

# Robust estimation and aggregation of ocean internal wave parameters using Lagrangian drifters

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**Abstract**—This paper considers a group of drogues whose objective is to estimate the physical parameters that determine the dynamics of ocean nonlinear internal waves. While underwater, individual drogues do not have access to absolute position information and only rely on inter-drogue measurements. Building on this data and the structure of the drogue dynamics under the flow induced by an internal wave, this paper improves in three different ways upon our previous strategy, termed `PARAMETER DETERMINATION STRATEGY`, which determines all wave parameters. The first is by showing that with sufficiently fast sampling, the extended algorithm determines the wave parameters. The second is by showing its applicability to situations where two internal waves are present simultaneously. With the extended algorithm, multiple estimates are calculated for each parameter. Thus, with the presence of noisy measurements, the third contribution is a method to aggregate parameter estimates to reduce the error. Simulations illustrate the algorithm performance under noisy measurements, the effect that the initial drogue locations has on robustness, and the effectiveness of parameter aggregation.

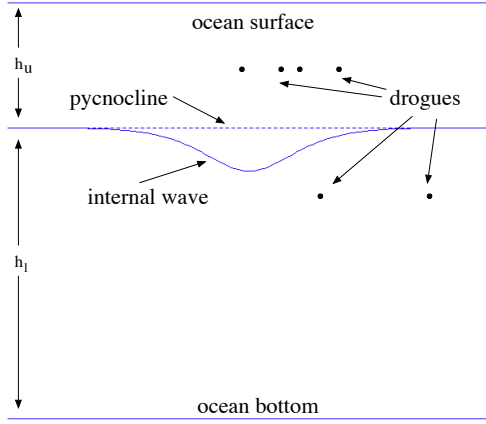
## I. INTRODUCTION

Internal waves travel within a fluid, rather than on its surface. We consider ones that are a moving oscillation in the boundary surface between two layers of a stratified fluid. A continuously stratified fluid (such as the ocean) can be modeled as a two-layer fluid, where the interface, called pycnocline, is the surface of constant density where the vertical rate of change in density is largest. This class of internal waves can be broadly categorized into linear and nonlinear, and we focus here on the latter. These waves have large amplitudes (relative to the depth of the water column), allowing them to be an agent of transport of small oceanic life. The nonlinear waves considered here are solitons, which are stable, solitary peaks (or troughs) that propagate along the pycnocline. Given that pycnoclines are typically deep below the ocean surface, it is difficult to measure them without placing sensors close by. We address this issue by using a group of drogues capable of drifting underwater near the internal wave’s interface. A drogue is a sensor-laden, Lagrangian drifter able to actuate its depth by changing its buoyancy. While underwater, drogues are subject to the flow induced by the motion of the internal wave and do not have access to exact location information. Figure 1 presents a schematic of the problem setup. Our basic premise is that drogues should be able to extract information from their inter-drogue measurements to characterize the internal wave.

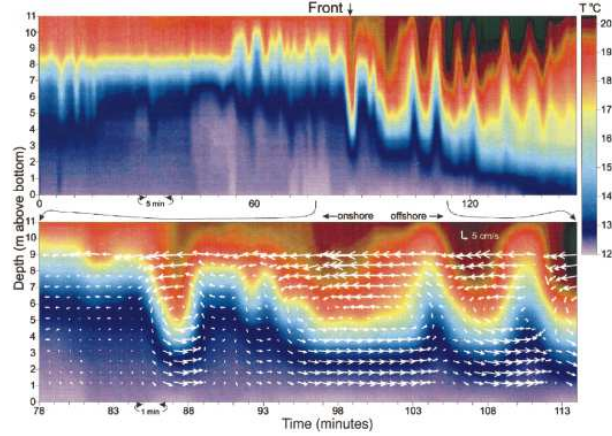
*Literature review:* Internal waves are associated with high concentrations of various types of planktonic organisms and small fishes [2], [3], as well as an agent of larval transport [1]. This makes their study important to

oceanographers, see e.g. [4], [5], [6]. In particular, striping of low/high densities in plankton can be well explained by small amplitude, linear internal waves [4]. However, nonlinear waves are needed to account for the advection required for larval transport [1]. Many models exist for nonlinear waves [7], [8] to account for the wide variety of conditions and bathymetries found in the ocean. Scientists widely use drogues drifting passively as monitoring platforms to gather relevant ocean data [9], [10], [11]. The use of autonomous underwater vehicles to detect and characterize internal waves is a relatively new approach. Whereas previous works use ocean measurements such as conductivity, temperature, pressure data [12], [13] or vertical flow velocity [14] to detect and analyze internal waves, our approach follows [15], [16] in using inter-vehicles measurements. This work is the conference version of our previous work [16], but contains several novel contributions of its own, including extending the applicability of the proposed algorithm to multiple instants of times, dealing with the estimation of two nonlinear waves simultaneously, and aggregating multiple parameter estimates for increased accuracy. Since the parameter distributions are implicitly defined and non-Gaussian, we adopt a mixture distribution approach to express the distribution as a sum of simpler distributions [17]. Finally, our work is also connected to the increasing literature that deals with cooperative networks of agents estimating spatial natural phenomena, including ocean [18], [19], [20], river [21], and hurricane sampling [22].

