

# Event-Triggered Control with Bounded Data Rate

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**Abstract**—This paper addresses the problem of asymptotic stabilization for linear time-invariant (LTI) systems using event-triggered control under finite data rate communication - both in the sense of finite precision data at each transmission and finite average data rate. Given a prescribed rate of convergence for asymptotic stability, we introduce an event-triggered control implementation that opportunistically determines the transmission instants and the finite precision data to be transmitted at each transmission. We show that our design exponentially stabilizes the origin while guaranteeing a positive lower bound on the inter-transmission times, ensuring that the number of bits transmitted at each transmission is upper bounded, and allowing for the possibility of transmitting fewer bits at any given time if more bits than prescribed were transmitted on a previous transmission. In our technical approach, we consider both the case of instantaneous and non-instantaneous transmissions. Several simulations illustrate the results.

## I. INTRODUCTION

In networked control systems (NCS), feedback information is communicated over communication channels with low, time varying and possibly unreliable channel capacity. Despite the many significant advances in the last decade, control over networks with communication constraints is still a challenging problem. Two particular themes of research that seek to address this problem have received wide interest - information theoretic approach to control under data rate constraints and event-triggered control. Although the two themes share common motivations, they address different aspects of the problem. We believe that combining the two themes into an integrated approach provides a more complete solution to the problem of control under communication. This paper is a contribution towards this end.

*Literature Review:* One of the earliest data rate results appeared in [1]–[3] which employed the idea of countering the information generated (the growth in the uncertainty of the system state) with a sufficiently high data rate of the encoded feedback. This approach was very successful in providing tight necessary and sufficient conditions on the bit rate of the encoded feedback for asymptotic stabilization in the discrete time setting. Subsequently similar ideas were used to provide data rate theorems also for cases such as time-varying feedback channels [4] and Markov feedback channels [5]. In the continuous time setting the problem has been mainly studied under the assumption of periodic sampling or aperiodic sampling with known upper and lower bounds [6], [7] (single input systems), [8] (nonlinear feedforward systems) and [9] (switched linear systems). More comprehensive accounts of this literature may be found in [10],

[11]. Focusing on the continuous-time setting, it is not known if and how a best sampling period may be designed or if state based aperiodic sampling as in event-triggered control provides any advantages. Additionally, the results in this theme do not provide (with the exception of [9]) convergence rates. On the other hand, the event-triggered control literature (see [12] for a fairly comprehensive list of references) is focused on goal-driven state-based aperiodic sampling to minimize the number of transmissions for efficient use of the communication resources, while largely ignoring the quantization aspect. Among the exceptions are [13], which considers static quantization, and [14]–[17], which consider dynamic quantization. All these works guarantee a positive lower bound on the inter-transmission times, as is typical in the event-triggered control literature, and a sufficient condition on the number of bits per transmission. While such guarantees are useful, nothing is known about either necessary or sufficient conditions on the required number of bits averaged over a finite or infinite time horizon. In fact, this has been a shortcoming of the event-triggered control literature on the whole.

*Statement of Contributions:* We address the problem of asymptotic stabilization of LTI systems with a prescribed rate of convergence using event-triggered control and under finite data rate communication. Our main contributions pertain to the design of event-triggered controllers that guarantee exponential convergence with a prescribed rate by adjusting the data rate in accordance with the state information. We consider increasingly realistic scenarios, ranging from instantaneous transmissions with arbitrary, but finite data rate, through instantaneous transmissions with uniformly bounded data rate, to finally non-instantaneous transmissions with bounded average data rate. In all cases, our design guarantees the existence of a positive lower bound on inter-transmission times and in the latter two cases, ensures that the number of bits transmitted at each transmission is upper bounded. From an event-triggered control perspective, our key contribution is that we adopt the information-theoretic approach to quantization, coding and triggering. This allows us to characterize sufficient bit rates averaged over time. It also allows the capability to transmit fewer bits if more bits than prescribed were transmitted on a previous transmission, a feature which is useful in NCS with time-varying channel capacity. From an information-theoretic perspective, our contribution is that we exploit state-based opportunistic sampling strategies to guarantee a specified convergence rate. We note that we do not consider overhead bits associated with addresses, error correction, or encryption. Proofs are omitted due to space limitations and will appear elsewhere.

*Notation:* We let  $\mathbb{R}$ ,  $\mathbb{R}_{\geq 0}$ ,  $\mathbb{N}$ , and  $\mathbb{N}_0$  denote the set of real, nonnegative real, positive integer, and nonnegative integer numbers, respectively. For a symmetric matrix  $P \in \mathbb{R}^{n \times n}$ , we let  $\lambda_m(P)$  and  $\lambda_M(P)$  denote its smallest and largest eigenvalues, respectively. We denote by  $\|\cdot\|_2$  and  $\|\cdot\|_\infty$  the Euclidean and infinity norm of a vector, respectively, or the corresponding induced norm of a matrix. For  $A \in \mathbb{R}^{n \times n}$ , note that  $\|e^{A\tau}\|_2 \leq e^{\|A\|_2\tau}$ . Finally, for  $f : \mathbb{R} \mapsto \mathbb{R}^n$  and  $t \in \mathbb{R}$ , we let  $f(t^-)$  denote the limit from the left,  $\lim_{s \uparrow t} f(s)$ .

