frac{1}{4\mathsf{d}_{\mathrm{out}}^i(t)}\sum_{j=1}^{N}\mathsf{a}_{ij}(t)|\hat{x}^i(t) - \hat{x}^j(t)|^2 + \frac{1}{4\mathsf{d}_{\mathrm{out}}^i(t)}\epsilon_i^2\}. \quad (20)$$

Then, for any $\alpha, \beta \in \mathbb{R}_{>0}$, the algorithm evolution starting from $x^i(0) \in \mathbb{R}$ and $v^i(0) \in \mathbb{R}$ with $\sum_{i=1}^{N} v^i(0) = 0$ satisfies

$$\limsup_{t\to\infty}\left|x^i(t) - \frac{1}{N}\sum_{j=1}^{N}\mathsf{u}^j(t)\right| \le \frac{\gamma}{\beta\underline{\lambda}_2} + \sqrt{\Big(\frac{\gamma}{\beta\underline{\lambda}_2}\Big)^2 + \frac{\|\boldsymbol{\epsilon}\|^2}{2\underline{\lambda}_2}}, \quad (21)$$

for $i \in \{1,\dots,N\}$. Furthermore, the inter-execution times of agent $i \in \{1,\dots,N\}$ are lower bounded by

$$\tau^i = \frac{1}{\alpha}\ln\Big(1 + \frac{\alpha\epsilon_i}{2c^i\sqrt{\overline{\mathsf{d}_{\mathrm{out}}^i}}}\Big), \quad (22)$$

where c^i is given in (13)*, with η substituted by*

$$\zeta = \max\Big\{\|\mathbf{\Pi}_N(\mathbf{x}(0)-\bar{\mathbf{u}}(0))\|, \tfrac{\alpha\|\mathbf{\Pi}_N(\mathbf{x}(0)-\mathbf{u}(0))+\mathbf{v}(0)\|}{2} + \tfrac{\gamma}{\beta\underline{\lambda}_2} + \sqrt{(\tfrac{\|\alpha\mathbf{\Pi}_N(\mathbf{x}(0)-\mathbf{u}(0))+\mathbf{v}(0)\|}{2} + \tfrac{\gamma}{\beta\underline{\lambda}_2})^2 + \tfrac{\|\boldsymbol{\epsilon}\|^2}{2\underline{\lambda}_2}}\Big\}.$$

PROOF. Given an initial condition, let $[0,T), T>0$ be the maximal interval on which there is no accumulation point in the set of event times $\{t_k\}_{k\in\mathbb{N}} = \cup_{i=1}^N \cup_{k\in\mathbb{N}} t_k^i$. The expressions in (14) are equally valid in this case. To bound $t \mapsto \mathbf{z}_{2:N}(t)$, consider the candidate Lyapunov function

$$V(\mathbf{z}_{2:N}) = \frac{1}{2}\mathbf{z}_{2:N}^\top \mathbf{z}_{2:N}.$$

The derivative of this candidate Lyapunov function along the trajectories of (8d) can be upper bounded, for $t\in[0,T)$, as

$$\dot{V} \leq \|\mathbf{z}_{2:N}\|\|\mathbf{q}_{2:N}(0)\|e^{-\alpha t} - \frac{1}{2}\beta\underline{\lambda}_2\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N} - \frac{1}{2}\beta(\mathbf{z}_{2:N}^\top\mathbf{R}^\top\mathbf{L}_t\mathbf{R}\mathbf{z}_{2:N} + 2\mathbf{z}_{2:N}^\top\mathbf{R}^\top\mathbf{L}_t\tilde{\mathbf{x}}) + \gamma\|\mathbf{z}_{2:N}\|.$$

For convenience, let

$$s = -\mathbf{z}_{2:N}^\top\mathbf{R}^\top\mathbf{L}_t\mathbf{R}\mathbf{z}_{2:N} - 2\mathbf{z}_{2:N}^\top\mathbf{R}^\top\mathbf{L}_t\tilde{\mathbf{x}}.$$