*Statement of contributions:* We consider the problem of estimating the physical parameters of a nonlinear internal wave that is propagating horizontally. A group of underwater Lagrangian drifters are subjected to the flow induced by the internal wave and can only measure inter-drogue distances and distance derivatives. Because the drogues only have access to these relative measurements, they must rely on the presence of other drogues to achieve their task. The benefit obtained here by ‘the power of many’ in the estimation of the ocean flow field is a key feature of the paper. Our starting point is the `PARAMETER DETERMINATION STRATEGY` introduced in the journal version [16] of this work. This algorithm is run on the drogues using only relative measurements and is capable of determining all of the internal wave parameters. Here, we extend its domain of applicability and improve the correctness result by showing that the algorithm can determine the wave parameters with noiseless inter-drogue measurements sampled sufficiently fast, as opposed to in continuous time. We also show that the method can be



(a) Schematic of drogues and internal wave



(b) Thermal fluctuations induced by an internal wave

Fig. 1. For an ocean nonlinear internal wave, (a) shows its spatial structure at a fixed instant of time whereas (b) shows its temporal structure at a fixed horizontal location. In (a), one can see a vertical cross-section of the ocean perpendicular to the wave propagation direction. A group of drogues float at constant depths (but not necessarily along a straight line) and do not have access to exact location information. Our objective is to provide drogues with mechanisms that rely only on the relative distances between them to determine the parameters that uniquely define the internal wave. In (b), one can see temperature and vertical/onshore-offshore current vectors data taken from a train of nonlinear soliton internal waves about one kilometer off the coast of La Jolla, CA on July 3, 1996. The bottom figure is a zoom-in of the top figure. Figure (b) is © (1999) by the Association for the Sciences of Limnology and Oceanography, Inc., see [1] for additional information.

extended to the case where there are two internal waves present. Because with our extension multiple estimates are now available for each parameter, we also describe a method of aggregating individual parameters to improve the accuracy of the estimation. Simulations illustrate the benefit of parameter aggregation in reducing the final estimation error.

## II. PRELIMINARIES

This section presents some notation and the model of nonlinear internal waves used. Let  $\mathbb{R}$ ,  $\mathbb{R}_{>0}$ , and  $\mathbb{R}_{\geq 0}$  denote the set of all, positive, and non-negative real numbers, respectively and let  $\mathbb{Z}_{\geq 1}$  be the set of all positive integers. Given independent random variables  $x_1$  and  $x_2$  with mean  $E[x_1] = E[x_2] = \mu$  and variances  $\text{Var}[x_1] = \sigma_1^2$ ,  $\text{Var}[x_2] = \sigma_2^2$ , consider the optimal aggregating function  $\text{OptAgg}$  by

$$\text{OptAgg}(x_1, \sigma_1^2, x_2, \sigma_2^2) = \left( \frac{\frac{x_1}{\sigma_1^2} + \frac{x_2}{\sigma_2^2}}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}}, \frac{1}{\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}} \right).$$

The first component of  $\text{OptAgg}$  is the new random variable and the second one its variance. This is the convex combination of  $x_1$  and  $x_2$  with the smallest variance.

### A. Modeling of nonlinear internal waves

This section describes a model of nonlinear internal waves following [8], [7]. As Figure 1 shows, an internal wave is a wave that travels beneath the surface of the ocean, along a pycnocline, which is a surface of constant water density. Before introducing the wave model, we first specify the reference frame  $\Sigma = (\mathbf{p}, \{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\})$  as follows: the origin  $p_g$  is an arbitrary point at the ocean surface; the vector  $\mathbf{e}_x$  corresponds to the direction of wave propagation, which we assume parallel to the ocean bottom, and  $\mathbf{e}_z$  is perpendicular to the ocean bottom, pointing from bottom to surface. The coordinates induced by  $\Sigma$  are denoted by  $\{x, y, z\}$ .

