Dynamic Average Consensus with Distributed Event-triggered Communication

Solmaz S. Kia Jorge Cortés Sonia Martínez

Abstract— **This paper analyzes distributed algorithmic solutions to dynamic average consensus implemented in continuous time and relying on communication at discrete instants of time. Our starting point is a distributed coordination strategy that, under continuous-time communication, achieves practical asymptotic tracking of the dynamic average of the time-varying agents' inputs. We propose two different distributed event-triggered communication laws, depending on whether the interaction topology is described by a strongly connected and weight-balanced digraph or an undirected connected graph. In both cases, we establish positive lower bounds on the inter-event times of each agent and characterize their dependence of the algorithm design parameters. We build on this result to rule out the presence of Zeno behavior and characterize the asymptotic correctness of the resulting implementations. Simulations illustrate the results.**

I. Introduction

Given a network of agents, each endowed with a time-varying input signal, the dynamic average consensus problem consists of designing a distributed algorithm that allows individual agents to track the dynamic average of the inputs. This problem has applications in numerous areas, including multi-robot coordination [1], sensor fusion [2], [3], distributed estimation [4], and distributed tracking [5]. Our aim is to study algorithmic solutions to dynamic average consensus which rely on agents autonomously deciding when to share information with their neighbors in an opportunistic fashion for greater efficiency and energy savings.

Literature review: Available algorithms focusing on dynamic consensus in the literature are either continuous-time [6], [2], [7], [8], [9] or discrete-time strategies with fixed periodic stepsizes [10], [9]. The continuous-time algorithms converge under the assumption of local continuous-time information sharing among agents. Although discrete-time algorithms are more amenable to practical implementation, they tie the communication and computation stepsizes together, resulting in a conservative stepsize for communication times. This can result into a costly operation, as in networked systems communication requires more energy than computation. Periodic communication is also unrealistic in the cyber-physical world, as processors are subject to natural delays and errors which deviate them from the perfect operational conditions these strategies are designed for. Finally, as periodic implementations are designed to account for worst-case situations, they result into conservative schemes which can lead to a wasteful use of resources. Event-triggered communication can address this shortcomings by prescribing the times for information sharing in an opportunistic way. In recent years, an increasing body of work that seeks to trade computation and decision-making for less communication, sensing or actuation effort while guaranteeing a desired level of performance has emerged, see e.g., [11], [12], [13]. Closest to the problem considered here are the works that study event-triggered communication laws for static average consensus, see e.g., [14], [15], [16] and references therein.

Statement of contributions: We propose novel algorithmic solutions to the dynamic average consensus problem that employ an opportunistic strategy for information sharing among neighboring agents. The basic idea is that agents share information with their neighbors when the uncertainty in the outdated information is such that the monotonic convergent behavior of the coordination algorithm can no longer be guaranteed. The benefits of this mode of operation are twofold. First, because communication is triggered as needed, the network operation is more efficient than with periodic communication schemes that need to account for worst-case scenarios. Second, as each agent decides autonomously its communication times, the algorithm is more in line with the practical challenges of real-time implementations. We propose and characterize the correctness of two different distributed event-triggered communication laws, depending on whether the interaction topology is described by a strongly connected and weight-balanced digraph or an undirected connected graph. By establishing positive lower bounds on the inter-event times of each agent, we also show that the proposed distributed event-triggered communication laws are free from Zeno behavior (i.e., an infinite amount of communication rounds in a finite amount of time). Finally, we analyze the dependence of the inter-event times to the algorithm's design parameters. Such characterization provides guidelines on the trade-offs between the minimum inter-event times for communication and the algorithms' performance. For reasons of space, we only present proof sketches of the results. A full technical treatment will appear elsewhere.

Organization: Section II gathers basic notation and graph-theoretic notions. Section III presents the network model and the dynamic average consensus problem. Section IV introduces our continuous-time algorithmic solutions with event-triggered communication. Section V presents simulations and Section VI gathers our conclusions and ideas for future work.

SSK is with the Department of Mechanical and Aerospace Engineering, University of California Irvine, Irvine, CA 92697, solmaz@uci.edu; JC and SM are with the Department of Mechanical and Aerospace Engineering, University of California San Diego, La Jolla, CA 92093, USA {cortes,soniamd}@ucsd.edu

II. Preliminaries

In this section, we introduce basic notation and concepts from graph theory used throughout the paper.

A. Notation

We let $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq 0}$ and $\mathbb{N}$ denote the set of real, positive real, nonnegative real and natural numbers, respectively. The transpose of a matrix $\mathbf{A}$ is $\mathbf{A}^\top$. We let $\mathbf{1}_n$ (resp. $\mathbf{0}_n$) denote the vector of n ones (resp. n zeros), and denote by $\mathbf{I}_n$ the $n\times n$ identity matrix. We let $\mathbf{\Pi}_n = \mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top$. When clear from the context, we do not specify the matrix dimensions. For $\mathbf{u}\in\mathbb{R}^d$, $\|\mathbf{u}\| = \sqrt{\mathbf{u}^\top\mathbf{u}}$ denotes the standard Euclidean norm. For vectors $\mathbf{u}_1,\dots,\mathbf{u}_m$, we let $\mathbf{u} = (\mathbf{u}_1,\dots,\mathbf{u}_m)$ represent the aggregated vector. In a networked system, we distinguish the local variables at each agent by a superscript. For $\mathbf{p}^i\in\mathbb{R}^d$, the aggregated $\mathbf{p}^i$'s of the network of N agents is represented by $\mathbf{p} = (\mathbf{p}^1,\dots,\mathbf{p}^N)\in(\mathbb{R}^d)^N$.

