Distributed online second-order dynamics for convex optimization over switching connected graphs

David Mateos-Núñez Jorge Cortés

Abstract— **This paper studies the regret of a family of distributed algorithms for online convex unconstrained optimization. A team of agents cooperate in a decision making process enabled by local interactions and the knowledge of each agent about the local cost functions associated with its decisions in previous rounds. We propose a class of online, second-order distributed coordination algorithms that combine subgradient descent on the local objectives revealed in the previous round and proportional-integral feedback on the disagreement among neighboring agents. The communication network is given by a time-varying sequence of connected graphs, and the local objectives can be adversarially adaptive to the agents' behavior. The goal of each agent is to incur a cumulative cost over time with respect to the sum of local objectives across the network that is competitive with the best fixed and centralized decision in hindsight. For this, we establish the classical logarithmic regret bound under strong convexity of the local objectives.**

I. Introduction

Networked multi-agent systems are being increasingly deployed in scenarios where information is dynamic and increasingly revealed over time as the interaction between the network and the environment progresses. Given the limited resources available to the network, such scenarios bring to the forefront the need for optimizing network behavior under uncertain and dynamic information. Motivated by these observations, this paper combines distributed and online optimization. On the one hand, distributed optimization exploits cooperation, while maintaining privacy, to solve optimization problems where computational and data-collection capabilities are distributed across a network. On the other hand, online optimization leverages streaming data to produce adaptive solutions in scenarios where information is increasingly revealed over time. Distributed online optimization arises from the combination of these two areas and allows the goal-directed fusion of information in both space –across a network of agents– and time –incorporating new information as it becomes available.

Literature review: Distributed optimization problems are pervasive in distributed and parallel computation and multi-agent systems [1], [2], [3], and in the convex case has motivated a growing body of work, see e.g., [4], [5], [6], [7], [8], [9], on the synthesis of distributed algorithms with asymptotic convergence guarantees. Online learning, on the other hand, is about sequential decision making given historical observations of the cost functions associated with previous decisions, even when the cost functions are adversarially adaptive to the behavior of the decision maker. Interestingly, in online convex optimization [10], [11], [12], [13] it is doable to compete with the best fixed decision in hindsight, which means that the regret, i.e., the difference between the cumulative cost over time and the cost of the best fixed decision in hindsight, is sublinear in the time horizon. This setup has applications in information theory [11], game theory [14], and supervised online machine learning, including interactive learning [15], online regression [12], instance ranking and AdaBoost [14], and in fact these areas inform each other [11] and offer different perspectives into problems like portfolio selection [13] and online advertisement placement, among others. Algorithmic approaches include online gradient descent [10], online Newton step [16], follow-the-approximate-leader [17], [16], and online alternating directions [18]. A few recent works have explored the combination of distributed and online convex optimization. The work [19] proposes distributed online strategies that rely on the computation and maintenance of spanning trees for global vector-sum operations, and makes statistical assumptions on the sequence of objectives. [20] studies decentralized online convex programming for groups of agents whose interaction topology is a chain. The works [21], [22] extend online projected subgradient descent and online dual averaging to scenarios similar to ours, that is, they study the notion of agent regret, which differs from the alternative notion of empirical risk [23, Th. 1], and make no statistical assumptions on the sequence of objectives. [21] shows regret $\mathcal{O}(\sqrt{T})$ under convexity of the cost functions and regret $\mathcal{O}(\log T)$ under strong convexity (always under the standard Lipschitz assumption), but both analyses require a projection step onto a compact convex set. In contrast, [22] shows $\mathcal{O}(\sqrt{T})$ regret under convexity using a general regularized projection. Both works consider fixed directed communication digraphs with doubly-stochastic adjacency matrices, whereas here we consider the case of switching undirected topologies. For this, we study a family of distributed saddle-point subgradient algorithms [24], [8] that in the static case converge with constant learning rates and enjoy robust asymptotic behavior in the presence of noise [25].

Statement of contributions: We consider a network of agents communicating over a sequence of time-dependent connected graphs. The network is involved in an online optimization scenario where each agent has access to a component of the sum of objective functions increasingly revealed to

The authors are with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, USA, {dmateosn,cortes}@ucsd.edu.

the network. We propose a class of distributed online algorithms that build on subgradient saddle-point (discrete-time) dynamics. Our algorithm design combines subgradient descent on the local objectives revealed in the previous round and proportional-integral distributed feedback on the disagreement among neighboring agents in a time-dependent communication network. We study the asymptotic convergence properties of the proposed algorithms and establish the classical logarithmic regret bound under strong convexity of the local objectives. Our technical approach builds on a concept that we term network regret, and on the cumulative disagreement of the collective estimates via the input-to-state stability properties of the second-order consensus component of the algorithm.

Organization: The paper is organized as follows. Section II introduces preliminary notions on matrix analysis, convex functions and graph theory. Section III formulates the networked online convex optimization problem. Section IV introduces our distributed algorithmic solution and Section V presents the convergence analysis leading up to the logarithmic regret bound. Finally, Section VI discusses our conclusions and ideas for future work. The appendix gathers auxiliary linear algebra results.

II. Preliminaries

Here we introduce notational conventions and basic notions on linear algebra, convex functions and graph theory.

Linear algebra: We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, by $\mathrm{I}_n \in \mathbb{R}^{n\times n}$ the identity matrix, by $e_{i|n}$ the ith column of I_n, and by $\mathbb{1}_n$ the column vector of ones in $\mathbb{R}^n$. A subspace $\mathcal{U} \subseteq \mathbb{R}^n$ is a subset of $\mathbb{R}^n$ which is itself a vector space. Two subspaces $\mathcal{U}$, $\mathcal{V} \subseteq \mathbb{R}^n$ are orthogonal, $\mathcal{U}\perp\mathcal{V}$, if $u^\top v = 0$ for any $u \in \mathcal{U}$, $v \in \mathcal{V}$. Given a matrix $A \in \mathbb{R}^{m\times n}$, we denote its nullspace by $\mathcal{N}(A) := \{x \in \mathbb{R}^n : Ax = 0\}$ and its column space by $\mathcal{C}(A)$. Given $A \in \mathbb{R}^{n\times n}$, we let $\mathrm{spec}(A)$ and $\rho(A)$ denote the set of eigenvalues and the spectral radius of A, respectively. The matrix A is diagonalizable if it can be written as $A = S_A D_A S_A^{-1}$, where $D_A \in \mathbb{R}^{n\times n}$ is a diagonal matrix with the eigenvalues of A as entries, and $S_A \in \mathbb{R}^{n\times n}$ contains in its columns the corresponding eigenvectors. If the eigenvalues are real, then the eigenvectors are real. Throughout the paper we use different labels for the eigenvalues with the exception that, when they are real and nonnegative, $\lambda^\star_{\min}$ denotes the minimum nonzero eigenvalue, and $\lambda_{\max}$ denotes the maximum eigenvalue. We let $\|.\|_2$ denote the Euclidean norm. For $B \in \mathbb{R}^{n\times m}$, we also denote $\|B\|_2 := \sigma_{\max}(B)$, the largest singular value of B, and $\kappa(B) := \|B\|_2\|B^{-1}\|_2 = \sigma_{\max}(B)/\sigma_{\min}(B)$, the condition number of B. The Kronecker product of $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times q}$ is denoted by $B \otimes C \in \mathbb{R}^{np\times mq}$.