## II. PROBLEM STATEMENT

Consider a linear time-invariant control system,

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$

where  $x \in \mathbb{R}^n$  is the plant state and  $u \in \mathbb{R}^m$  is the control input, while  $A \in \mathbb{R}^n \times \mathbb{R}^n$  and  $B \in \mathbb{R}^n \times \mathbb{R}^m$  are the system matrices. We assume that the pair  $(A, B)$  is stabilizable, i.e., there exists a control gain  $K \in \mathbb{R}^m \times \mathbb{R}^n$  such that the matrix  $\bar{A} = A + BK$  is Hurwitz. Under these assumptions,  $u(t) = Kx(t)$  renders the origin globally exponentially stable.

*Description of the NCS:* We assume that the plant is equipped with a sensor and an actuator that are not co-located with each other. Further, we assume that the sensor can measure the state exactly, and that the actuator can exert the input to the plant with infinite precision. However, the sensor can transmit state information to the controller at the actuator only at discrete time instants (of its choice) and using only a finite number of bits. In this sense, we refer to the sensor as the encoder and the actuator as the decoder. We let  $\{t_k\}_{k \in \mathbb{N}}$  be the sequence of *transmission (or encoding) times* at which the sensor samples the plant state, encodes it, and transmits it. We denote by  $np_k$  the number of bits used to encode the plant state at the transmission time  $t_k$ . The process of encoding, transmission by the sensor, reception of a complete packet of encoded data at the controller, and decoding may take non-zero time. Thus, we let  $\{r_k\}_{k \in \mathbb{N}}$  be the sequence of *reception (or update) times* at which the decoder receives a complete packet of data, decodes it, and updates the controller state. Therefore,  $r_k \geq t_k$ . The  $k^{\text{th}}$  communication time  $\Delta_k \triangleq r_k - t_k$  is then a function of  $t_k$  and the packet size (of  $np_k$  bits) represented by  $p_k$ ,

$$\Delta_k = r_k - t_k \triangleq \Delta(t_k, p_k).$$

We use the term instantaneous communication to refer to the case  $\Delta \equiv 0$ . For simplicity, we assume that the encoder and the decoder have synchronized clocks and that they synchronously update their states at update times  $\{r_k\}_{k \in \mathbb{N}}$ . This assumption is justified in situations where the function  $t \mapsto \Delta(t, p)$  is independent of  $t$  or where the encoder and decoder send short synchronization signals to indicate the start of encoding and the end of decoding, respectively.

*Coding Scheme:* We use dynamic quantization as the basis for finite-bit transmissions from the encoder to the decoder. In dynamic quantization, there are two distinct phases: the zoom out stage, during which no control is applied while the quantization domain is expanded until it captures the system state at time  $r_0 = t_0 \in \mathbb{R}_{\geq 0}$ ; and the zoom in stage,

during which the encoded feedback is used to asymptotically stabilize the system. A detailed description of the zoom out stage can be found in the literature, e.g., [18]. In this paper, we focus exclusively on the zoom-in stage, i.e., for  $t \geq t_0$ . For the task of asymptotic stabilization, we use a hybrid dynamic controller. We assume that both the encoder and the decoder have perfect knowledge of the plant system matrices. The state of the encoder/decoder is composed of the controller state  $\hat{x} \in \mathbb{R}^n$  and an upper bound  $d_e \in \mathbb{R}_{\geq 0}$  on the encoding error  $x_e \triangleq x - \hat{x}$ . Thus, the actual input to the plant is given by  $u(t) = K\hat{x}(t)$ . During inter-update times, the state of the dynamic controller evolves as

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) = \bar{A}\hat{x}(t), \quad t \in [r_k, r_{k+1}). \quad (2a)$$

Let us denote the encoding and decoding functions for the  $k^{\text{th}}$  iteration by  $q_{E,k} : \mathbb{R}^n \times \mathbb{R}^n \mapsto G_k$  and  $q_{D,k} : G_k \times \mathbb{R}^n \mapsto \mathbb{R}^n$ , respectively, where  $G_k$  is a finite set of  $2^{p_k}$  symbols. At  $t_k$ , the encoder encodes the plant state as  $z_{E,k} \triangleq q_{E,k}(x(t_k), \hat{x}(t_k^-))$ , where  $\hat{x}(t_k^-)$  is the controller state just prior to the encoding time  $t_k$ , and sends it to the controller. This signal is decoded as  $z_{D,k} \triangleq q_{D,k}(z_{E,k}, \hat{x}(t_k^-))$  by the decoder at time  $r_k$ . Then at the update time  $r_k$ , the sensor and the controller update  $\hat{x}$  using the jump map,

$$\begin{aligned} \hat{x}(r_k) &= e^{\bar{A}\Delta_k} \hat{x}(t_k^-) + e^{A\Delta_k} (z_{D,k} - \hat{x}(t_k^-)) \\ &\triangleq q_k(x(t_k), \hat{x}(t_k^-)). \end{aligned} \quad (2b)$$

where the notation  $q_k : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$  represents the quantization that occurs as a result of the finite-bit coding. We allow the quantization domain, the number of bits and the resulting quantizer,  $q_k$ , at each transmission instant  $t_k \in \mathbb{R}_{\geq 0}$  to be variable. Note that the evaluation of the map  $q_k$  is inherently from the encoder's perspective because it depends on the plant state  $x(t_k)$ , which is unknown to the decoder. Also, while the encoder could store  $\hat{x}(t_k^-)$ , the decoder has to infer its value if  $\Delta_k > 0$ . We detail the specifics of the decoder's procedure to implement (2b) when communication is not instantaneous later.