Using $\mathbf{R}\mathbf{R}^\top = \mathbf{\Pi}_N$, $\mathbf{L}_t\mathbf{\Pi}_N = \mathbf{\Pi}_N\mathbf{L}_t = \mathbf{L}_t$, and (7), we obtain

$$s = -\mathbf{x}^\top\mathbf{L}_t\mathbf{x} - 2\mathbf{x}^\top\mathbf{L}_t\tilde{\mathbf{x}}.$$

The application of Lemma A.2 implies that $s \leq \frac{1}{2}\|\boldsymbol{\epsilon}\|^2$ for $t\in[0,T)$ under the event-triggered communication law law (20). Therefore, for $\theta\in(0,1)$, we have

$$\begin{aligned}\dot{V} &\leq -\frac{\beta\underline{\lambda}_2(1-\theta)}{2}\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N} + \frac{\beta\underline{\lambda}_2}{2}\Big(\frac{2}{\beta\underline{\lambda}_2}\|\mathbf{z}_{2:N}\|\,\|\mathbf{q}_{2:N}(0)\|e^{-\alpha t} \\ &\quad - \theta\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N} + \frac{1}{2\underline{\lambda}_2}\|\boldsymbol{\epsilon}\|^2 + \frac{2\gamma}{\beta\underline{\lambda}_2}\|\mathbf{z}_{2:N}\|\Big) \\ &\leq -\frac{\beta\underline{\lambda}_2}{2}(1-\theta)\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N} + \frac{\beta\underline{\lambda}_2}{2}r, \end{aligned} \tag{23}$$

where $r = \frac{2}{\beta\underline{\lambda}_2}\|\mathbf{z}_{2:N}\|\,\|\mathbf{q}_{2:N}(0)\| - \theta\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N} + \frac{1}{2\underline{\lambda}_2}\|\boldsymbol{\epsilon}\|^2 + \frac{2\gamma}{\beta\underline{\lambda}_2}\|\mathbf{z}_{2:N}\|$. Notice that $r<0$ for

$$\|\mathbf{z}_{2:N}\| \geq \frac{\|\mathbf{q}_{2:N}(0)\|}{\beta\underline{\lambda}_2\theta} + \frac{\gamma}{\beta\underline{\lambda}_2\theta} + \sqrt{\Big(\frac{\|\mathbf{q}_{2:N}(0)\|}{\beta\underline{\lambda}_2\theta} + \frac{\gamma}{\beta\underline{\lambda}_2\theta}\Big)^2 + \frac{\|\boldsymbol{\epsilon}\|^2}{2\underline{\lambda}_2\theta}} = \hat{\zeta}.$$

Hence, for $t\in[0,T)$, as long as $\|\mathbf{z}_{2:N}(t)\| \geq \hat{\zeta}$, one has

$$\dot{V} \leq -\frac{1}{2}\beta\underline{\lambda}_2(1-\theta)\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N}.$$

Combining this inequality with the definition of V, and considering the limiting case $\theta\to1$, we deduce that, for any $\mathbf{z}_{2:N}(0)\in\mathbb{R}^{N-1}$ and $t\in[0,T)$,

$$\|\mathbf{z}_{2:N}(t)\| \leq \max\{\|\mathbf{z}_{2:N}(0)\|, \hat{\zeta}\} = \zeta. \tag{24}$$

Next, following the same arguments as in the proof of Theorem 4.2, one can establish a lower bound on the inter-execution times of any agent $i\in\{1,\dots,N\}$. To do this, we determine a lower bound on the time it takes i to have $|\hat{x}^i - x^i|$ evolve from 0 to $\epsilon_i/(2\sqrt{\mathsf{d}_{\text{out}}^i})$ (note the conservativeness in this step as we disregard the first term in the righthand side of (20)). The application of Lemma A.1 yields

$$|\hat{x}^i - x^i(t)| \leq \frac{c^i}{\alpha}(\mathrm{e}^{\alpha(t-t_k^i)} - 1), \quad t\geq t_k^i.$$