B. Graph theory

In the following, we review some basic concepts from algebraic graph theory following [17]. A *directed graph*, or simply a *digraph*, is a pair $\mathcal{G} = (\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{1,\dots,N\}$ is the *node set* and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the *edge set*. An edge from i to j, denoted by (i,j), means that agent j can send information to agent i. For an edge $(i,j)\in\mathcal{E}$, i is called an *in-neighbor* of j and j is called an *out-neighbor* of i. We denote the set of out-neighbors of an agent $i\in\{1,\dots,N\}$ by $\mathcal{N}^i$. A graph is *undirected* if $(i,j)\in\mathcal{E}$ anytime $(j,i)\in\mathcal{E}$. A *directed path* is a sequence of nodes connected by edges. A digraph is called *strongly connected* if for every pair of vertices there is a directed path connecting them.

A *weighted digraph* is a triplet $\mathcal{G} = (\mathcal{V},\mathcal{E},\mathbf{A})$, where $(\mathcal{V},\mathcal{E})$ is a digraph and $\mathbf{A}\in\mathbb{R}^{N\times N}$ is a weighted *adjacency* matrix with the property that $\mathsf{a}_{ij} > 0$ if $(i,j)\in\mathcal{E}$ and $\mathsf{a}_{ij} = 0$, otherwise. A weighted digraph is *undirected* if $\mathsf{a}_{ij} = \mathsf{a}_{ji}$ for all $i,j\in\mathcal{V}$. We refer to a strongly connected and undirected graph as a *connected graph*. The *weighted out-degree* and *weighted in-degree* of a node i, are respectively, $\mathsf{d}^i_{\text{in}} = \sum_{j=1}^N \mathsf{a}_{ji}$ and $\mathsf{d}^i_{\text{out}} = \sum_{j=1}^N \mathsf{a}_{ij}$. A digraph is *weight-balanced* if at each node $i\in\mathcal{V}$, the weighted out-degree and weighted in-degree coincide (although they might be different across different nodes). The *(out-) Laplacian* matrix is $\mathbf{L} = \mathbf{D}^{\text{out}} - \mathbf{A}$, where $\mathbf{D}^{\text{out}} = \text{Diag}(\mathsf{d}^1_{\text{out}},\cdots,\mathsf{d}^N_{\text{out}})\in\mathbb{R}^{N\times N}$. Note that $\mathbf{L}\mathbf{1}_N = \mathbf{0}$. A digraph is weight-balanced if and only if $\mathbf{1}_N^\top\mathbf{L} = \mathbf{0}$ if and only if $\text{Sym}(\mathbf{L}) = (\mathbf{L}+\mathbf{L}^\top)/2$ is positive semi-definite. Based on the structure of $\mathbf{L}$, at least one of the eigenvalues of $\mathbf{L}$ is zero and the rest of them have nonnegative real parts. We denote the eigenvalues of $\mathbf{L}$ and $\text{Sym}(\mathbf{L})$ by λ_i and $\hat{\lambda}_i$, $i\in\{1,\dots,N\}$, respectively. For a strongly connected and weight-balanced digraph, zero is a simple eigenvalue of both $\mathbf{L}$ and $\text{Sym}(\mathbf{L})$. In this case, we order the eigenvalues of $\text{Sym}(\mathbf{L})$ as $\hat{\lambda}_1 = 0 < \hat{\lambda}_2 \leq \hat{\lambda}_3 \leq \cdots \leq \hat{\lambda}_N$. For connected graphs, we order the eigenvalues of $\mathbf{L}$ as $\lambda_1 = 0 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_N$.

III. Network Model and Problem Statement

Here, we formalize the problem of interest. Consider a network of N agents with single-integrator dynamics,

$$\dot{x}^i = g^i, \quad i\in\{1,\dots,N\},$$

where $x^i\in\mathbb{R}$ is the *agreement state* and $g^i\in\mathbb{R}$ is the *driving command* of agent i. Each agent $i\in\{1,\dots,N\}$ has access to a time-varying input signal $\mathsf{u}^i:\mathbb{R}_{\geq 0}\to\mathbb{R}$. The network interaction topology is modeled by a weighted digraph $\mathcal{G}$ that models the capability of agents to transmit information to other agents through wireless communication. Given that communication occurs at discrete instants of time, we let $\hat{x}^i$ denote the last known state of agent $i\in\{1,\dots,N\}$ transmitted to its in-neighbors. We let $\{t^i_k\}\subset\mathbb{R}_{\geq 0}$ denote the sequence of times at which agent i communicates with its in-neighbors, so that $\hat{x}^i(t) = x^i(t^i_k)$ for $t\in[t^i_k, t^i_{k+1})$. The variable $\tilde{x}^i(t) = \hat{x}^i(t) - x^i(t)$ denotes the mismatch between the last transmitted state and the state of agent i at time t.

Under the network model described above, our goal is to design a distributed algorithm that allows each agent to asymptotically track the average of the inputs $\frac{1}{N}\Sigma_{j=1}^N \mathsf{u}^j(t)$ across the group. The algorithm design amounts to specifying, for each agent $i\in\{1,\dots,N\}$, a suitable distributed driving command $g^i:\mathbb{R}^{\mathcal{N}^i}\to\mathbb{R}$ together with a mechanism for triggering communication with its in-neighbors in an opportunistic fashion. By distributed, we mean that each agent only needs to receive information from its out-neighbors to evaluate g^i and the communication triggering law. By opportunistic, we mean that the transmission of information to its in-neighbors should happen at times when it is needed to preserve the stability and convergence of the coordination algorithm. A key requirement on the communication triggering mechanism is that the resulting network evolution is free from Zeno behavior, i.e., does not exhibit an infinite amount of communication rounds in any finite amount of time.

IV. Continuous-time computation with distributed discrete-time communication

Here, we present our solution to the problem stated in Section III. Our starting point is the following continuous-time algorithm for dynamic average consensus proposed in our previous work [9],

$$\begin{aligned} \dot{v}^i &= \alpha\beta\textstyle\sum_{j=1}^N \mathsf{a}_{ij}(x^i - x^j),\\ \dot{x}^i &= \dot{\mathsf{u}}^i - \alpha(x^i - \mathsf{u}^i) - \beta\textstyle\sum_{j=1}^N \mathsf{a}_{ij}(x^i - x^j) - v^i, \end{aligned} \tag{1}$$

for $i\in\{1,\dots,N\}$. Note that the execution of this algorithm requires agents to continuously interchange state information with their neighbors. The following result summarizes for reference the asymptotic correctness guarantees of (1).