Convex functions: Given a convex set $\mathcal{C} \subseteq \mathbb{R}^n$, a function $f : \mathcal{C} \to \mathbb{R}$ is convex if $f(\alpha x+(1-\alpha)y) \leq \alpha f(x)+(1-\alpha)f(y)$, for all $\alpha \in [0, 1]$ and $x, y \in \mathcal{C}$. A vector $\xi_x \in \mathbb{R}^n$ is called a subgradient of f at $x \in \mathcal{C}$ if $f(y) - f(x) \geq \xi_x^\top(y - x)$, for all $y \in \mathcal{C}$, and we denote by $\partial f(x)$ the set of all such subgradients. The characterization [26, Lemma 3.1.6] asserts that a function $f : \mathcal{C} \to \mathbb{R}$ is convex if and only if $\partial f(x)$ is nonempty for each $x \in \mathcal{C}$. Equivalently, note that f is convex if $\partial f(x)$ is nonempty and for each $x \in \mathcal{C}$ and $\xi_x \in \partial f(x)$,

$$f(y) - f(x) \geq \xi_x^\top(y - x) + \tfrac{p(x,y)}{2}\|y - x\|_2^2,$$

for all $y \in \mathcal{C}$, where the function $p : \mathcal{C} \times \mathcal{C} \to \mathbb{R}_{\geq 0}$ is the modulus of strong convexity (whose value may be 0). Furthermore, f is p-strongly convex in $\mathcal{C}$, for some $p > 0$, if $p(x, y) = p$ for all $x, y \in \mathcal{C}$. Finally, a convex function $f : \mathbb{R}^n \to \mathbb{R}$ has H-bounded subgradient sets if there exists $H \in \mathbb{R}_{>0}$ such that $\|\xi_x\|_2 \leq H$ for all $\xi_x \in \partial f(x)$ and $x \in \mathbb{R}^n$.

Graph theory: We follow the exposition in [27]. An (undirected) graph $\mathcal{G} = (\mathcal{I}, \mathcal{E})$ is composed of a vertex set $\mathcal{I} = \{1, \ldots, N\}$ and an edge set $\mathcal{E} \subseteq \mathcal{I} \times \mathcal{I}$. The edge $(i, j) \in \mathcal{E}$ is considered unordered, meaning that i is a neighbor of j and vice versa. A weighted graph $\mathcal{G} = (\mathcal{I}, \mathcal{E}, \mathsf{A})$ is a graph together with a symmetric adjacency matrix $\mathsf{A} \in \mathbb{R}^{N\times N}_{\geq 0}$ with the property that $\mathsf{a}_{ij} > 0$ if and only if i and j are neighbors. The Laplacian matrix $\mathsf{L} := \mathrm{diag}(\mathsf{A}\mathbb{1}_N) - \mathsf{A}$ satisfies $\mathsf{L}\mathbb{1}_N = 0_N$ and is symmetric and positive semidefinite. The complete graph is the graph where every vertex is a neighbor of every other vertex. For convenience, we let $\mathsf{L}_\mathcal{K}$ denote the Laplacian of the complete graph with edge weights equal to $1/N$, that is, $\mathsf{L}_\mathcal{K} := \mathrm{I}_N - \mathrm{M}$, where $\mathrm{M} := \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\top$. We note that $\mathsf{L}_\mathcal{K}$ is idempotent, i.e., $\mathsf{L}_\mathcal{K}^2 = \mathsf{L}_\mathcal{K}$. A path is an ordered sequence of vertices such that any pair of vertices appearing consecutively is an edge. A graph is connected if there is a path between any pair of distinct vertices. A weighted graph $\mathcal{G}$ is connected if and only if $\mathrm{rank}(\mathsf{L}) = N - 1$. Since the Laplacian is symmetric, it can be factorized as $\mathsf{L} = S_\mathsf{L} D_\mathsf{L} S_\mathsf{L}^{-1}$, where $D_\mathsf{L} \in \mathbb{R}^{N\times N}$ is a diagonal matrix with the eigenvalues of L in increasing order and S_L is an orthogonal matrix whose columns form an orthonormal basis of the corresponding eigenvectors.

III. Problem statement

This section introduces the problem of interest. We begin by describing the online convex optimization problem for one player and then present the networked version, which is the focus of the paper. In online convex optimization, given a time horizon $T \in \mathbb{Z}_{\geq 1}$, in each round $t \in \{1, \ldots, T\}$ a player chooses a point $x_t \in \mathbb{R}^d$. After committing to this choice, a convex cost function $f_t : \mathbb{R}^d \to \mathbb{R}$ is revealed. Consequently, the 'cost' incurred by the player is $f_t(x_t)$. Given the temporal sequence of objectives $\{f_t\}_{t=1}^T$, the regret of the player using $\{x_t\}_{t=1}^T$ with respect to a competing choice $u \in \mathbb{R}^d$ over a time horizon T is defined by

$$\mathcal{R}(u, \{f_t\}_{t=1}^T) := \sum_{t=1}^T f_t(x_t) - \sum_{t=1}^T f_t(u), \qquad (1)$$

i.e., the difference between the total cost and the cost of the fixed decision u. A common selection of u is the best

decision had all the information been available a priori, i.e.,

$$u = x_T^* \in \arg\min_{x\in\mathbb{R}^d} \sum_{t=1}^{T} f_t(x).$$

In the case when no information is available about the evolution of the functions $\{f_t\}_{t=1}^T$, one is interested in designing algorithms whose worst-case regret is upper bounded sublinearly in the time horizon T. This implies that, on the average, the algorithm performs as well as the best fixed decision in hindsight.

In this paper, we are interested in a distributed version of the online convex optimization problem described above where the online player is replaced by a network of N agents. In the round $t \in \{1,\dots,T\}$, agent $i \in \{1,\dots,N\}$ chooses a point x_t^i corresponding to what it thinks the network as a whole should have chosen. After committing to this choice, the agent has access to a convex cost function $f_t^i : \mathbb{R}^d \to \mathbb{R}$, and the network cost is then given by the evaluation of

$$f_t(x) := \sum_{i=1}^{N} f_t^i(x). \tag{2}$$

Since information is now distributed across the network as opposed to centralized, agents must collaborate with each other to determine their choices for the next round. Assume that the network communication topology is time-dependent and described by a sequence of connected weighted graphs $\{\mathcal{G}_t\}_{t=1}^T = \{(\{1,\dots,N\},\mathcal{E}_t,\mathsf{A}_t)\}_{t=1}^T$. Through communication, agents become aware of the choices made by their neighbors in the previous round. In this scenario, the regret of agent $j \in \{1,\dots,N\}$ using $\{x_t^j\}_{t=1}^T$ with respect to a competing choice $u \in \mathbb{R}^d$ over a time horizon T is

$$\mathcal{R}^j(u,\{f_t\}_{t=1}^T) := \sum_{t=1}^{T}\sum_{i=1}^{N} f_t^i(x_t^j) - \sum_{t=1}^{T}\sum_{i=1}^{N} f_t^i(u).$$

Note that this regret is not directly computable by agent j even in hindsight, because it does not know the local cost functions of the other agents in the network. The complexity of this problem formulation stems from combining the distributed computation aspect and the online optimization aspect. That is, information is distributed across the agents (who act as decision makers) and is not known a priori but incrementally revealed to the agents.

IV. Saddle-point dynamics for online distributed optimization

In this section we introduce a distributed coordination algorithm to solve the networked online convex optimization problem described in Section III. In each round $t \in \{1,\dots,T\}$, agent $i \in \{1,\dots,N\}$ performs

$$\begin{aligned} x_{t+1}^i &= x_t^i + \sigma\Big(a\sum_{j=1}^{N}\mathsf{a}_{ij,t}(x_t^j - x_t^i) + \sum_{j=1}^{N}\mathsf{a}_{ij,t}(z_t^j - z_t^i)\Big) - \eta_t g_{x_t^i} \\ z_{t+1}^i &= z_t^i - \sigma\sum_{j=1}^{N}\mathsf{a}_{ij,t}(x_t^j - x_t^i), \end{aligned} \tag{3}$$

where $g_{x_t^i} \in \partial f_t^i(x_t^i)$ and $\mathsf{a}_{ij,t} := (\mathsf{A}_t)_{ij}$. Here $\sigma \in \mathbb{R}_{>0}$ and $a \in \mathbb{R}_{>0}$ are design parameters and $\eta_t \in \mathbb{R}_{>0}$ is the learning rate. Agent i is responsible for the variables x^i, z^i, and shares their values with its neighbors according to the time-dependent communication graph $\mathcal{G}_t$. Note that (3) is consistent with the incremental access to information of individual agents (in the round $t+1$, agent i is aware of f_t^i) and is distributed over $\mathcal{G}_t$.