A classical equation used to model weakly nonlinear long internal waves is the Korteweg-de Vries (KdV) equation,

$$\frac{\partial \eta}{\partial t} - \frac{3}{2}c \frac{h_l - h_u}{h_u h_l} \eta \frac{\partial \eta}{\partial x} + \frac{1}{6}ch_u h_l \frac{\partial^3 \eta}{\partial x^3} = 0, \quad (1)$$

where  $\eta$  is the distance that the internal wave is displacing the pycnocline,  $c = \sqrt{g \frac{|\rho_l - \rho_u|}{\rho_l} \frac{h_u h_l}{h_u + h_l}}$ ,  $\rho_u$ ,  $h_u$  and  $\rho_l$ ,  $h_l$  are the density and depth of the upper and lower layers, respectively, and  $g$  is the acceleration due to gravity. The stable soliton solution to (1) is [7]

$$\eta(x, t) = A \text{sech}^2(k(x - \chi_0) - \omega t).$$

The wave's amplitude, wavenumber, and frequency are

$$A = -\frac{2Ch_u h_l}{h_l - h_u}, \quad k = \frac{1}{2} \sqrt{\frac{6C}{ch_u h_l}}, \quad \omega = \frac{1}{2} \sqrt{\frac{6C}{ch_u h_l}} C,$$

and  $C = \frac{\omega}{k}$  is the wave celerity and  $\chi_0$  is the initial location of the center of the wave. As the wave propagates, it induces motion in the nearby water. The model assumes that the vertical velocity varies linearly with depth. Coupled with the conservation of mass law for an incompressible fluid yields the horizontal  $u_u$  and vertical  $v_u$  velocities of the upper layer,

$$u_u(x, t) = -\frac{2CA}{h_u} \text{sech}^2(k(x - \chi_0) - \omega t),$$

$$v_u(x, z, t) = \frac{2\omega Az}{h_u} \text{sech}^2(k(x - \chi_0) - \omega t) \cdot \tanh(k(x - \chi_0) - \omega t),$$

and the horizontal  $u_l$  and vertical  $v_l$  velocities of the lower layer

$$u_l(x, t) = -\frac{h_u}{h_l} u_u(x, t), \quad v_l(x, z, t) = \frac{h_u(h_u + h_l - z)}{zh_l} v_u(x, t).$$

For convenience, we define the upper and lower velocity amplitudes as  $B_u = -\frac{2CA}{h_u}$  and  $B_l = \frac{2CA}{h_l}$ . Motivated by

practical considerations, we assume there exists a closed and bounded interval with the lower bound strictly positive for each wave parameter that the parameter is guaranteed to fall within. We refer to a given parameter's bounds with subscripts min and max.

### III. PROBLEM STATEMENT

This section presents the model for the drogue drifters and their interaction with the nonlinear internal wave, and the formal statement of the problem of interest.

#### A. Drogue model

A drogue is a submersible buoy that can drift in the ocean, unattached to the ocean floor or a boat, and is able to change its depth in the water by controlling its buoyancy. While underwater, a drogue can measure the relative distance, the distance derivative, and orientation to other drogues through sensing (e.g., via acoustic or optical sensors and an onboard compass). A drogue can also measure its depth. However, it does not have access to absolute position because GPS is unavailable underwater. Consider a group of  $N$  drogues, where each has a local reference frame aligned with the wave's reference frame. This assumption is merely for ease of exposition (our previous work [15] shows how each drogue  $i$  can determine the angle  $\theta_i$  between its local reference frame and the wave's reference frame using relative distance and distance derivative measurements). We make the simplifying assumption that the drogues' dynamics are Lagrangian, i.e., the drogue's velocity is equal to ocean's velocity at its current location. Furthermore, we assume that the drogues maintain the same prescribed depth by means of buoyancy control. Thus, the dynamics of drogue  $i \in \{1, \dots, N\}$  in the upper layer is  $\dot{\mathbf{p}}_i = (\dot{x}_i, \dot{y}_i, \dot{z}_i) = (u_u(x, t), 0, 0)$  and can be similarly defined for drogues in the lower layer. Drogue  $i$  senses inter-drogue measurements with the  $M$  closest neighbors. For each neighbor  $j$ , drogue  $i$  has access to

$$\mathbf{d}_{i,j} = (d_{i,j}^x, d_{i,j}^y, 0) = \mathbf{x}_j - \mathbf{x}_i, \quad \dot{\mathbf{d}}_{i,j} = (\dot{d}_{i,j}^x, 0, 0) = \dot{\mathbf{x}}_j - \dot{\mathbf{x}}_i.$$

We assume the drogues have continuous access to these quantities. In Section IV, we elaborate on the fact that a large enough, finite sampling rate will also produce noiseless parameter estimates. Since the internal wave causes no motion in the  $y$ -direction (due to choice of coordinates), we ease notation by letting  $d_{i,j} = d_{i,j}^x$  and  $\dot{d}_{i,j} = \dot{d}_{i,j}^x$ .

#### B. Problem description

A team of  $N$  drogues is deployed in the ocean and their motion is governed by an internal wave. The drogues may control their depth through buoyancy changes, and each one can measure the relative distance and orientation to the closest  $M$  drogues in their own coordinate frame. Our objective is to design an algorithm that allows the drogues to collectively determine the physical parameters  $C$ ,  $\frac{|\rho_u - \rho_l|}{\rho_l}$ ,  $h_u$ , and  $h_l$  that define the internal wave.