The evolution of the plant state  $x$  and the encoding error  $x_e$  on the time interval  $[r_k, r_{k+1})$  can be written as

$$\dot{x}(t) = \bar{A}x(t) - BKx_e(t), \quad (3a)$$

$$\dot{x}_e(t) = Ax_e(t). \quad (3b)$$

Note that while the controller state  $\hat{x}$  is known to both the encoder and the decoder, the plant state (equivalently, the encoding error  $x_e$ ) is known only to the encoder. However, at  $t_0$ , if a bound on  $\|x_e(t_0)\|_\infty$  is available, then both the encoder and the decoder can compute a bound  $d_e(t)$  on  $\|x_e(t)\|_\infty$  for any  $t \in \mathbb{R}_{\geq 0}$ , as we explain later.

Finally, in order to formalize the control goal, we select an arbitrary symmetric positive definite matrix  $Q \in \mathbb{R}^{n \times n}$ . Because  $\bar{A}$  is Hurwitz, there exists a symmetric positive definite matrix  $P$  that satisfies the Lyapunov equation

$$P\bar{A} + \bar{A}^T P = -Q. \quad (4)$$

Consider then the associated candidate Lyapunov function  $x \mapsto V(x) = x^T P x$ . Given a desired "control performance"

$$V_d(t) = V_d(t_0) e^{-\beta(t-t_0)} \quad (5)$$

with  $\beta > 0$  (rate of convergence) a constant, the *control objective* is as follows: recursively determine the sequence of transmission times  $\{t_k\}_{k \in \mathbb{N}} \subset \mathbb{R}_{>0}$  and finite bit messages encoding  $\hat{x}(t_k)$  so that  $V(x(t)) \leq V_d(t)$  holds for all  $t \geq t_0$ , while also ensuring that the inter-transmission times  $\{t_k - t_{k-1}\}_{k \in \mathbb{N}}$  are uniformly lower bounded by a positive quantity and that the number of bits transmitted at any instant is uniformly upper bounded. We structure our solution to this problem in two stages, depending on whether communication is instantaneous (Section III) or non-instantaneous (Section IV).

### III. EVENT-TRIGGERED CONTROL WITH BOUNDED DATA RATE AND INSTANTANEOUS COMMUNICATION

Consider the system defined by (3) where the controller state evolves according to (2). Given the control goal of Section II, our task is to determine the sequences of transmission times  $\{t_k\}$  and encoded transmissions  $\{\hat{x}(t_k)\}$ .

#### A. Requirements on the Encoding Scheme

First, we specify the requirements of the encoding scheme essential for our purposes. Assume that, at  $t = t_0 \in \mathbb{R}_{\geq 0}$ , the beginning of the zoom in stage, the encoder and decoder have a common knowledge of a constant  $d_e(t_0)$  such that  $\|x_e(t_0)\|_\infty \leq d_e(t_0)$ . Given this common knowledge, the encoder and the decoder *inductively* construct a signal  $d_e(\cdot)$  such that  $\|x_e(t)\|_\infty \leq d_e(t)$  is satisfied for all  $t \geq t_0$  as follows. First, note that as a consequence of (3b), we have

$$\|x_e(t)\|_\infty \leq \|e^{A(t-t_k)}\|_\infty \|x_e(t_k)\|_\infty.$$

Now, assuming that the encoder and the decoder know  $d_e(t_k) \geq 0$  at time  $t_k$  such that  $\|x_e(t_k)\|_\infty \leq d_e(t_k)$ , then both can compute

$$d_e(t) \triangleq \|e^{A(t-t_k)}\|_\infty d_e(t_k), \quad (6a)$$

for  $t \in [t_k, t_{k+1})$ . The above discussion guarantees that  $\|x_e(t)\|_\infty \leq d_e(t)$  for  $t \in [t_k, t_{k+1})$ . Next, at time  $t_{k+1}$ , if  $np_{k+1}$  is the number of bits used to quantize and transmit information, then the encoder and the decoder update the value of  $d_e(t_{k+1})$  by the jump,

$$d_e(t_{k+1}) = \frac{1}{2^{p_{k+1}}} d_e(t_{k+1}^-). \quad (6b)$$

Assuming the quantization at time  $t_k$  is such that  $\|x_e(t_k)\|_\infty \leq d_e(t_k)$  given  $\|x_e(t_k)\|_\infty \leq d_e(t_k^-)$ , then it is straightforward to verify by induction that the so constructed signal  $d_e$  ensures  $\|x_e(t)\|_\infty \leq d_e(t)$  for all  $t \geq t_0$ .