where c^i is given in the statement (i.e., the expression in (13) with η substituted by ζ as defined in (24)). Therefore, the time it takes $|\hat{x}^i - x^i|$ to reach $\epsilon_i/(2\sqrt{\mathsf{d}_{\text{out}}^i})$ is lower bounded by $\tau^i>0$ as given in (22). This fact also implies that $T=\infty$. Finally, (19) together with (14) and (24), imply that for $i\in\{1,\dots,N\}$,

$$|x^i(t) - \frac{1}{N}\sum_{j=1}^N \mathsf{u}^j(t)| \leq \sqrt{\zeta^2 + |\mathbf{r}^\top(\mathbf{x}(0)-\bar{\mathbf{u}}(0))|^2},$$

for $t\in\mathbb{R}_{\geq0}$. Moreover, since $T=\infty$, from (23), we have

$$\dot{V} \leq -\frac{1}{2}\beta\underline{\lambda}_2(1-\theta)\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N} + \frac{\beta\underline{\lambda}_2}{2}\bar{r}(t),$$

for $t\in\mathbb{R}_{\geq0}$, where $\bar{r}(t) = \frac{2}{\beta\underline{\lambda}_2}\|\mathbf{z}_{2:N}\|\|\mathbf{q}_{2:N}(0)\|e^{-\alpha t} - \theta\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N} + \frac{1}{2\underline{\lambda}_2}\|\boldsymbol{\epsilon}\|^2 + \frac{2\gamma}{\beta\underline{\lambda}_2}\|\mathbf{z}_{2:N}\|$. Note that $\bar{r}(t)<0$ for

$$\|\mathbf{z}_{2:N}\| \geq \frac{\|\mathbf{q}_{2:N}(0)\|e^{-\alpha t}}{\beta\underline{\lambda}_2\theta} + \frac{\gamma}{\beta\underline{\lambda}_2\theta} + \sqrt{\Big(\frac{\|\mathbf{q}_{2:N}(0)\|e^{-\alpha t}}{\beta\underline{\lambda}_2\theta} + \frac{\gamma}{\beta\underline{\lambda}_2\theta}\Big)^2 + \frac{\|\boldsymbol{\epsilon}\|^2}{2\underline{\lambda}_2\theta}} = \bar{\zeta}(t).$$

Therefore, for $t\in\mathbb{R}_{\geq0}$, as long as $\|\mathbf{z}_{2:N}(t)\| \geq \bar{\zeta}(t)$,

$$\dot{V} \leq -\frac{1}{2}\beta\underline{\lambda}_2(1-\theta)\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N}.$$

As a result,

$$\limsup_{t\to\infty}\|\mathbf{z}_{2:N}(t)\| \leq \frac{\gamma}{\beta\underline{\lambda}_2} + \sqrt{\Big(\frac{\gamma}{\beta\underline{\lambda}_2}\Big)^2 + \frac{\|\boldsymbol{\epsilon}\|^2}{2\underline{\lambda}_2}}.$$