### IV. PARAMETER DETERMINATION STRATEGY

In this section, we extend the strategy proposed in [16], termed the PARAMETER DETERMINATION STRATEGY, to estimate the nonlinear wave parameters. The following informal rationale describes the basic idea behind its design.

*[Rationale]: The strategy for determining the physical parameters that define the internal wave are based on first determining the phase of the wave relative to the drogues at some time. Our method leverages the fact that, when the crest of the wave is directly between two drogues, their inter-drogue distance derivative momentarily becomes zero and the drogues can then determine the phase. Using this insight, one can create equations between inter-drogue measurements and the parameters of interest. The crux of the analysis is to ensure that only the true set of parameters solve the constructed set of equations.*

Our discussion below extends the range of applicability of the algorithm by considering times when the inter-drogue distance derivative is sufficiently close to zero, rather than exactly zero. The basic algorithm methodology remains the same, yet the framework and the results must be extended because, among other things, the specific set of constructed equations used to determine the parameters differ. The algorithm requires the capability for measuring both inter-drogue distance and its derivative. It is written in terms of drogue  $i$  using measured inter-drogue data between itself and nearest neighbors with identities  $j_1, j_2, j_3, j_4, j_5$ , and  $j_6$ . Before introducing it, we comment briefly on some assumptions on drogue locations that make the presentation easier.

*Remark 4.1: (Assumptions on drogue locations)* For concreteness in the algorithm's presentation, we make the assumption that drogues  $i, j_1, j_2, j_3, j_4$  are in the same ocean layer. The algorithm also requires at least one drogue in the lower layer and one in the upper layer, we assume are drogues  $j_5$  and  $j_6$ , respectively. •

#### A. Determination of the wavenumber

Because drogues do not have absolute measurements, we write their dynamics in terms of the distance between them. To completely describe the drogues evolution, we also need to add the state  $v_i$ , which is the position of the wave relative to drogue  $i$ ,  $v_i = k(x - \chi_0) - \omega t$ . Thus, the dynamics are

$$\dot{d}_{i,j_m} = B(\text{sech}^2(kd_{i,j_m} + v_i) - \text{sech}^2(v_i)), \quad \forall m \neq i \quad (2a)$$

$$\dot{v}_i = Bk \text{sech}^2(v_i) - \omega. \quad (2b)$$

The relative phase  $v_i$  is unobservable, however, at times when inter-drogue distance derivatives momentarily vanish, one can gain insight, as we show next. At all other times, there exists an implicit function that describes the relative phase.

*Lemma 4.2: (Relative wave position when distance derivative vanishes)* For three drogues  $i, j_1$  and  $j_2$  at initial positions  $x_i(0) \neq x_{j_1}(0) \neq x_{j_2}(0)$ , if  $x_i(0), x_j(0) > \chi_0$ , then there exists time  $t_{\text{cr}} > 0$  when  $\dot{d}_{i,j}(t_{\text{cr}}) = 0$  and  $v_i(t_{\text{cr}}) = -\frac{kd_{i,j_1}(t_{\text{cr}})}{2}$ . Furthermore, for all  $t > 0$ , there



exists an implicit function for  $v_i(k, d_{i,j_1}, d_{i,j_2}, \dot{d}_{i,j_1}, \dot{d}_{i,j_2})$  determined by the equation

$$\frac{\dot{d}_{i,j_1}}{\dot{d}_{i,j_2}} - \frac{\operatorname{sech}^2(kd_{i,j_1} + v_i) - \operatorname{sech}^2(v_i)}{\operatorname{sech}^2(kd_{i,j_2} + v_i) - \operatorname{sech}^2(v_i)} = 0. \quad (3)$$

From (2), the dynamics of an inter-drogue distance between drogues  $i$  and  $j$  in the wave propagation direction contain the unknown parameters  $B$  and  $k$ , as well as unmeasurable state  $v_i$ . However, using Lemma 4.2 to write the ratio of two equations in (2a) for  $i,j_3$  and  $i,j_4$  specifically at the  $t_{\text{cr}}$  when  $\dot{d}_{i,j_1}(t_{\text{cr}}) = 0$ , one gets

$$\frac{\dot{d}_{i,j_3}}{\dot{d}_{i,j_4}} - \frac{\operatorname{sech}^2(k(d_{i,j_3} - \frac{d_{i,j_1}}{2})) - \operatorname{sech}^2(k\frac{d_{i,j_1}}{2})}{\operatorname{sech}^2(k(d_{i,j_4} - \frac{d_{i,j_1}}{2})) - \operatorname{sech}^2(k\frac{d_{i,j_1}}{2})} = 0,$$

and, more generally, for all times,

$$\frac{\dot{d}_{i,j_3}}{\dot{d}_{i,j_4}} - \frac{\operatorname{sech}^2(kd_{i,j_3} - v_i) - \operatorname{sech}^2(v_i)}{\operatorname{sech}^2(kd_{i,j_4} - v_i) - \operatorname{sech}^2(v_i)} = 0,$$