As an example, we next specify (up to the number of bits) an encoding scheme that satisfies the above requirements. Given  $d_e(t_k)$  such that  $\|x_e(t_k)\|_\infty \leq d_e(t_k)$ , for  $k \in \mathbb{N}_0$ , the plant state satisfies

$$x(t) \in S(\hat{x}(t), d_e(t)) = \{\xi \in \mathbb{R}^n : \|\xi - \hat{x}(t)\|_\infty \leq d_e(t)\},$$

for all  $t \in [t_k, t_{k+1})$ . At time  $t_{k+1}$ , the sensor/encoder encodes the plant state and transmits using  $np_{k+1}$  bits. In this encoding scheme, the set  $S(\hat{x}(t_{k+1}^-), d_e(t_{k+1}^-))$  is divided uniformly into  $2^{np_{k+1}}$  hypercubes and  $\hat{x}(t_{k+1})$  is chosen as the centroid of the hypercube containing the plant state

$x(t_{k+1})$ . This results in  $d_e(t_{k+1})$  being updated as in (6b). Formally, we can express the quantization at time  $t_k$  as

$$q_k(x(t_k), \hat{x}(t_k^-)) \in \underset{\xi \in \mathcal{X}_k}{\operatorname{argmin}} \{\|x(t_k) - \xi\|_\infty\}, \quad (7)$$

where  $\mathcal{X}_k$  is the set of centroids of the  $2^{np_k}$  hypercubes that the set  $S(\hat{x}(t_k^-), d_e(t_k^-))$  is divided into. We assume that if  $x(t_k)$  lies on the boundary of two or more hypercubes, then the encoder and decoder choose the value of  $q_k(x(t_k), \hat{x}(t_k^-))$  according to a common deterministic rule. As a result, given  $\hat{x}(t_0)$  and  $d_e(t_0)$  at time  $t_0$ ,  $\hat{x}(t)$  and  $d_e(t)$  are known to both the encoder and the decoder at all times  $t \geq t_0$ .

In the remainder of the paper, we make no reference to this specific encoding scheme. Instead it is sufficient for us to use the properties of the encoding scheme specified by (6).

#### B. Event-Triggered Design with Arbitrary Finite Data Rate

We first address the problem under the condition of arbitrary finite data rate, without imposing an explicit uniform bound across all transmissions. Consider the Lyapunov function  $V(x) = x^T P x$ , and let

$$h_1(t) = V(x(t)) - V_d(t). \quad (8)$$

Next, let the time instants  $\{t_k\}$  be given recursively as

$$t_{k+1} = \min\{t \geq t_k : h_1(t) \geq 0, \dot{h}_1(t) \geq 0\}. \quad (9)$$

Now, we are ready to present the first result.

##### Theorem 3.1 (Control with Arbitrary Finite Data Rate):

Consider the system (1) under the feedback law  $u(t) = K\hat{x}(t)$ , where  $t \mapsto \hat{x}(t)$  evolves according to (2) with the sequence  $\{r_k\}$  identical to  $\{t_k\}$  and determined recursively by (9). Assume that the encoding scheme is such that (6) holds for all  $t \geq 0$ . Further assume that  $V(x(t_0)) \leq V_d(t_0)$  and that

$$W \triangleq \frac{\lambda_m(Q)}{\lambda_M(P)} - a\beta > 0, \quad (10)$$

where  $a > 1$  is an arbitrary constant. If the number of bits  $np_n$  transmitted at time  $t_k$  satisfies

$$p_k \geq \underline{p}_k \triangleq \left\lceil \log_2 \left( \frac{d_e(t_k^-)}{c\sqrt{V_d(t_k)}} \right) \right\rceil, \quad (11)$$

where  $c \triangleq \frac{W\sqrt{\lambda_m(P)}}{2\sqrt{n}\|PBK\|_2}$ , then the following hold:

- (i) the inter-transmission times  $\{t_{k+1} - t_k\}$  for  $k \in \mathbb{N}$  have a uniform positive lower bound,
- (ii) the origin is exponentially stable for the closed loop system with  $V(x(t)) \leq V_d(t)$  for all  $t \geq t_0$ .

The quantity  $\underline{p}_k$  in Theorem 3.1 can be interpreted as the minimum number of bits to be transmitted sufficient to ensure that, after transmission,  $\dot{h}_1(t_k) < 0$ . The recursive nature of the inequalities (11) can be leveraged to better understand the relationship across different times among the bounds on the number of bits sufficient for stability. The following result says that the upper bound on  $\underline{p}_k$  at time  $t_k$  would be smaller if more bits were transmitted in the past.

*Corollary 3.2:* (Less bits are sufficient now if more bits were transmitted before). Under the assumptions of Theorem 3.1, the following holds for any  $k \in \mathbb{N}$ ,

$$\underline{p}_{k+1} \leq \log_2 \left( e^{(\|A\|_2 + \frac{\beta}{2})(t_{k+1} - t_0)} \right) + (k+1) - \sum_{i=1}^k p_i.$$

The next result gives insight into the total number of bits sufficient for stability as a function of time.