Theorem 4.1 (Convergence of (1) over strongly connected and weight-balanced digraphs [9]): *Assume the agent inputs satisfy $\|\mathbf{\Pi}_N\dot{\mathbf{u}}\|_{ess} = \gamma < \infty$. Then, for any $\alpha,\beta > 0$, the trajectories of the algorithm (1) executed on a strongly connected*

and weight-balanced digraph $\mathcal{G}$ initialized at $z^i(0), v^i(0) \in \mathbb{R}$ with $\sum_{i=1}^N v^i(0) = 0$ are bounded and satisfy

$$\lim_{t\to\infty} \left| x^i(t) - \frac{1}{N}\sum_{j=1}^N \mathsf{u}^j(t) \right| \leq \frac{\gamma}{\beta\hat{\lambda}_2}, \quad i \in \{1,\dots,N\}. \quad (2)$$

Given the network model of Section III, where the transmission of information is limited to discrete instants of time, we propose here the following implementation of (1) with discrete-time communication,

$$\begin{aligned} \dot{v}^i &= \alpha\beta \textstyle\sum_{j=1}^N \mathsf{a}_{ij}(\hat{x}^i - \hat{x}^j), \\ \dot{x}^i &= \dot{\mathsf{u}}^i - \alpha(x^i - \mathsf{u}^i) - \beta \textstyle\sum_{j=1}^N \mathsf{a}_{ij}(\hat{x}^i - \hat{x}^j) - v^i, \end{aligned} \quad (3)$$

for each $i \in \{1,\dots,N\}$. Our remaining task is to provide individual agents with triggers that allow them to determine in an opportunistic fashion when to transmit information to their in-neighbors. The design of such triggers is challenging because of the following requirements: triggers need to be distributed, so that agents can check them with the information available to them from their out-neighbors, they must guarantee the absence of Zeno behavior, and ensure the network achieves dynamic average consensus even though agents operate with outdated information about the state of each other and the inputs may be changing with time.

A. Compact-form algorithm representations

Here we present two equivalent compact-form representations of the algorithm (3) for analysis purposes. For the first representation, let $\bar{\mathbf{u}} = \frac{1}{N}\Sigma_{j=1}^N \mathsf{u}^j \mathbf{1}_N$, and consider the change of variables

$$\mathbf{y} = \mathbf{x} - \bar{\mathbf{u}}, \quad (4a)$$
$$\mathbf{w} = \mathbf{v} - \alpha\mathbf{\Pi}_N\mathbf{u}. \quad (4b)$$

In these new variables, the dynamics looks like

$$\dot{\mathbf{y}} = -\alpha\mathbf{y} - \beta\mathbf{L}\mathbf{y} - \beta\mathbf{L}\tilde{\mathbf{x}} + \mathbf{\Pi}_N\dot{\mathbf{u}} - \mathbf{w}, \quad (5a)$$
$$\dot{\mathbf{w}} = \alpha\beta\mathbf{L}\mathbf{y} + \alpha\beta\mathbf{L}\tilde{\mathbf{x}} - \alpha\mathbf{\Pi}_N\dot{\mathbf{u}}, \quad (5b)$$

where we have used $\mathbf{L}\hat{\mathbf{x}} = \mathbf{L}(\mathbf{x} + \tilde{\mathbf{x}}) = \mathbf{L}\mathbf{y} + \mathbf{L}\tilde{\mathbf{x}}$. Next, consider the following change of variables,

$$q_1 = \mathbf{r}^\top\mathbf{w}, \quad \mathbf{q}_{2:N} = \alpha\mathbf{R}^\top\mathbf{y} + \mathbf{R}^\top\mathbf{w}, \quad \mathbf{z} = \mathbf{T}^\top\mathbf{y}. \quad (6)$$

We partition the new variable $\mathbf{z}$ as $(z_1, \mathbf{z}_{2:N})$, where $z_1 \in \mathbb{R}$. Then, if the network interaction topology is weight-balanced, the algorithm (5) can be written as,

$$\dot{q}_1 = 0, \quad (7a)$$
$$\dot{\mathbf{q}}_{2:N} = -\alpha\mathbf{q}_{2:N}, \quad (7b)$$
$$\dot{z}_1 = -\alpha z_1 - q_1, \quad (7c)$$
$$\dot{\mathbf{z}}_{2:N} = -\beta\mathbf{R}^\top\mathbf{L}\mathbf{R}\mathbf{z}_{2:N} - \beta\mathbf{R}^\top\mathbf{L}\tilde{\mathbf{x}} + \mathbf{R}^\top\dot{\mathbf{u}} - \mathbf{q}_{2:N}. \quad (7d)$$

We close this section by describing the relationship between the initial conditions of the variables for each representation. Note that $\mathbf{q}_{2:N} = \mathbf{R}^\top(\alpha\mathbf{y} + \mathbf{w}) = \mathbf{R}^\top(\alpha(\mathbf{x} - \mathbf{u}) + \mathbf{v})$. Then, given $\mathbf{x}(0), \mathbf{v}(0) \in \mathbb{R}^N$ with $\sum_{i=1}^N v^i(0) = 0$, and using $\mathbf{r}^\top\mathbf{\Pi}_N = \mathbf{0}$ and $\mathbf{R}\mathbf{R}^\top = \mathbf{\Pi}_N = \mathbf{\Pi}_N^2$,

$$q_1(0) = \mathbf{r}^\top\mathbf{w}(0) = \mathbf{r}^\top\mathbf{v}(0) = 0, \quad (8a)$$
$$\|\mathbf{q}_{2:N}(0)\| = \|\alpha\mathbf{\Pi}_N(\mathbf{x}(0) - \mathbf{u}(0)) + \mathbf{v}(0)\|, \quad (8b)$$
$$z_1(0) = \mathbf{r}^\top\mathbf{y}(0) = \mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0)), \quad (8c)$$
$$\|\mathbf{z}_{2:N}(0)\| = \|\mathbf{\Pi}_N(\mathbf{x}(0) - \bar{\mathbf{u}}(0))\|. \quad (8d)$$

B. Strongly connected and weight-balanced digraphs

In this section, for networks with strongly connected and weight-balanced digraph interactions, we introduce a distributed event-trigger mechanism that agents can employ to determine their sequence of communication times. For each agent, the execution of this mechanism relies merely on local variables. This naturally results in asynchronous schedules of communication, which poses additional challenges for analysis. Nevertheless, the following result states that the closed-loop network execution is free from Zeno behavior and guaranteed to achieve practical dynamic average consensus. For brevity, we only provide an sketch of the proofs here and the full technical treatment will appear elsewhere.