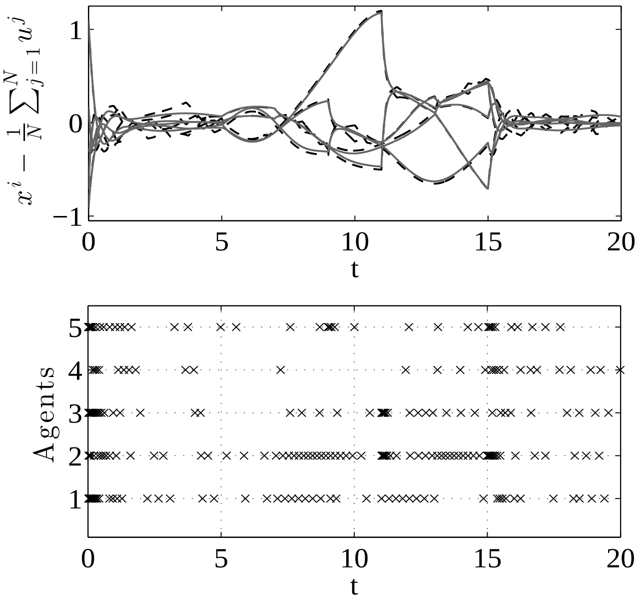

Fig. 1. Executions of (4) with the event-triggered communication law (10) and of (2) with continuous-time communication. The network is a weight-balanced time-varying digraph of 5 agents with unit weights, where for $t \in [0, 5)$ it is a ring digraph, for $t \in [5, 15)$ every 2 seconds it is a single connected pair of nodes and then for $t \in [15, \infty)$ it is a ring digraph again. The inputs are $u^1(t) = 0.5\sin(0.8t)$, $u^2(t) = 0.5\sin(0.7t) + 0.5\cos(0.6t)$, $u^3(t) = \sin(0.2t) + 1$, $u^4(t) = \text{atan}(0.5t)$, $u^5(t) = 0.1\cos(2t)$. The top plot shows the tracking error and the bottom one shows the communication times of each agent. Black dashed (resp. gray solid) lines correspond to the event-triggered communication law (10) with $\epsilon_i = 0.1$ (resp. continuous-time communication (2)). In both cases, $\alpha = 1$ and $\beta = 4$.

Here, we used $\theta \to 1$. On the other hand, $\lim_{t\to\infty} z_1(t) = 0$. Combining these facts with (19) yields (21), concluding the proof. □

We should point out that, as observed in the proof of Proposition 4.4, the guaranteed lower bound (22) on the inter-event-times is more conservative than strictly necessary because we have neglected the effect of the term $\frac{1}{4\mathsf{d}^i_{\text{out}}(t)}\sum_{j=1}^{N} \mathsf{a}_{ij}(t)|\hat{x}^i(t) - \hat{x}^j(t)|^2$ in (20). The simulations of Section 5 show the implementation of (20) resulting in inter-event times longer than the ones of the event-triggered law (10).

5 Simulations

In this section, we illustrate the performance of the coordination algorithm (4) under the event-triggered communication laws (10) and (20) over a recurrently jointly strongly connected digraph (cf. Fig. 1), a ring graph (cf. Fig. 2) and a time-varying connected graph (cf. Fig. 3). Figure 1 shows a small degradation between the tracking performance of the algorithm (4) with the event-triggered communication law (10) and the algorithm (2) with continuous-time communication. In the event-triggered implementation, the number of times that agents $\{1, 2, 3, 4, 5\}$ communicate in the time interval $[0, 20]$ is $(54, 74, 46, 29, 42)$, respectively. The large error observed in the time interval $[5, 15]$ is expected as in this time period every two seconds only two agents are communicating with each other. Naturally, these two agents tend to converge to the average of their inputs and the rest of the agents, being oblivious to the inputs of the other agents, follow their own input.

Figure 2 compares the algorithm (4) with event-triggered communication (20) and the Euler discretizations of the algorithm (2) and the proportional-integral (PI) dynamic average consensus algorithm proposed in [Freeman et al., 2006]. We set the parameters of the PI algorithm so that its ultimate tracking error is similar to that of (2). For the discretizations, we use the largest possible fixed stepsize $\delta = 0.039$ for the PI algorithm (beyond this value the algorithm diverges) and we use the stepsize $\delta = 0.12$ for the algorithm (2) (from [Kia et al., 2014b], convergence is guaranteed if $\delta \in (0, \min\{\alpha^{-1}, \beta^{-1}(\text{d}^{\text{out}}_{\text{max}})^{-1}\})$, which for this example results in $\delta \in (0, 0.125)$). The number of times that agents $\{1, 2, 3, 4, 5\}$ communicate in the time interval $[0, 20]$ is $(39, 40, 42, 40, 39)$, respectively, when implementing event-triggered communication (20). This is significantly less than the communication used by each agent in the Euler discretizations of (2) ($20/0.12 \simeq 166$ rounds) and of the PI algorithm ($20/0.039 \simeq 512$ rounds).