*Corollary 3.3:* (Upper bound on the bit rate sufficient for stability). Under the assumptions of Theorem 3.1, the following holds for any  $k \in \mathbb{N}$ ,

$$\begin{aligned} n(\underline{p}_k + \sum_{i=1}^{k-1} p_i) \\ \leq n \left( \|A\|_2 + \frac{\beta}{2} \right) \log_2(e)(t_k - t_0) + n \log_2 \left( \frac{d_e(t_0)}{c\sqrt{V_d(t_0)}} \right) + n. \end{aligned}$$

### C. Event-Triggered Design with Uniform Data Rate

Note that while Theorem 3.1 guarantees that the number of bits per transmission is finite, it does not guarantee a uniform bound on it. The result is not useful if the communication channel imposes an upper bound on the number of bits per transmission. This is the problem that we next address. We start by relaxing the triggering condition so that at time instants  $t_k$ ,  $V(x(t_k)) \leq V_d(t_k)$ , i.e.  $b(t_k) \leq 1$  where

$$b(t) \triangleq \frac{V(x(t))}{V_d(t)} \quad (12)$$

First we want to estimate the time it takes  $b(t)$  to evolve from  $b(t_k) \leq 1$  to 1 given  $\epsilon(t_k)$ , where  $\epsilon(\cdot)$  is defined as

$$\epsilon(t) \triangleq \frac{d_e(t)}{c\sqrt{V_d(t)}}. \quad (13)$$

It is easy to show that the Lie derivative of  $V$  along the trajectories of the closed loop system, between discrete updates, is bounded as

$$\begin{aligned} \dot{V} \leq & -\frac{\lambda_m(Q)}{\lambda_M(P)} V(x(t)) + \\ & + W\sqrt{V(x(t))} e^{\alpha(A)(t-t_k)} \epsilon(t_k) \sqrt{V_d(t_k)} \end{aligned}$$

as a result of which, we have

$$\dot{b} = \frac{\dot{V}V_d - V\dot{V}_d}{V_d^2} = \frac{\dot{V} + \beta V}{V_d} \leq -wb + W\epsilon(t_k)e^{\theta\tau}\sqrt{b}$$

where  $w \triangleq \left( \frac{\lambda_m(Q)}{\lambda_M(P)} - \beta \right)$ ,  $\theta \triangleq \alpha(A) + (\beta/2)$  and  $\tau \triangleq (t - t_k)$ . Assuming (10) holds, both  $w > 0$  and  $W > 0$ . Then for  $b \in [0, 1]$ , our region of interest, we have

$$\dot{b} \leq -wb + W\epsilon(t_k)e^{\theta\tau}$$

Then, by the Comparison Lemma, we have that  $b(\tau + t_k) \leq \tilde{b}(\tau, b(t_k), \epsilon(t_k))$ , where

$$\tilde{b}(\tau, b_0, \epsilon_0) \triangleq e^{-w\tau}b_0 + \frac{W\epsilon_0}{w+\theta}e^{-w\tau}[e^{(w+\theta)\tau} - 1] \quad (14)$$

If we fix  $\tau$  and  $\epsilon(t_k)$ , then  $\tilde{b}(\tau)$  is an increasing function of the initial condition  $b(t_k)$ . Similarly, if we fix  $\tau$  and  $b(t_k)$  then  $\tilde{b}(\tau)$  is an increasing function of  $\epsilon(t_k)$ . Now, the following definition will be useful

$$\tilde{\Gamma}_1(b_0, \epsilon_0) \triangleq \min\{\tau \geq 0 : \tilde{b}(\tau, b_0, \epsilon_0) = 1, \dot{b} \geq 0\}, \quad (15)$$

which is a lower bound on the time it takes  $b$  to evolve to 1 starting from  $b(t_k) = b_0$  and  $\epsilon(t_k) = \epsilon_0$ . Then, for all  $b_0 \in [0, 1]$ , we have

$$\tilde{\Gamma}_1(b_0, \epsilon_0) \geq \tilde{\Gamma}_1(1, \epsilon_0) \geq \tilde{\Gamma}_1(1, 1) > 0, \quad \forall \epsilon_0 \in [0, 1],$$

which implies that if  $b(t_k) \in [0, 1]$  and  $\epsilon(t_k) \in [0, 1]$  then  $b(t) \leq 1$  for at least all  $t \in [t_k, t_k + \tilde{\Gamma}_1(1, 1)]$ . Thus, we would like our trigger to ensure that ‘the number of bits required to have the value of  $\epsilon$  smaller than or equal to 1 just after transmission’ is no more than  $n\bar{p}$ , the upper bound imposed by the channel. Therefore, we define  $\{t_k\}$  recursively as

$$t_{k+1} = \min\{t \geq t_k : h_1(t) \geq 0, \dot{h}_1(t) \geq 0 \text{ OR } \epsilon(t) \geq 2^{\bar{p}}\} \quad (16)$$

where  $\bar{p}n$  is the upper bound on the number of bits that can be transmitted per transmission. Analogous to  $\tilde{\Gamma}_1$ , we define the following function that gives us the time it takes for  $\epsilon$  to evolve to  $2^{\bar{p}}$  from an initial value  $\epsilon_0$ .