The role of the auxiliary states $z^1,\dots,z^N$ is best understood examining the rationale behind the synthesis of the coordination algorithm (3). Our design builds upon the strategy for distributed optimization of a sum of convex functions studied in [24], [8]. Consider the collective network state $\boldsymbol{x} := (x^1,\dots,x^N) \in (\mathbb{R}^d)^N$. For $t \in \{1,\dots,T\}$, let the convex function $\tilde{f}_t : (\mathbb{R}^d)^N \to \mathbb{R}$ be defined by

$$\tilde{f}_t(\boldsymbol{x}) := \sum_{i=1}^{N} f_t^i(x^i). \tag{4}$$

Note that $\tilde{f}_t(\mathbb{1}_N \otimes x) = f_t(x)$, i.e., when all agents agree on the same choice, we recover the value of the network cost function (2). Given the connectivity of $\mathcal{G}_t$, this agreement constraint can be expressed, in each round t, in terms of the Laplacian, $(\mathsf{L}_t \otimes \mathrm{I}_d)\boldsymbol{x} = 0_{dN}$. Motivated by these observations, consider the time-dependent augmented Lagrangian $\mathcal{L}_t : (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \to \mathbb{R}$,

$$\mathcal{L}_t(\boldsymbol{x},\boldsymbol{z}) := \tfrac{\eta_t}{\sigma}\tilde{f}_t(\boldsymbol{x}) + \tfrac{a}{2}\boldsymbol{x}^\top\mathbf{L}_t\boldsymbol{x} + \boldsymbol{z}^\top\mathbf{L}_t\boldsymbol{x},$$

where $\mathbf{L}_t$ is shorthand notation for $\mathsf{L}_t \otimes \mathrm{I}_d$. The network auxiliary state $\boldsymbol{z} := (z^1,\dots,z^N)$ corresponds to the Lagrange multipliers associated to the agreement constraint, and the coordination algorithm (3) corresponds to a first-order Euler discretization of the saddle-point dynamics associated to the Lagrangian function $\mathcal{L}_t$.

In compact form, the dynamics (3) can be expressed as

$$\begin{bmatrix}\boldsymbol{x}_{t+1}\\ \boldsymbol{z}_{t+1}\end{bmatrix} = \begin{bmatrix}\boldsymbol{x}_t\\ \boldsymbol{z}_t\end{bmatrix} + \sigma\begin{bmatrix}-a\mathbf{L}_t & -\mathbf{L}_t\\ \mathbf{L}_t & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_t\\ \boldsymbol{z}_t\end{bmatrix} - \eta_t\begin{bmatrix}\tilde{g}_{\boldsymbol{x}_t}\\ 0\end{bmatrix}, \tag{5}$$

where $\tilde{g}_{\boldsymbol{x}_t} = (g_{x_t^1},\dots,g_{x_t^N}) \in \partial\tilde{f}_t(\boldsymbol{x}_t)$. This representation shows that the coordination algorithm (3) is a combination of a second-order linear consensus algorithm together with a subgradient descent. We will build on this interpretation later in our convergence analysis.

Our main result states that, under (3), the agent regret is logarithmic in the time horizon.

Theorem IV.1. (Logarithmic agent regret under (3)**):** For $T \in \mathbb{Z}_{\geq 1}$, let $\{f_t^1,\dots,f_t^N\}_{t=1}^T$ be convex functions on $\mathbb{R}^d$ with H-bounded subgradient sets and nonempty sets of minimizers. Assume that the sets of minimizers are bounded uniformly in T and the functions $\{f_t^1,\dots,f_t^N\}_{t=1}^T$ are p-strongly convex in a sufficiently large set. Let $\{\mathcal{G}_t\}_{t=1}^T$ be a sequence of connected weighted graphs such that

$$0 < \underline{\Lambda} \leq \min_{1\leq t\leq T}\{\lambda_{\min}^\star(\mathsf{L}_t)\}, \quad \max_{1\leq t\leq T}\{\lambda_{\max}(\mathsf{L}_t)\} \leq \overline{\Lambda}, \tag{6}$$

and let $\eta_t = \frac{1}{pt}$, $a \in (2,\infty)$ and $\sigma \in (0, \sigma^*)$, where

$$\sigma^* := \frac{2}{\big(\frac{a}{2} + \sqrt{(\frac{a}{2})^2 - 1}\,\big)\Lambda}. \tag{7}$$

Then the sequence $\{\boldsymbol{x}_t = (x_t^1, \dots, x_t^N)\}_{t=1}^T$ generated by the coordination algorithm (3) satisfies the following regret bound:

$$\mathcal{R}^j\big(u, \{f_t\}_{t=1}^T\big) \leq C(\|u\|_2^2 + 1 + \log T), \tag{8}$$

for any $j \in \{1, \dots, N\}$ and $u \in \mathbb{R}^d$, for some $C > 0$.

The proof of this result is in the following section and provides an explicit characterization of the constant C in terms of the initial conditions and the network parameters.

V. Regret analysis

This section presents our technical approach to establish the $\mathcal{O}(\log T)$ regret stated in Theorem IV.1. Our proof strategy uses the following auxiliary notion. For any sequence (of network states) $\{\boldsymbol{x}_t\}_{t=1}^T \subseteq (\mathbb{R}^d)^N$, we define the network regret [19], [23] with respect to a competing choice $u \in \mathbb{R}^d$ over the time horizon T as

$$\mathcal{R}_{\mathcal{N}}(u, \{\tilde{f}_t\}_{t=1}^T) := \sum_{t=1}^T \tilde{f}_t(\boldsymbol{x}_t) - \sum_{t=1}^T \tilde{f}_t(\mathbb{1}_N \otimes u).$$

Based on this concept, our proof strategy to establish Theorem IV.1 consists of the following steps.

- in Section V-A, we bound the difference between network and agent regret as a function of the cumulative disagreement of the collective estimates over time, and then we similarly bound the network regret;
- in Section V-B, we bound the cumulative disagreement in terms of the learning rates and the size of the subgradients.
- Finally, combining the previous bounds with a bound on the trajectories independent of the time horizon, we make a choice of learning rates that yields $\mathcal{O}(\log T)$ regret under local strong convexity.

A. Bounds on network and agent regret

We start with a result that is independent of the algorithm.

Lemma V.1. (Bound on the difference between agent and network regret): For $T \in \mathbb{Z}_{\geq 1}$, let $\{f_t^1, \dots, f_t^N\}_{t=1}^T$ be convex functions on $\mathbb{R}^d$ with H-bounded subgradient sets. Then, any sequence $\{\boldsymbol{x}_t\}_{t=1}^T \subset (\mathbb{R}^d)^N$ satisfies

$$\mathcal{R}^j(u, \{f_t\}_{t=1}^T) \leq \mathcal{R}_{\mathcal{N}}(u, \{\tilde{f}_t\}_{t=1}^T) + \sqrt{2}\, NH \sum_{t=1}^T \|\mathbf{L}_{\mathcal{K}} \boldsymbol{x}_t\|_2,$$

where $\mathbf{L}_{\mathcal{K}} := \mathsf{L}_{\mathcal{K}} \otimes \mathrm{I}_d$, for any $j \in \{1, \dots, N\}$ and $u \in \mathbb{R}^d$.

The proof is omitted and will appear elsewhere.

Next, we bound the network regret in terms of the learning rates and the cumulative disagreement. The bound holds regardless of the connectivity of the communication network.

Lemma V.2. (Bound on network regret): For $T \in \mathbb{Z}_{\geq 1}$, let $\{f_t^1, \dots, f_t^N\}_{t=1}^T$ be convex functions on $\mathbb{R}^d$ with H-bounded subgradient sets. Let the sequence $\{\boldsymbol{x}_t\}_{t=1}^T$ be generated by the coordination algorithm (3) over a sequence of arbitrary weighted graphs $\{\mathcal{G}_t\}_{t=1}^T$. Then, for any $u \in \mathbb{R}^d$ and any sequence of learning rates $\{\eta_t\}_{t=1}^T \subset \mathbb{R}_{>0}$,

$$2\mathcal{R}_{\mathcal{N}}\big(u, \{\tilde{f}_t\}_{t=1}^T\big) \leq \sum_{t=2}^T \|\mathbf{M}\boldsymbol{x}_t - \boldsymbol{u}\|_2^2 \big(\tfrac{1}{\eta_t} - \tfrac{1}{\eta_{t-1}} - p_t(\boldsymbol{u}, \boldsymbol{x}_t)\big)$$
$$+ 2\sqrt{N}H \sum_{t=1}^T \|\mathbf{L}_{\mathcal{K}} \boldsymbol{x}_t\|_2 + NH^2 \sum_{t=1}^T \eta_t + \tfrac{1}{\eta_1}\|\mathbf{M}\boldsymbol{x}_1 - \boldsymbol{u}\|_2^2,$$

where $\mathbf{M} := \mathrm{M} \otimes \mathrm{I}_d$, $\boldsymbol{u} := \mathbb{1}_N \otimes u$ and each function $p_t : (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \to \mathbb{R}_{\geq 0}$ is the modulus of strong convexity.