Theorem 4.2 (Convergence of (3) over strongly connected and weight-balanced digraphs with asynchronous distributed event-triggered communication): *Assume that the input of each agent $i \in \{1,\dots,N\}$ satisfies $|\dot{\mathsf{u}}^i|_{ess} = \kappa^i < \infty$, and the input differences satisfy $\|\mathbf{\Pi}_N\dot{\mathbf{u}}\|_{ess} = \gamma < \infty$. For $\boldsymbol{\epsilon} \in \mathbb{R}^N_{>0}$, consider an implementation of the algorithm (3) over a strongly connected and weight-balanced digraph $\mathcal{G}$, where agent $i \in \{1,\dots,N\}$ communicates with its neighbors at times $\{t_k^i\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$, starting at $t_1^i = 0$, determined by*

$$t_{k+1}^i = \operatorname{argmax}\{t \in [t_k^i, \infty) \,|\, |x^i(t_k^i) - x^i(t)| \leq (\epsilon^i)^2\}. \quad (9)$$

Then, for any $\alpha, \beta > 0$, the algorithm evolution starting from $x^i(0) \in \mathbb{R}$ and $v^i(0) \in \mathbb{R}$ with $\sum_{i=1}^N v^i(0) = 0$ satisfies

$$\lim_{t\to\infty}\sup \left| x^i(t) - \frac{1}{N}\sum_{j=1}^N \mathsf{u}^j(t) \right| \leq \frac{(\gamma + \beta\|\mathbf{L}\|\|\boldsymbol{\epsilon}\|^2)}{\beta\hat{\lambda}_2}, \quad (10)$$

for $i \in \{1,\dots,N\}$ with a exponential rate of convergence of $\min\{\alpha, \beta\hat{\lambda}_2\}$. Furthermore the inter-execution times of agent $i \in \{1,\dots,N\}$ are lower bounded by

$$\tau^i = \frac{1}{\alpha}\ln\left(1 + \frac{\alpha(\epsilon^i)^2}{c^i}\right), \quad (11)$$

where

$$\begin{aligned} c^i = \kappa^i &+ (\alpha + 2\beta\mathsf{d}_{\text{out}}^i)\sqrt{\eta^2 + |\mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0))|^2} \\ &+ \|\mathbf{\Pi}_N(\alpha(\mathbf{x}(0) - \mathbf{u}(0)) + \mathbf{v}(0))\| + \alpha\eta, \end{aligned} \quad (12)$$

and

$$\eta = \frac{(\gamma + \beta\|\mathbf{L}\|\|\boldsymbol{\epsilon}\|^2)}{\beta\hat{\lambda}_2} + \|\mathbf{\Pi}_N(\mathbf{x}(0) - \bar{\mathbf{u}}(0))\| + \|\mathbf{q}_{2:N}(0)\| \times$$
$$\begin{cases} \frac{1}{\alpha - \beta\hat{\lambda}_2}\left(\left(\frac{\beta\hat{\lambda}_2}{\alpha}\right)^{\frac{\beta\hat{\lambda}_2}{\alpha - \beta\hat{\lambda}_2}} - \left(\frac{\beta\hat{\lambda}_2}{\alpha}\right)^{\frac{\alpha}{\alpha - \beta\hat{\lambda}_2}}\right), & \text{if } \beta\hat{\lambda}_2 \neq \alpha, \\ \frac{1}{\beta\hat{\lambda}_2 e}, & \text{if } \beta\hat{\lambda}_2 = \alpha. \end{cases}$$

Sketch of the proof: Consider the equivalent representation (7) of (3). From (7a)-(7c), for $t \in \mathbb{R}_{\geq 0}$, for given initial conditions, the system trajectories are given by, respectively,

$$q_1(t) = q_1(0),\ \mathbf{q}_{2:N}(t) = \mathbf{q}_{2:N}(0)\mathrm{e}^{-\alpha t},\ z_1(t) = z_1(0)\mathrm{e}^{-\alpha t}. \quad (13)$$

Now consider (7d). Given an initial condition, let $[0, T)$ be the maximal interval on which there is no accumulation point in the set of event times $\{t_k\}_{k\in\mathbb{N}} = \cup_{i=1}^N \cup_{k\in\mathbb{N}} t_k^i$. Note that $T > 0$, since the number of agents is finite and, for each $i \in \{1, \ldots, N\}$, $\epsilon^i > 0$ and $\tilde{x}^i(0) = \hat{x}^i(0) - x^i(0) = \mathbf{0}$. The dynamics (7d), under the event-triggered communication scheme (9), has a unique solution in the time interval $[0, T)$. Consider the Lyapunov function

$$V = \frac{1}{2}\mathbf{z}_{2:N}^\top \mathbf{z}_{2:N}. \quad (14)$$

By upper-bounding the Lie derivative of (14) along the trajectories of (7d) by an appropriate bound and applying the Comparison Lemma (cf. [18]), we can establish