$$\Gamma_2(\epsilon_0) \triangleq \min\{\tau \geq 0 : \epsilon(\tau + t_k) = 2^{\bar{p}}, \epsilon(t_k) = \epsilon_0\} \quad (17)$$

$$= \min\{\tau \geq 0 : \|e^{A\tau}\|_{\infty} e^{(\beta/2)\tau} \epsilon_0 = 2^{\bar{p}}\}. \quad (18)$$

We are now ready to present the next result, whose proof mainly relies on showing that the functions  $\tilde{\Gamma}_1$  and  $\Gamma_2$  are uniformly lower bounded for  $b_0, \epsilon_0 \in [0, 1]$ .

*Theorem 3.4 (Control under Bounded Channel Capacity):* Consider the system (1) under the feedback law  $u(t) = K\hat{x}(t)$ , where  $t \mapsto \hat{x}(t)$  evolves according to (2) with the sequence  $\{r_k\}$  identical to  $\{t_k\}$  and determined recursively by (16). Assume that the encoding scheme is such that (6) holds for all  $t \geq 0$ . Further assume that  $V(x(t_0)) \leq V_d(t_0)$ ,  $\epsilon(t_0) \leq 2^{\bar{p}}$  and that (10) holds. Suppose that the number of bits  $p_k n$  transmitted at time  $t_k$  satisfies  $p_k \in \mathbb{N} \cap [p_k, \bar{p}]$  with  $p_k$  given by (11) and where  $\bar{p}$  is the uniform upper bound on  $p_k$  imposed by the channel. Then, the following hold:

- (i) the inter-transmission times  $\{t_{k+1} - t_k\}$  for  $k \in \mathbb{N}$  have a uniform positive lower bound,
- (ii) the origin is exponentially stable for the closed loop system with  $V(x(t)) \leq V_d(t)$  for all  $t \geq t_0$ .

## IV. EVENT-TRIGGERED CONTROL WITH BOUNDED DATA RATE AND NON-INSTANTANEOUS COMMUNICATION

Here we design event-triggered laws for deciding the transmission times and the number of bits used per transmission when communication is not instantaneous. This corresponds to the setup of Section II in its full generality.

The encoder quantizes at  $t_k$  as described earlier and sends  $np_k$  bits which are received completely by the decoder at  $r_k \geq t_k$ . However, the discrete update of  $\hat{x}$  and  $d_e$  are performed synchronously by the encoder and the decoder at time instants  $r_k$  according to Algorithms 1 and 2, respectively.

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**Algorithm 1:** Update of encoder variables

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At  $t = t_k$ , store the encoder variable  $\hat{x}(t_k)$ , encode the plant state  $x(t)$  using  $n p_k$  bits.  $\delta_k$  is the new bound on  $x_e(t_k)$ .

- 1: Set  $z_k \leftarrow \hat{x}(t_k)$
- 2:  $z_{E,k} \leftarrow q_{E,k}(x(t_k), z_k)$
- 3: Set  $\delta_k \leftarrow d_e(t_k^-)/2^{p_k}$

At  $t = r_k$ , decode the message  $z_{E,k}$  and map forward in time to obtain  $\hat{x}(r_k)$  and  $d_e(r_k)$ .

- 4:  $z_{D,k} \leftarrow q_{D,k}(z_{E,k}, z_k)$
  - 5:  $\hat{x}(r_k) \leftarrow e^{A\Delta_k} z_k + e^{A\Delta_k}(z_{D,k} - z_k)$
  - 6:  $d_e(r_k) \leftarrow \|e^{A\Delta_k}\|_\infty \delta_k$
- 

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**Algorithm 2:** Update of decoder variables

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At  $t = r_k$ , map  $\hat{x}(r_k^-)$  and  $d_e(r_k^-)$  back in time to obtain  $z_k$  and  $\delta_k$ , decode the received message  $z_{E,k}$  and map forward in time to obtain  $\hat{x}(r_k)$  and  $d_e(r_k)$ .

- 1: Set  $z_k \leftarrow e^{-\bar{A}\Delta_k} \hat{x}(r_k^-)$
  - 2: Set  $\delta_k \leftarrow \|e^{A(t_k - r_{k-1})}\|_\infty d_e(r_{k-1})/2^{p_k}$
  - 3: Execute steps 4 through 6 of the encoder algorithm
- 

Note that Step 5 in Algorithm 1 is a consequence of the fact that  $x(t) = \hat{x}(t) + x_e(t)$  and (2a) and (3b). The decoder makes all the computations at  $t = r_k$  as in Algorithm 2.