The proof is omitted and will appear elsewhere.

B. Bound on cumulative disagreement

Here we study the evolution of the disagreement among agents. Our analysis builds on the representation (5) of the coordination algorithm (3) as a combination of a second-order linear consensus algorithm with subgradient descent, that we treat here as a perturbation. Consequently, consider the general dynamics

$$\boldsymbol{v}_{t+1} = (\mathrm{I}_{2Nd} + \sigma \mathbb{L}_t)\boldsymbol{v}_t + u_t, \tag{9}$$

where $\boldsymbol{v}_t := (\boldsymbol{x}_t, \boldsymbol{z}_t) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$, $u_t \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ is arbitrary, and

$$\mathbb{L}_t := \begin{bmatrix} -a\mathbf{L}_t & -\mathbf{L}_t \\ \mathbf{L}_t & 0 \end{bmatrix} = \begin{bmatrix} -a & -1 \\ 1 & 0 \end{bmatrix} \otimes \mathbf{L}_t. \tag{10}$$

Note that we recover the dynamics (5) with the choice $u_t = -\eta_t(\tilde{g}_{\boldsymbol{x}_t}, 0)$. Next, we left-multiply (9) by $\hat{\mathbf{L}}_{\mathcal{K}} := \mathrm{I}_2 \otimes \mathbf{L}_{\mathcal{K}}$ to describe the evolution of the disagreement

$$\begin{aligned} \hat{\mathbf{L}}_{\mathcal{K}}\boldsymbol{v}_{t+1} &= \hat{\mathbf{L}}_{\mathcal{K}}\big(\mathrm{I} + \sigma\mathbb{L}_t\big)\boldsymbol{v}_t + \hat{\mathbf{L}}_{\mathcal{K}}u_t \\ &= \big(\hat{\mathbf{L}}_{\mathcal{K}} + \sigma\mathbb{L}_t\big)\boldsymbol{v}_t + \hat{\mathbf{L}}_{\mathcal{K}}u_t = \big(\hat{\mathbf{L}}_{\mathcal{K}} + \sigma\mathbb{L}_t\big)\hat{\mathbf{L}}_{\mathcal{K}}\boldsymbol{v}_t + \hat{\mathbf{L}}_{\mathcal{K}}u_t, \end{aligned} \tag{11}$$

where we have used that $\hat{\mathbf{L}}_{\mathcal{K}}\mathbb{L}_t = \mathbb{L}_t$ in the second equality and $\hat{\mathbf{L}}_{\mathcal{K}}^2 = \hat{\mathbf{L}}_{\mathcal{K}}$ and $\mathbb{L}_t = \mathbb{L}_t\hat{\mathbf{L}}_{\mathcal{K}}$ in the third equality. Given the disagreement dynamics (11), the next result examines the properties of $\hat{\mathbf{L}}_{\mathcal{K}} + \sigma\mathbb{L}_t$, to which we refer as second-order disagreement matrix for the graph $\mathcal{G}_t$. Its proof is given in the Appendix.

Proposition V.3. (Properties of the second-order disagreement matrix): Let $\mathcal{G}$ be a connected graph with associated Laplacian matrix L, and define, for $a > 2$,

$$\mathbb{L} := \begin{bmatrix} -a & -1 \\ 1 & 0 \end{bmatrix} \otimes \mathsf{L} \otimes \mathrm{I}_d. \tag{12}$$

Using the shorthand notation $h(a) := \sqrt{(\frac{a}{2})^2 - 1}$, the following holds:

(i) for any $\sigma \in \mathbb{R}$, the eigenvalues of $\hat{\mathbf{L}}_{\mathcal{K}} + \sigma\mathbb{L}$ are 0, with multiplicity $2d$, and

$$\big\{\, 1 + \sigma\big(-\tfrac{a}{2} \pm h(a)\big)\lambda \;:\; \lambda \in \mathrm{spec}(\mathsf{L}) \setminus \{0\} \,\big\} \tag{13}$$

(each with multiplicity d). Consequently,

$$\rho(\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}) = \max\big\{\,|1+\sigma\big(-\tfrac{a}{2}+h(a)\big)\,\lambda^{\star}_{\min}(\mathsf{L})|, \\ |1+\sigma\big(-\tfrac{a}{2}-h(a)\big)\,\lambda_{\max}(\mathsf{L})|\big\}.$$

(ii) $\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}$ is diagonalizable, $\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L} = SDS^{-1}$, where

$$D = \Big(\mathrm{I}_{2N} + \sigma \underbrace{\mathrm{diag}\,\big(-\tfrac{a}{2}+h(a), -\tfrac{a}{2}-h(a)\big)}_{D_G} \otimes D_{\mathsf{L}} \\ - \mathrm{I}_2 \otimes \big(e_{1|N}\, e_{1|N}^{\top}\big)\Big) \otimes \mathrm{I}_d,$$

$$S = \underbrace{\begin{bmatrix} -\frac{a}{2}+h(a) & -\frac{a}{2}-h(a) \\ 1 & 1 \end{bmatrix}}_{S_G} \otimes S_{\mathsf{L}} \otimes \mathrm{I}_d.$$

(iii) $\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}$ is convergent (i.e., $\rho(\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}) < 1$) for

$$\sigma \in \Big(\,0\,,\, \frac{2}{(\frac{a}{2}+h(a))\,\lambda_{\max}(\mathsf{L})}\,\Big).$$

We are now ready to bound the cumulative disagreement of the collective estimates over time. Our result is a consequence of the input-to-state stability property of the dynamics (9), which we also establish.

Proposition V.4. (Input-to-state stability of the disagreement dynamics under switching connected topologies): Let $\{\mathcal{G}_t\}_{t=1}^{T}$ be a sequence of connected weighted graphs and let $a \in (2,\infty)$ and $\sigma \in (0,\sigma^*)$, where σ^* is defined in (7). Then, the dynamics (9) is input-to-state stable with respect to $\mathcal{N}(\hat{\mathbf{L}}_{\mathcal{K}})$. Specifically, for any $t \in \mathbb{Z}_{\geq 1}$ and any $\{u_s\}_{s=1}^{t-1} \subset (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$,

$$\|\hat{\mathbf{L}}_{\mathcal{K}}\boldsymbol{v}_t\|_2 \leq \kappa(S_G)\,\rho_\sigma^{t-1}\,\|\boldsymbol{v}_1\|_2 \\ + \frac{\kappa(S_G)}{1-\rho_\sigma}\max_{1\leq s\leq t-1}\|u_s\|_2, \tag{14}$$

where

$$\rho_\sigma := \max\big\{\,|1+\sigma\big(-\tfrac{a}{2}+h(a)\big)\underline{\Lambda}|, \\ |1+\sigma\big(-\tfrac{a}{2}-h(a)\big)\overline{\Lambda}|\,\big\} < 1. \tag{15}$$

Furthermore, the following bound for the cumulative disagreement for $T \in \mathbb{Z}_{\geq 1}$ holds:

$$\sum_{t=1}^{T}\|\mathbf{L}_{\mathcal{K}}\boldsymbol{x}_t\|_2 \leq \sum_{t=1}^{T}\|\hat{\mathbf{L}}_{\mathcal{K}}\boldsymbol{v}_t\|_2 \\ \leq \frac{\kappa(S_G)}{1-\rho_\sigma}\Big(\|\boldsymbol{v}_1\|_2 + \sum_{t=1}^{T-1}\|u_t\|_2\Big). \tag{16}$$