$$\|\mathbf{z}_{2:N}(t)\| \leq \eta, \quad t \in [0, T), \quad (15)$$

where the constant η is given in the statement. Next, we show that $T = \infty$. We start by establishing a lower bound on the inter-execution times of any agent by determining a lower bound on the amount of time it takes for $|\hat{x}^i - x^i|$ to evolve from 0 to $(\epsilon^i)^2$ at each agent $i \in \{1, \ldots, N\}$. To this end, we first use (13) and (15) (recall (6) and (8)) to obtain $\|\mathbf{y}\| \leq \sqrt{\eta^2 + |\mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0))|^2}$ and $\|\mathbf{w}\| \leq \|\alpha \mathbf{\Pi}_N(\mathbf{x}(0) - \mathbf{u}(0)) + \mathbf{v}(0)\| + \alpha\eta$ for $t \in [0, T)$. Then, using (3) and (4b), $|y^i| \leq \|\mathbf{y}\|$ and $|w^i| \leq \|\mathbf{w}\|$, we obtain

$$\frac{d}{dt}|\hat{x}^i - x^i| \leq \alpha|\hat{x}^i - x^i| + c^i, \quad i \in \{1, \ldots, N\},$$

where c^i is given in (12). As a result, using the Comparison Lemma and the fact that $|\hat{x}^i - x^i(t_k^i)| = 0$, we deduce

$$|\hat{x}^i - x^i(t)| \leq \frac{c^i}{\alpha}(\mathrm{e}^{\alpha(t - t_k^i)} - 1), \quad t \geq t_k^i. \quad (16)$$

Then, the time it takes $|\hat{x}^i - x^i|$ to reach $(\epsilon^i)^2$ is lower bounded by $\tau^i > 0$ given by (11). Next, to show $T = \infty$, we proceed by contradiction. Suppose that $T < \infty$. Then, the sequence of events $\{t_k\}_{k\in\mathbb{N}}$ has an accumulation point at T. Because we have a finite number of agents, this means that there must be an agent $i \in \{1, \ldots, N\}$ for which $\{t_k^i\}_{k\in\mathbb{N}}$ has an accumulation point at T, implying that agent i transmits infinitely often in the time interval $[T - \Delta, T)$ for any $\Delta \in (0, T]$. However, this is in contradiction with the fact that inter-event times are lower bounded by $\tau^i > 0$ on $[0, T)$. Having established $T = \infty$, note that this fact implies that under the event-triggered communication law (9), the algorithm (3) does not exhibit Zeno behavior. Then, given (13) and (15), for $t \in \mathbb{R}_{\geq 0}$, from $|x^i - \frac{1}{N}\Sigma_{j=1}^N \mathsf{u}^j| \leq \|\mathbf{x} - \bar{\mathbf{u}}\| = \|\mathbf{y}\| = \|\mathbf{z}\|$ we can establish the following bound, for $i \in \{1, \ldots, N\}$,

$$|x^i(t) - \frac{1}{N}\Sigma_{j=1}^N \mathsf{u}^j(t)| \leq \sqrt{\eta^2 + |\mathbf{r}^\top(\mathbf{x}(0) - \bar{\mathbf{u}}(0))|^2}.$$

By studying the limiting behavior of the trajectories of the algorithm (3), we can establish a tighter bound on the limiting value of $|x^i - \frac{1}{N}\Sigma_{j=1}^N \mathsf{u}^j|$ as in (10) and show that the convergence is exponential. ∎

Not surprisingly, the ultimate convergence error bound (10) obtained under event-triggered discrete-time communication is worse than the bound (2) obtained when agents communicate continuously. The trigger (9) does not use the full state of the agent and hence can be interpreted as an output feedback event-triggered controller, see e.g., [19], for which guaranteeing the existence of lower-bounded inter-execution times is in general difficult.

Remark 4.1 (Inter-event times as a function of the design parameters): The lower bound τ^i in (11) on the inter-event times allows a designer to compute bounds on the maximum energy spent by each agent $i \in \{1, \ldots, N\}$ (and hence the network) on communication during any given time interval. It is interesting to analyze how this lower bound depends on the various problem ingredients: τ^i is an increasing function of ϵ^i and a decreasing function of α and c^i. Through the latter, the bound also depends on the graph topology and the design parameter β. Given the definition of c^i, one can deduce that the faster an input of an agent is changing (larger κ^i) or the farther the agent initially starts from the average of the inputs, the more often that agent would need to trigger communication. The connection between the network performance and the communication overhead can also be observed here. Increasing β or decreasing ϵ^i to improve the ultimate tracking error bound (10) results in smaller inter-event times. Given that the rate of convergence of (3) under (9) is $\min\{\alpha, \beta\hat{\lambda}_2\}$, decreasing α to increase the inter-event times slows down the convergence. •

C. Connected undirected graphs

Here, we obtain an alternative distributed event-triggered communication for the algorithm (3) over connected undirected graphs. While the results of the previous section are of course valid for these topologies, here we show that using the structural properties of the Laplacian matrix in the undirected case we can design an event-triggered law which allows agents to have longer inter-event communication times with almost the same tracking performance.

Theorem 4.3 (Convergence of (3) over connected graphs with asynchronous distributed event-triggered communication): *Assume that the input of each agent $i \in \{1, \ldots, N\}$ satisfies $|\dot{\mathsf{u}}^i|_{ess} = \kappa^i < \infty$, and the aggregated inputs satisfy $\|\mathbf{\Pi}_N \dot{\mathbf{u}}\|_{ess} = \gamma < \infty$. For $\boldsymbol{\epsilon} \in \mathbb{R}_{>0}^N$, consider an implementation of the algorithm (3) over a connected graph $\mathcal{G}$, where agent $i \in \{1, \ldots, N\}$ communicates with its neighbors at times $\{t_k^i\}_{k\in\mathbb{N}} \subset \mathbb{R}_{\geq 0}$, starting at $t_1^i = 0$, determined by*

$$t_{k+1}^i = \operatorname{argmax}\{t \in [t_k^i, \infty) \mid |\hat{x}^i(t) - x^i(t)|^2 \leq \frac{1}{4\mathsf{d}_{\text{out}}^i}\sum_{j=1}^N \mathsf{a}_{ij}|\hat{x}^i(t) - \hat{x}^j(t)|^2 + \frac{1}{4\mathsf{d}_{\text{out}}^i}(\epsilon^i)^2\}. \quad (17)$$