Notice that the variables  $v$ ,  $\delta$  and  $\tilde{v}$  computed by the encoder and the decoder are the same, and hence  $\hat{x}(t)$  and  $d_e(t)$  at the encoder and decoder are synchronized for all time. In fact, the decoder algorithm, between the discrete updates, evolves the coding variables as

$$d_e(t) \triangleq \|e^{A(t-t_k)}\|_\infty \delta_k, \quad \forall t \in [r_k, r_{k+1}) \quad (19a)$$

$$\delta_{k+1} = \frac{1}{2^{p_{k+1}}} d_e(t_{k+1}^-) \quad (19b)$$

As a result, given  $\hat{x}(t_0)$  and  $d_e(t_0)$ ,  $\hat{x}(t)$  and  $d_e(t)$  are known to both the encoder and the decoder at all times  $t \geq t_0$ .

Next, we make an assumption regarding the function  $\Delta$ .

- (A1)** For any given  $t$ , if  $s_1 \leq s_2$  then  $\Delta(t, s_1) \leq \Delta(t, s_2)$ . There exists  $\Delta_m > 0$  such that  $\Delta_m \leq \Delta(t, 1)$  for all  $t \geq 0$ . Given  $\bar{p} \in \mathbb{N}$ , there exists  $\Delta^* \in \mathbb{R}_{\geq 0}$  with  $\Delta^* \leq \min\{\tilde{\Gamma}_1(1, 1), \Gamma_2(1)\}$ , such that  $\Delta(t, \bar{p}) \leq \Delta^*$  for all  $t \geq 0$ .

The basic idea behind the design of event-triggering rule in this scenario is to anticipate the zero crossings of  $h_1(t)$  and  $\epsilon(t) - 2^{\bar{p}}$  functions at least  $\Delta^*$  units of time ahead. Thus, the problem reduces to the checking of the zero crossings of the functions  $\tilde{\Gamma}_1(b(t), \epsilon(t)) - \Delta^*$  and  $\Gamma_2(\epsilon(t)) - \Delta^*$ , respectively. While the latter function is precisely computable the former requires solving a transcendental equation. To overcome this, we make the following observation.

*Lemma 4.1:* Suppose that  $\Delta > 0$  be any number. For any  $b_0 \in [0, 1]$  and  $\epsilon_0 \in [0, 1]$ ,  $\tilde{\Gamma}_1(b_0, \epsilon_0) > \Delta$  if and only if  $\tilde{b}(\Delta, b_0, \epsilon_0) < 1$ . Further, the corresponding statement with the inequalities reversed and the one in which the inequalities are replaced by equality are true. ■

Since  $\epsilon$  evolves monotonously between discrete updates, the condition  $\Gamma_2(\epsilon(t)) \geq \Delta^*$  is equivalent to  $R(\Delta^*)\epsilon(t) \geq 2^{\bar{p}}$ , where  $R(\Delta) \triangleq \|e^{A\Delta}\|_\infty e^{(\beta/2)\Delta}$ . Hence, we recursively

determine the transmission time instants  $\{t_k\}$  as

$$t_{k+1} = \min\{t \geq r_k : \tilde{b}(\Delta^*, b(t), \epsilon(t)) \geq 1, \text{ OR } R(\Delta^*)\epsilon(t) \geq 2^{\bar{p}}\}. \quad (20)$$

Now we present the final result.

*Theorem 4.2 (Bounded Data and Communication Rate):* Consider the system (1) under the feedback law  $u(t) = K\hat{x}(t)$ , where  $t \mapsto \hat{x}(t)$  evolves according to (2) with  $\{t_k\}$  determined recursively by (20) where  $\Delta^*$  is as given in Assumption (A1). Let  $\{r_k\}$  be given as  $r_0 = t_0$  and  $r_k = t_k + \Delta_k$  for  $k \in \mathbb{N}$ . Assume that the encoding scheme is such that (19) holds for all  $t \geq 0$ . Further assume that  $V(x(t_0)) \leq V_d(t_0)$ ,  $\epsilon(t_0) \leq 2^{\bar{p}}$  and that (10) hold. Let  $\underline{p}_k$  be given by

$$\underline{p}_k \triangleq \left\lceil \log_2(R(\Delta^*)\epsilon(t_k)) \right\rceil. \quad (21)$$

Then, the following hold:

- (i)  $\underline{p}_1 \leq \bar{p}$ . Further for each  $k \in \mathbb{N}$ , if  $p_k \in \mathbb{N} \cap [\underline{p}_k, \bar{p}]$ , then  $\underline{p}_{k+1} \leq \bar{p}$ .
- (ii) the inter-transmission times  $\{t_{k+1} - t_k\}$  and inter-reception times  $\{r_{k+1} - r_k\}$  for  $k \in \mathbb{N}$  have a uniform positive lower bound,
- (iii) the origin is exponentially stable for the closed loop system with  $V(x(t)) \leq V_d(t)$  for all  $t \geq t_0$ .