which is now only a function of the unknown parameter  $k$  (because  $v_i = v_i(k, d_{i,j_1}, d_{i,j_2}, \dot{d}_{i,j_1}, \dot{d}_{i,j_2})$ , as implicitly defined in Lemma 4.2). We now wish to show that only the actual value of  $k$  satisfies this equation. With this in mind, we define the function  $f$  as

$$f(k, d_{i,j_3}, d_{i,j_4}, \dot{d}_{i,j_3}, \dot{d}_{i,j_4}, v_i(k, d_{i,j_1}, d_{i,j_2}, \dot{d}_{i,j_1}, \dot{d}_{i,j_2})) = \frac{\dot{d}_{i,j_3}}{\dot{d}_{i,j_4}} - \frac{\operatorname{sech}^2(kd_{i,j_3} - v_i) - \operatorname{sech}^2(v_i)}{\operatorname{sech}^2(kd_{i,j_4} - v_i) - \operatorname{sech}^2(v_i)} \quad (4)$$

and examine the number of roots in the next result.

**Lemma 4.3: (Uniqueness of spatial wavenumber)** Given noiseless measurements of  $d_{i,j}(t)$  and  $\dot{d}_{i,j}(t)$ , for  $j \in \{j_1, j_2, j_3, j_4\}$ , where  $t$  is sufficiently close to  $t_{\text{cr}}$ , the time when  $\dot{d}_{i,j_1}(t_{\text{cr}}) = 0$ . If  $d_{i,j_1}(t)$  is sufficiently small, then  $\kappa = k$  is the only root to (4).

### B. Correctness analysis

Once  $k$  has been determined, we wish to leverage it to calculate other parameters. This is what the PARAMETER DETERMINATION STRATEGY accomplishes using the dynamics that defines the internal wave. The strategy is formally presented in Algorithm 1. We also refer back to Remark 4.1, which explains the assumptions on drogue locations that are needed to make the algorithm more concrete.

The next result establishes the algorithm correctness. Its proof follows from Lemma 4.3, the form of the inter-drogue distance derivative equation, and algebraic relations between parameters in the nonlinear soliton model in Section II-A.

**Proposition 4.4: (Correctness of PARAMETER DETERMINATION STRATEGY)** At times  $t$  sufficiently close to  $t_{\text{cr}}$  when  $\dot{d}_{i,j_1} = 0$  and if  $d_{i,j_1}(t)$  is sufficiently small, then given noiseless knowledge of  $d_{i,j_m}(t)$  and  $\dot{d}_{i,j_m}(t)$  for all  $m \in \{j_1, j_2, j_3, j_4, j_5, j_6\}$ , the PARAMETER DETERMINATION STRATEGY, presented in Algorithm 1, can be used to determine all of the internal wave physical parameters.

Note that Step 2 can be solved using a variety of root-finding methods. Since  $f$  is a monotonic function in  $k$ , a gradient descent method would be sufficient, for example.

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### Algorithm 1: PARAMETER DETERMINATION STRATEGY

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- 1 Set  $t$  such that  $\dot{d}_{i,j_1}(t)$  sufficiently close to 0
  - 2  $k$  uniquely solves  $f(k, d_{i,j_3}, d_{i,j_4}, \dot{d}_{i,j_3}, \dot{d}_{i,j_4}, v_i(k, d_{i,j_1}, d_{i,j_2}, \dot{d}_{i,j_1}, \dot{d}_{i,j_2})) = 0$
  - 3 Set  $v_i(t) = v_i(k, d_{i,j_1}(t), d_{i,j_2}(t), \dot{d}_{i,j_1}(t), \dot{d}_{i,j_2}(t))$
  - 4 Set  $B_l = \frac{\dot{d}_{i,j_5}(t)}{\operatorname{sech}^2(kd_{i,j_5}(t) + v_i(t)) - \operatorname{sech}^2(v_i(t))}$
  - 5 Set  $B_u = \frac{\dot{d}_{i,j_6}(t) + B_l \operatorname{sech}^2(v_i(t))}{\operatorname{sech}^2(kd_{i,j_6}(t) + v_i(t))}$
  - 6 Set  $t_2$  not equal to  $t$
  - 7 Set  $v_i(t_2) = v_i(k, d_{i,j_1}(t_2), d_{i,j_2}(t_2), \dot{d}_{i,j_1}(t_2), \dot{d}_{i,j_2}(t_2))$
  - 8 Set  $\omega = \frac{k(d_{i,j_1}(t_2) - d_{i,j_1}(t)) - v_i(t_2) + v_i(t)}{t_2 - t}$  and  $C = \frac{\omega}{k}$
  - 9 Set  $h_u = \frac{h_{\text{ocean}}}{1 - \frac{B_u}{B_l}}$  and  $h_l = h_{\text{ocean}} - h_u$
  - 10 Set  $c = \frac{3C}{2k^2 h_u h_l}$  and  $\frac{|\rho_l - \rho_u|}{\rho_l} = \frac{c^2 h_{\text{ocean}}}{g h_u h_l}$
- 