Proof. The disagreement dynamics (11) yields

$$\hat{\mathbf{L}}_{\mathcal{K}}\boldsymbol{v}_{t+1} = \boldsymbol{\Phi}(t,1)\hat{\mathbf{L}}_{\mathcal{K}}\boldsymbol{v}_1 \\ + \sum_{s=1}^{t-1}\boldsymbol{\Phi}(t,s+1)\hat{\mathbf{L}}_{\mathcal{K}}u_s + \hat{\mathbf{L}}_{\mathcal{K}}u_t, \tag{17}$$

where, for any $k \geq s$, we define the transition matrix $\boldsymbol{\Phi}(k,s)$ for the second-order disagreement dynamics as

$$\boldsymbol{\Phi}(k,s) := (\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}_k)\ldots(\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}_s).$$

By Proposition V.3(ii), we have $\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}_k = S_k D_k S_k^{-1}$, where

$$S_k := S_G \otimes S_{\mathsf{L}_k} \otimes \mathrm{I}_d,$$
$$D_k := \Big(\mathrm{I}_{2N} + \sigma D_G \otimes D_{\mathsf{L}_k} - \mathrm{I}_2 \otimes \big(e_{1|N}\, e_{1|N}^{\top}\big)\Big) \otimes \mathrm{I}_d.$$

Note that

$$S_k^{-1}S_{k-1} = \big(S_G^{-1} \otimes S_{\mathsf{L}_k}^{-1} \otimes \mathrm{I}_d\big)\big(S_G \otimes S_{\mathsf{L}_{k-1}} \otimes \mathrm{I}_d\big) \\ = \mathrm{I}_2 \otimes S_{\mathsf{L}_k}^{-1}S_{\mathsf{L}_{k-1}} \otimes \mathrm{I}_d.$$

Using this fact and the sub-multiplicativity of the norm, together with [28, Fact 9.12.22] for the norms of Kronecker products, we get

$$\|\boldsymbol{\Phi}(k,s)\|_2 = \|\big(S_kD_kS_k^{-1}\big)\big(S_{k-1}D_{k-1}S_{k-1}^{-1}\big)\times\cdots \\ \times\big(S_{s+1}D_{s+1}S_{s+1}^{-1}\big)\big(S_sD_sS_s^{-1}\big)\|_2$$
$$\leq \|S_G\|_2\|S_{\mathsf{L}_k}\|_2\|D_k\|_2\|S_{\mathsf{L}_k}^{-1}\|_2\Big(\prod_{r=s+1}^{k-1}\|S_{\mathsf{L}_r}\|_2\|D_r\|_2\|S_{\mathsf{L}_r}^{-1}\|_2\Big) \\ \times\|S_{\mathsf{L}_s}\|_2\|D_s\|_2\|S_G^{-1}\|_2\|S_{\mathsf{L}_s}^{-1}\|_2.$$

Grouping the condition numbers and noting that $\kappa(S_{\mathsf{L}_t}) = 1$ because S_{L_t} is orthogonal for each $t \in \mathbb{Z}_{\geq 1}$, we get

$$\|\boldsymbol{\Phi}(k,s)\|_2 \leq \kappa(S_G)\prod_{r=s}^{k}\kappa(S_{\mathsf{L}_r})\|D_r\|_2 \\ = \kappa(S_G)\prod_{r=s}^{k}\rho(\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}_r).$$

Using this bound in (17) in conjunction with the triangular inequality, and noting that $\|\hat{\mathbf{L}}_{\mathcal{K}}\|_2 = 1$ (because $\hat{\mathbf{L}}_{\mathcal{K}}$ is symmetric and its nonzero eigenvalues equal 1), we obtain

$$\|\hat{\mathbf{L}}_{\mathcal{K}}\boldsymbol{v}_{t+1}\|_2 \leq \kappa(S_G)\,\rho_\sigma^{t}\|\boldsymbol{v}_1\|_2 \\ + \kappa(S_G)\sum_{s=1}^{t}\rho_\sigma^{t-s}\|u_s\|_2, \tag{18}$$

where, from (6) and by Proposition V.3(i), one can see that ρ_σ, given in (15), satisfies

$$\rho_\sigma \geq \max_{1\leq r\leq T}\rho(\hat{\mathbf{L}}_{\mathcal{K}}+\sigma\mathbb{L}_r).$$

Moreover, $\rho_\sigma \in (0,1)$ because

$$-1 < 1+\sigma\big(-\tfrac{a}{2}-h(a)\big)\overline{\Lambda} \leq 1+\sigma\big(-\tfrac{a}{2}+h(a)\big)\underline{\Lambda} < 1$$

for $\sigma \in (0,\sigma^*)$. Equation (14) now follows from (18) by noting that $\sum_{k=0}^{\infty}\rho^k = \frac{1}{1-\rho}$ for $\rho \in (0,1)$. To bound the cumulative disagreement, we sum (18) over the time horizon T to get

$$\sum_{t=1}^{T}\|\hat{\mathbf{L}}_{\mathcal{K}}\boldsymbol{v}_t\|_2 \leq \frac{\kappa(S_G)\|\boldsymbol{v}_1\|_2}{1-\rho_\sigma} + \kappa(S_G)\sum_{t=1}^{T}\sum_{s=1}^{t-1}\rho_\sigma^{t-1-s}\|u_s\|_2.$$

Finally, noting that

$$\sum_{t=1}^{T}\sum_{s=1}^{t-1}\rho^{t-1-s}\|u_s\|_2 = \sum_{s=1}^{T-1}\sum_{t=s+1}^{T}\rho^{t-1-s}\|u_s\|_2$$
$$\sum_{s=1}^{T-1}\|u_s\|_2\sum_{t=s+1}^{T}\rho^{t-1-s} \leq \frac{1}{1-\rho}\sum_{s=1}^{T-1}\|u_s\|_2,$$

for any $\rho \in (0,1)$, equation (16) follows. □

C. General bound on agent regret

We now combine the previous results to bound the agent regret in terms of the learning rates.

Proposition V.5. (General bound on agent regret): For $T \in \mathbb{Z}_{\geq 1}$, let $\{f_t^1, \dots, f_t^N\}_{t=1}^T$ be convex functions on $\mathbb{R}^d$ with H-bounded subgradient sets for each $t \in \{1, \dots, T\}$. Let the sequence $\{\boldsymbol{x}_t\}_{t=1}^T$ be generated by the execution of the coordination algorithm (3) over a sequence of connected weighted graphs $\{\mathcal{G}_t\}_{t=1}^T$, and with $a > 2$ and $\sigma \in (0,\, \sigma^*)$. Then the following bound for the agent regret holds for any $j \in \{1, \dots, N\}$, $u \in \mathbb{R}^d$, and $\{\eta_t\}_{t=1}^T \subset \mathbb{R}_{>0}$:

$$
\begin{aligned}
&2\mathcal{R}^j(u, \{f_t\}_{t=1}^T) \qquad (19)\\
&\leq \sum_{t=2}^{T} \|\mathbf{M}\boldsymbol{x}_t - \mathbb{1}_N \otimes u\|_2^2 \Big(\tfrac{1}{\eta_t} - \tfrac{1}{\eta_{t-1}} - p_t(\mathbb{1}_N \otimes u, \boldsymbol{x}_t) \Big)\\
&\quad + 2\sqrt{N} H \big(1 + \sqrt{2N}\,\big) \frac{\kappa(S_G)}{1-\rho_\sigma} \sqrt{\|\boldsymbol{x}_1\|_2^2 + \|\boldsymbol{z}_1\|_2^2}\\
&\quad + NH^2 \Big(2\big(1 + \sqrt{2N}\,\big) \frac{\kappa(S_G)}{1-\rho_\sigma} + 1 \Big) \sum_{t=1}^{T} \eta_t\\
&\quad + \tfrac{1}{\eta_1} \|\mathbf{M}\boldsymbol{x}_1 - \mathbb{1}_N \otimes u\|_2^2,
\end{aligned}
$$

where each function $p_t : (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \to \mathbb{R}_{\geq 0}$ is the modulus of strong convexity (whose value may be 0).