Figure 3 shows the execution of (4) with the event-triggered communication laws (10) and (20) over a time-varying connected graph. For each agent $i \in \{1, 2, 3, 4, 5\}$, we choose ϵ_i for each law so that the summand in the right-hand side of the trigger (ϵ_i for (10), $\epsilon_i/(2\sqrt{\mathsf{d}^i_{\text{out}}})$ for (20)) amounts to the same quantity. The plots show similar tracking performance for both algorithms, with the law (20) inducing less than half communication than (10). In fact, the number of times that agents $\{1, 2, 3, 4, 5\}$ communicate in the time interval $[0, 12]$ is $(42, 55, 71, 57, 54)$ under (10) and $(20, 29, 32, 26, 22)$ under (20).

6 Conclusions

We have studied the multi-agent dynamic average consensus problem over networks where inter-agent communication takes place at discrete time instants in an opportunistic fashion. Our starting point has been our previously developed continuous-time dynamic average consensus algorithm which is known to converge exponentially to a small neighborhood of the network's inputs average. We have proposed two different distributed event-triggered laws that agents can employ to trigger communication with neighbors, depending on whether the

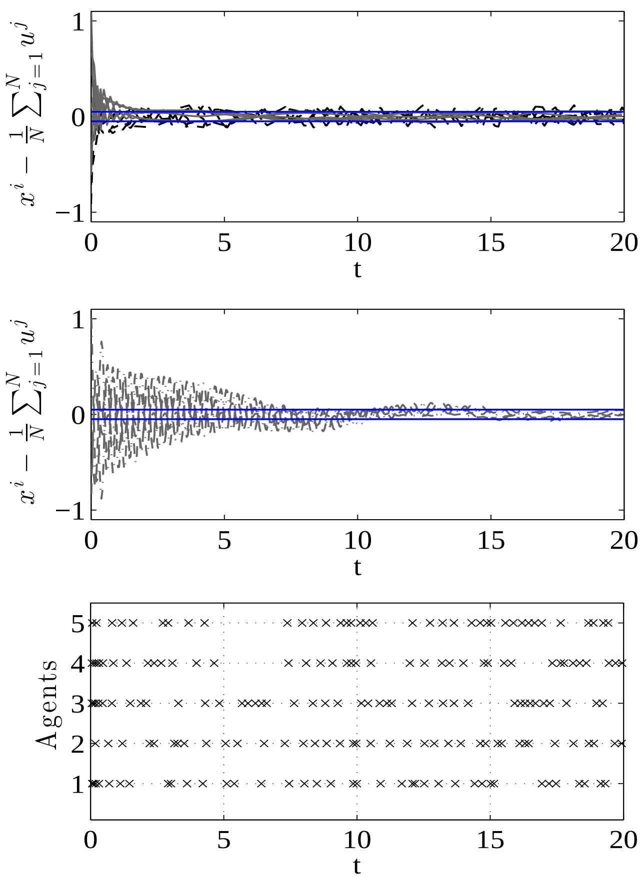

Fig. 2. Comparison between the algorithm (4) employing the event-triggered communication law (20) and the Euler discretizations of the algorithm (2) and the proportional-integral (PI) dynamic average consensus algorithm proposed in [Freeman et al., 2006]. For the first two, we set $\alpha = 1$ and $\beta = 4$. For the latter, we set $\gamma = 5$, $\mathbf{L}_{\mathrm{P}} = \mathbf{L}$ and $\mathbf{L}_{\mathrm{I}} = 4\mathbf{L}$. The network is a connected ring graph of 5 agents with unit weights and the inputs are the same of Figure 1. In the top plot, black (resp. gray) lines correspond to the event-triggered law (20) with $\epsilon_i/(2\sqrt{\mathsf{d}^i_{\mathrm{out}}}) = 0.1$ (resp. the Euler discretization of the algorithm (2) with fixed stepsize $\delta = 0.12$). The middle plot shows the response of the Euler discretization of the PI algorithm with fixed stepsize $\delta = 0.039$. The horizontal lines in both the top and middle plots show the ± 0.05 error bound for reference. The bottom plot shows the communication times of each agent under the event-triggered communication law (20).