Then, for any $\alpha, \beta > 0$, the algorithm evolution starting from $x^i(0) \in \mathbb{R}$ and $v^i(0) \in \mathbb{R}$ with $\sum_{i=1}^N v^i(0) = 0$ satisfies

$$\lim_{t\to\infty}\left|x^i(t)-\frac{1}{N}\sum_{j=1}^{N}\mathsf{u}^j(t)\right|\le\frac{\gamma}{\beta\lambda_2}+\sqrt{\left(\frac{\gamma}{\beta\lambda_2}\right)^2+\frac{\|\boldsymbol{\epsilon}\|^2}{2\lambda_2}},\quad(18)$$

for $i\in\{1,\dots,N\}$. Furthermore, the inter-execution times of agent $i\in\{1,\dots,N\}$ are lower bounded by

$$\tau^i=\frac{1}{\alpha}\ln\left(1+\frac{\alpha\epsilon^i}{2c^i\sqrt{\mathsf{d}^i_{\text{out}}}}\right),\quad(19)$$

where c^i is given in (12), now with $\eta=\max\{\|\mathbf{\Pi}_N(\mathbf{x}(0)-\bar{\mathbf{u}}(0))\|,\frac{\alpha\|\mathbf{\Pi}_N(\mathbf{x}(0)-\mathbf{u}(0))+\mathbf{v}(0)\|}{2}+\frac{\gamma}{\beta\lambda_2}+\sqrt{(\frac{\|\alpha\mathbf{\Pi}_N(\mathbf{x}(0)-\mathbf{u}(0))+\mathbf{v}(0)\|}{2}+\frac{\gamma}{\beta\lambda_2})^2+\frac{1}{2\lambda_2}\|\boldsymbol{\epsilon}\|^2}\}$.

Sketch of the proof: The proof is similar to the proof of Theorem 4.2 and for brevity, in the following, we only outline the parts that are different. We use the same Lyapunov function candidate (14) and taking its Lie derivative along the trajectories of (7d) we can show that, for $t\in[0,T)$,

$$\dot V\le\|\mathbf{z}_{2:N}\|\|\mathbf{q}_{2:N}(0)\|e^{-\alpha t}-\frac{\beta\lambda_2}{2}\mathbf{z}_{2:N}{}^\top\mathbf{z}_{2:N}-\frac{\beta}{2}s+\gamma\|\mathbf{z}_{2:N}\|,$$

where $s=-\mathbf{z}_{2:N}^\top\mathbf{R}^\top\mathbf{L}\mathbf{R}\mathbf{z}_{2:N}-2\mathbf{z}_{2:N}^\top\mathbf{R}^\top\mathbf{L}\tilde{\mathbf{x}}$. Using $\mathbf{x}=\hat{\mathbf{x}}-\tilde{\mathbf{x}}$, $\mathbf{D}_{\text{out}}+\mathbf{A}\ge0$, and $\hat{\mathbf{x}}^\top\mathbf{L}\hat{\mathbf{x}}=\frac{1}{2}\Sigma_{i=1}^N\Sigma_{j=1}^N\mathsf{a}_{ij}|\hat{x}^i-\hat{x}^j|^2$, we obtain

$$s=\frac{1}{2}\Sigma_{i=1}^N\left(4\mathsf{d}^i_{\text{out}}|\hat{x}^i-x^i|^2-\Sigma_{j=1}^N\mathsf{a}_{ij}|\hat{x}^i-\hat{x}^j|^2\right),$$

which, together with (17), implies $s\le\frac{1}{2}\|\boldsymbol{\epsilon}\|^2$ for $t\in[0,T)$. As such, we can write

$$\dot V\le-\frac{\beta\lambda_2}{2}(1-\theta)\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N}+\frac{2}{\beta\lambda_2}r,$$

where $0<\theta<1$ and $r=\|\mathbf{z}_{2:N}\|\|\mathbf{q}_{2:N}(0)\|-\theta\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N}+\frac{1}{2\lambda_2}\|\boldsymbol{\epsilon}\|^2+\frac{2\gamma}{\beta\lambda_2}\|\mathbf{z}_{2:N}\|$. Here we used $\|\mathbf{q}_{2:N}(0)\|e^{-\alpha t}\le\|\mathbf{q}_{2:N}(0)\|$ for all $t\in\mathbb{R}_{\ge0}$. Notice that $r<0$ for

$$\|\mathbf{z}_{2:N}\|\ge\frac{1}{2\theta}\|\mathbf{q}_{2:N}(0)\|+\frac{\gamma}{\beta\lambda_2\theta}+\sqrt{(\frac{1}{2\theta}\|\mathbf{q}_{2:N}(0)\|+\frac{\gamma}{\beta\lambda_2\theta})^2+\frac{1}{2\lambda_2\theta}\|\boldsymbol{\epsilon}\|^2}=\bar\eta.$$

Therefore, for $t\in[0,T)$,

$$\dot V\le-\frac{1}{2}\beta\lambda_2(1-\theta)\mathbf{z}_{2:N}^\top\mathbf{z}_{2:N},\quad\|\mathbf{z}_{2:N}\|\ge\bar\eta.\quad(20)$$

Recall the Lyapunov function candidate (14). Then, for any given initial condition $\mathbf{z}_{2:N}(0)\in\mathbb{R}^{N-1}$, regardless of value of T, from (20), for $t\in[0,T)$, we have

$$\|\mathbf{z}_{2:N}(t)\|\le\max\{\|\mathbf{z}_{2:N}(0)\|,\bar\eta\}=\eta.$$

The rest of the proof follows the proof of Theorem 4.2 to establish $T=\infty$ by establishing a lower bound (here (19)) on the amount of time it takes $|\hat{x}^i-x^i|$ of agent $i\in\{1,\dots,N\}$ to evolve from 0 to $\epsilon^i/(2\sqrt{\mathsf{d}^i_{\text{out}}})$, and using the same argument to rule out the Zeno behavior. To establish (18), we study the limiting behavior of the trajectories of the algorithm (3) when $t\to\infty$ and $\theta\to1$. ∎