Proof. Using Lemmas V.1 and V.2, we can write

$$
\begin{aligned}
&2\mathcal{R}^j(u, \{f_t\}_{t=1}^T) \leq \sum_{t=2}^{T} \|\mathbf{M}\boldsymbol{x}_t - \boldsymbol{u}\|_2^2 \big(\tfrac{1}{\eta_t} - \tfrac{1}{\eta_{t-1}} - p_t(\boldsymbol{u}, \boldsymbol{x}_t) \big)\\
&+ \big(2\sqrt{N}H + 2^{3/2} NH \big) \sum_{t=1}^{T} \|\mathbf{L}_{\mathcal{K}} \boldsymbol{x}_t\|_2 + NH^2 \sum_{t=1}^{T} \eta_t\\
&+ \big(\tfrac{1}{\eta_1} - p_1 \big) \|\mathbf{M}\boldsymbol{x}_1 - \boldsymbol{u}\|_2^2 - \sum_{t=1}^{T} p_t \|\mathbf{L}_{\mathcal{K}} \boldsymbol{x}_t\|_2^2. \qquad (20)
\end{aligned}
$$

According to Proposition V.4, the cumulative disagreement of the collective estimates over time is bounded as

$$
\sum_{t=1}^{T} \|\mathbf{L}_{\mathcal{K}} \boldsymbol{x}_t\|_2 \leq \frac{\kappa(S_G)}{1-\rho_\sigma} \big(\|\boldsymbol{v}_1\|_2 + \sum_{t=1}^{T-1} \eta_t \sqrt{N} H \big),
$$

where we have taken $u_s = -\eta_s(\tilde{g}_{\boldsymbol{x}_s}, 0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ whose norm is bounded by $\|u_s\|_2 = \eta_s \|\tilde{g}_{\boldsymbol{x}_s}\|_2 \leq \eta_s \sqrt{N} H$. The result now follows by substituting this inequality into (20) and bounding by 0 the negative terms. □

Proposition V.6. (Boundedness of the online estimates): For $T \in \mathbb{Z}_{\geq 1}$, let $\{f_t^1, \dots, f_t^N\}_{t=1}^T$ be convex functions on $\mathbb{R}^d$ with H-bounded subgradient sets and nonempty sets of minimizers. In addition, assume that the sets of minimizers are contained in a closed ball centered at the origin with some radius independent of T, and further assume that the functions $\{f_t^1, \dots, f_t^N\}_{t=1}^T$ are p-strongly convex in any open neighborhood of that ball. Let the sequence $\{\boldsymbol{x}_t = (x_t^1, \dots, x_t^N)\}_{t=1}^T$ be generated by the coordination algorithm (3) over a sequence of connected weighted graphs $\{\mathcal{G}_t\}_{t=1}^T$ under condition (6) and with $\eta_t = \frac{1}{p\,t}$, $a \in (2, \infty)$ and $\sigma \in (0, \sigma^*)$, where σ^* is defined in (7). Then there exists $D > 0$, independent of the time horizon T, such that $\|\boldsymbol{x}_t\|_2 \leq D$ for all $t \in \{1, \dots, T\}$.

The proof is omitted and will appear elsewhere.

Theorem IV.1 then is derived from Proposition V.5 assuming that the region of p-strong convexity of the functions $\{f_t^1, \dots, f_t^N\}_{t=1}^T$ is the ball centered at the origin of radius D, where D is given in Proposition V.6. Under this assumption, we have $p_t(\boldsymbol{u}, \boldsymbol{x}_t) = p$ for all $\boldsymbol{u} \in (\mathbb{R}^d)^N$ and $t \in \{1, \dots, T\}$, regardless of the time horizon T. Hence, the logarithmic regret bound (8) follows choosing $\eta_t = \frac{1}{\tilde{p}\,t}$ in (19), for any $\tilde{p} \in (0, p\,]$, because

$$
\tfrac{1}{\eta_t} - \tfrac{1}{\eta_{t-1}} - p_t(\boldsymbol{u}, \boldsymbol{x}_t) = \tilde{p}\,t - \tilde{p}\,(t-1) - p = \tilde{p} - p \leq 0,
$$

and also noting that $\sum_{t=1}^{T} \frac{1}{t} < 1 + \log T$. Figure 1 illustrates a particular case of Theorem IV.1.

VI. Conclusions

We have considered networked convex optimization problems where a team of agents generate local decisions over time that achieve sublinear regret with respect to the best fixed centralized decision in hindsight. We have proposed a class of distributed online algorithms that allow agents to fuse their local estimates and incorporate new information as it becomes available. Our algorithm design uses only first-order local information about the local objectives, in the form of subgradients, and only requires local communication of estimates among neighboring agents in a time-dependent connected graph. We have established the classical logarithmic agent regret bound under strong convexity, relying on a boundedness property of the trajectories as opposed to a projection onto a compact set. Future work will consider directed topologies and bounded interval connectivity, the convex case, and the characterization of the algorithm performance under noise.

Acknowledgments

This research was partially supported by NSF Award CMMI-1300272.

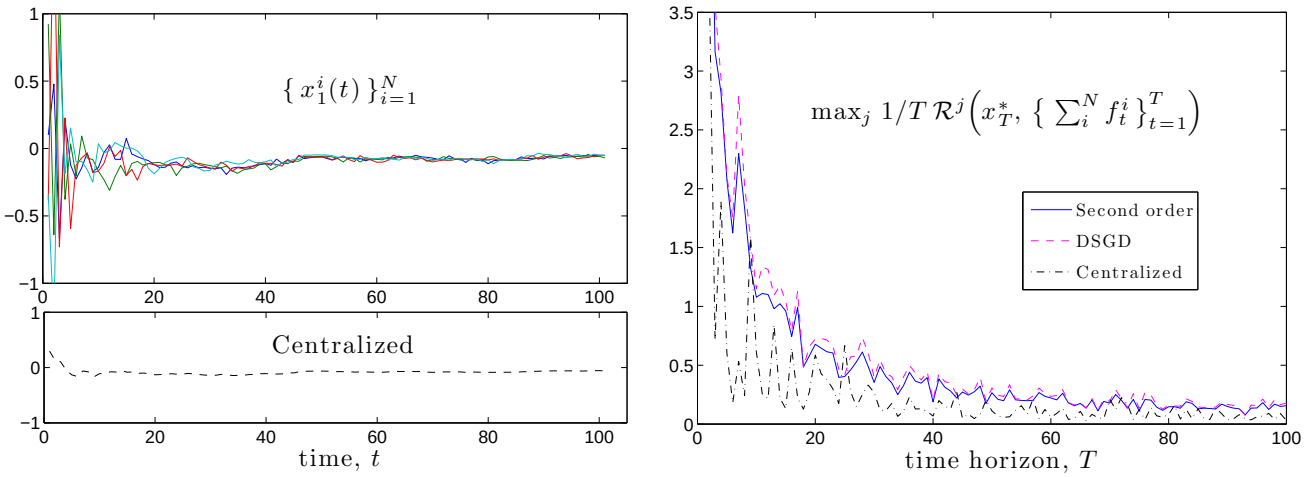
References

[1] D. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Athena Scientific, 1997.

[2] J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans, "Distributed asynchronous deterministic and stochastic gradient optimization algorithms," *IEEE Transactions on Automatic Control*, vol. 31, no. 9, pp. 803–812, 1986.

[3] N. A. Lynch, *Distributed Algorithms*. Morgan Kaufmann, 1997.

[4] A. Nedic and A. Ozdaglar, "Distributed subgradient methods for multi-agent optimization," *IEEE Transactions on Automatic Control*, vol. 54, no. 1, pp. 48–61, 2009.