interaction topology is described by a weight-balanced and recurrently jointly strongly connected digraph or a time-varying connected undirected graph. In both cases, we have established the correctness of the algorithm and showed that a positive lower bound on the inter-event times of each agent exists, ruling out the presence of Zeno behavior. Future work will be devoted to further relaxing the connectivity requirements on the interaction topology tying them in to the evolution of the dynamic inputs available to the agents, the use of agent abstractions in the development of self-triggered communication laws,

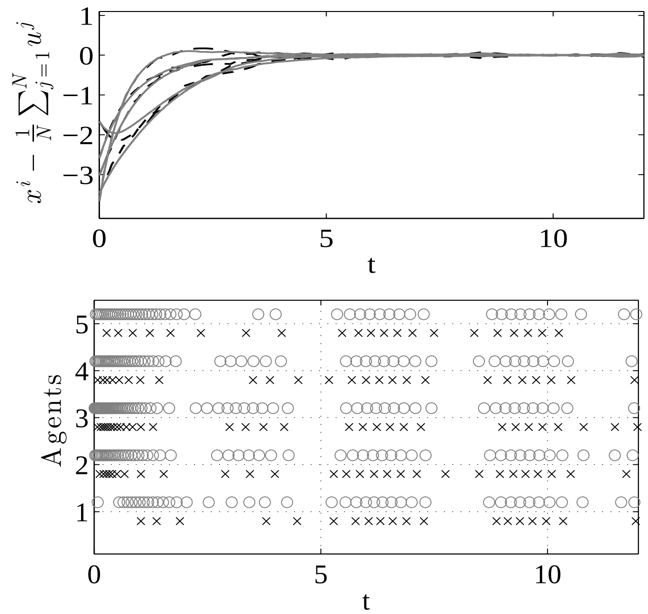

Fig. 3. Executions of (4) with the event-triggered communication laws (10) and (20). The network is a time-varying graph of 5 agents corresponding to a connected ring graph with unit weights where one edge breaks every 3 seconds. The inputs are $u^1(t) = 0.5\sin(t) + 1/(t+2) + 2$, $u^2(t) = 0.5\sin(t) + 1/(t+2)^2 + 4$, $u^3(t) = 0.5\sin(t) + 1/(t+2)^3 + 5$, $u^4(t) = 0.5\sin(t) + \mathrm{e}^{-t} + 4$, $u^5(t) = 0.5\sin(t) + \mathrm{atan}(t) - 1.5$. The top plot shows the tracking error with the gray solid (resp. black dashed) lines correspond to the law (10) with $\epsilon_i = 0.1$ (resp. (20) with $\epsilon_i/(2\sqrt{\mathsf{d}^i_{\mathrm{out}}}) = 0.1$). The bottom plot shows the communication times of each agent with o (resp. $\times$) markers correspond to the law (10) (resp. (20)). In both cases, $\alpha = \beta = 1$.

and the synthesis of other distributed triggers that individual agents can evaluate autonomously and lead to a more efficient use of the limited network resources.

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A Auxiliary Results

Here, we present some supplementary material that we use in the main body of the paper. The following lemma is invoked in the proofs of Theorem 4.2 and Proposition 4.4.

Lemma A.1 *Under the algorithm* (4)*, assume* $\eta > 0$ *is such that* $\|\mathbf{y}(t)\| \leq \sqrt{\eta^2 + |\mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0))|^2}$ *and* $\|\mathbf{w}(t)\| \leq \|\alpha\mathbf{\Pi}_N(\mathbf{x}(0)-\mathbf{u}(0))+\mathbf{v}(0)\|+\alpha\eta$*, for* $t \in [0,T)$ *with* $T \in (0,\infty)$*. Then, for* $i \in \{1,\dots,N\}$*,*

$$\frac{d}{dt}|\hat{x}^i - x^i| \leq \alpha|\hat{x}^i - x^i| + c^i,$$

where $c^i = \big(\kappa^i + (\alpha+2\beta\bar{\mathsf{d}}^i_{\text{out}})\sqrt{\eta^2 + |\mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0))|^2} + \|\mathbf{\Pi}_N(\alpha(\mathbf{x}(0) - \mathbf{u}(0)) + \mathbf{v}(0))\| + \alpha\eta\big)$.