**Remark 4.5 (Robustness against noise):** Here we comment on the algorithm performance when errors are present, specifically noise in the sensor measurements. We assume that this noise is unbiased, Gaussian, and that noise at different time instances and for different measurements are uncorrelated. The fact that all the functions that appear in the equations employed in Algorithm 1 have a continuous dependence on the variables makes the PARAMETER DETERMINATION STRATEGY naturally robust against errors, in the sense that the estimated parameters are still unique and remain close to the true parameters for small enough errors. Figure 2 illustrates the algorithm robustness for three different initial drogue configurations. Note that our method has a linear relationship on a log-log plot between relative errors in measurements and relative errors in the wavenumber. Three drogues are located at 0, 1, and 2 meters and the fourth drogue's position varies; in three trials it is located at 10, 100, and 200 meters. As the largest inter-drogue distance grows, the algorithm robustness improves. •

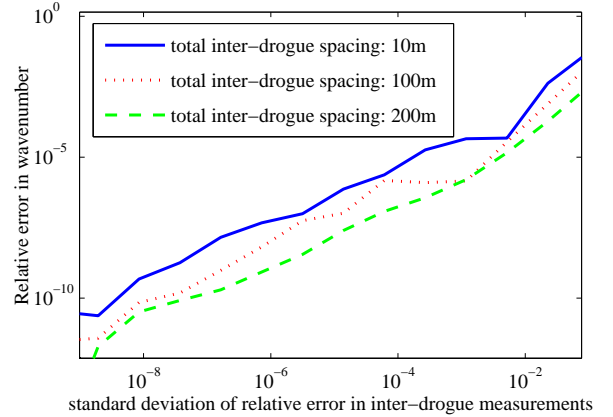


Fig. 2. Relative error in estimates of the wavenumber as a function of the relative errors in the inter-drogue distance and its derivative measurements for several different initial drogue locations. The plot shows the effect of changing the largest inter-drogue distance on the performance of PARAMETER DETERMINATION STRATEGY. The true value of the wavenumber is  $k = .0065 \frac{1}{m}$  and the first three drogues were located at 0, 1, 2 meters. The fourth drogue was at 10, 100, and 200 meters. Each point is the average of 3000 runs. Robustness improves as the largest inter-drogue distance grows.

### C. Extension to two nonlinear waves

In this section, we discuss how our previous algorithm design and analysis can be extended to situations where two nonlinear internal waves are simultaneously present. The basic idea relies on choosing a coordinate system such that, in one of the directions, the drogues only feel the effect of one of the two waves. One can apply the extended PARAMETER DETERMINATION STRATEGY in the direction where only one wave is felt to solve for the parameters of that wave. Once that wave has been characterized, its effect can be removed from the drogue's measurements, allowing one to again apply the PARAMETER DETERMINATION STRATEGY to determine the other wave's parameters.

Consider the situation where drogues are floating in the presence of two nonlinear internal waves, each with its own set of wave parameters and propagation directions. We assume that these propagation directions are known a priori. Choosing a horizontal  $x - y$  coordinate system with the  $x$ -direction aligned with wave 1's direction makes the angle between the  $x$ -direction and wave 1 equal to zero, i.e.,  $\theta_1 = 0$ . Summing each wave's individual effect yields the following planar dynamics for drogue  $i$ ,

$$\begin{aligned} \dot{x}_i &= \sum_{n=1}^2 -\frac{2C_n A_n}{h_u} \operatorname{sech}^2(k_n(x_i \cos(\theta_n) + y_i \sin(\theta_n) - \chi_{0n} - \omega_n t) \cos(\theta_n), \\ \dot{y}_i &= -\frac{2C_2 A_2}{h_u} \operatorname{sech}^2(k_2(x_i \cos(\theta_2) + y_i \sin(\theta_2) - \chi_{02} - \omega_2 t) \sin(\theta_2). \end{aligned}$$