(a) Agents' estimates evolution versus online centralized gradient descent

(b) Comparison of regrets for three online algorithms

Fig. 1: Evolution of the online second-order distributed algorithm (3) over a group of $N = 4$ agents communicating over the undirected graph with edge set $\mathcal{E} = \big\{\{1,2\},\{2,3\},\{3,4\},\{4,1\}\big\}$ and weights equal to 1. The local objective functions, defined on $\mathbb{R}^3$, are given by $f_t^i(x) = \max\big\{0,\, 1 - y_t^i\, x^\top w_t^i\big\} + 1/2\|x\|_2^2$, where the data points $w_t^i \in \mathbb{S}^2$, $y_t^i \in \{-1, 1\}$, for $i \in \{1,\dots,N\}$ and $t \in \{1,\dots,T\}$, are generated randomly. These functions are nonsmooth and 1-strongly convex. The initial condition $\boldsymbol{x}_1 \in (\mathbb{R}^3)^4$ is also randomly generated and we take $\boldsymbol{z}_1 = \mathbb{1}_4 \otimes \mathbb{1}_3$. The design parameters are $a = 3$, $\sigma = 0.1$ and $\eta_t = 1/t$. Plot (a) shows the evolution of the first coordinate of each agent's estimate versus the evolution of an online centralized gradient descent algorithm [10]. Although the local objective functions change with time, the agents' estimates achieve agreement under (3) thanks to the second-order consensus component and the fact that the learning rates decay with time. Plot (b) compares the temporal average of the maximum agent regret over increasing time horizons for three algorithms: our second-order distributed algorithm, the online centralized gradient descent algorithm, and the distributed subgradient descent (DSGD) algorithm proposed in [21]. The latter requires self-loops, so it has been implemented with the doubly stochastic matrix $\mathsf{A}^* = (\mathsf{D}_{\text{out}} + \mathrm{I}_N)^{-1}(\mathsf{A} + \mathrm{I}_N)$, where $\mathsf{D}_{\text{out}} = \operatorname{diag}\big(\mathsf{A}\mathbb{1}_N\big)$ is the out-degree matrix. The global optimal solution in hindsight, x_T^*, for each time horizon T, is computed offline with a centralized gradient descent algorithm running over 100 iterations with step size 0.01.

[5] B. Johansson, M. Rabi, and M. Johansson, "A randomized incremental subgradient method for distributed optimization in networked systems," *SIAM Journal on Control and Optimization*, vol. 20, no. 3, pp. 1157–1170, 2009.

[6] J. C. Duchi, A. Agarwal, and M. J. Wainwright, "Dual averaging for distributed optimization: convergence analysis and network scaling," *IEEE Transactions on Automatic Control*, vol. 57, no. 3, pp. 592–606, 2012.

[7] M. Zhu and S. Martínez, "On distributed convex optimization under inequality and equality constraints," *IEEE Transactions on Automatic Control*, vol. 57, no. 1, pp. 151–164, 2012.

[8] B. Gharesifard and J. Cortés, "Distributed continuous-time convex optimization on weight-balanced digraphs," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 781–786, 2014.

[9] D. P. Bertsekas, "Incremental gradient, subgradient, and proximal methods for convex optimization: a survey," in *Optimization for Machine Learning* (S. Sra, S. Nowozin, and S. J. Wright, eds.), pp. 85–120, Cambridge, MA: MIT Press, 2011.

[10] M. Zinkevich, "Online convex programming and generalized infinitesimal gradient ascent," in *Proceedings of the Twentieth International Conference on Machine Learning*, (Washighton, D.C.), pp. 928–936, 2003.

[11] N. Cesa-Bianchi and G. Lugosi, *Prediction, learning, and games*. Cambridge University Press, 2006.

[12] S. Shalev-Shwartz, *Online Learning and Online Convex Optimization*, vol. 12 of *Foundations and Trends in Machine Learning*. Now Publishers Inc, 2012.

[13] E. Hazan, "The convex optimization approach to regret minimization," in *Optimization for Machine Learning* (S. Sra, S. Nowozin, and S. J. Wright, eds.), pp. 287–304, Cambridge, MA: MIT Press, 2011.

[14] S. Shalev-Shwartz and Y. Singer, "Convex repeated games and fenchel duality," in *Advances in Neural Information Processing Systems* (B. Schölkopf, J. Platt, and T. Hoffman, eds.), vol. 19, (Cambridge, MA), MIT Press, 2007.

[15] S. Ross, *Interactive Learning for Sequential Decisions and Predictions*. PhD thesis, The Robotics Institute, Carnegie Mellon University, Pittsburgh, Pennsylvania, 2013.

[16] E. Hazan, A. Agarwal, and S. Kale, "Logarithmic regret algorithms for online convex optimization," *Machine Learning*, vol. 69, no. 2-3, pp. 169–192, 2007.

[17] A. T. Kalai and S. Vempala, "Efficient algorithms for online optimization," *Journal of Computer and System Sciences*, vol. 71, no. 3, pp. 291–307, 2005.

[18] H. Wang and A. Banerjee, "Online alternating direction method," in *International Conference on Machine Learning*, (Edinburgh, Scotland), pp. 1119–1126, July 2012.

[19] O. Dekel, R. Gilad-Bachrach, O. Shamir, and L. Xiao, "Optimal distributed online prediction using mini-batches," *Journal of Machine Learning Research*, vol. 13, no. Jan, pp. 165–202, 2012.

[20] M. Raginsky, N. Kiarashi, and R. Willett, "Decentralized online convex programming with local information," in *American Control Conference*, (San Francisco, CA), pp. 5363–5369, 2011.

[21] F. Yan, S. Sundaram, S. V. N. Vishwanathan, and Y. Qi, "Distributed autonomous online learning: regrets and intrinsic privacy-preserving properties," *IEEE Transactions on Knowledge and Data Engineering*, vol. 25, no. 11, pp. 2483–2493, 2013.

[22] S. Hosseini, A. Chapman, and M. Mesbahi, "Online distributed optimization via dual averaging," in *IEEE Conf. on Decision and Control*, (Florence, Italy), 2013.

[23] K. I. Tsianos and M. G. Rabbat, "Distributed strongly convex optimization," 2012. arXiv:1207.3031v2.

[24] J. Wang and N. Elia, "A control perspective for centralized and distributed convex optimization," in *IEEE Conf. on Decision and Control*, (Orlando, Florida), pp. 3800–3805, 2011.

[25] D. Mateos-Núñez and J. Cortés, "Noise-to-state stable distributed convex optimization on weight-balanced digraphs," in *IEEE Conf. on Decision and Control*, (Florence, Italy), pp. 2781–2786, 2013.
[26] Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course*, vol. 87 of *Applied Optimization*. Springer, 2004.
[27] F. Bullo, J. Cortés, and S. Martínez, *Distributed Control of Robotic Networks*. Applied Mathematics Series, Princeton University Press, 2009. Electronically available at http://coordinationbook.info.
[28] D. S. Bernstein, *Matrix Mathematics*. Princeton, NJ: Princeton University Press, 2005.

APPENDIX

Here we prove Proposition V.3. We start with a simple result from linear algebra whose proof is standard.

Lemma A.1. (Rank-one additions to the nullspace of a special class of matrices): Let $B \in \mathbb{R}^{n\times n}$ be a diagonalizable matrix with real eigenvalue-eigenvector pairs $\{(\lambda_i, u_i)\}_{i=1}^n$. Without loss of generality, assume $\mathcal{N}(B)$ is generated by the orthonormal vectors $\{u_i\}_{i=1}^k$. If $\mathcal{N}(B)\perp\mathcal{C}(B)$, then, for any $(\mu_1,\dots,\mu_k)\in\mathbb{R}^k$,

$$B + \sum_{i=1}^{k} \mu_i u_i u_i^\top = S_B(D_B + D_\mu)S_B^{-1},$$

where $S_B = [u_1 \cdots u_n]$, $D_B = \mathrm{diag}(0,...,0,\lambda_{k+1},...,\lambda_n) \in \mathbb{R}^{n\times n}$ correspond to the eigenvalue decomposition $B = S_B D_B S_B^{-1}$, and $D_\mu := \mathrm{diag}(\mu_1,...,\mu_k,0,..,0)\in\mathbb{R}^{n\times n}$.

Our next result describes the properties of the matrix that defines the second-order linear consensus algorithm (9).