We close this section by pointing out that the guaranteed lower bound (19) on the inter-event-times is conservative because to obtain it in the proof of Theorem 4.3 we have neglected the effect of the term $\frac{1}{4\mathsf{d}^i_{\text{out}}}\sum_{j=1}^N\mathsf{a}_{ij}|\hat{x}^i(t)-\hat{x}^j(t)|^2$ in (17) (recall that τ^i is an estimate on the time it takes for $|\tilde{x}^i|$ to evolve from 0 to $\epsilon^i/(2\sqrt{\mathsf{d}^i_{\text{out}}})$). The simulations of Section V show that the implementation of (17) results in longer inter-event times in comparison to the ones associated with the event-triggered law (9).

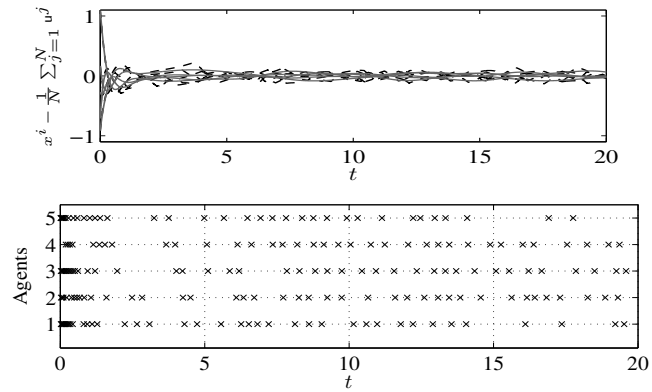

Fig. 1: Executions of (3) using $\alpha=1$ and $\beta=4$ over a ring digraph of 5 agents with unit weights and inputs $u^1(t)=0.5\sin(0.8t)$, $u^2(t)=0.5\sin(0.7t)+0.5\cos(0.6t)$, $u^3(t)=\sin(0.2t)+1$, $u^4(t)=\text{atan}(0.5t)$, $u^5(t)=0.1\cos(2t)$: the top plot shows the tracking error and the bottom one shows the communication times of each agent. Black (resp. gray) lines correspond to the event-triggered communication law (9) with $(\epsilon^i)^2=0.1$ (resp. continuous-time communication (1)).

V. Simulations

Here, we illustrate the performance of (3) under the event-triggered communication laws (9) and (17) over a ring digraph (cf. Figures 1 and 2) and a ring graph (cf. Figure 3). Figure 1 shows a small degradation between the tracking performance of the algorithm (3) with the event-triggered law (9) or with continuous-time communication. In the event-triggered implementation, the number of times that agents $\{1,2,3,4,5\}$ communicate during $[0,20]$ is $(42,40,42,36,38)$, respectively. Figure 2 compares the performance of (3) with the event-triggered communication law (9) and the Euler discretization of (1). For the latter, we choose the stepsize $\delta=0.19$ (from [9], convergence is guaranteed if $\delta\in(0,\min\{\alpha^{-1},\beta^{-1}(\mathsf{d}^{\text{out}}_{\max})^{-1}\})$, which for this example results in $\delta\in(0,0.2)$). The number of times that agents $\{1,2,3,4,5\}$ communicate in the time interval $[0,20]$ is $(54,42,46,39,45)$, respectively, when implementing the event-triggered communication scheme. This is about half the communication used by each agent in the Euler discretization implementation. Figure 3 shows the execution of (3) with the event-triggered communication laws (9) and (17). For each agent $i\in\{1,2,3,4,5\}$, we choose ϵ^i for each law so that the summand in the right-hand side of the trigger ($(\epsilon^i)^2$ for (9), $\epsilon^i/(2\sqrt{\mathsf{d}^i_{\text{out}}})$ for (17)) amounts to the same quantity. The plots show similar tracking performance for both algorithms, with (17) inducing less communication than (9). The number of times that agents $\{1,2,3,4,5\}$ communicate during $[0,12]$ is $(105,139,177,148,143)$ under (9) and $(50,60,66,57,53)$ under (17).

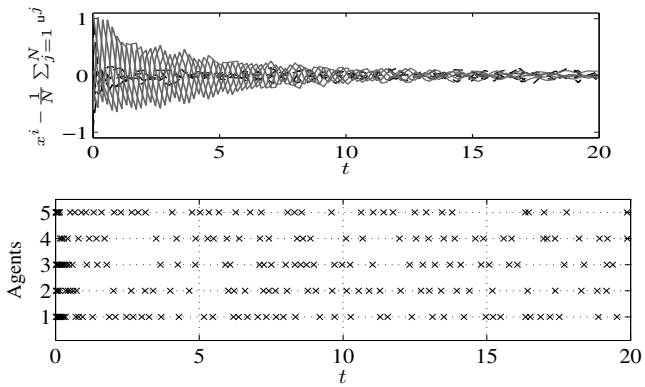

Fig. 2: Comparison between the algorithm (3) with the event-triggered communication law (9) and the Euler discretization of the algorithm (1) with parameters $\alpha = 1$ and $\beta = 5$ over a ring digraph of 5 agents with unit weights and inputs $u^1(t) = 0.5\sin(0.8t)$, $u^2(t) = 0.5\sin(0.7t) + 0.5\cos(0.6t)$, $u^3(t) = \sin(0.2t)+1$, $u^4(t) = \text{atan}(0.5t)$, $u^5(t) = 0.1\cos(2t)$: Black (resp. gray) lines correspond to the event-triggered communication law (9) with $(\epsilon^i)^2 = 0.1$ (resp. the Euler discretization with fixed stepsize $\delta = 0.19$).