Note that in the  $y$  direction, the drogue only feels the effect of the second wave. Writing the dynamics for the  $y$  direction in terms of inter-drogue distances between drogue  $i$  and  $j$  as well as the relative phase  $v_i$  gives

$$\begin{aligned} \dot{d}_{i,j_1}^y &= -\frac{2C_2 A_2}{h_u} \sin(\theta_2) \cdot \\ &(\operatorname{sech}^2(k_2(d_{i,j}^x \cos(\theta_2) + d_{i,j}^y \sin(\theta_2) + v_i^2) - \omega_2 t) - \operatorname{sech}^2(v_i^2)), \end{aligned}$$

where  $v_i^2 = k_2(x_i \cos(\theta_2) + y_i \sin(\theta_2) - \chi_{02}) - \omega_2 t$ . Now, the PARAMETER DETERMINATION STRATEGY can be used to determine the parameters of internal wave 2 using 4.4. With the internal wave 2's parameters known, we define

$$\begin{aligned} \widetilde{\dot{d}_{i,j_1}^x} &= \dot{d}_{i,j_1}^x - \dot{d}_{i,j_1}^y \cot(\theta) \\ &= -\frac{2C_1 A_1}{h_u} (\operatorname{sech}^2(k_1 d_{i,j}^x + v_i^1) - \operatorname{sech}^2(v_i^1)), \end{aligned}$$

as the part of the distance derivative in the  $x$ -direction that is due to the internal wave 1. Here,  $v_i^1 = k_1(x_i - \chi_{01}) - \omega_1 t$ .

Again, one can now apply the PARAMETER DETERMINATION STRATEGY using 4.4. to show the following result. Thus, one can write the repeated application of 4.4 formally in the following result.

*Proposition 4.6 (Determination of both waves):* At times  $t$  and  $t_2$  sufficiently close to  $t_{\text{cr}}$  when  $\widetilde{\dot{d}_{i,j_1}^y}(t_{\text{cr}}) = 0$  and  $t_{\text{cr}2}$  when  $\widetilde{\dot{d}_{i,j_1}^x}(t_{\text{cr}2}) = 0$ , respectively, if  $\|d_{i,j_1}(t)\|, \|d_{i,j_1}(t_2)\|$  are sufficiently small, then given noiseless knowledge of

$\widetilde{\dot{d}_{i,j_m}^y}(t), \widetilde{\dot{d}_{i,j_m}^x}(t), \widetilde{\dot{d}_{i,j_m}^y}(t_2),$  and  $\widetilde{\dot{d}_{i,j_m}^x}(t_2)$  for all  $m \in \{j_1, j_2, j_3, j_4, j_5, j_6\}$ , the PARAMETER DETERMINATION STRATEGY in Algorithm 1, can be used to determine all of the internal wave physical parameters of both internal waves.

### V. AGGREGATION OF ESTIMATES

The PARAMETER DETERMINATION STRATEGY can be executed at various instants of times, as described in the previous section. Each execution gives rise to an estimate of the parameters. Therefore, a natural question is whether drogues could aggregate their individual estimates, as well as potentially use multiple sets of data from many different waves, to improve estimates of the parameters. Since the parameters are solutions to (implicit) nonlinear equations, even with the assumption of Gaussian measurements, the resulting distributions of the parameters are, in general, non-Gaussian and only implicitly defined. Therefore, we employ a Mixture Distribution approach to represent each parameter's distribution as a truncated Taylor series expansion. For concreteness and simplicity, we define the procedure for aggregating estimates of one parameter, the wavenumber  $k$ .

Using (4) and the implicit function theorem, it can be shown that in a neighborhood around true measurements, noisy measurements produce a unique estimate. Since this function is only implicitly defined, one can resort to calculating successive terms of its Taylor series expansion. In this way, we estimate the implicit distribution of the wavenumber as a function of the measurement error.

Letting  $D = (d_{i,j_1}, d_{i,j_2}, d_{i,j_3}, d_{i,j_4}, \dot{d}_{i,j_1}, \dot{d}_{i,j_2}, \dot{d}_{i,j_3}, \dot{d}_{i,j_4})$ , we substitute the implicit function  $k(D)$  into (4) and differentiate with respect to  $D$ , which yields  $\frac{\partial k(D)}{\partial D} = \frac{-\frac{\partial f(k,D)}{\partial D}}{\frac{\partial f(k,D)}{\partial k}}$ .

Therefore, for a set of noisy measurements  $\hat{D}$ , we have the following distribution  $k_D(\hat{D}) = k + \frac{\partial k(D)}{\partial D}(\hat{D} - D) + \mathcal{O}((\hat{D} - D)^2)$ . Repeated differentiation of (4) can produce higher-order terms in this Taylor series expansion. For simplicity, we consider only the first-order term. Then, our approximation of this distribution is  $\hat{k}_D(\hat{D}) = k + \frac{\partial k(D)}{\partial D}(\hat{D} - D) = k + \hat{e}_D(\hat{D})$ . By assumption,  $\hat{D} - D$  is a zero-mean, Gaussian random variable with a diagonal covariance matrix, making our approximation of the parameter's distribution a sum of Gaussian distributions. Given parameter estimates  $\{k_\ell^{\text{est}} \mid \ell \in \mathbb{Z}_{\geq 1}\}$  and measurements  $\{\hat{D}_\ell \mid \ell \in \mathbb{Z}_{\geq 1}\}$ , we define the following aggregation scheme:

$$\begin{aligned} (k_{\ell+1}^{\text{agg}}, \text{Var}[k_{\ell+1}^{\text{agg}}]) &= \text{OptAgg} \left( k_\ell^{\text{agg}}, \text{Var}[k_\ell^{\text{agg}}], \right. \\ &\left. k_{\ell+1}^{\text{est}} - \text{E}[\hat{e}_{D_{\ell+1}}(\hat{D}_{\ell+1})], \text{Var}[\hat{e}_{D_{\ell+1}}(\hat{D}_{\ell+1})] \right), \end{aligned} \quad (5)$$

where  $k_1^{\text{agg}} = k_1^{\text{est}} - \text{E}[\hat{e}_D(\hat{D}_1)]$  and  $\text{Var}[k_1^{\text{agg}}] = \text{Var}[\hat{e}_D(\hat{D}_1)]$ . The following result is now a consequence of [15, Prop. 5.6].

*Proposition 5.1 (Wavenumber aggregation):* Given a sequence of noisy measurements  $\{\hat{D}_\ell \mid \ell \in \mathbb{Z}_{\geq 1}\}$ , assume there exist  $\epsilon_E \geq 0$  and  $\epsilon_V \geq 0$  such that the following bounds hold uniformly for all  $\ell \in \mathbb{Z}_{\geq 1}$

$$|\text{E}[k_{D_\ell}(\hat{D}_\ell) - \hat{e}_{D_\ell}(\hat{D}_\ell)] - k| \leq \epsilon_E \quad \text{Var}[k_{D_\ell}(\hat{D}_\ell)] \leq \epsilon_V.$$

Then, with the estimates  $\{k_\ell^{\text{est}} \mid \ell \in \mathbb{Z}_{\geq 1}\}$  generated by the PARAMETER DETERMINATION STRATEGY using  $\{\hat{D}_\ell\}$ , the iterates  $\{k_\ell^{\text{agg}} \mid \ell \in \mathbb{Z}_{\geq 1}\}$  of the aggregation scheme defined in (5) satisfy the following:

$$\lim_{\ell \rightarrow \infty} \Pr[|k_\ell^{\text{agg}} - k| \leq \epsilon_E + \epsilon] = 1 \quad \forall \epsilon > 0.$$

Figure 3 depicts the aggregation method discussed above. It plots the relative error in estimates of the wavenumber, both for individual estimates and the aggregated estimate. Note that the aggregated estimate converges to a relative error significantly smaller than the individual estimates.

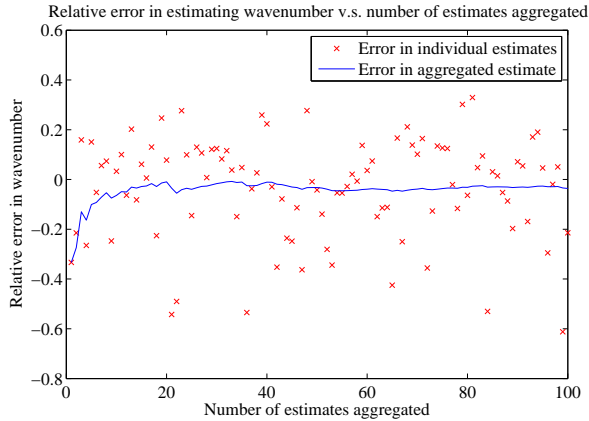


Fig. 3. This figure plots the relative error in estimates of the wavenumber, both for individual estimates and the aggregated estimate. Note that the aggregated estimate converges to a relative error significantly smaller than the individual estimates. The true value for the wavenumber was  $.014 \frac{1}{\text{m}}$  and the drogues were located at 0, 1, 2, and 10 meters from the origin. The relative error in inter-drogue measurements is .05.

## VI. CONCLUSIONS

We have considered the problem of estimating the physical parameters of a horizontally-propagating nonlinear internal wave. Because of the lack of absolute position information, a group of underwater drogues subject to the flow induced by the internal wave only have access to relative measurements (inter-drogue distances and distance derivatives) with respect to each other. We have extended the PARAMETER DETERMINATION STRATEGY proposed in our previous work [16], which is capable of determining the wave parameters given noiseless measurements in continuous time, in three ways. First, the extended method is applicable to instants of time when the inter-drogue distance derivative is sufficiently close to zero, making it implementable with sufficiently fast sampling. Second, we augmented the algorithm in a way that made it extendable to the case where there are two different nonlinear internal waves present simultaneously. Third, we developed a method to aggregate individual parameter estimates obtained with this extended method. Simulations illustrate the robustness to noisy measurements, the effect of initial drogue locations on parameter sensitivity, and the benefit of aggregating estimates in reducing the estimated parameter error. Ideas for future work include the development of analytic results regarding the algorithm robustness, the explicit characterization of the minimum sampling rate,

the extension to scenarios with three or more internal waves, and the consideration of second-order drogue dynamics.

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