Lemma A.2. (Properties of the second-order consensus matrix): Let $\mathcal{G}$ be a connected graph with associated Laplacian matrix $\mathsf{L}\in\mathbb{R}^{N\times N}$. The following properties hold regarding the matrix $\mathbb{L}$ defined in (12) for $a\in\mathbb{R}_{>0}$:

(i) The matrix $\mathbb{L}$ has eigenvalues

$$\{(-\tfrac{a}{2} \pm h(a))\lambda \,:\, \lambda\in\mathrm{spec}(\mathsf{L})\},$$

and $\mathrm{rank}(\mathbb{L}) = 2(N-1)d$. In addition, if $a\in\mathbb{R}_{>0}$, then $\mathbb{L}$ is semistable and, if $a\in(2,\infty)$, then all its eigenvalues are real.

(ii) The nullspace of $\mathbb{L}$ is given by

$$\mathcal{N}(\mathbb{L}) = \{(\mathbb{1}_N\otimes b, \mathbb{1}_N\otimes c) \,:\, b,c\in\mathbb{R}^d\},$$

and is orthogonal to $\mathcal{C}(\mathbb{L})$.

(iii) If $a\neq 2$, then $\mathbb{L}$ is diagonalizable, $\mathbb{L} = S_{\mathbb{L}} D_{\mathbb{L}} S_{\mathbb{L}}^{-1}$, where $S_{\mathbb{L}} = S_G\otimes S_{\mathsf{L}}\otimes \mathrm{I}_d$ and

$$D_{\mathbb{L}} = \mathrm{diag}\big(-\tfrac{a}{2} + h(a), -\tfrac{a}{2} - h(a)\big)\otimes D_{\mathsf{L}}\otimes \mathrm{I}_d.$$

(iv) For any $\sigma\in\mathbb{R}$, $a>2$, the matrix $\hat{\mathbb{L}} := \sigma\mathbb{L} - \mathrm{I}_2\otimes\mathbf{M}$ is diagonalizable, $\hat{\mathbb{L}} = S_{\mathbb{L}} D_{\hat{\mathbb{L}}} S_{\mathbb{L}}^{-1}$, where

$$D_{\hat{\mathbb{L}}} = \sigma D_{\mathbb{L}} - \mathrm{I}_2\otimes\big(e_{1|N}\, e_{1|N}^\top\big)\otimes \mathrm{I}_d.$$

Proof. We start by noting that the matrix

$$G := \begin{bmatrix} -a & -1 \\ 1 & 0 \end{bmatrix}$$

has eigenvalues $-\frac{a}{2}\pm h(a)$. Since $\mathcal{G}$ is connected, then $\mathrm{rank}(\mathsf{L}) = N-1$ and hence 0 is an eigenvalue of $\mathbf{L}$ with multiplicity d. Fact (i) follows by noting that the eigenvalues of a Kronecker product are the product of the eigenvalues of the individual matrices, cf. [28, Prop. 7.1.10]. Regarding fact (ii), the expression for the nullspace follows by noting that $\dim(\mathcal{N}(\mathbb{L})) = 2d$ and $\mathbb{L}(\mathbb{1}_N\otimes b, \mathbb{1}_N\otimes c)^\top = 0$ for any $b,c\in\mathbb{R}^d$. The orthogonality property $\mathcal{N}(\mathbb{L})\perp\mathcal{C}(\mathbb{L})$ is a consequence of the fact that $v^\top\mathbb{L} = 0$ for any $v\in\mathcal{N}(\mathbb{L})$. Regarding fact (iii), note that if $a\neq 2$, then the eigenvalues of G are different and $G = S_G D_G S_G^{-1}$ with $D_G := \mathrm{diag}(-\frac{a}{2} + h(a), -\frac{a}{2} - h(a))$. Additionally,

$$\begin{aligned}\mathbf{L} &= \mathsf{L}\otimes \mathrm{I}_d = S_{\mathsf{L}} D_{\mathsf{L}} S_{\mathsf{L}}^{-1}\otimes \mathrm{I}_d \\ &= (S_{\mathsf{L}}\otimes \mathrm{I}_d)(D_{\mathsf{L}}\otimes \mathrm{I}_d)(S_{\mathsf{L}}\otimes \mathrm{I}_d)^{-1},\end{aligned}$$

Therefore, we can write

$$\begin{aligned}G\otimes\mathbf{L} &= S_G D_G S_G^{-1}\otimes(S_{\mathsf{L}}\otimes \mathrm{I}_d)(D_{\mathsf{L}}\otimes \mathrm{I}_d)(S_{\mathsf{L}}\otimes \mathrm{I}_d)^{-1} \\ &= (S_G\otimes S_{\mathsf{L}}\otimes \mathrm{I}_d)(D_G\otimes D_{\mathsf{L}}\otimes \mathrm{I}_d)(S_G\otimes S_{\mathsf{L}}\otimes \mathrm{I}_d)^{-1}.\end{aligned}$$

Finally, to show fact (iv), we resort to Lemma A.1. Since $a>2$, the matrix $\sigma\mathbb{L}$ is diagonalizable by (iii). Also, according to (i), its eigenvalues are real and its nullspace has dimension $k = 2d$. In addition, $\mathcal{N}(\mathbb{L})\perp\mathcal{C}(\mathbb{L})$ by (ii). To apply Lemma A.1, we select the orthonormal basis $\{u_l\}_{l=1}^{2d}$ of $\mathcal{N}(\mathbb{L})$ as

$$\big\{\tfrac{1}{\sqrt{N}}\, e_{i|2}\otimes\mathbb{1}_N\otimes e_{j|d} \,:\, 1\le i\le 2,\ 1\le j\le d\big\},$$

and $\mu_l = -1$ for all $l\in\{1,\dots,2d\}$. It should be noted that, with respect to the eigenvalue decomposition of $\mathbb{L}$ presented in (iii), the orthonormal eigenvectors above correspond to the columns $1,\dots,d,Nd+1,\dots,Nd+d$ of $S_{\mathbb{L}} = S_G\otimes S_{\mathsf{L}}$. Since $\sum_{l=1}^{2d}\mu_l\, u_l u_l^\top = \sum_{l=1}^{2d} -u_l u_l^\top = -\mathrm{I}_2\otimes\mathbf{M}$, the result follows noting that $D_\mu\in\mathbb{R}^{2Nd\times 2Nd}$ in Lemma A.1 is

$$D_\mu = \mathrm{I}_2\otimes\mathrm{diag}(-\mathbb{1}_d, 0_{(N-1)d}) = -\mathrm{I}_2\otimes\big(e_{1|N}\, e_{1|N}^\top\big)\otimes \mathrm{I}_d,$$

whose zero diagonal entries are located according with the rearrangement of eigenvectors described above. □

The previous results allow us to prove Proposition V.3.

Proof of Proposition V.3. We start by noting that

$$\hat{\mathbf{L}}_{\mathcal{K}} + \sigma\mathbb{L} = \mathrm{I}_{2Nd} + \hat{\mathbb{L}},$$

where $\hat{\mathbb{L}}$ is defined in Lemma A.2(iv). Therefore, $\hat{\mathbf{L}}_{\mathcal{K}} + \sigma\mathbb{L}$ is diagonalizable with the same eigenvectors as $\hat{\mathbb{L}}$ and eigenvalues shifted by the addition of unity. Facts (i) and (ii) now follow from Lemma A.2(i) and (iv). Regarding matrix convergence, since the eigenvalues are real by Lemma A.2(i), all of them lie strictly inside the unit circle if

$$-1 < 1 + \sigma\big(-\tfrac{a}{2}\pm h(a)\big)\lambda < 1$$

for all $\lambda\in\mathrm{spec}(\mathsf{L})\backslash\{0\}$. The right inequality is automatically satisfied for $\sigma>0$ because $-\frac{a}{2} - h(a) < -\frac{a}{2} + h(a) < 0$ and $\mathrm{spec}(\mathsf{L})\backslash\{0\}\subset(0,\infty)$. For the left inequality, note that

$$\sigma\big(-\tfrac{a}{2} - h(a)\big)\lambda > -2 \;\Leftrightarrow\; \sigma < \frac{2}{(\frac{a}{2} + h(a))\lambda},$$

for each $\lambda\in\mathrm{spec}(\mathsf{L})\backslash\{0\}$, and fact (iii) follows. □