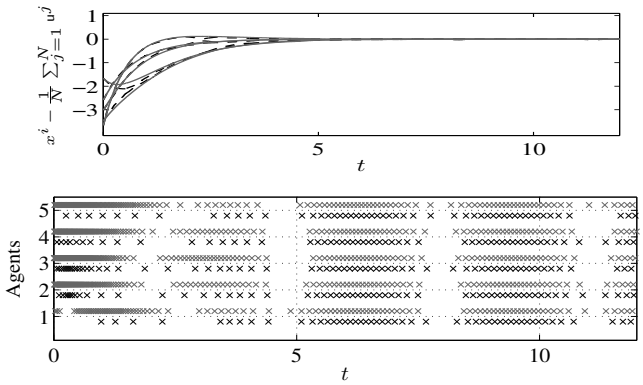

Fig. 3: Executions of (3) with parameters $\alpha = \beta = 1$ over a connected ring graph of 5 agents with unit weights and inputs $u^1(t) = 0.5\sin(t) + 1/(t+2) + 2$, $u^2(t) = 0.5\sin(t) + 1/(t+2)^2 + 4$, $u^3(t) = 0.5\sin(t) + 1/(t+2)^3 + 5$, $u^4(t) = 0.5\sin(t) + \mathrm{e}^{-t} + 4$, $u^5(t) = 0.5\sin(t) + \text{atan}(t) - 1.5$. The top plot shows the tracking error and the bottom one shows the communication times of each agent. Black (resp. gray) lines correspond to the event-triggered communication law (9) with $(\epsilon^i)^2 = 0.04$ (resp. (17) with $\epsilon^i/(2\sqrt{\mathrm{d}^i_{\text{out}}}) = 0.04$).

Acknowledgments

This work was supported by L3 Communications through the UCSD Cymer Center for Control Systems and Dynamics, NSF Award ECCS-1307176 and AFOSR - 11RSL548.

VI. Conclusions

We have studied the multi-agent dynamic average consensus problem over strongly connected and weight-balanced digraphs where inter-agent communication takes place at discrete instants of time in an opportunistic fashion. Our starting point has been a continuous-time dynamic average consensus algorithm which is known to converge exponentially to a small neighborhood of the network's inputs average. We have proposed two different distributed event-triggered laws to trigger communication with neighbors, depending on whether the interaction topology is described by a strongly connected and weight-balanced digraph or an undirected connected graph. In both cases, we established the correctness of the algorithm and showed that a positive lower bound on the inter-event times of each agent exists, ruling out the presence of Zeno behavior. Future work will focus on the exploration of abstractions about other agents' behaviors to develop self-triggered communication laws and the extension of the results to dynamic network topologies.

References

[1] P. Yang, R. Freeman, and K. Lynch, "Multi-agent coordination by decentralized estimation and control," *IEEE Transactions on Automatic Control*, vol. 53, no. 11, pp. 2480–2496, 2008.

[2] R. Olfati-Saber and J. Shamma, "Consensus filters for sensor networks and distributed sensor fusion," in *IEEE Conf. on Decision and Control and European Control Conference*, (Seville, Spain), pp. 6698–6703, December 2005.

[3] R. Olfati-Saber, "Distributed Kalman filtering for sensor networks," in *IEEE Conf. on Decision and Control*, (New Orleans, LA, USA), pp. 5492–5498, Dec. 2007.

[4] S. Meyn, *Control Techniques for Complex Networks*. Cambridge University Press, 2007.

[5] P. Yang, R. Freeman, and K. Lynch, "Distributed cooperative active sensing using consensus filters," in *IEEE Int. Conf. on Robotics and Automation*, (Roma, Italy), pp. 405–410, April 2007.

[6] D. P. Spanos, R. Olfati-Saber, and R. M. Murray, "Dynamic consensus on mobile networks," in *IFAC World Congress*, (Prague, Czech Republic), July 2005.

[7] R. A. Freeman, P. Yang, and K. M. Lynch, "Stability and convergence properties of dynamic average consensus estimators," in *IEEE Conf. on Decision and Control*, pp. 338–343, IEEE, 2006.

[8] H. Bai, R. Freeman, and K. Lynch, "Robust dynamic average consensus of time-varying inputs," in *IEEE Conf. on Decision and Control*, (Atlanta, GA, USA), pp. 3104–3109, December 2010.

[9] S. S. Kia, J. Cortés, and S. Martínez, "Dynamic average consensus under limited control authority and privacy requirements," *International Journal on Robust and Nonlinear Control*, 2014. To appear.

[10] M. Zhu and S. Martínez, "Discrete-time dynamic average consensus," *Automatica*, vol. 46, no. 2, pp. 322–329, 2010.

[11] W. Heemels, K. Johansson, and P. Tabuada, "An introduction to event-triggered and self-triggered control," in *IEEE Conf. on Decision and Control*, (Maui, HI), pp. 3270–3285, 2012.

[12] M. Mazo and P. Tabuada, "Decentralized event-triggered control over wireless sensor/actuator networks," *IEEE Transactions on Automatic Control*, vol. 56, no. 10, pp. 2456–2461, 2011.

[13] X. Wang and M. D. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 586–601, 2011.

[14] D. Dimarogonas, E. Frazzoli, and K. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1291–1297, 2012.

[15] E. Garcia, Y. Cao, H. Yuc, P. Antsaklis, and D. Casbeer, "Decentralised event-triggered cooperative control with limited communication," *International Journal of Control*, vol. 86, no. 9, pp. 1479–1488, 2013.

[16] C. Nowzari and J. Cortés, "Zeno-free, distributed event-triggered communication and control for multi-agent average consensus," in *American Control Conference*, (Portland, OR), pp. 2148–2153, 2014.

[17] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*. Applied Mathematics Series, Princeton University Press, 2009. Available at http://www.coordinationbook.info.

[18] H. K. Khalil, *Nonlinear Systems*. Prentice Hall, 3 ed., 2002.

[19] M. Donkers and W. Heemels, "Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralized event-triggering," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1362–1376, 2012.