PROOF. Using (4) and (5b), for each $i \in \{1,\dots,N\}$,

$$\begin{aligned}
&\frac{d}{dt}|\hat{x}^i - x^i| = -\frac{(\hat{x}^i - x^i)^\top \dot{x}^i}{|\hat{x}^i - x^i|} \leq |\dot{x}^i| \\
&= |\dot{\mathsf{u}}^i - \alpha(x^i - \mathsf{u}^i) - \beta\sum_{j=1}^N \mathsf{a}_{ij}(\hat{x}^i - \hat{x}^j) - v^i| \\
&= |\dot{\mathsf{u}}^i - \alpha(x^i - \frac{1}{N}\sum_{j=1}^N \mathsf{u}^j) - \beta\sum_{j=1}^N \mathsf{a}_{ij}(\hat{y}^i - \hat{y}^j) - w^i| \\
&\leq \kappa^i + \alpha|\hat{x}^i - x^i| + \alpha|\hat{y}^i| + \beta\sum_{j=1}^N \mathsf{a}_{ij}(|\hat{y}^i| + |\hat{y}^j|) + |w^i|.
\end{aligned}$$

The result follows by noting that $|y^i| \leq \|\mathbf{y}\|$ and $|w^i| \leq \|\mathbf{w}\|$, and using the known bounds on $\|\mathbf{y}\|$ and $\|\mathbf{w}\|$. □

The next result is used in the proof of Proposition 4.4.

Lemma A.2 *Let* $\mathcal{G}$ *be a connected graph and* $s = -\mathbf{x}^\top\mathbf{L}\mathbf{x} - 2\mathbf{x}^\top\mathbf{L}\tilde{\mathbf{x}}$*. Then, given* (20)*, we have* $s < \frac{1}{2}\|\boldsymbol{\epsilon}\|^2$*.*

PROOF. Since $\mathbf{x} = \hat{\mathbf{x}} - \tilde{\mathbf{x}}$, then

$$\begin{aligned}
s &= -(\hat{\mathbf{x}} - \tilde{\mathbf{x}})^\top\mathbf{L}(\hat{\mathbf{x}} - \tilde{\mathbf{x}}) - 2(\hat{\mathbf{x}} - \tilde{\mathbf{x}})^\top\mathbf{L}\tilde{\mathbf{x}} \\
&= -\hat{\mathbf{x}}^\top\mathbf{L}\hat{\mathbf{x}} + \tilde{\mathbf{x}}^\top\mathbf{L}\tilde{\mathbf{x}}.
\end{aligned}$$

Using $\mathbf{D}_{\text{out}} + \mathbf{A} \geq 0$, we have

$$\begin{aligned}
\tilde{\mathbf{x}}^\top\mathbf{L}\tilde{\mathbf{x}} &= \tilde{\mathbf{x}}^\top\mathbf{D}_{\text{out}}\tilde{\mathbf{x}} - \tilde{\mathbf{x}}^\top\mathbf{A}\tilde{\mathbf{x}} \leq 2\tilde{\mathbf{x}}^\top\mathbf{D}_{\text{out}}\tilde{\mathbf{x}} \\
&= 2\sum_{i=1}^N \mathsf{d}^i_{\text{out}}|\tilde{x}^i|^2 = 2\sum_{i=1}^N \mathsf{d}^i_{\text{out}}|\hat{x}^i - x^i|^2.
\end{aligned}$$

Using $\hat{\mathbf{x}}^\top\mathbf{L}\hat{\mathbf{x}} = \frac{1}{2}\sum_{i=1}^N\sum_{j=1}^N \mathsf{a}_{ij}|\hat{x}^i - \hat{x}^j|^2$, we deduce

$$s \leq \frac{1}{2}\sum_{i=1}^N \Big(4\mathsf{d}^i_{\text{out}}|\hat{x}^i - x^i|^2 - \sum_{j=1}^N \mathsf{a}_{ij}|\hat{x}^i - \hat{x}^j|^2\Big),$$

which, together with (20), implies the result. □