## V. SIMULATIONS

Here, we present an example with simulations only for the scenario corresponding to Theorem 3.4, due to space constraints. Consider the system given by (1) with

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad K = [2 \quad -8]$$

The plant matrix  $A$  has eigenvalues at 2 and 3, while the control gain matrix  $K$  places the eigenvalues of the matrix  $\bar{A} = (A + BK)$  at  $-1$  and  $-2$ . The other parameters and the initial conditions have been chosen as follows.

$$Q = I_2, \quad \beta = 0.9 \frac{\lambda_m(Q)}{\lambda_M(P)}, \quad a = 1.1,$$

$$x(t_0) = [6 \quad -4]^T, \quad \hat{x}(t_0) = [5 \quad -3]^T$$

$$d_e(t_0) = 1.1 \|x(t_0) - \hat{x}(t_0)\|_\infty, \quad V_d(t_0) = 1.2V(x(t_0))$$

We present simulations for two cases,  $\bar{p} = 12$  and  $\bar{p} = 20$ , where  $n\bar{p}$  (here  $n = 2$ ) is the uniform upper bound on the number of bits per transmission that is imposed by the communication channel. Figures 1a and 1b show the evolution of  $V(x(t))$  and  $V_d(t)$  for  $\bar{p} = 12$  and  $\bar{p} = 20$  respectively. We see that in each case, the desired convergence rate is guaranteed. Note that in the simulation for  $\bar{p} = 12$ ,  $V(x(t))$  is always strictly lesser than  $V_d(t)$ . Hence, in this case  $p_k = \bar{p} = 12$  for each transmission  $k$ . For any  $\bar{p} \leq 12$  similar behavior was observed, and since in these cases only the second condition in the triggering rule (16) is ever satisfied, this results in periodic transmission instants. However, in the case of  $\bar{p} = 20$ ,  $p_k$  for each  $k$  was chosen as  $\underline{p}_k$  (see (11)). In this simulation, this quantity is strictly less than  $\bar{p}$  on all

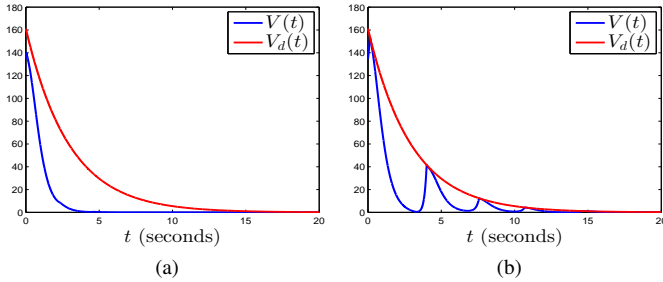


Fig. 1: The evolution of the Lyapunov function  $V(x(t))$  and  $V_d(t)$  for (a)  $\bar{p} = 12$  and (b)  $\bar{p} = 20$ .

transmissions except one (see Figure 2a). Figure 2b shows the interpolated plot of the total number of bits transmitted in each of the cases with  $\bar{p} = 12$  and  $\bar{p} = 20$ . In reality,

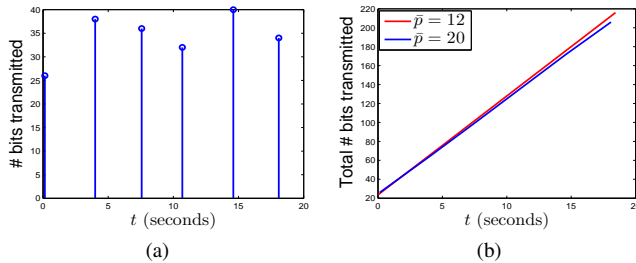


Fig. 2: (a) Number of bits transmitted on each transmission in the case of  $\bar{p} = 20$ . (b) Interpolated plot of the total number of bits transmitted for cases with  $\bar{p} = 12$  and  $\bar{p} = 20$ .

the total number of bits transmitted as a function of time is piecewise constant. However, the interpolated plots enable a more insightful comparison.

In the case of  $\bar{p} = 20$ , after having transmitted more bits initially than for  $\bar{p} = 12$ , the gap in the cumulative bit counts is not only recovered but also reversed as time progresses. From Figure 1 we see that in the case of  $\bar{p} = 12$ , the control goal is over-met at the cost of increased communication of information as seen in Figure 2b. On the other hand, with  $\bar{p} = 20$  lesser communication resources are used over time while still meeting the control goal satisfactorily.

## VI. CONCLUSIONS

We have addressed the problem of asymptotic stabilization for LTI systems with a prescribed rate of convergence using event-triggered control under finite data rate communication. Our design exponentially stabilizes the origin with a prescribed rate of convergence, guarantees a positive lower bound on inter-transmission and inter-reception communication times, and ensures that the number of bits transmitted at each transmission is upper bounded. These guarantees are valid for instantaneous transmissions with finite precision data as well as for non-instantaneous transmissions with finite average data rate. The combination of elements from event-triggered control and information theory has allowed us to (i) guarantee an arbitrarily prescribed convergence rate (something not typically ensured in the information-theoretic approach) and (ii) characterize sufficient conditions on the time-average usage of the network resources (an issue

mostly overlooked in the event-triggered control literature). Future work will be devoted to address the limitations of our approach (specifically, the conservativeness in some of the bounds, the restriction  $\epsilon(t_k) \leq 1$ , and the requirement of synchronized clocks between the encoder and the decoder to maintain a synchronized quantization domain) and the extension to stochastic time-varying communication channels, and in general, to further understanding the trade-offs between system performance and timeliness and size of transmissions.

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